Applied Mathematical Sciences, Vol. 9, 2015, no. 86, 4255 - 4260
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.5128

Existence of Fixed Points of Some Classes of Nonlinear Mappings in Spaces with Weak Uniform Normal Structure

Godwin Amechi Okeke

Department of Mathematics, College of Science and Technology,
Covenant University, Canaanland, KM 10 Idiroko Road,
P.M.B. 1023 Ota, Ogun State, Nigeria

Mufutau Adesina Olabiyi

Department of Mathematics
University of Lagos, Akoka, Lagos, Nigeria

Abstract

In this paper, we prove some fixed point results for some classes of nonlinear mappings recently introduced by Okeke and Olaleru [5]. Our results improves several other known results in literature, including the results of Sahu *et al.* [8] and Sahu [7].

Mathematics Subject Classification: 47H10, 47H09

Keywords: Existence of fixed point, nonlinear mappings, ϕ-nearly Lipschitzian mappings

1 Introduction and Preliminaries

Let C be a nonempty subset of a Banach space X and $S : C \to C$ a Lipschitzian mapping, we use the symbol $\sigma(S)$ to denote the exact Lipschitz constant of S, i.e.,

$$\sigma(S) = \inf\{k \in [0, \infty] : \|Sx - Sy\| \leq k\|x - y\| \text{ for all } x, y \in C\}. \quad (1.1)$$

A mapping $T : C \to C$ is said to be
(a) *nonexpansive* if $\sigma(T) = 1$,
(b) *asymptotically nonexpansive* if $\sigma(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \sigma(T^n) = 1$,
(c) *uniformly L-Lipschitzian* if $\sigma(T^n) = L$ for all $n \in \mathbb{N}$ and for some $L \in (0, \infty)$.

Sahu [7] recently introduced the following classes of nonlinear mappings as intermediate classes between the class of asymptotically nonexpansive mappings and that of mappings of asymptotically nonexpansive type (see, Goebel and Kirk [3], Kirk [4]).

Definition 1.1 **[7]** Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ will be called *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \;\; \forall \;\; x, y \in C. \quad (1.2)$$

The infimum of constants k_n for which (2.18) holds will be denoted by $\eta(T^n)$ and called *nearly Lipschitz constant.* Notice that

$$\eta(T^n) = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in C, x \neq y\right\}. \quad (1.3)$$

A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be
(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,
(ii) *nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,
(iii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,
(iv) *nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,
(v) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Inspired by the facts above, Okeke and Olaleru [5] introduced the following classes of nonlinear mappings.

Definition 1.2 Let C be a nonempty subset of a Banach space E, $\phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$ and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ will be called *ϕ-nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n.\phi(\|x - y\| + a_n) \;\; \forall \;\; x, y \in C. \quad (1.4)$$

The infimum of constants k_n for which (1.6) holds will be denoted by $\eta(T^n)$ and called *ϕ-nearly Lipschitz constant.* Notice that

$$\eta(T^n) = \sup\left\{\frac{\|T^n x - T^n y\|}{\phi(\|x-y\| + a_n)} : x, y \in C, x \neq y\right\}. \tag{1.5}$$

A ϕ-nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *ϕ-nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,

(ii) *ϕ-nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,

(iii) *ϕ-nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,

(iv) *ϕ-nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(v) *ϕ-nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Observe that if ϕ is identity in Definition 1.2, then we obtain the concepts introduced by Sahu [7] (see Definition 1.1 above).

Our purpose in this paper is to prove some fixed point results for the classes of nonlinear mappings defined by Okeke and Olaleru [5], as given in Definition 1.2 above.

The following definitions and lemma will be needed in this study.

Definition 1.3 [7] Let C be a nonempty subset of a Banach space E and $T : C \to C$ a mapping. T is said to be *demicontinuous* if whenever a sequence $\{x_n\}$ in C converges strongly to $x \in C$, then $\{Tx_n\}$ converges weakly to Tx.

Definition 1.4 [2] The normal structure coefficient $N(E)$ of a Banach space E is defined by

$$N(E) = \inf\{\tfrac{diam(C)}{r_C(C)} : C \text{ is nonempty bounded convex subset of } E \text{ with } diam\ C > 0\},$$

where $r_C(C) = \inf_{x\in C}\{\sup_{y\in C} \|x-y\|\}$ is the *Chebyshev radius of C* relative to itself and *diam* $(C) = \sup_{x,y\in C} \|x-y\|$ is diameter of C. The space E is said to have the *uniform normal structure* if $N(E) > 1$. A weakly convergent sequence coefficient of E is defined by

$$WCS(E) = \sup\{k : k \limsup_{n\to\infty} \|x_n\| < diam_a(\{x_n\}) \text{ for all } \{x_n\} \text{ in } E \text{ with } x_n \rightharpoonup 0\}.$$

The space E is said to have the *weak uniform normal structure* if $WCS(E) > 1$.

Definition 1.5 [1] Let C be a nonempty subset of a Banach space E. A nonempty closed convex subset D of C is said to satisfy property (ω) with respect to a mapping $T : C \to C$ if

$$\omega_T(x) \subset D \quad \text{for every} \quad x \in D, \tag{1.6}$$

where $\omega_T(x)$ denotes the set of all weak subsequential limits of $\{T^n x : n \in \mathbb{N}\}$. Moreover, T is said to satisfy the (ω)-*fixed point property* if T has a fixed point in every nonempty closed convex subset D of C which satisfies property (ω).

Lemma 1.6 [**8**] Let C be a nonempty closed convex subset of a Banach space and $T : C \to C$ a mapping such that $T^n u \to v$ as $n \to \infty$ for some $u, v \in C$. Suppose that T is demicontinuous at v. Then v is a fixed point of T in C.

2 Main Results

Theorem 2.1 Let E be a Banach space with weak uniform normal structure, C a nonempty weakly compact convex subset of E and $T : C \to C$ a ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\limsup_{n\to\infty} \eta(T^n) < \sqrt{WCS(E)}$. Also suppose that there exists a nonempty closed convex subset M of C which satisfies property (ω) with respect to T. Then

(a) for an arbitrary $x_0 \in M$, there exists an iterative sequence $\{x_m\}$ in M defined by

$$x_m = w - \lim_{n\to\infty} T^n x_{m-1} \quad \forall m \in \mathbb{N}, \tag{2.1}$$

(b) if T is asymptotically regular on C, then there exists an element $v \in M$ such that

$\{x_m\}$ converges strongly to $v \in M$. Further, if T is demicontinuous at v, then

$v \in F(T)$.

Proof. (a) We can easily construct a nonempty closed convex separable subset C_0 of C which is invariant under each T^n (i.e. $T^n(C_0) \subset C_0$ for $n = 1, 2, \cdots$), we suppose that C itself is separable.

Due to the separability of C_0, we can select a subsequence $\{T^n x\}$ such that $\{T^n x\}$ is weakly convergent for each $x \in C$. For every $x_0 \in M \subset C$, we consider a sequence $\{T^n x_0\}$ in C. Suppose that $w - \lim_{n\to\infty} T^n x_0 = x_1 \in C$. Using property (ω) we have that $x_1 \in M$. By induction, we can construct a sequence $\{x_m\}$ in M defined by (2.1).

(b) Suppose that T is asymptotically regular on C. The weak asymptotic regularity of T ensures that $x_m = w - \lim_{n\to\infty} T^{n+r} x_{m-1}$ for each $r \in \mathbb{N}$. We are to show that $\{x_m\}$ converges strongly to a fixed point T. We set $L := \limsup_{n\to\infty} \eta(T^n)$, $D_m := \limsup_{n\to\infty} \|x_m - T^n x_m\|$ and $R_m := \limsup_{n\to\infty} \|x_{m+1} - T^n x_m\|$ for all $m = 0, 1, 2, \cdots$ Using the property of $WCS(E)$, we obtain

$$R_m = \limsup_{n\to\infty} \|x_{m+1} - T^n x_m\| \le \frac{1}{WCS(E)} D[\{T^n x_m\}]. \tag{2.2}$$

Using the asymptotic regularity of T and the w-$l.s.c.$ of the norm $\|.\|$, we obtain

$$\begin{aligned} D[\{T^n x_m\}] &= \limsup_{n\to\infty} (\limsup_{r\to\infty} \|T^n x_m - T^r x_m\|) \\ &\le \limsup_{n\to\infty}(\limsup_{r\to\infty}(\|T^n x_m - T^{n+r} x_m\| \\ &\quad + \|T^{n+r} x_m - T^r x_m\|)) \\ &\le \limsup_{n\to\infty} (\limsup_{r\to\infty}(\eta(T^n).\phi(\|x_m - T^r x_m\| + a_n))) \\ &= L \limsup_{r\to\infty} (\phi(\|x_m - T^r x_m\|)) \\ &\le L \limsup_{r\to\infty} (\phi(\limsup_{s\to\infty}(\|T^s x_{m-1} - T^r x_m\|))) \\ &\le L \limsup_{r\to\infty}(\phi(\limsup_{s\to\infty}(\|T^s x_{m-1} - T^{r+s} x_{m-1}\| \\ &\quad + \|T^{r+s} x_{m-1} - T^r x_m\|))) \\ &\le L \limsup_{r\to\infty}(\phi(\limsup_{s\to\infty}(\|T^s x_{m-1} - T^{r+s} x_{m-1}\| \\ &\quad + \eta(T^r)(\|T^s x_{m-1} - x_m\| + a_r)))) \\ &\le L^2 \limsup_{s\to\infty}(\phi(\|T^s x_{m-1} - x_m\|)) = L^2 \times \phi(R_{m-1}). \end{aligned} \tag{2.3}$$

We set $\lambda := \frac{L^2}{WCS(E)} < 1$. Using (2.2), we have

$$\phi(R_m) \le \lambda \times \phi(R_{m-1}) \le \lambda^2 \times \phi(R_{m-2}) \le \cdots \le \lambda^m \times \phi(R_0) \to 0 \tag{2.4}$$

as $m \to \infty$. For each $m \in \mathbb{N}$, we obtain

$$\begin{aligned} \|x_{m+1} - x_m\| &\le \limsup_{n\to\infty}(\|x_{m+1} - T^n x_m\| + \|T^n x_m - x_m\|) \\ &\le R_m + \limsup_{n\to\infty}(\limsup_{r\to\infty} \|T^n x_m - T^r x_{m-1}\|) \\ &\le R_m + \limsup_{n\to\infty}(\limsup_{r\to\infty} \|T^n x_m - T^{n+r} x_{m-1}\| \\ &\quad + \|T^{n+r} x_{m-1} - T^r x_{m-1}\|)) \\ &\le R_m + \limsup_{n\to\infty}(\phi(\limsup_{r\to\infty}(\eta(T^n)\times \\ &\quad (\|x_m - T^r x_{m-1}\| + a_n)))) \\ &\le (\lambda + L).\phi(R_{m-1}) \\ &\quad \cdots \\ &\le (\lambda + L)\lambda^{m-1} \times \phi(R_0). \end{aligned} \tag{2.5}$$

We see that $\{x_m\}$ is a Cauchy sequence in M and hence there exists an element $v \in M$ such that $\lim_{m\to\infty} x_m = v$. Clearly,

$$\begin{aligned} \|v - T^n v\| &\le \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \|T^n x_m - T^n v\| \\ &\le \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \eta(T^n)\times \\ &\quad \phi(\|x_m - v\| + a_n). \end{aligned} \tag{2.6}$$

Taking limit superior as $n \to \infty$ on both sides, we obtain

$$\limsup_{n\to\infty} \|v - T^n v\| \le \|v - x_{m+1}\| + \phi(R_m) + L\|x_m - v\| \to 0,$$

as $m \to \infty$. Hence, we have that $T^n v \to v$ as $n \to \infty$. Furthermore, we assume that T is demicontinuous at v. Therefore, using Lemma 1.6, we obtain $v \in F(T)$. □

Remark 2.2 The results of Theorem 2.1 improves and generalizes several other known results in literature, including the results of Sahu *et al.* [8] and Sahu [7].

References

[1] T. D. Benavides, P. L. Ramirez, *Structure of the fixed point set and common fixed points of asymptotically nonexpansive mappings,* Proc. Amer. Math. Soc. 129 (2001) 3549-3557. http://dx.doi.org/10.1090/s0002-9939-01-06141-x

[2] W. L. Bynum, *Normal structure coefficient for Banach spaces,* Pacific J. Math. 86 (1980) 427-436. http://dx.doi.org/10.2140/pjm.1980.86.427

[3] K. Goebel, W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings,* Proc. Amer. Math. Soc. 35 (1972) 171-174. http://dx.doi.org/10.1090/s0002-9939-1972-0298500-3

[4] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type,* Israel J. Math. 17 (1974) 339-346. http://dx.doi.org/10.1007/bf02757136

[5] G. A. Okeke, J. O. Olaleru, *Fixed points of demicontinuous of ϕ-nearly Lipschitzian mappings in Banach spaces,* (to appear).

[6] G. A. Okeke, J. O. Olaleru, H. Akewe, *Convergence theorems on asymptotically generalized Φ-hemicontractive mappings in the intermediate sense,* Int. Journal of Math. Analysis, Vol. 7, 2013, no. 40, 1991-2003. http://dx.doi.org/10.12988/ijma.2013.35117

[7] D. R. Sahu, *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces,* Comment. Math. Univ. Carolin. 46, 4 (2005) 653-666.

[8] D. R. Sahu, Z. Liu, S. M. Kang, *Existence and approximation of fixed points of nonlinear mappings in spaces with weak uniform normal structure,* Computers and Mathematics with Applications 64 (2012) 672-685. http://dx.doi.org/10.1016/j.camwa.2011.12.081

Received: January 21, 2015; Published: June 6, 2015