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ON A HYBRID NUMERICAL ALGORITHM FOR THE SOLUTIONS OF HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. A hybrid Adam-Moulton type one step numerical algorithm is proposed in this paper. The numerical algorithm is implemented in the block mode. Characterization of the method in terms of convergence and region of stability is given. Numerical experiments performed reveals the convergence of the method at very reasonable cost.

Keywords: convergence, hybrid, block method, algorithm, stability.

AMS Subject Classification: 65D30, 65L05, 65L06, 65L07.

1. Introduction

We propose a numerical algorithm for the solution of higher order ordinary differential equations(ODE), of the form:

$$\begin{cases} \ddot{y} = f\left(x, y(x), \dot{y}(x)\right), \quad x \in [a, T] \\ y(a) = \xi_0, \\ \dot{y}(a) = \xi_1, \end{cases} \tag{1}$$

where $T > 0,\ f : [a, T] \times \Re^m \times \Re^m \to \Re^m$ is a smooth function and $\xi_0, \xi_1 \in \Re^m$. The existence and uniqueness of the solutions of (1) has been discussed in [12].

The mathematical modeling of physical phenomena in science and engineering often give rise to problems of this nature.

In order to avoid increasing the dimension of the problem, a situation associated with reducing (1) to a system of first order equations, (See [11]), we intend to solve the problem directly. This latter approach is preferred due to its advantages over the former, [4, 5].

In what follows, we shall develop an implicit single step method which is a hybrid of the continuous linear multistep method proposed in [4]. This we hope to achieve by the inclusion of offstep points. The motivation for the development of this method comes from the desire to combine the advantages in single step methods with the efficiency of multistep methods, (see [7] and [8]).

The derivation and specification of the algorithm is presented in section two. In section three, we characterize the algorithm in terms of convergence and region of stability and then the implementation is discussed in section four. Lastly, numerical experiments are conducted to determine the efficiency of the algorithm

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2. Derivation of the method

Let the polynomial of the form

$$y(x) = \sum_{i=0}^{m} a_i x^i \tag{2}$$

approximate the exact solution of (1) in the partition $a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_N = T$, such that the step-size h, is defined as $h = x_{n+1} - x_n, \ \ n = 0, 1, \cdots, N-1$.

Now, consider the one step design: $x_n < x_{n+\nu_s} < x_{n+1}, \ s = 1, \dots, t$, where $\nu_s, \ \forall \ s$, are offstep points and $\nu \in \mathbb{Q}$. Then, by interpolation of (2) at points $x_{n+\nu_{t-1}}$, and $x_{n+\nu_t}$ and collocation of (1) at $x_{n+r}, r = 0, \nu_s, 1$, we obtained the system

$$y_{n+\nu_s} = \sum_{i=0}^{8} a_i x^i_{n+\nu_s}, \ s = t-1, t \tag{3a}$$

$$f_{n+r} = \sum_{i=0}^{8} i(i-1) a_i x^{i-2}_{n+r}, \ r = 0, \nu_s, 1. \tag{3b}$$

Equations (3a) and (3b) are respectively interpolation and collocation equations. The system (3) is solved for the unknown coefficients, a_i which are substituted into (2). Using a scaling factor, $z \in \mathbb{Z}$, defined as $z = \frac{x - x_{n+\nu_t}}{h}$, such that $z \leq 1$, a continuous representation of our implicit hybrid algorithm and its first derivative is obtained as follows:

$$y_{n+r} = \Phi_{n+\nu}(z) + h^2 \Big[\sum_{i=0}^{1} \beta_i(z) f_{n+i} + \sum_{s=1}^{t} \beta_{\nu_s}(z) f_{n+\nu_s} \Big] \tag{4a}$$

$$\dot{y}_{n+r} = \dot{\Phi}_{n+\nu}(z) + h \Big[\sum_{i=0}^{1} \dot{\beta}_i(z) f_{n+i} + \sum_{s=1}^{t} \dot{\beta}_{\nu_s}(z) f_{n+\nu_u} \Big] \tag{4b}$$

where $\Phi_{n+\nu}(z) = \alpha_{\nu_{(t-1)}}(z) x_{n+\nu_{(t-1)}} + \alpha_{\nu_{(t)}}(z) x_{n+\nu_{(t)}}$. Obviously, $\alpha_r(z)$ and $\beta_r(z)$, are continuous coefficients of z. By way of definition, $y_{n+j} = y(x_{n+jh})$, is the approximation of the exact solution at the point x_{n+j}, $f_{n+j} = f(x_{n+j}, y(x_{n+j}), \dot{y}(x_{n+j}))$, for arbitrary j.

In particular, choosing $\nu = \frac{s}{6}, \ s = 1, \dots, 5$, i.e $t = 5$ the procedure described above yields the discrete schemes

$$y_{n+r} = \alpha_{\frac{2}{3}}(z) y_{n+\frac{2}{3}} + \alpha_{\frac{5}{6}}(z) y_{n+\frac{5}{6}} + h^2 \Big[\sum_{i=0}^{1} \beta_i(z) f_{n+i} + \sum_{s=1}^{5} \beta_{\nu_u}(z) f_{n+\nu} \Big] \tag{5a}$$

$$\dot{y}_{n+r} = \dot{\alpha}_{\frac{2}{3}}(z) y_{n+\frac{2}{3}} + \dot{\alpha}_{\frac{5}{6}}(z) y_{n+\frac{5}{6}} + h \Big[\sum_{i=0}^{1} \dot{\beta}_i(z) f_{n+i} + \sum_{s=1}^{5} \dot{\beta}_{\nu_s}(z) f_{n+\nu_s} \Big] \tag{5b}$$

Evaluating (5a) for $z = -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}$ and $\frac{1}{6}$ yield discrete schemes whose coefficients are given in Table 1. Also, evaluating (5b) for $z = -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}$, $-\frac{1}{6}, 0$ and $\frac{1}{6}$ yield discrete derivative schemes described by their coefficients in Table 2.

Our main numerical algorithm and its first derivative are obtained as follows:

$$\begin{aligned} y_{n+1} = 2y_{n+\frac{5}{6}} - y_{n+\frac{2}{3}} + \frac{h^2}{2177280} \Big[& 4315 f_{n+1} + 53994 f_{n+\frac{5}{6}} - 2307 f_{n+\frac{2}{3}} \\ & + 7948 f_{n+\frac{1}{2}} - 4827 f_{n+\frac{2}{3}} + 1578 f_{n+\frac{1}{6}} - 221 f_n \Big] \end{aligned} \tag{6}$$

$$\begin{aligned} y'_{n+1} = 6y_{n+\frac{5}{6}} - 6y_{n+\frac{2}{3}} + \frac{h}{241920} \Big[& 12313 f_{n+1} + 55246 f_{n+\frac{5}{6}} - 18689 f_{n+\frac{2}{3}} \\ & + 19460 f_{n+\frac{1}{2}} - 10921 f_{n+\frac{2}{3}} + 3470 f_{n+\frac{1}{6}} - 479 f_n \Big] \end{aligned} \tag{7}$$

Table 1. The coefficients for the method (5a)

r	$\alpha_{\frac{2}{3}}$	$\alpha_{\frac{5}{6}}$	β_0	$\beta_{\frac{1}{6}}$	$\beta_{\frac{1}{3}}$	$\beta_{\frac{1}{2}}$	$\beta_{\frac{2}{3}}$	$\beta_{\frac{5}{6}}$	β_1
0	5	-4	$\frac{409}{217728}$	$\frac{6366}{217728}$	$\frac{11679}{217728}$	$\frac{18532}{217728}$	$\frac{21255}{217728}$	$\frac{2334}{217728}$	$\frac{-95}{217728}$
$\frac{1}{6}$	4	-3	$\frac{-11}{362880}$	$\frac{758}{362880}$	$\frac{10503}{362880}$	$\frac{19708}{362880}$	$\frac{26883}{362880}$	$\frac{2754}{362880}$	$\frac{-95}{362880}$
$\frac{1}{3}$	3	-2	$\frac{31}{725760}$	$\frac{-207}{725760}$	$\frac{2313}{725760}$	$\frac{19708}{725760}$	$\frac{35073}{725760}$	$\frac{3762}{725760}$	$\frac{-137}{725760}$
$\frac{1}{2}$	2	-1	$\frac{31}{2177280}$	$\frac{-186}{2177280}$	$\frac{213}{2177280}$	$\frac{5426}{2177280}$	$\frac{49353}{2177280}$	$\frac{5862}{2177280}$	$\frac{-221}{2177280}$
1	-1	2	$\frac{-221}{2177280}$	$\frac{1578}{2177280}$	$\frac{-4827}{2177280}$	$\frac{7948}{2177280}$	$\frac{-2307}{2177280}$	$\frac{53994}{2177280}$	$\frac{4315}{2177280}$

Table 2. The coefficients for the method (5b)

r	$\alpha_{\frac{2}{3}}$	$\alpha_{\frac{5}{6}}$	β_0	$\beta_{\frac{1}{6}}$	$\beta_{\frac{1}{3}}$	$\beta_{\frac{1}{2}}$	$\beta_{\frac{2}{3}}$	$\beta_{\frac{5}{6}}$	β_1
0	-6	6	$\frac{-12287}{241920}$	$\frac{-58738}{241920}$	$\frac{-19697}{241920}$	$\frac{-58876}{241920}$	$\frac{-26465}{241920}$	$\frac{-6962}{241920}$	$\frac{585}{241920}$
$\frac{1}{6}$	-6	6	$\frac{1313}{725760}$	$\frac{-45990}{725760}$	$\frac{-149013}{725760}$	$\frac{-101620}{725760}$	$\frac{-119817}{725760}$	$\frac{-8262}{725760}$	$\frac{29}{725760}$
$\frac{1}{3}$	-6	6	$\frac{-413}{725760}$	$\frac{4266}{725760}$	$\frac{-55035}{725760}$	$\frac{-134132}{725760}$	$\frac{-105219}{725760}$	$\frac{-12438}{725760}$	$\frac{571}{725760}$
$\frac{1}{2}$	-6	6	$\frac{43}{241920}$	$\frac{-418}{241920}$	$\frac{2201}{241920}$	$\frac{-19708}{241920}$	$\frac{-39587}{241920}$	$\frac{-3074}{241920}$	$\frac{63}{241920}$
$\frac{2}{3}$	-6	6	$\frac{-253}{725760}$	$\frac{1962}{725760}$	$\frac{-6939}{725760}$	$\frac{15884}{725760}$	$\frac{-57123}{725760}$	$\frac{-14742}{725760}$	$\frac{731}{725760}$
$\frac{5}{6}$	-6	6	$\frac{289}{725760}$	$\frac{-2214}{725760}$	$\frac{7659}{725760}$	$\frac{-16628}{725760}$	$\frac{36855}{725760}$	$\frac{35514}{725760}$	$\frac{-995}{725760}$
1	-6	6	$\frac{-479}{241920}$	$\frac{3470}{241920}$	$\frac{-10921}{241920}$	$\frac{19460}{241920}$	$\frac{-18689}{241920}$	$\frac{55246}{241920}$	$\frac{12393}{241920}$

3. Analysis of the methods

In this section, we set out to test the convergence and determine the region of stability of the new algorithm, (6) .

3.1. **Order and error constant.** We begin by rewriting (6) as a linear difference operator of the form:

$$\mathcal{L}[y(x);h] = y(x) - \alpha_{\nu_{t-1}} y(x_n + \nu_{(t-1)}h) - \alpha_{\nu_n} y(x_n + \nu_t h) - h^2 \left[\sum_{i=0}^{1} \beta_i \ddot{y}(x_n + ih) + \right. \\ \left. + \sum_{s=1}^{t} \beta_{\nu_s} \ddot{y}(x_n + \nu_s h) \right], \tag{8}$$

where $y(x) \in C^d[a,T],\ d \in \mathbb{R}$, is an arbitrary test function. Then expand $y(x_n + rh)$ and $\ddot{y}(x_n + rh), r = 0, \nu_u, 1$ for all r respectively in Taylor series about x_n and collecting terms in powers of y gives

$$\mathcal{L}[y(x);h] = C_0 y(x) + C_1 h y^{(1)}(x) + C_2 h^2 y^{(2)}(x) + \cdots + C_p h^p y^{(p)}(x) + \\ + C_{p+1} h^{p+1} y^{(p+1)}(x) + \cdots \tag{9}$$

Where the constant coefficients $C_q, q = 0, 1, 2, \ldots$ are defined as follows:

$$\begin{aligned} &C_0 = \textstyle\sum_{i=0}^{k} \alpha_i \\ &C_1 = \textstyle\sum_{i=0}^{k} i\alpha_i \\ &\vdots \\ &C_q = \tfrac{1}{q!}\left[\textstyle\sum_{i=1}^{k} i^q\alpha_i - q(q-1)\left(\textstyle\sum_{i=1}^{k} i^{q-2}\beta_i + \textstyle\sum_{i=1}^{k} \mu_i^{(q-2)}\beta_{\mu_i}\right)\right] \end{aligned} \tag{10}$$

Definition 3.1. (1) *The difference operator $\mathcal{L}$ and the associated method is said to be of order p if $C_0 = C_1 = \cdots = C_p = 0$ and $C_{p+2} \neq 0$*

(2) *The term C_{p+2} is called the error constant and it implies that the local truncation error(l.t.e) is defined by:*

$$l.t.e = C_{p+2}h^{p+2}y^{p+2} + O(h^{p+3}) \tag{11}$$

We have established from our computation that methods (6) has order $p = 7$ and error constants $C_{p+2} = -3.1173 \times 10^{-10}$.

3.2. **Zero stability and consistency.**

Definition 3.2. *The method (6) is said to be zero stable as $h \to 0$ if its first characteristic polynomial $\rho(z)$, satisfies the condition $|z_\phi| \leq 1$. Furthermore, those roots for which $|z_\phi| = 1$ have multiplicity not exceeding 2.*

The first characteristic polynomial of the algorithm is obtained as follows;

$$\rho(z) = z - 2z^{5/6} + z^{2/3} = 0. \tag{12}$$

It can be seen that the conditions in Definition (3.2) are satisfied by (12). Therefore, we conclude that the numerical algorithm (6) is zero stable. The consistency of the method is established also, by the fact that the order $p = 7$ of the algorithm is greater than 1, that is, $\ddot{\rho}(1) = 2!\sigma(1)$ (see [10]).

3.3. **Convergence.** Since we have established the consistency and zero stability of the algorithm, we can conclude that the numerical algorithm (6) is convergent (see [9]).

3.4. **Region of absolute stability.** Absolute stability for the algorithms is determined by means of the boundary locus method. Consider the stability polynomial

$$\Pi(z, \bar{h}) = \rho(z) - \bar{h}\sigma(z) = 0 \tag{13}$$

where $\bar{h} = h^2\lambda^2$ and $\lambda = \dfrac{df}{dy}$ are assumed constant.

The polynomial (13) is obtained by applying the continuous implicit one step hybrid algorithm (6) to the scalar test problem;

$$y'' = -\lambda^2 y \tag{14}$$

Definition 3.3. *The algorithm, (6) is said to be absolutely stable if for a given $\bar{h}$ all the roots z_ϕ of (13) satisfy $|z_\phi| < 1$, $\phi = 1, 2, \ldots, (r-1)$.*

Definition 3.4. *The region $\mathcal{R}$ of the complex $\bar{h}$-plane such that the roots of the polynomial $\Pi(z, \bar{h})$ lie within the unit circle whenever $\bar{h}$ lies in the interior of the region is called the region of absolute stability.*

We established from our computation using the boundary locus method, that our algorithm, (6) has interval of absolute stability to be $(-9.87, 0)$.

3.5. **Implementation of the method.** The method (6), we use simultaneous solutions of a zero stable block formula of the form (the zero stability of the block formula has been established in [3]). :

$$h^{\gamma}\mathbf{A}Y_m^{(n)} = h^{\gamma}\mathbf{B}y_m^{(n)} + h^{\mu-\gamma}\left[\mathbf{C}\mathbf{F}(Y_m)\right] \tag{15}$$

where $Y_m^{(n)} = (y_{n+\nu_1}, \ldots, y_{n+\nu_{(t-1)}}, y_{n+\nu_{(t)}}, y_{n+1}, \dot{y}_{n+\nu_1}, \ldots, \dot{y}_{n+\nu_{(t-1)}}, \dot{y}_{n+\nu_{(t)}}, \dot{y}_{n+1})^T$;
$y_m^{(n)} = (y_{n-\nu_1}, \ldots, y_{n-\nu_{(t-1)}}, y_{n-\nu_{(t)}}, y_n, \dot{y}_{n-\nu_1}, \ldots, \dot{y}_{n-\nu_{(t-1)}}, \dot{y}_{n-\nu_{(t)}}, \dot{y}_n)^T$;
$\mathbf{F}(Y_m) = (f_n, f_{n+\nu_1}, \ldots, f_{n+\nu_{(t-1)}}, f_{n+\nu_{(t)}}, f_{n+1})^T$; n represents the order of the derivative of (6); $\gamma = n$ is the power of h; μ is the order of (1); $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are constant coefficient matrices obtained as follows: $\mathbf{A}$ is 12×12 identity matrix,

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

and

$$\mathbf{C} = \begin{bmatrix}
\frac{28549}{4354560} & \frac{275}{20736} & -\frac{5717}{483840} & \frac{10621}{1088640} & -\frac{7703}{1451520} & \frac{403}{241920} & \frac{-199}{870912} \\
\frac{1027}{68040} & \frac{97}{1890} & -\frac{2}{81} & \frac{197}{8505} & \frac{97}{7560} & \frac{23}{5670} & -\frac{19}{34120} \\
\frac{253}{10752} & \frac{165}{1792} & -\frac{267}{17920} & \frac{5}{128} & -\frac{363}{17920} & \frac{57}{8960} & -\frac{47}{53760} \\
\frac{272}{8505} & \frac{376}{2835} & -\frac{2}{945} & \frac{656}{8505} & -\frac{2}{81} & \frac{8}{945} & -\frac{2}{1701} \\
\frac{35225}{870912} & \frac{8375}{48384} & \frac{3125}{290304} & \frac{25625}{217728} & -\frac{625}{96768} & \frac{275}{20736} & -\frac{1375}{870912} \\
\frac{41}{840} & \frac{3}{14} & \frac{3}{140} & \frac{17}{105} & \frac{3}{280} & \frac{3}{70} & 0 \\
\frac{19087}{362880} & \frac{2703}{15120} & -\frac{15487}{120960} & \frac{293}{2835} & -\frac{6737}{120960} & \frac{263}{15120} & -\frac{863}{362880} \\
\frac{1139}{22680} & \frac{47}{189} & \frac{11}{7560} & \frac{166}{2835} & -\frac{269}{7560} & \frac{11}{945} & -\frac{37}{22680} \\
\frac{137}{2688} & \frac{27}{112} & \frac{387}{4480} & \frac{17}{105} & -\frac{243}{4480} & \frac{9}{560} & -\frac{29}{13440} \\
\frac{143}{2835} & \frac{232}{945} & \frac{64}{945} & \frac{752}{2835} & \frac{29}{945} & \frac{8}{945} & -\frac{4}{2835} \\
\frac{3715}{7257} & \frac{725}{3024} & \frac{2125}{24192} & \frac{125}{567} & \frac{3875}{24192} & \frac{235}{3024} & -\frac{275}{72576} \\
\frac{41}{840} & \frac{9}{25} & \frac{9}{280} & \frac{34}{105} & \frac{9}{280} & \frac{9}{35} & \frac{41}{840}
\end{bmatrix}$$

The procedure is a block by block procedure where starting values for subsequent blocks are obtained from the previous block for the implementation of the method over the subintervals: $[x_0, x_1], [x_1, x_2], \cdots, [x_{N-1}, x_N]$.

4. Numerical experiment

The derived method is experimented on some sample problem to illustrate its performance. Our results are compared with previous results obtained from other authors.

Problem 4.1. $y'' - x(y')^2 = 0,\ y(0) = 1,\ y'(0) = \dfrac{1}{2};\ \ h = 0.01$
Theoretical solution:
$y = 1 + \dfrac{1}{2}ln\left(\dfrac{2+x}{2-x}\right)$

Problem 4.2. $y'' = \dfrac{(y')}{2y} - 2y,\ \ y\left(\dfrac{\pi}{6}\right) = \dfrac{1}{4},\ y'\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2};\ \ h = 0.01$
Theoretical Solution:
$y(x) = (sinx)^2$

Table 3. Absolute errors in the new method to errors in [1, 2] for Problem 4.1

X	Exact Result	Computed Result	Error in method (6)	Error in [1]	Error in [2] -
0.1	1.050041728	1.050041728	6.2172E-15	7.5028(-13)	4.8627(-14)
0.2	1.100335345	1.100335345	2.4425E-14	9.7410(-12)	2.1604(-13)
0.3	1.151140433	1.151140433	5.6843E-14	3.7638(-11)	5.2557(-13)
0.4	1.202732549	1.202732549	1.0347E-13	9.7765(-11)	1.0254(-12)
0.5	1.255412806	1.255412806	1.6742E-13	2.0825(-10)	1.8032(-12)
0.6	1.309519597	1.309519597	2.5091E-13	3.9604(-10)	3.0078(-12)
0.7	1.365443745	1.365443745	3.6016E-13	7.0460(-10)	4.8991(-12)
0.8	1.423648920	1.423648920	5.0493E-13	1.2095(-09)	7.9460(-12)
0.9	1.484700266	1.484700266	6.9522E-13	2.0511(-09)	1.3702(-11)
1.0	1.549306129	1.549306129	9.4836E-13	3.5066(-09)	2.1885(-11)

Table 4. Absolute errors in method (6) to errors in [1, 6] for Problem 4.2

X	Exact Result	Computed Result	Error in Method (6)	Error in [1]	Error in [6]
1.1035988	0.7971525560	0.7971525560	3.9946E-11	1.8811(-10)	2.8047(-10)
1.2035988	0.8711181669	0.8711181669	3.3286E-11	2.4539(-10)	2.7950(-10)
1.3035988	0.9302884674	0.9302884673	2.5298E-11	3.0306(-10)	2.1490(-10)
1.4035988	0.9723045242	0.9723045242	1.6302E-11	3.5819(-10)	5.4975(-11)
1.5035988	0.9954912899	0.9954912899	6.6553E-12	4.0838(-10)	1.1545(-10)
1.6035988	0.9989243811	0.9989243811	3.2564E-12	4.5128(-10)	4.4825(-10)
1.7035988	0.9824669315	0.9824669315	1.3038E-11	5.0696(-10)	1.1840(-09)
1.8035988	0.9467750474	0.9467750475	2.2300E-11	5.0696(-10)	1.1840(-09)
1.9035988	0.8932716519	0.8932716519	3.0673E-11	5.1697(-10)	1.6318(-09)
2.0035988	0.8240897564	0.8240897565	3.7823E-11	5.1381(-10)	2.0567(-09)

5. Conclusion

Hybrid numerical algorithm of a one step method of order $p = 7$ has been developed in this paper. The algorithm gives convergent solutions with very low errors and is absolutely stable. Implementation of the algorithm is by block method which apart from generating simultaneous solutions also gives derivative solutions. The experiment performed with the numerical algorithm proposed in this paper shows that it performs better than the methods proposed in earlier [1, 2] and [6].

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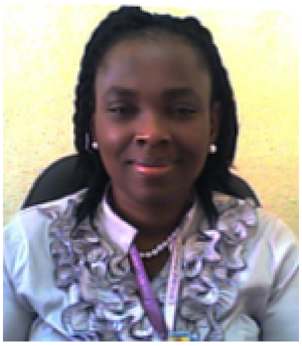

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