



Dynamic Response of Mindlin Elastic Plate Supported by Pasternak Foundation under Uniform Partially Distributed Moving Load

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The research is financed by Covenant University

Abstract

Various transport structures, ranging from railways, roads and bridges to space vehicles and submarines, are usually subjected to moving loads which vary in both space and time. All branches of transport have experienced great advances, characterised by increasing high speed and weights of railway vehicles. Structures and media on which the railway vehicles move have, therefore, been subjected to vibration and dynamic stress more than ever before. The motivation for this paper is from the observation that most of the works available in the literature are concerned with plates for which the effects of both rotatory inertia and shear deformation are neglected. Also the plates are assumed not resting on any foundation. In this paper, the dynamic response of Mindlin plate, continuously supported by Pasternak foundation and traversed by moving load is investigated. Finite difference method is used to transform the set of coupled partial differential equations to a set of algebraic equations. The desired solutions are obtained with the aid of computer programs developed in conjunction with MATLAB. This shows that the elastic foundation, rotatory inertia and shear deformation have significant effect on the dynamic response of the plate, to the moving load. In particular, it is observed that the deflection of the plate decreases as the foundation moduli increase.

Keywords: Mindlin plate, finite difference method, dynamic response, Pasternak foundation, moving load.

1. Introduction

The dynamic analysis of an elastic system (plate) which supports moving loads is fundamental in the design of highway and railway bridges. A few studies concerning dynamic analysis of rectangular Mindlin plates on elastic foundation have been carried out. The problem of assessing the response of elastic structures neglecting the effects of Shear deformation and rotatory inertia, with or without, elastic foundation has continued to motivate a considerable number of researches [1-7]. However, such assumption does not realistically model the physical situations. In an attempt to model such physical situations in realistic manner, one has to consider the effects of Shear deformation and rotatory inertia on the response of the plate to a moving load. Gborashi [8] has investigated many cases of moving load problems. The vibration of an Euler Bernoulli beam traversed by uniform partially distributed moving mass has, also, been studied. In addition, Gbadeyan and Dada [10] studied the dynamic response of elastic plate on Pasternak type of foundation under distributed loads. The same authors extended the work by considering the dynamic response of a Mindlin elastic rectangular plate subjected to distributed moving load. Most of the publications [1-4] on moving load dynamic response of isotopic plate to the best knowledge of the authors, involved either non – Mindlin or plates that are not resting on any elastic foundation. The present paper consider the dynamic response of Mindlin elastic type of plates under the influence of a partially uniform distributed moving load and supported by a Pasternak foundation. A set of partial differential equations satisfying the Mindlin elastic rectangular plate, resting on Pasternak foundation and subjected to a partially distributed moving load was transformed into its equivalent non – dimensional form. Using the finite difference technique, a new set of linear algebraic

equations was obtained and subsequently solved in order to present the results. Numerical discussions of bending and Shear deformations are also given.

2. Problem Definition

A rectangular Mindlin plate supported by Pasternak foundation, and traversed by a partially distributed moving load is considered. U is the velocity of a load (M_L) of rectangular dimensions ϵ by μ with one of its lines of symmetry moving along $y = y_1$, the plate is L_x by L_y in dimensions and $\xi = UT + \epsilon/2$ as shown in fig. 1.

2.1 Assumptions

$W(x,y,t) = W$ = deflection of the Mindlin plate

No damping in the system

Uniform gravitational field, g .

M = constant mass moving on the plate

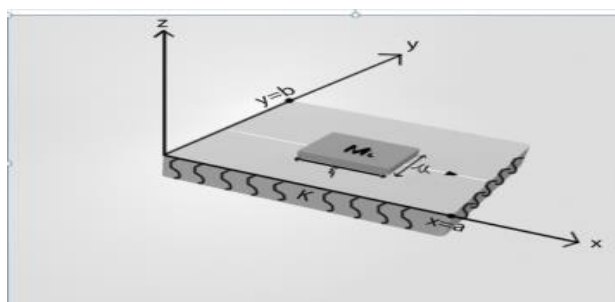


Figure 1. A moving rectangular load on a Mindlin plate supported by pasternak foundation

2.2. Initial Conditions

$$W(x, y, 0) = 0 = \frac{\partial W}{\partial T}(x, y, 0)$$

2.3. Boundary Conditions

$$W(x,y,t) = M_x(x,y,t) = \varphi_y(x,y,t) = 0,$$

for $x=0$ and $x=a$

$$W(x,y,t) = M_y(x,y,t) = \varphi_x(x,y,t) = 0,$$

for $y=0$ and $y=b$

The non – dimensional boundary conditions:

$$d_t = m_x = \varphi_{yt} = 0 \text{ (at } x=0 \text{ and } x = 1)$$

$$d_t = m_y = \varphi_{xt} = 0 \text{ (at } y=0 \text{ and } y = L_x/L_y)$$

3. Problem Solution

The set of dynamic equilibrium equations which governs behaviour of Mindlin plate supported by Pasternak foundation, and traversed by a partially distributed moving load can be written as [10, 11]:

$$Q_x - \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = \frac{\rho h^3}{12} \frac{\partial^2 \varphi_x}{\partial T^2} + \frac{\rho_L h_1^3}{12} \left[\frac{\partial^2 \varphi_x}{\partial T^2} + U \frac{\partial^2 \varphi_x}{\partial_x \partial T^2} + \frac{U}{D(\nu^2-1)} \left\{ \frac{\partial M_x}{\partial T} + U \frac{\partial M_x}{\partial x} \right\} - \frac{U\nu}{D(\nu^2-1)} \left\{ \frac{\partial M_y}{\partial T} + U \frac{\partial M_y}{\partial y} \right\} \right] B \quad (1)$$

$$Q_y - \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} = \frac{\rho h^3}{12} \frac{\partial^2 \varphi_y}{\partial T^2} + \frac{\rho_L h_1^3}{12} \left[\frac{\partial^2 \varphi_y}{\partial T^2} + U \frac{\partial^2 \varphi_y}{\partial_x \partial T^2} + \frac{U}{D(\nu^2-1)} \left\{ \frac{\partial M_x}{\partial T} + U \frac{\partial M_x}{\partial x} \right\} - \frac{U\nu}{D(\nu^2-1)} \left\{ \frac{\partial M_y}{\partial T} + U \frac{\partial M_y}{\partial y} \right\} \right] B \quad (2)$$



$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + kW + (M_f - \rho h) \frac{\partial D_T}{\partial T} + \frac{M_L}{A} \left[g + \frac{\partial D_T}{\partial T} + U \frac{\partial D_T}{\partial T} + G_1 \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} \right) + \left\{ \frac{\partial \psi_x}{\partial T} + \frac{U}{D(v^2-1)} M_x - \frac{Uv}{D(v^2-1)} M_y \right\} - \frac{U}{\alpha Gh} \left\{ \frac{\partial Q_x}{\partial T} + U \frac{\partial Q_x}{\partial x} \right\} \right] B = \rho h \frac{\partial^2 \varphi_x}{\partial T^2} \quad (3)$$

where ψ_x and ψ_y are local rotations in the x – and y – directions respectively. M_x and M_y bending moments in the x- and y- directions respectively, M_{xy} is the twisting moments, Q_x and Q_y are the traversed Shearing forces in x – and y – directions respectively, h and h_1 are thickness of the plate and load respectively, ρ and ρ_L are the densities of the plate and the load per unit volume respectively $W(x,y,T)$ is the traverse displacement of the plate at time T , $P(x,y,T)$ is the applied dynamic load (force) and the last terms in equation (1) and (2) account for inertia effects of the load in x – and y – directions respectively. It is the velocity of a load (M_L) of rectangular dimensions E by U with one of its lines of symmetry moving along $Y=Y_1$. The plate is L_x and L_y in dimensions and $\xi = UT + \frac{\epsilon}{2}$ as shown in figure1, also $B = B_x B_y$, where $B_x =$

$$\left\{ \begin{array}{l} 1 - H\left(x - \xi + \frac{\epsilon}{2}\right) \dots \dots \dots 0 < 1 < \frac{\epsilon}{2} \\ H\left(x - \xi + \frac{\epsilon}{2}\right) - H\left(x - \xi - \frac{\epsilon}{2}\right) \dots \dots \dots \frac{\epsilon}{2} < 1 < \frac{L_x}{u} \\ H\left(\xi + \frac{\epsilon}{2}\right) \dots \dots \dots \frac{L_x}{u} \leq T < (L_x + \epsilon)/U \\ 0 \dots \dots \dots (L_x + \epsilon)/U \leq T \end{array} \right.$$

$$B_y = H\left(y - y_1 + \frac{\mu}{2}\right) - H\left(y - y_1 - \frac{\mu}{2}\right)$$

$H(x)$ is the Heaviside function defined as

$$H(x) = \begin{cases} 1 & x > 0 \\ 0.5 & x = 0 \\ 0 & x < 0 \end{cases}$$

K is the foundation stiffness, G_1 is the foundation Shear modulus and M_f is the mass of the foundation. The equations for the bending moments, twisting moments and Shear force are given as follows [5]:

$$M_x = -D \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) \quad (4)$$

$$M_y = -D \left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \right) \quad (5)$$

$$M_{xy} = \left(\frac{1-\nu}{2} \right) D \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) \quad (6)$$

$$Q_x = -K^2 Gh \left(\psi_x - \frac{\partial W}{\partial x} \right) \quad (7)$$

$$Q_y = -K^2 Gh \left(\psi_y - \frac{\partial W}{\partial y} \right) \quad (8)$$

$$\frac{\partial W}{\partial T} = D_T \quad (9)$$

$$\frac{\partial W}{\partial x} = D_x \quad (10)$$

$$\frac{\partial W}{\partial y} = D_y \quad (11)$$

Where G is the modulus of rigidity of the plate, D is the flexural rigidity of the plate defined by $D = \frac{1}{12} E h^3 (1-\nu^2) = Gh^3/6(1-\nu)$ for isotopic plate K^2 is the Shear correction factor and ν is the Poisson's ratio of the plate.

3.1 Non – Dimensional Form

The dimensional forms of the first order partial differential equations version of the above system of dynamic equilibrium second order partial differential equation which governs behaviour of Mindlin



plate supported by Pasternak foundation, and traverse by a partially distributed moving load can be written as:

$$(\alpha Gh)q_x + (-\alpha ghr - \frac{\rho_L h_1^3 r^3 c^2 U^2 \alpha Gh}{12L_x D(v^2-1)} Bn) \frac{\partial M_x}{\partial x} - (\alpha Ghr) \frac{\partial M_{xy}}{\partial y} - (\frac{\rho_L h_1^3 r^3 c^2}{12L_x^2} Bn + \frac{\rho_L h_1^3 r^4 c^3 U Bn}{12L_x^2} + \frac{\rho_L h_1^3 r^3 c^2}{12L_x^2}) \frac{\partial \Psi_{xt}}{\partial t} - (\frac{Bn \rho_L h_1^3 r^3 c^2 U \alpha Gh}{12L_x D(v^2-1)}) \frac{\partial M_x}{\partial t} + (\frac{\rho_L h_1^3 r^3 c^2 U \alpha Gh Bn}{12D(v^2-1)}) \frac{\partial M_y}{\partial t} + (\frac{\rho_L h_1^3 r^3 c^2 U \alpha Gh Bn}{12D(v^2-1)}) \frac{\partial M_{xy}}{\partial x} = 0 \quad (12)$$

$$(\alpha Gh)q_y - (\alpha Ghr) \frac{\partial M_{xy}}{\partial x} + (-\alpha Ghr - \frac{\rho_L h_1^3 r^3 c^2 U^2 \alpha Gh Bn}{12D(v^2-1)}) \frac{\partial M_{xy}}{\partial y} - (\frac{r^3 h^2 c^2 \rho}{12L_x^2} + \frac{Bn \rho_L h_1^3 r^3 c^2}{12L_x^2}) \frac{\partial \Psi_{yt}}{\partial t} - (\frac{\rho_L h_1^3 r^3 c^2 U}{12L_x^2} Bn) \frac{\partial \Psi_y}{\partial y} - (\frac{\rho_L h_1^3 r^3 c^2 U \alpha Gh}{12L_x D(v^2-1)}) \frac{\partial M_y}{\partial t} + (\frac{\rho_L h_1^3 r^3 c^2 U \alpha Gh Bn}{12D(v^2-1)}) \frac{\partial M_x}{\partial t} + (\frac{\rho_L h_1^3 r^3 c^2 U^2 \alpha Gh Bn}{12D(v^2-1)}) \frac{\partial M_x}{\partial y} = 0 \quad (13)$$

$$(\frac{\alpha Gh}{L_x} - \frac{\rho hmr^2 c^2 u^2}{L_x \alpha} Bn) \frac{\partial q_x}{\partial x} + (\frac{\alpha Gh}{L_x}) \frac{\partial q_y}{\partial y} + (kh) w + (\frac{\rho h^2 r^2 c^2}{L_x^2} + \frac{\rho h^2 mr^2 c^2}{\alpha L_x^2} Bn + \frac{\rho h^2 mr^3 c^3 u}{\alpha L_x^2} Bn) \frac{\partial d_t}{\partial t} + (G_1 \frac{L_x}{h}) \frac{\partial d_x}{\partial x} + (G_1 \frac{L_x}{h}) \frac{\partial d_y}{\partial y} + \frac{\rho hmr^3 c^2}{\alpha L_x} Bn g_n + (\frac{\rho hmr^3 c^2 u}{\alpha L_x} Bn) \Psi_{xt} + (\frac{\rho h^2 mr^2 c u \alpha G L_x Bn}{\alpha D(v^2-1)}) M_x - (\frac{\rho h^2 r^3 c^2 u^2 \alpha G L_x Bn}{\alpha D(v^2-1)}) M_y - (\frac{\rho hmr^3 c^2 u}{\alpha L_x}) \frac{\partial q_x}{\partial t} = 0 \quad (14)$$

$$\frac{\partial m_x}{\partial t} = -N_1 \frac{\partial \Psi_{xt}}{\partial t} - \nu N_1 \frac{\partial \Psi_{yt}}{\partial y} \quad (15)$$

$$\frac{\partial m_y}{\partial t} = -N_1 \frac{\partial \Psi_{yt}}{\partial y} - \nu N_1 \frac{\partial \Psi_{xt}}{\partial x} \quad (16)$$

$$\frac{\partial m_{xy}}{\partial t} = -N_1 (\frac{1-\nu}{2}) (\frac{\partial \Psi_{xt}}{\partial y} + \frac{\partial \Psi_{yt}}{\partial x}) \quad (17)$$

$$\frac{\partial q_x}{\partial t} = \Gamma (\Psi_{xt} \frac{\partial d_t}{\partial x}) \quad (18)$$

$$\frac{\partial q_y}{\partial t} = \Gamma (\Psi_{xt} \frac{\partial d_t}{\partial y}) \quad (19)$$

$$d_t = \frac{\partial w}{\partial t} \quad (20)$$

$$d_x = \frac{\partial w}{\partial x} \quad (21)$$

$$d_y = \frac{\partial w}{\partial y} \quad (22)$$

The set of first order partial differential equations (12) - (22), where

$$N_1 = \frac{D}{\alpha Gh L_x^2}$$

are the simplified partial differential equations to be solved for the following eleven dependent variables M_x , M_y , M_{xy} , q_x , q_y , Ψ_{xt} , Ψ_{yt} , w , d_t , d_x and d_y . A numerical procedure, finite difference method, can be used to solve the system of equations (12) - (22) [10] Rearranging them in matrix form results in

$$R_{i,j+1} S'_{i,j+1} + P_{i+1,j+1} S'_{i+1,j+1} = -T_{ij} S'_{ij} - Y_{i+1,j} S_{i+1,j} + Z_k \quad (23)$$

$i=1, 2, 3 \dots N-1; \quad j = 1, 2, 3 \dots M-1$

Where N and M are the number of the nodal points along x and y axes respectively, Z_k is a matrix representing the right hand side of equation (12) - (22) defined by

$$Z_k = A_{i,j} S^0_{i,j} + P_{i+1,j} S^0_{i+1,j} + G_{i+1,j} S^0_{i+1,j} + D_{i+1,j+1} S^0_{i+1,j+1} + E_1 \quad (24)$$

Each term in equations (23) and (24) is an 11 x 11 matrix

4. Kirchhoff, Shear and rotating plate resting on Pasternak foundation.

In order to compare the effects of Shear deformation and rotatory inertia on the deflection of plate under a moving load supported by Pasternak foundation, the following types of plates are considered;

the Shear plate (no rotatory inertia effect), the rotatory plate (no Shear deformation effect), and Kirchhoff plate (non – Mindlin plate) [5,10]

5. Results Discussion:

The numerical calculations were carried out for a simply supported rectangular plate resting on a Pasternak foundation and subject to a moving load. Damping effect was neglected.

In figure 2, the dimensionless time history of the mid – plate deflections for the Mindlin, Shear, rotatory and Kirchhoff plate cases for $K = 100$, $G = 0.09$, $A_{rp} = 0.02$, $U_p = 1.5$ are presented. It is observed that the shear plate produces the maximum deflection for fixed values of K , G , U and A_{rp} . It is also observed that there is no clear cut difference between the deflection of non – Mindlin and rotatory plates. In other words, the effect of rotatory inertia is minimal when compared with the effect of shear deformation.

In figure 3, the deflection of the plate for different values of K and G , keeping the contact area, A_{rp} , constant, is plotted as a function of time. Evidently, it can be noticed that the response amplitude of the plate continuously supported by a subgrade is less than that of the plate not resting on any elastic subgrade (i.e. $K=0$, $G=0$). It can also be seen that as K and G increase the response amplitude decreases. Deflection profiles of the Mindlin plate for various values of the contact area A_{rp} ($A_{rp}=0.02$, 0.125 and 0.5) are shown in figures 4, 5 and 6 respectively. In figure 4, the response curves of the plate is shown for $K=0$ and with the contact area A_{rp} , as a parameter. The corresponding profiles for $K=100$ and $K=200$ are depicted in figures 5 and figure 6 respectively. It is found from these figures that as A_{rp} increases, the response maximum amplitude increases for fixed values of K and G . For various values of the foundation reaction modulus K , the deflection of the plate for the various values of the subgrade's shear modulus G (i.e $G=0$, $G=0.09$ and $G=0.9$), considered were calculated and are plotted in figures 7, 8 and 9 as function of time. Specifically in figure 7, the deflection profile of the Mindlin plate is depicted for $K=0$ and with the subgrade's shear modulus G as a parameter, The corresponding curves for $K=100$ and 200 are shown in figures 8 and 9 respectively. Clearly, from the figures, the response maximum amplitude decreases with an increase in the value of G for fixed values of K , A_{rp} and U_p .

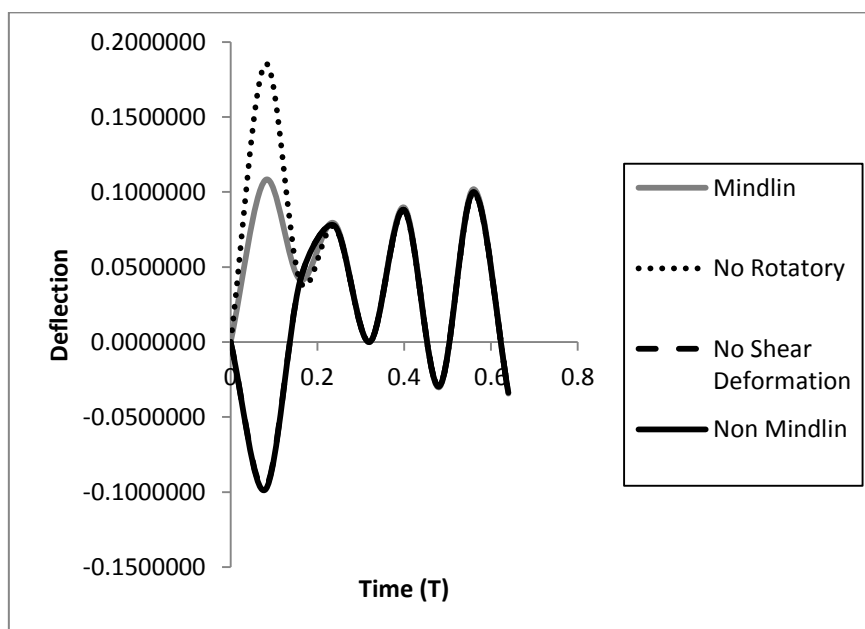


Fig. 2: Deflection of Mindlin, Non-Mindlin, Rotatory and Shear Plates for $K = 100$, $G = 0.09$, $A_{rp} = 0.02$ and $v = 1.5$ at various values of time.

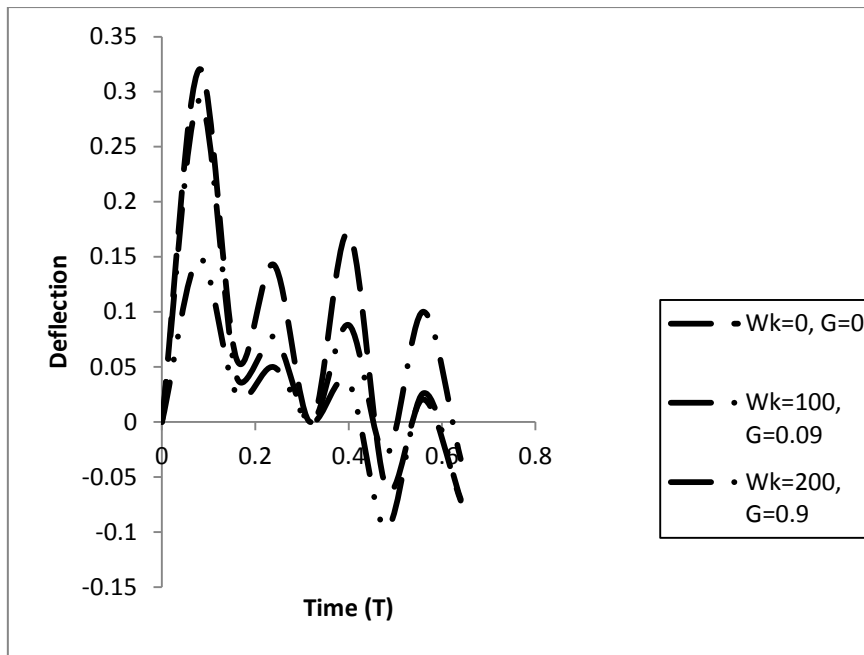


Fig.3: Deflection of the plates at $A_{rp} = 0.5$ and different values of K, G at various values of time.

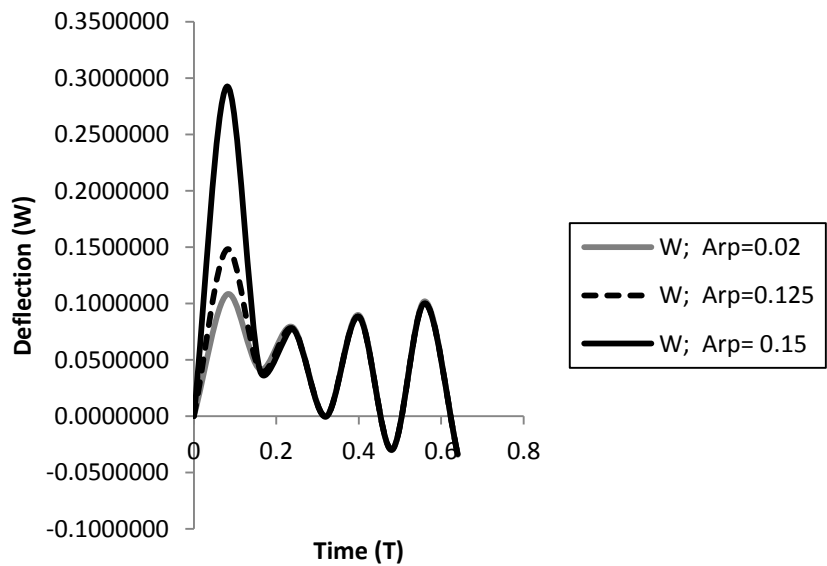


Fig. 4: Deflection of the Plates at $K = 0, G = 0.09$ for various values of A_{rp} and time.

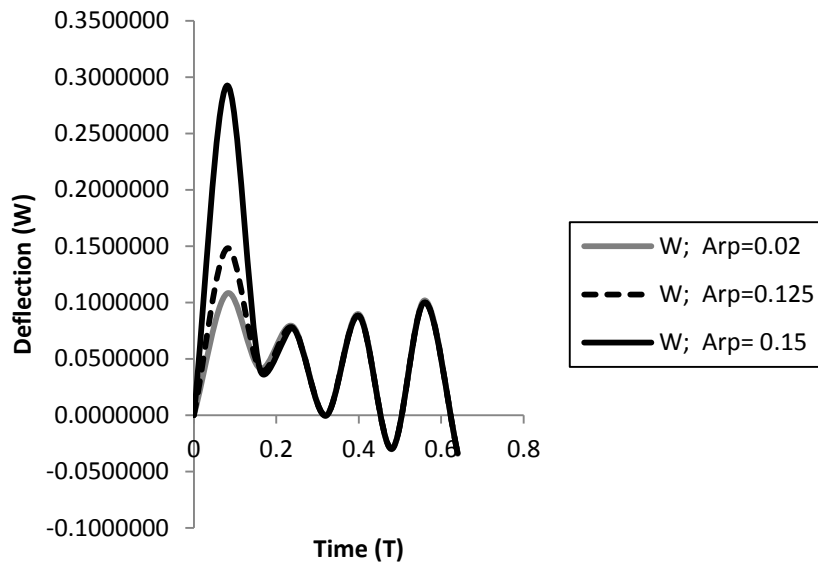


Fig. 5: Deflection of the plates at $K = 100$, $G = 0.09$ for various values of A_{rp} and time.

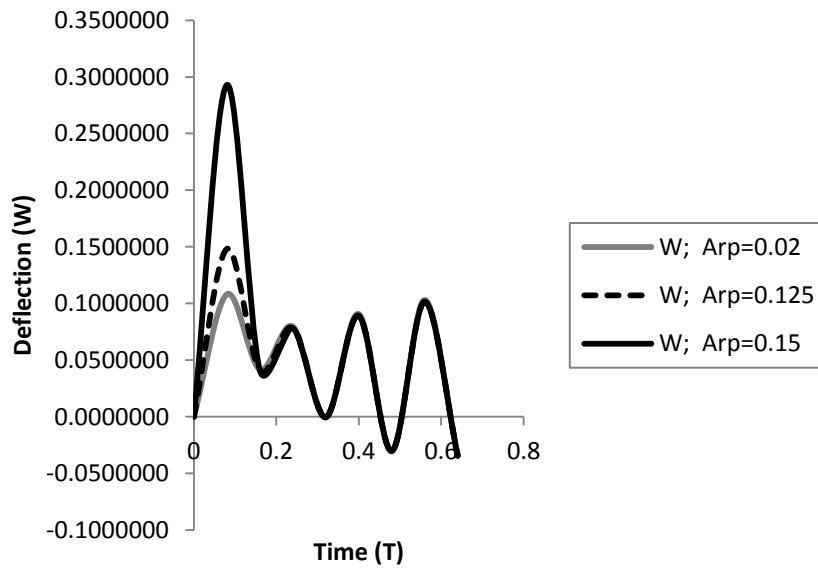


Fig. 6: Deflection of the plates at $K = 200$, $G = 0.09$ for various values of A_{rp} and time.

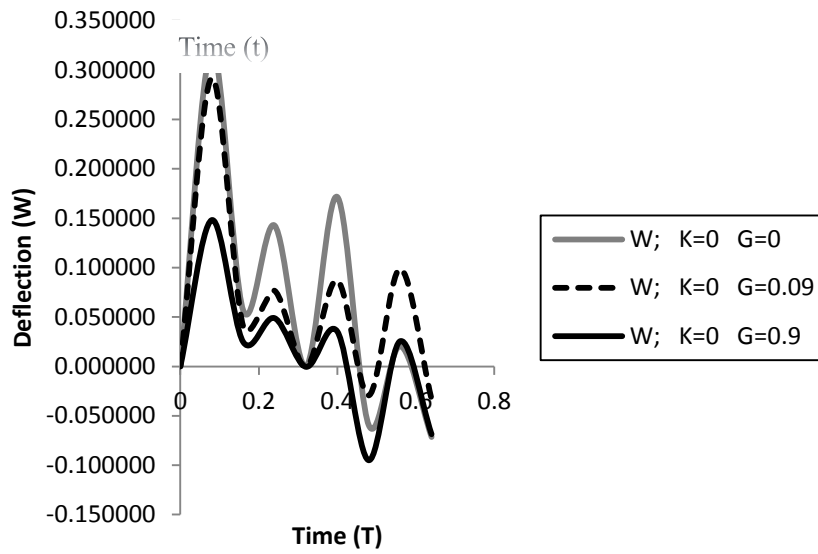


Fig. 7: Deflection of the plates at $K = 0, A_{rp} = 0.5$ for various values of G and time.

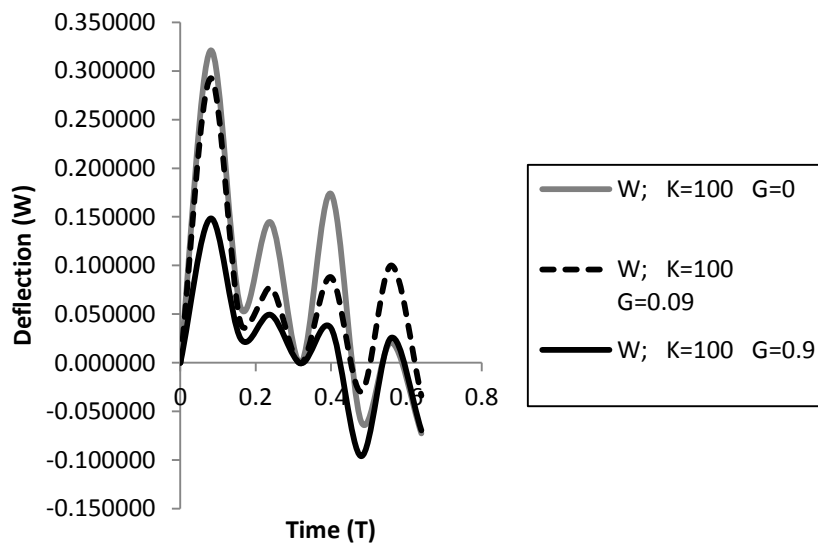


Fig. 8: Deflection of the plates at $K = 100, A_{rp} = 0.5$ for various values of G and time.

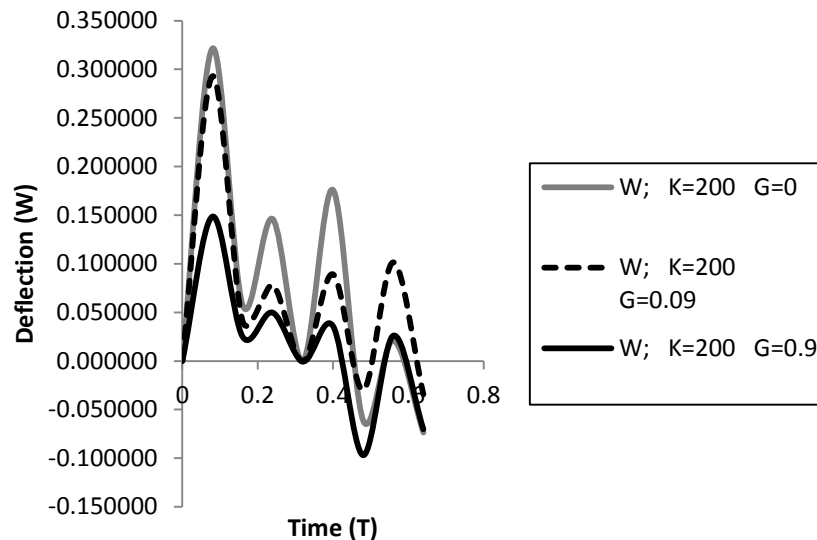


Fig. 9: Deflection of the plates at $K = 200$, $A_{rp} = 0.5$ for various values of G and time.

6. Conclusion

The dynamic behaviour of a Mindlin plate carrying a uniform partially distributed moving load, supported by a Pasternak foundation, has been analysed. The non-dimensionalized equations of motion were transformed into equivalent finite difference ones, and then solved. Results have been presented not only for the deflection but also for the velocity, bending and twisting moments, shearing force for all instants of time and at selected space nodes. Hence all the components composing the dynamic response of the system have been obtained. The formulation for the Kirchoff plate is deduced by neglecting both effects of rotatory inertia and shear deformation. A numerical example of simply supported rectangular plate is presented. It is shown that the elastic subgrade, on which the Mindlin plate rests has a significant effect on the dynamic response of the plate to a partially distributed load. The effect of rotatory inertia and shear deformation on the dynamic response of the Mindlin plate to the moving load give a more realistic results for practical application, especially when such plate is considered to rest on a foundation.

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