Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 2299-2313

http://www.ripublication.com

On the Application of the Open Jackson Queuing Network

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Abstract

In real life, waiting for service is a common phenomenon. As a system gets congested, service delay is inevitable; as the service delay increases, waiting time in the queue gets longer. In a typical hospital, the network is made up of various departments (nodes). In this study we considered the inflow and outflow of an hospital network; this is depicted in the schematic diagram. For an efficient hospital planning, a good patient flow means that patient queuing time is minimized, while poor patient flow means the patient suffer considerable queuing delays. This paper presents the results of a study carried out in a University Hospital Centre; the queuing model adopted used the Open Jackson Queuing Network to minimize the waiting times in the queues. The data collection was done for a period of two weeks, with a week interval in order to observe the system for any anomaly. For each node, the number of arrivals and departures together with the service times were recorded at an interval of five minutes. The study showed that for a good hospital planning, the more the personnel (servers) are made to focus on their assignments, the lesser the time the patients will spend on the queue and this leads to more efficient patient flow.

KeyWords: Queuing theory, Jackson Open Network, Hospital network, Waiting time, Hospital planning, Patient flow.

Introduction

Waiting for service in real life is a common phenomenon. We wait for service in hospitals, gas stations, bars and bus stops; we queue up for service in banks, schools, supermarkets, post offices, etc. Service delay is inevitable as a system gets congested. According to Kandenmir-Cavas [1], queues forms when the demand for service exceeds its supply. In hospitals, patients can wait for minutes, hours, days or months to receive medical services – waiting before, during or after being attended to. As portrayed by Obamiro [2], patients or customers, waiting in lines or queuing is annoying; this is further emphasized in [3].

Network of queues are used to model possible conflict of queuing when a set of resources are shared. It is a model in which jobs departing from one queue arrive at another; it describes a situation where the input from one queue is the output of one or more queues. Examples of where queuing network can be applied are, machine shops, communication network, hospital system, movement of memos within an organisation just to mention a few.

In [4], Moss described those internal operational factors of a hospital, such as arrival pattern of patients, drug prescription, percentage of staff at work, interaction between the pharmacy services providers amongst others, in determining the outpatient waiting time cumulatively. Skakutis and Boyle [5], explained the effect of queuing during hospital visit in relation to time spent for patients to access treatments in hospital as becoming a major source of concern to a modern society growing in technological advancement and speed. The purpose of this study is to investigate an existing hospital network that describes the hospital's general processes and then formulate a mathematical model that describes the hospital network and proffer a better departure rate that minimizes the waiting times in the network.

Literatures on queuing ([6], [7], [8]) show that queue causes inconvenience to economic cost to individuals and organisations; most organisations try to minimize the total waiting cost. It therefore implies that speed of service is increasingly becoming a very important competitive parameter [6]. According to Sitzia and Wood [7], patients' perception of health care has gained increasing attention over the years. Patients' evaluation of service quality is now affected not only by actual waiting time but also by the perceived waiting time. The amount of time patients spend waiting can significantly influence their satisfaction [8]. In [9], it was clearly stated that recent research has demonstrated that customer satisfaction is affected not just by waiting time but also by customers' expectations or attribution of the causes for the waiting. As stated in [10], any system in which arrivals place demand upon finite capacity resources may be termed as a queuing system. In the case of the hospital system, it is observed that patients arrive randomly. In this study we looked at a typical hospital system that consists of a record department, a nursing station, a consultation room, a medical laboratory department and a pharmacy department. We assumed in this study that patients who come into the hospital for services will start by going first to the record department to register and then proceed to the nursing station, from where the patients move to see the doctor at the consultation room; this procedures are followed until the patient depart from the system (hospital).

We have, in this study, modelled the hospital as a collection of systems and the Open Jackson queuing network was adopted in investigating the system. The Open Jackson queuing network with only one customer class and unlimited overall number of jobs is the simplest form of queuing network; see [11] and [12]. The model assumes that the external arrival pattern is identified by a Poisson arrival process. All systems have one or more servers with exponential service time. The service rates depend on the number of patients at the system. In all the systems, patients are served in order of arrival, first in first out (FIFO). The system in the network can be considered as an independent $M|M|m|FIFO|\infty$ queuing system.

We have three classifications of network of queues. Open network, closed network and mixed network. We shall describe these networks briefly.
A queuing network is open if patients enter from outside the network, circulate among the service centres (or nodes) for service, and depart from the network. This type of network receives patients from an external source and sends them to an external destination. Here, P_{01} and P_{02} are the weights (or conditional probabilities) in which patients goes to the 1st and 2nd server respectively, also P_{23}, P_{24}, are the weights (or conditional probabilities) in which patients goes from the 2nd server to the 3rd and 4th server respectively while P_{31} is the weight (or conditional probability) of a patients moving from the 3rd server to the 1st server. P_{30} is the weight (or conditional probability) of a patients moving from the 3rd server out of the system given the servers are server 1, 2, 3 and 4. Open queuing networks can be further divided into two categories: open feed forward queuing networks and open feedback queuing networks. In an open feed forward queuing network, a patient cannot appear in the same queue for more than one time. In an open feedback queuing network, after a patient is served by a queue, it may re-enter the same queue.

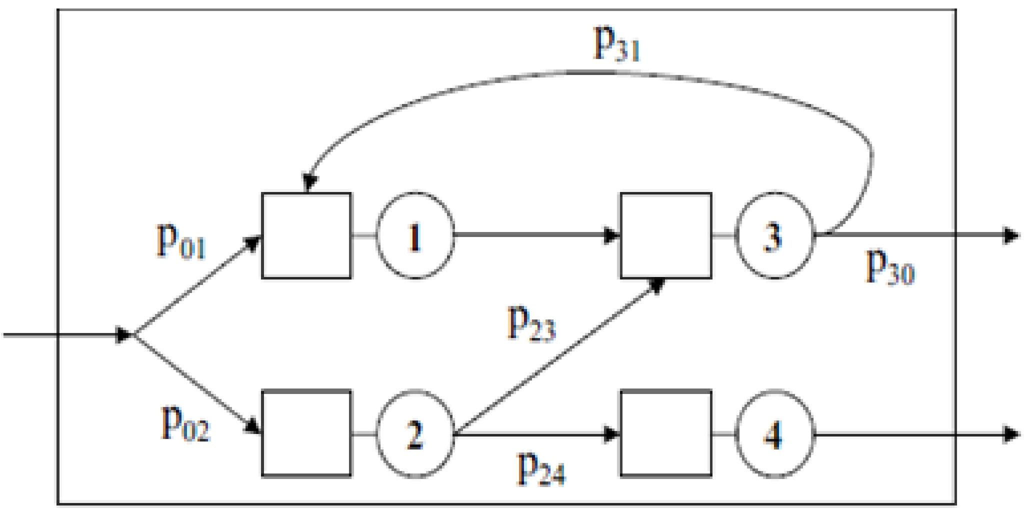

Fig. 1: An open network.

The assumptions for Jackson's Theorem are:

- The network is composed of K FCFS, single-server queues
- The arrival processes for the K queues are Poisson
- The service times of customers at j^{th} queue are exponentially distributed and independent of the arrival processes;
- Once a patient is served at queue *i*, it joins each queue *j* with probability $P_{i,j}$ or leaves the system.

However, the Implications of Jackson's Theorem are as follows:

- Once flow balance has been solved, the individual queues may be considered in isolation.
- The queues behave as if they are independent of each other (even though they really are not independent of each other) and the joint state distribution may be obtained as the continued product of the individual state distributions (product-form solution).
- The flows entering the individual queues behave as if they are Poisson, even though they may not really be Poisson in nature (i.e. if there is feedback in the network).

A queuing network is closed if a fixed number of patients circulate indefinitely among the queues. In closed models, there are fixed populations of patients in the network i.e. the number of patients in the system is an independent variable and the throughput is a dependent variable. Here, P_{01} and P_{02} are the weights(or conditional probabilities) in which patients goes to the 1st and 2nd server respectively while 1, 2, 3 are the servers in the system.
Closed network are usually used in applications due to the fact that it can be assumed that a new patients enters the system to replace an old one whenever the old one has received all the services it requires. In contrast to Jackson's open network, Gordon and Newell [13] considered a closed network of Markovian queues in which a fixed and finite number of customers say K circulate through the network, there being no external Input or departure from the network.

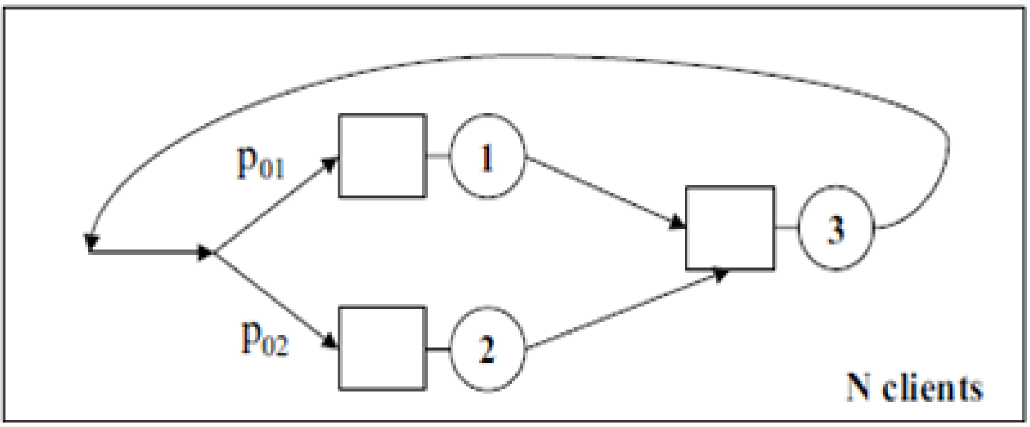

Fig. 2: A Closed Network.

A queuing network is mixed if some patients enters from outside the network and eventually leave, and if some customers always remain in the network. This network is named after the authors of the paper where the network was first described see [14]. In there work, the network consists of 4 servers and the job entering the system to server 1 or server 2 are λ_1 and λ_2 respectively. This network is a significant extension to a Jackson network allowing virtually arbitrary customer routing and service time distributions, subject to particular service disciplines.

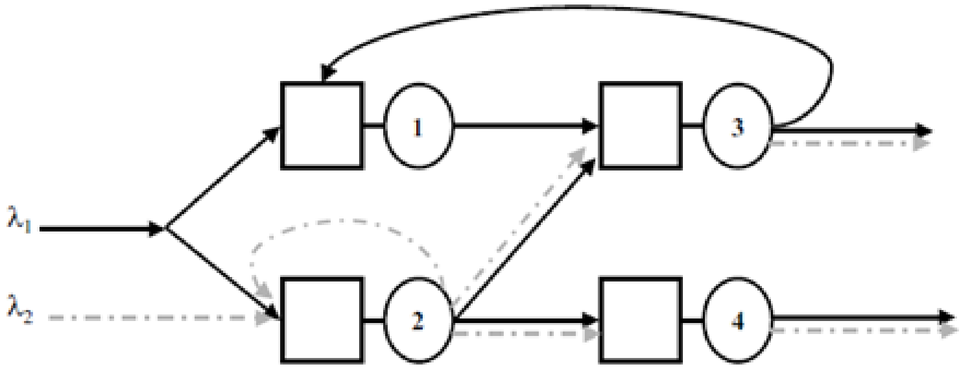

Fig. 3: A mixed network.

Mathematical Framework

Consider an open network consisting of $K\ M|M|1$ queues. Patients arrive from outside the system (hospital) joining queue i according to a Poisson process with rate λ_i^0. After service at queue i, which is exponentially distributed with parameter μ_i, the patient either leaves the hospital with probability Φ_{i0}, or goes to queue j, with probability Φ_{ij}. Clearly, $\sum_{j=0}^{k} \Phi_{ij} = 1$, since each patient must go somewhere.

In this study, we consider a hospital network that describes the Jackson's open network. The hospital network under consideration, constitutes of the various departments such as the records keeping department, the nursing station(for pre-test and treatment), the consultation department, the lab department and the pharmacy department. The schmatic diagram of the network below in figure 1.

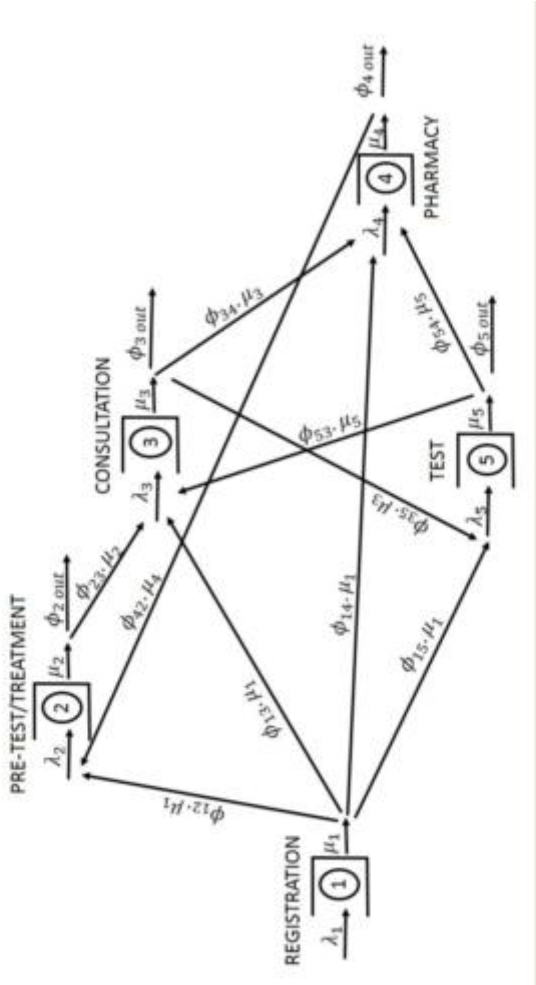

Fig. 1: A SCHEMATIC DIAGRAM OF THE HOSPITAL NETWORK

Where:
λ_i, for i = 1,…,5 is the arrival rate into the system
μ_i, for i = 1,…,5 is the departure rate out of the system
ϕ_{ij} 's are the weight of moving from server i to server j.
m_i *for* $i = 1, \dots ,5$, is the number of servers at the various node points in the system.

From the schematic diagram above, the system of linear equations which captures the probabilities at which a patient enters a particular node and also leaves a node to any other node in the system or out of the system is given below:

$$\lambda_2 = \phi_{12}\mu_1 + \phi_{42}\mu_4 \quad (1)$$

$$\lambda_3 = \phi_{23}\mu_2 + \phi_{13}\mu_1 + \phi_{33}\mu_5 \quad (2)$$

$$\lambda_4 = \phi_{34}\mu_3 + \phi_{14}\mu_1 + \phi_{54}\mu_5 \quad (3)$$

$$\lambda_5 = \phi_{35}\mu_3 + \phi_{15}\mu_1 \quad (4)$$

$$\mu_1 = \phi_{12}\mu_1 + \phi_{13}\mu_1 + \phi_{14}\mu_1 + \phi_{15}\mu_1 \tag{5}$$

$$\mu_2 = \phi_{23}\mu_2 + \phi_{2out}\mu_2 \tag{6}$$

$$\mu_3 = \phi_{3out}\mu_3 + \phi_{35}\mu_3 + \phi_{34}\mu_3 \tag{7}$$

$$\mu_4 = \phi_{42}\mu_4 + \phi_{4out}\mu_4 \tag{8}$$

$$\mu_5 = \phi_{5out}\mu_5 + \phi_{54}\mu_5 + \phi_{53}\mu_5 \tag{9}$$

Where $\phi_{12}, \phi_{13}, \phi_{14}, \phi_{15}, \phi_{23}, \phi_{2out}, \phi_{34}, \phi_{35}, \phi_{3out}, \phi_{42}, \phi_{4out}, \phi_{53}, \phi_{54}, \phi_{5out}$ are to be determined. Equations, (1 – 9) could be expressed as:

$$\lambda_2 = \mu_1\phi_{12} + 0\phi_{13} + 0\phi_{14} + 0\phi_{15} + 0\phi_{23} + 0\phi_{2out} + 0\phi_{34} + 0\phi_{35} + 0\phi_{3out} + \mu_4\phi_{42} + 0\phi_{4out} + 0\phi_{53} + 0\phi_{54} + 0\phi_{5out} \tag{10}$$

$$\lambda_3 = 0\phi_{12} + \mu_1\phi_{13} + 0\phi_{14} + 0\phi_{15} + \mu_2\phi_{23} + 0\phi_{2out} + 0\phi_{34} + 0\phi_{35} + 0\phi_{3out} + 0\phi_{42} + 0\phi_{4out} + 0\phi_{54} + \phi_{53}\mu_5 + 0\phi_{5out} \tag{11}$$

$$\lambda_4 = 0\phi_{12} + 0\phi_{13} + \mu_1\phi_{14} + 0\phi_{15} + 0\phi_{23} + 0\phi_{2out} + \mu_3\phi_{34} + 0\phi_{35} + 0\phi_{3out} + 0\phi_{42} + 0\phi_{4out} + 0\phi_{53} + \mu_5\phi_{54} + 0\phi_{5out} \tag{12}$$

$$\lambda_5 = 0\phi_{12} + 0\phi_{13} + 0\phi_{14} + \mu_1\phi_{15} + 0\phi_{23} + 0\phi_{2out} + 0\phi_{34} + \mu_3\phi_{35} + 0\phi_{3out} + 0\phi_{42} + 0\phi_{4out} + 0\phi_{53} + 0\phi_{54} + 0\phi_{5out} \tag{13}$$

$$\mu_1 = \mu_1\phi_{12} + \mu_1\phi_{13} + \mu_1\phi_{14} + \mu_1\phi_{15} + 0\phi_{23} + 0\phi_{2out} + 0\phi_{34} + 0\phi_{35} + 0\phi_{3out} + \mu_4\phi_{42} + 0\phi_{4out} + 0\phi_{53} + 0\phi_{54} + 0\phi_{5out} \tag{14}$$

$$\mu_2 = 0\phi_{12} + 0\phi_{13} + 0\phi_{14} + 0\phi_{15} + \mu_2\phi_{23} + \mu_2\phi_{2out} + 0\phi_{34} + 0\phi_{35} + 0\phi_{3out} + 0\phi_{42} + 0\phi_{4out} + 0\phi_{53} + 0\phi_{54} + 0\phi_{5out} \tag{15}$$

$$\mu_3 = 0\phi_{12} + 0\phi_{13} + 0\phi_{14} + 0\phi_{15} + 0\phi_{23} + 0\phi_{2out} + \mu_3\phi_{34} + \mu_3\phi_{35} + \mu_3\phi_{3out} + 0\phi_{42} + 0\phi_{4out} + 0\phi_{53} + 0\phi_{54} + 0\phi_{5out} \tag{16}$$

$$\mu_4 = 0\phi_{12} + 0\phi_{13} + 0\phi_{14} + 0\phi_{15} + 0\phi_{23} + 0\phi_{2out} + 0\phi_{34} + 0\phi_{35} + 0\phi_{3out} + \mu_4\phi_{42} + \mu_4\phi_{4out} + 0\phi_{53} + 0\phi_{54} + 0\phi_{5out} \tag{17}$$

$$\mu_5 = 0\phi_{12} + 0\phi_{13} + 0\phi_{14} + 0\phi_{15} + 0\phi_{23} + 0\phi_{2out} + 0\phi_{34} + 0\phi_{35} + 0\phi_{3out} + 0\phi_{42} + 0\phi_{4out} + \mu_5\phi_{53} + \mu_5\phi_{54} + \mu_5\phi_{5out} \tag{18}$$

Equations (10-18), above can be represented in the matrix form as:

$$\begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_5 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & \mu_5 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & \mu_3 & \mu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & \mu_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_5 & \mu_5 & \mu_5 \end{pmatrix} \begin{pmatrix} \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{23} \\ \phi_{2out} \\ \phi_{34} \\ \phi_{35} \\ \phi_{3out} \\ \phi_{42} \\ \phi_{4out} \\ \phi_{53} \\ \phi_{54} \\ \phi_{5out} \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} \tag{19}$$

The matrix above is obviously non-singular and hence its determinant is zero. Therefore, in order to solve the above matrix, we shall introduce a pseudo inverse matrix called the Moore–Penrose pseudo inverse.

Mathematical formulation for new departure rates

Here, our aim is to minimize the waiting time which in turn will reduce the length of queues in the system. The way a patient enters the system cannot be restricted but the departing rate out of the system can be made faster; so, we introduce a new departure rate for each node this is based on the schematic diagram of the network and mathematical model from equations (1-5), from which we have equations (20 -23) below:

$$\lambda_2 = \phi_{12}\mu_1 + \phi_{42}\mu_4 \tag{20}$$

$$\lambda_3 = \phi_{23}\mu_2 + \phi_{13}\mu_1 + \phi_{53}\mu_5 \tag{21}$$

$$\lambda_4 = \phi_{34}\mu_3 + \phi_{14}\mu_1 + \phi_{54}\mu_5 \tag{22}$$

$$\lambda_5 = \phi_{35}\mu_3 + \phi_{15}\mu_1 \tag{23}$$

Where all ϕ_i's and λ_i's are known.

Equations (20-23) can be written as:

$$\lambda_2 = \phi_{12}\mu_1 + 0\mu_2 + 0\mu_3 + \phi_{42}\mu_4 + 0\mu_5 \tag{24}$$

$$\lambda_3 = \phi_{13}\mu_1 + \phi_{23}\mu_2 + 0\mu_3 + 0\mu_4 + \phi_{53}\mu_5 \tag{25}$$

$$\lambda_4 = \phi_{14}\mu_1 + 0\mu_2 + \phi_{34}\mu_3 + 0\mu_4 + \phi_{54}\mu_5 \tag{26}$$

$$\lambda_5 = \phi_{15}\mu_1 + 0\mu_2 + \phi_{35}\mu_3 + 0\mu_4 + 0\mu_5 \tag{27}$$

Equations (24-27) are represented in a matrix form as:

$$\begin{pmatrix} \phi_{12} & 0 & 0 & \phi_{42} & 0 \\ \phi_{13} & \phi_{23} & 0 & 0 & \phi_{53} \\ \phi_{14} & 0 & \phi_{34} & 0 & \phi_{54} \\ \phi_{15} & 0 & \phi_{35} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} \tag{28}$$

The matrix in equation (28) is solved using R.

Collection of Data

The data collection was carried out at a University Health Centre for the period of two (2) weeks with a week interval in order to be able to watch the system for a while before taking another data collection. This data collection was done from Monday through Sunday, in a day; collection of data was for a total of six (6) hours at different times of the day. For each node, the number of arrivals and departures together with the service times were recorded at intervals of five (5) minutes. The arrival rate was gotten by the average number of five (5) minutes arrivals of patients into a node (λ), while the departure rate was gotten also by the average number of 5 minutes departures of patients (μ) at that particular node. Each of the nodes was observed for a period of one (1) hour daily within the recommended time frame for the collection of the data.

The notations used for the presentation of the data are Node 1, Node 2, Node 3, Node 4, and Node 5. They are defined as follows:

Node 1 represents the total data collected at the **registration point** all through the period of 14 days, which includes the number of arrivals and number of departures with an interval of 5 minutes.
Node 2 represents the total data collected at the **nursing station (i.e for pre test and treatment) point** all through the period of 14 days, which includes the the number of arrivals and number of departures with an interval of 5 minutes.
Node 3 represents the total data collected at the **consultation point** all through the period of 14 days, which includes the the number of arrivals and number of departures with an interval of 5 minutes.
Node 4 represents the total data collected at the **pharmacy point** all through the period of 14 days, which includes the the number of arrivals and number of departures with an interval of 5 minutes.
Node 5 represents the total data collected at the **test (labouratory) point** all through the period of 14 days, which includes the the number of arrivals and number of departures with an interval of 5 minutes.

Analysis of data

For each node, the mean arrival and departure rates, maximum arrival and departure rates frequency of arrival and departure were calculated using the R software and this is presented in table 1.

Table 1.

	Node 1	**Node 2**	**Node 3**	**Node 4**	**Node 5**
Mean arrival rate	1.804	1	1.833	1.696	1.788
Mean departure rate	1.731	1	1.286	1.417	1.327
Max. arrival rate	8.0	1	16	5	6
Max. departure rate	6.0	1	3	3	3
Freq. of arrival	30	18	26	14	25
Freq. of departure	32	15	21	16	39

With the information from table 1, we can calculate the arrival and departure rates for each node, expected number in the queue, expected number in the system, expected waiting time in the queue and expected waiting time in the sytem.

For node 1, the arrival rate $\lambda_1 = \dfrac{1}{mean\ number\ of\ arrival} = \dfrac{1}{1.804} = 0.55$ persons per minute.

The departure rate for node 1 is given as $\mu_1 = \dfrac{1}{mean\ number\ of\ departure} = \dfrac{1}{1.731} = 0.578$ persons per minute.

$\rho = \frac{\lambda_1}{\mu_1} = \frac{0.55}{0.578} = 0.95.$ With this information, the expected number in the queue is given as

$l_q = \frac{\rho}{m-\rho} = \frac{0.95}{1-0.95} = 19$ patients, where m represents the number of servers at the node. The expected waiting time in the queue is given as

$wq_1 = \frac{l_q}{\lambda_1} = \frac{19}{0.55} = 34.55$ minutes.

The expected number in the system is given as $l = l_q + \rho = 19 + 0.95 = 19.95 = 20$ patients.

Finally, the expected waiting time in the system for node 1 is given as $w_1 = \frac{1}{\lambda_1} = \frac{1}{0.55} = 36.4$ minutes.

These parameters calculated for node1 can be done for nodes 2 – 5.

The expected waiting time in node 1 through node 5 were calculated as $w_1 = 36.4$ minutes for node 1, $w_2 = 2$ minutes for node 2, $w_3 = 7.27$ minutes, for node 3, $w_4 = 2.41$ minutes for node 4 and $w_5 = 1.07$ minutes for node 5.

Therefore, the total expected waiting time in the system before modification is $W = w_1 + w_2 + w_3 + w_4 + w_5 = 58.25 \approx 59$.

Recall that from table 1**,** all the departure rates (μ's) and arrival rates (λ's) for each node in the system have been calculated, therefore by putting these values in equation (19), we have

$$\begin{pmatrix} 1.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.42 & 0 & 0 & 0 & 0 \\ 0 & 1.73 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1.33 & 0 & 0 \\ 0 & 0 & 1.73 & 0 & 0 & 0 & 1.29 & 0 & 0 & 0 & 0 & 0 & 1.33 & 0 \\ 0 & 0 & 0 & 1.73 & 0 & 0 & 0 & 1.29 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.73 & 1.73 & 1.73 & 1.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.29 & 1.29 & 1.29 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.42 & 1.42 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.33 & 1.33 & 1.33 \end{pmatrix} \begin{pmatrix} \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{23} \\ \phi_{2out} \\ \phi_{34} \\ \phi_{35} \\ \phi_{3out} \\ \phi_{42} \\ \phi_{4out} \\ \phi_{53} \\ \phi_{54} \\ \phi_{5out} \end{pmatrix} = \begin{pmatrix} 1.00 \\ 1.83 \\ 1.70 \\ 1.79 \\ 1.73 \\ 1.00 \\ 1.29 \\ 1.42 \\ 1.33 \end{pmatrix} \quad (29)$$

Equation (29) can be written as

$$\begin{pmatrix} \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{23} \\ \phi_{2out} \\ \phi_{34} \\ \phi_{35} \\ \phi_{3out} \\ \phi_{42} \\ \phi_{4out} \\ \phi_{53} \\ \phi_{54} \\ \phi_{5out} \end{pmatrix} = \begin{pmatrix} 1.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.42 & 0 & 0 & 0 & 0 \\ 0 & 1.73 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1.33 & 0 & 0 \\ 0 & 0 & 1.73 & 0 & 0 & 0 & 1.29 & 0 & 0 & 0 & 0 & 0 & 1.33 & 0 \\ 0 & 0 & 0 & 1.73 & 0 & 0 & 0 & 1.29 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.73 & 1.73 & 1.73 & 1.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.29 & 1.29 & 1.29 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.42 & 1.42 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.33 & 1.33 & 1.33 \end{pmatrix}^{-1} \begin{pmatrix} 1.00 \\ 1.83 \\ 1.70 \\ 1.79 \\ 1.73 \\ 1.00 \\ 1.29 \\ 1.42 \\ 1.33 \end{pmatrix} \tag{30}$$

In order to solve non-symmetry matrix in equation (30), we introduce a Moore–Penrose pseudo inverse to multiply the right hand side of equation (30) using R. Hence the results for the values of the weights are represented in equation (31) below.

$$\begin{pmatrix} \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{23} \\ \phi_{2out} \\ \phi_{34} \\ \phi_{35} \\ \phi_{3out} \\ \phi_{42} \\ \phi_{4out} \\ \phi_{53} \\ \phi_{54} \\ \phi_{5out} \end{pmatrix} = \begin{pmatrix} -0.04488675 \\ 0.19397816 \\ 0.18433544 \\ 0.66657314 \\ 0.75136801 \\ 0.24863199 \\ 0.50225539 \\ 0.63682846 \\ -0.13908385 \\ 0.75891132 \\ 0.24108868 \\ 0.55868404 \\ 0.55127085 \\ -0.10995488 \end{pmatrix} \tag{31}$$

The values in equation (31) are probabilities, since we do not have negative probabilities, we used R to normalize the parameters, that is, taking the absolute value of the parameter estimates and rescaling them so that their sum at each node sum up to 1, we have:

$$\begin{pmatrix} \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{23} \\ \phi_{2out} \\ \phi_{34} \\ \phi_{35} \\ \phi_{3out} \\ \phi_{42} \\ \phi_{4out} \\ \phi_{53} \\ \phi_{54} \\ \phi_{5out} \end{pmatrix} = \begin{pmatrix} 0.03687021 \\ 0.17880038 \\ 0.16991215 \\ 0.61441726 \\ 0.75136800 \\ 0.24863200 \\ 0.39266810 \\ 0.49787860 \\ 0.10945330 \\ 0.75891130 \\ 0.24108870 \\ 0.45795467 \\ 0.45187806 \\ 0.09016727 \end{pmatrix} \quad (32)$$

From equation (32), based on the results, the following deductions are made:

- At the point of registration (node 1), the weights $\Phi_{12}, \Phi_{13}, \Phi_{14}, \Phi_{15}$ which are respectively 0.03687021, 0.17880038, 0.16991215 and 0.61441726 shows that there is a high tendency of a patient leaving node 1 (registration) to join the queue for service at node 5 (test) than any other node. The least tendency is that a patient leaves node 1 to node 2.
- At the point of pre- test/treatment (node 2), the weights Φ_{23} *and* Φ_{2out} which are 0.751368 and 0.248632 shows that there is a high tendency that a patient leaves node 2 and goes directly to node 3.
- At the point of consultation (node 3), the weights $\Phi_{34}, \Phi_{35}, \Phi_{3out}$ which are respectively 0.3926681, 0.4978786 and 0.1094533 shows that there is a high tendency that a patient leaves node 3 to join the queue for service either at node 4 or node 5. The least tendency is that a patient leaves node 3 and moves out of the system.
- At the point of pharmacy (node 4), the weights Φ_{42} *and* Φ_{4out} which are 0.7589113 and 0.2410887 respectively shows that of the 2 weights, there is a higher tendency that a patient leaves node 4 and go back to node 2 (i.e. for treatment) than out of the system.
- At the laboratory test point (node 5), the weights , Φ_{53}, Φ_{54} *and* Φ_{5out} which are 0.45795467, 0.45187806 and 0.09016727 respectively shows that there is a high tendency that a patient leaves node 5 to join the queue for service either at node 3 or node 4.the least tendency here is that a patients leaves node 5 and goes out of the system.

3.2 Results (b) - SOLUTION FOR NEW DEPARTURE RATES

Recall the matrix in equation (28). In order for us to minimize the waiting time such that a patient's maximum waiting time do not exceed 30 minutes, the solution for the departure rates $\mu_i, i = 1...5$. This is represented in equation (33).

$$\begin{pmatrix} \phi_{12} & 0 & 0 & \phi_{42} & 0 \\ \phi_{13} & \phi_{23} & 0 & 0 & \phi_{53} \\ \phi_{14} & 0 & \phi_{34} & 0 & \phi_{54} \\ \phi_{15} & 0 & \phi_{35} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} \qquad (33)$$

After the necessary substitution, the solution to the matrix in equation (33) is given in equation (34).

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} = \begin{pmatrix} 20.21590 \\ 16.93377 \\ 11.89091 \\ 19.89332 \\ 24.94177 \end{pmatrix} \qquad (34)$$

We therefore have from equation (34) that in order to serve a patient within the space interval of 5 minutes, we have to divide the values in equation (34) by 5, the recommended number of servers therefore for each node is given in equation (35).

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix} = \begin{pmatrix} 4.0 \\ 3.4 \\ 2.4 \\ 4.0 \\ 5.0 \end{pmatrix} \qquad (35)$$

The values in equation (35) indicates that in order for us to reduce the waiting time in the system to an acceptable level, node one will require a total of four or more servers, node two will require a total of four or more servers, node three will require a total of three or more servers, node four will require a total of four or more servers and node five will require a total of five or more servers.

With the new estimated number of servers at each node given by equation (35), a new departure rate with a time constraint of thirty minutes is used to calculate the following queuing parameters.

The arrival rates for each are assumed to remain the same since we do not have control over it. A new departure rates and the expected waiting time for each node is estimated.

The new departure rate for node 1 is given as $\mu_1 = \dfrac{recommeded\ \ depature\ \ rate\ \ per\ \ 5\min s}{5} = \dfrac{4}{5} = 0.8$ persons per minute, similarly, we can calculate for node 2 to node 5.

The new expected waiting time in the system is as $w_1 = 1.8$ minutes, $w_2 = 2$ minutes, $w_3 = 1.82$ minutes, $w_4 = 1.69$ minutes and $w_5 = 1.23$ minutes. Hence, the total expected waiting time in the system is given as $W = w_1 + w_2 + w_3 + w_4 + w_5 = 1.8 + 2 + 1.82 + 1.69 + 1.23 = 8.54 = 9$ minutes.

CONCLUSION

For a patient to come into the system and not spend more than thirty minutes, this shows that the University Health Centre would need more servers at each node thereby improving the waiting time for better efficiency of the system. For instance, with one server at the registration point, the expected waiting time in the queue is thirty five minutes but with the recommended number of servers (i.e. 4), the expected waiting time in the queue is reduced to less than a minute.

From the results of the analysis, one can easily deduce that the more the number of service channels, the less the waiting time in the queue and thereby making the amount of waiting time in the service less. For instance, the total waiting time in the system was about sixty minutes, but with the recommended number of servers, the total waiting time reduced to about ten minutes. To enable the Health Centre continually maintain its high standard of providing adequate health care to her patients, the management of the University Health Centre is advised to implement the recommendations given below:

- The number of servers at the registration point should be four or more servers.
- The number of servers at the nursing station (for both pretest and treatment) should be four or more servers for proper efficiency.
- The number of servers at the consultation point should be three or more servers at each time of the day.
- The number of servers at the pharmacy point should be four or more servers as this is seen to be the most congested node in the system.
- The number of servers at the laboratory (test) point should also be five or more servers as this is also a major node of congestion in the system.
- There should also be a convenient waiting room and a friendly environment for patients as this will help them relax as they wait for services.
- Servers in each node of the system should be restricted from doing things outside their normal job to enable them concentrate more and thereby carry out their job efficiently.

Therefore, if these recommendations are taken cognizance of, the University Health Centre will be able to provide better and efficient service to its patients.

Caveat, this study is based on the data collected at a period of time.

Conflict of Interests:

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgement:
We thank the management and staff of Covenant University Health Centre for the support granted us.

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