Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 180–187
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Modified Noor iterations with errors for nonlinear equations in Banach spaces

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Communicated by R. Saadati

Abstract

We introduce a new three step iterative scheme with errors to approximate the unique common fixed point of a family of three strongly pseudocontractive (accretive) mappings on Banach spaces. Our results are generalizations and improvements of results obtained by several authors in literature. In particular, they generalize and improve the results of Mogbademu and Olaleru [A. A. Mogbademu and J. O. Olaleru, Bull. Math. Anal. Appl., **3** (2011), 132-139], Xue and Fan [Z. Xue and R. Fan, Appl. Math. Comput., **206** (2008), 12-15] which is in turn a correction of Rafiq [A. Rafiq, Appl. Math. Comput., **182** (2006), 589-595].

Keywords: Three-step iterative scheme with errors, Banach spaces, strongly pseudocontractive operators, unique common fixed point, strongly accretive.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Let E be a real Banach space with dual E^* and D is a nonempty closed convex subset of E. We denote by J the normalized duality from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \tag{1.1}$$

where $\langle .,.\rangle$ denotes the generalized duality pairing. We shall also denote the single-valued duality mapping by j.

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Received 2012-3-5

Definition 1.1 [20]. A map $T : E \to E$ is called strongly accretive if there exists a constant $k > 0$ such that, for each $x, y \in E$, there is a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \tag{1.2}$$

Definition 1.2 [20]. An operator T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ and a constant $0 < k < 1$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2. \tag{1.3}$$

The class of strongly accretive operators is closely related to the class of strongly pseudocontractive operators. It is well known that T is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive, where I denotes the identity operator. Browder [1] and Kato [8] indepedently introduced the concept of accretive operators in 1967. One of the early results in the theory of accretive operators credited to Browder states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0 \tag{1.4}$$

is solvable if T is locally Lipschitzian and accretive in an appropriate Banach space.
These class of operators have been studied extensively by several authors (see [2], [3], [9], [10], [11], [15], [16], [18], [20], [25]).

Definition 1.3 [20]. A mapping $T : E \to E$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \forall\ x, y \in D(T). \tag{1.5}$$

In 1953, Mann [10] introduced the Mann iterative scheme and used it to prove the convergence of the sequence to the fixed points for which the Banach principle is not applicable. Later in 1974, Ishikawa [6] introduced an iterative process to obtain the convergence of a Lipschitzian pseudocontractive operator when Mann iterative scheme failed to converge. In 2000 Noor [14] gave the following three-step iterative scheme (or Noor iteration) for solving nonlinear operator equations in uniformly smooth Banach spaces.
Let D be a nonempty convex subset of E and let $T : D \to D$ be a mapping. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0 \end{cases} \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying some conditions.

If $\gamma_n = 0$ and $\beta_n = 0$, for each $n \in \mathbb{Z}$, $n \geq 0$, then (1.6) reduces to:

the iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{Z},\ n \geq 0, \tag{1.7}$$

which is called the one-step (or Mann iterative scheme), introduced by Mann [9].

For $\gamma_n = 0$, (1.6) reduces to:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0 \end{cases} \tag{1.8}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0,1]$ satisfying some conditions. Equation (1.8) is called the two-step (or Ishikawa iterative process) introduced by Ishikawa [6].

In 1989, Glowinski and Le-Tallec [4] used a three-step iterative process to solve elastoviscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge *et al.* [5] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [4] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also lead to highly parallelized algorithms under certain conditions. Hence, we can conclude by observing that three-step iterative scheme play an important role in solving various problems in pure and applied sciences. Studies in nonlinear functional analysis reveals that several problems in sciences, engineering and management sciences can be converted and solved as a fixed point problem of the form $x = Tx$, where T is a mapping. Several authors in literature have obtained some interesting fixed points results (see, e.g. [1, 7, 8, 12, 13, 21, 19, 24, 26, 27]).

Rafiq [20] recently introduced the following modified three-step iterative scheme and used it to approximate the unique common fixed point of a family of strongly pseudocontractive operators.

Let $T_1, T_2, T_3 : D \to D$ be three given mappings. For a given $x_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n \\ \quad y_n = (1-\beta_n)x_n + \beta_n T_2 z_n \\ \quad z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 0, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying some conditions. Observe that iterative schemes (1.6)-(1.8) are special cases of (1.9).

More recently, Suantai [22] introduced the following three-step iterative schemes.

Let E be a normed space, D be a nonempty convex subset of E and $T : D \to D$ be a given mapping. Then for a given $x_1 \in D$, compute the sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ by the iterative scheme

$$\begin{cases} \quad z_n = a_n T^n x_n + (1-a_n)x_n \\ \quad y_n = b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n)x_n \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n)x_n, \quad n \geq 1, \end{cases} \tag{1.10}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in $[0,1]$.

Motivated by the facts above, we now introduce the following modified three-step iterative scheme with errors which we shall use in this paper to approximate the unique common fixed point of a family of strongly pseudocontractive maps.

Let E be a real Banach space, D be a nonempty convex subset of E and $T_1, T_2, T_3 : D \to D$ be a family of three maps. Then for a given $x_0, u_0, v_0, w_0 \in D$, compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{cases} x_{n+1} = (1-\alpha_n-\beta_n-e_n)x_n + \alpha_n T_1 y_n + \beta_n T_1 z_n + e_n u_n \quad n \geq 0 \\ \quad y_n = (1-a_n-b_n-e'_n)x_n + a_n T_2 z_n + b_n T_2 x_n + e'_n v_n \\ \quad z_n = (1-c_n-e''_n)x_n + c_n T_3 x_n + e''_n w_n, \end{cases} \tag{1.11}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{e_n\}_{n=0}^{\infty}$, $\{e'_n\}_{n=0}^{\infty}$, $\{e''_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying certain conditions and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ are bounded sequences in D.

Observe that (1.6)-(1.10) and the modified three step iteration process with errors introduced by Mogbademu and Olaleru [11] are special cases of (1.11). In this paper, we shall use algorithm (1.11) to approximate the unique common fixed point of a family of three strongly pseudocontractive operators in Banach spaces. Our results are generalizations and improvements of the results of Mogbademu and Olaleru [11], Xue and Fan [25] which in turn is a correction of Rafiq [20].

Rafiq [20] proved the following theorem

Theorem R [20]. Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be strongly pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_3 be uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n \\ y_n = (1-\beta_n)x_n + \beta_n T_2 z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying the conditions:
$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Xue and Fan [25] obtained the following convergence results which in turn is a correction of Theorem R.

Theorem XF [25]. Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2 and T_3 be strongly pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_2 and T_3 uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.9), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ which satisfy the conditions: $\alpha_n, \beta_n \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2 and T_3.

In this study, we use our newly introduced iterative scheme (1.11) to prove some convergence results. Our results are generalizations and improvements of the results of Mogbademu and Olaleru [11], Xue and Fan [25] which in turn is a correction of Rafiq [20].

The following lemmas will be useful in this study.

Lemma 1.1 [20]. Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall\, j(x+y) \in J(x+y). \tag{1.12}$$

Lemma 1.2 [23]. Let $\{\rho\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality:

$$\rho_{n+1} \leq (1-\lambda_n)\rho_n + \sigma_n, \quad n \geq 0, \tag{1.13}$$

where $\lambda_n \in (0,1)$, $n = 0, 1, 2, \cdots, \sum_{n=0}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Main Results

Theorem 2.1 Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2 and T_3 be strongly pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_2 and T_3 uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.11), where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{e_n\}_{n=0}^{\infty}$, $\{e'_n\}_{n=0}^{\infty}$, $\{e''_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying the conditions: $a_n, b_n, c_n, e_n, e'_n, e''_n, \alpha_n, \beta_n \longrightarrow 0$ as $n \to \infty$, $\alpha_n + \beta_n + e_n < 1$, $a_n + b_n + e'_n < 1$, $c_n + e''_n < 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in D. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2 and T_3.

Proof. Since T_1, T_2, T_3 are strongly pseudocontractive, there exists a constant $k = \max\{k_1, k_2, k_3\}$ so that

$$\langle T_i x - T_i y, j(x-y)\rangle \leq k\|x-y\|^2, \quad i = 1, 2, 3, \tag{2.1}$$

where k_1, k_2 and k_3 are constants for operators T_1, T_2 and T_3 respectively. Assume that $p \in F(T_1) \cap F(T_2) \cap F(T_3)$, using the fact that T_i is strongly pseudocontractive for each $i = 1, 2, 3$ we obtain $F(T_1) \cap F(T_2) \cap F(T_3) = p \neq \emptyset$. Since T_1 has a bounded range, we let

$$M_1 = \|x_0 - p\| + \sup_{n\geq 0} \|T_1 y_n - p\| + \sup_{n\geq 0} \|T_1 z_n - p\| + \|u_n - p\|. \tag{2.2}$$

We shall prove by induction that $\|x_n - p\| \leq M_1$ holds for all $n \in \mathbb{N}$. We observe from (2.2) that $\|x_0 - p\| \leq M_1$. Assume that $\|x_n - p\| \leq M_1$ holds for all $n \in \mathbb{N}$. We will prove that $\|x_{n+1} - p\| \leq M_1$. Using (1.11), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n - \beta_n - e_n)(x_n - p) + \alpha_n(T_1 y_n - p) + \\
&\quad \beta_n(T_1 z_n - p) + e_n(u_n - p)\| \\
&\leq (1 - \alpha_n - \beta_n - e_n)\|x_n - p\| + \alpha_n \|T_1 y_n - p\| + \\
&\quad \beta_n \|T_1 z_n - p\| + e_n \|u_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - e_n)M_1 + \alpha_n M_1 + \beta_n M_1 + e_n M_1 \\
&= M_1.
\end{aligned} \tag{2.3}$$

Using the uniform continuity of T_3, we obtain that $\{T_3 x_n\}_{n=0}^{\infty}$ is bounded. We now set

$$M_2 = \max\left\{ M_1, \sup_{n\geq 0}\{\|T_3 x_n - p\|\}, \sup_{n\geq 0}\{\|w_n - p\|\} \right\}, \tag{2.4}$$

hence

$$\begin{aligned}
\|z_n - p\| &= \|(1 - c_n - e''_n)(x_n - p) + c_n(T_3 x_n - p) + e''_n(w_n - p)\| \\
&\leq (1 - c_n - e''_n)\|x_n - p\| + c_n\|T_3 x_n - p\| + e''_n\|w_n - p\| \\
&\leq (1 - c_n - e''_n)M_1 + c_n M_2 + e''_n M_2 \\
&\leq (1 - c_n - e''_n)M_2 + c_n M_2 + e''_n M_2 \\
&= M_2.
\end{aligned} \tag{2.5}$$

By the uniform continuity of T_2, we obtain $\{T_2 z_n\}_{n=0}^{\infty}$ and $\{T_2 x_n\}_{n=0}^{\infty}$ are bounded. Set

$$M = \sup_{n\geq 0} \|T_2 z_n - p\| + \sup_{n\geq 0} \|x_n - p\| + \sup_{n\geq 0} \|v_n - p\| + M_2. \tag{2.6}$$

Using Lemma 1.1 and (1.11), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n - e_n)(x_n - p) + \alpha_n(T_1 y_n - p) + \\
&\quad \beta_n(T_1 z_n - p) + e_n(u_n - p)\|^2 \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 \\
&\quad + 2\langle \alpha_n(T_1 y_n - p) + \beta_n(T_1 z_n - p) + e_n(u_n - p), j(x_{n+1} - p)\rangle \\
&= (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 + 2\alpha_n\langle T_1 y_n - p, j(x_{n+1} - p)\rangle \\
&\quad + 2\beta_n\langle T_1 z_n - p, j(x_{n+1} - p)\rangle + 2e_n\langle u_n - p, j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 \\
&\quad + 2\alpha_n\langle T_1 x_{n+1} - T_1 p, j(x_{n+1} - p)\rangle \\
&\quad + 2\alpha_n\langle T_1 y_n - T_1 x_{n+1}, j(x_{n+1} - p)\rangle \\
&\quad + 2\beta_n\langle T_1 x_{n+1} - T_1 p, j(x_{n+1} - p)\rangle \\
&\quad + 2\beta_n\langle T_1 z_n - T_1 x_{n+1}, j(x_{n+1} - p)\rangle + 2e_n\langle u_n - p, j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 + 2\alpha_n k\|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n\|T_1 y_n - T_1 x_{n+1}\|.\|x_{n+1} - p\| + 2\beta_n k\|x_{n+1} - p\|^2 \\
&\quad + 2\beta_n\|T_1 z_n - T_1 x_{n+1}\|.\|x_{n+1} - p\| + 2e_n M \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 + 2\alpha_n k\|x_{n+1} - p\|^2 \\
&\quad + 2\beta_n k\|x_{n+1} - p\|^2 + 2\alpha_n\delta_n M_1 + 2\beta_n\tau_n M_1 + 2e_n M,
\end{aligned} \tag{2.7}$$

where $\delta_n = \|T_1y_n - T_1x_{n+1}\| \longrightarrow 0$ as $n \to \infty$ and $\tau_n = \|T_1z_n - T_1x_{n+1}\| \longrightarrow 0$ as $n \to \infty$. But,

$$\begin{aligned}
\|y_n - x_{n+1}\| &= \|(1-a_n-b_n-e'_n)x_n + a_nT_2z_n + b_nT_2x_n + e'_nv_n \\
&\quad -(1-\alpha_n-\beta_n-e_n)x_n - \alpha_nT_1y_n - \beta_nT_1z_n - e_nu_n\| \\
&= \|a_n(T_2z_n - x_n) + b_n(T_2x_n - x_n) + e'_n(v_n - x_n) \\
&\quad +\alpha_n(x_n - T_1y_n) + \beta_n(x_n - T_1z_n) + e_n(x_n - u_n)\| \\
&\le a_n\|T_2z_n - x_n\| + b_n\|T_2x_n - x_n\| + e'_n\|v_n - x_n\| \\
&\quad +\alpha_n\|x_n - T_1y_n\| + \beta_n\|x_n - T_1z_n\| + e_n\|x_n - u_n\| \\
&\le a_nM + b_nM + e'_nM + \alpha_nM_1 + \beta_nM_1 + e_nM_1 \\
&= M(a_n + b_n + e'_n) + M_1(\alpha_n + \beta_n + e_n) \\
&\le M(a_n + b_n + e'_n + \alpha_n + \beta_n + e_n) \longrightarrow 0, \qquad (2.8)
\end{aligned}$$

as $n \to \infty$.

$$\begin{aligned}
\|z_n - x_{n+1}\| &= \|(1-c_n-e''_n)x_n + c_nT_3x_n + e''_nw_n - (1-\alpha_n-\beta_n-e_n)x_n \\
&\quad -\alpha_nT_1y_n - \beta_nT_1z_n - e_nu_n\| \\
&= \|c_n(T_3x_n - x_n) + e''_n(w_n - x_n) + \alpha_n(x_n - T_1y_n) \\
&\quad +\beta_n(x_n - T_1z_n) + e_n(x_n - u_n)\| \\
&\le c_n\|T_3x_n - x_n\| + e''_n\|w_n - x_n\| + \alpha_n\|x_n - T_1y_n\| \\
&\quad +\beta_n\|x_n - T_1z_n\| + e_n\|x_n - u_n\| \\
&\le c_nM_2 + e''_nM + \alpha_nM_1 + \beta_nM_1 + e_nM \\
&\le M(c_n + e''_n + \alpha_n + \beta_n + e_n) \longrightarrow 0,
\end{aligned}$$

as $n \to \infty$. This implies that $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$ and $\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0$ since $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} e'_n = 0$, $\lim_{n\to\infty} e''_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} e_n = 0$. Using the uniform continuity of T_1, we obtain $\delta_n = \|T_1y_n - T_1x_{n+1}\| \longrightarrow 0$ as $n \to \infty$ and $\tau_n = \|T_1z_n - T_1x_{n+1}\| \longrightarrow 0$ as $n \to \infty$. Hence, there exists a positive integer N such that $\alpha_n, \beta_n < \min\{\frac{1}{2k}, \frac{1-k}{(1-k)^2+k^2}\}$ for all $n \ge N$. Hence, from (2.7), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \left(\tfrac{(1-\alpha_n-\beta_n-e_n)^2}{1-2\alpha_nk-2\beta_nk}\right)\|x_n - p\|^2 + \tfrac{2\alpha_n\delta_nM_1+2\beta_n\tau_nM_1+2e_nM_1}{1-2\alpha_nk-2\beta_nk} \\
&\le \left(\tfrac{(1-\alpha_n)^2}{1-2\alpha_nk-2\beta_nk}\right)\|x_n - p\|^2 + M\left(\tfrac{2\alpha_n\delta_n+2\beta_n\tau_n+2e_n}{1-2\alpha_nk-2\beta_nk}\right) \\
&= \left(\tfrac{1-2\alpha_n+\alpha_n^2-2\alpha_nk-2\beta_nk+2\alpha_nk+2\beta_nk}{1-2\alpha_nk-2\beta_nk}\right)\|x_n - p\|^2 \\
&\quad +M\left(\tfrac{2\alpha_n\delta_n+2\beta_n\tau_n+2e_n}{1-2\alpha_nk-2\beta_nk}\right) \\
&= \left(1 - \tfrac{2\alpha_n-\alpha_n^2-2\alpha_nk}{1-2\alpha_nk-2\beta_nk}\right)\|x_n - p\|^2 + M\left(\tfrac{2\alpha_n\delta_n+2\beta_n\tau_n+2e_n}{1-2\alpha_nk-2\beta_nk}\right) \\
&= \left(1 - \tfrac{2-\alpha_n-2k}{1-2\alpha_nk-2\beta_nk}\alpha_n\right)\|x_n - p\|^2 + M\left(\tfrac{2\alpha_n\delta_n+2\beta_n\tau_n+2e_n}{1-2\alpha_nk-2\beta_nk}\right) \\
&\le (1-(1-k)\alpha_n)\|x_n - p\|^2 + M\left(\tfrac{2\alpha_n\delta_n+2\beta_n\tau_n+2e_n}{1-2\alpha_nk-2\beta_nk}\right). \qquad (2.9)
\end{aligned}$$

Next, set $\rho_n = \|x_n - p\|$, $\lambda_n = (1-k)\alpha_n$ and $\sigma_n = M\left(\frac{2\alpha_n\delta_n+2\beta_n\tau_n+2e_n}{1-2\alpha_nk-2\beta_nk}\right)$. Using Lemma 1.2, we have $\|x_n - p\| \longrightarrow 0$ as $n \to \infty$. The proof of Theorem 2.1 is completed. □

Corollary 2.2 Let E be a real Banach space, D a nonempty closed and convex subset of E. Let T_1, T_2, T_3 be self maps of D with $T_1(D)$ bounded such that $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and T_1, T_2 and T_3 uniformly continuous. Suppose T_1, T_2, T_3 are strongly pseudocontractive mappings. For $x_0, u_0, v_0, w_0 \in D$, the three step iteration with errors $\{x_n\}$ defined as follows

$$\begin{cases} x_{n+1} = a_nx_n + b_nT_1y_n + c_nu_n \\ y_n = a'_nx_n + b'_nT_2z_n + c'_nv_n \\ z_n = a''_nx_n + b''_nT_3x_n + c''_nw_n \quad n \ge 0, \end{cases} \qquad (2.10)$$

where $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are arbitrary bounded sequences in D. $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$ and $\{c''_n\}$ are real sequences in $[0,1]$ satisfying the following conditions:
(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$
(ii) $b_n, b'_n, c_n, c'_n \longrightarrow 0$ as $n \to \infty$.
(iii) $\sum_{n=1}^{\infty} b_n = \infty$
(iv) $\lim_{n\to\infty} \frac{c_n}{b_n} = 0$,
converges strongly to the unique common fixed point of T_1, T_2 and T_3.

Remark 2.3 Corollary 2.2 is Theorem 2.1 of Mogbademu and Olaleru [11]. Observe that if $\beta_n = b_n \equiv 0$ for all $n = 0, 1, 2, \cdots$ in Theorem 2.1, then we obtain Theorem 2.1 of [11]. Similary, if $\beta_n = e_n = b_n = e'_n = e''_n \equiv 0$ for all $n = 0, 1, 2, \cdots$ in Theorem 2.1, then we obtain Theorem 2.1 of Xue and Fan [25]. Hence, Theorem 2.1 is an improvement and a generalization of Mogbademu and Olaleru [11], Xue and Fan [25] which in turn is a correction of Rafiq [20].

Theorem 2.4 Let E be a real Banach space, $T_1, T_2, T_3 : E \to E$ be uniformly continuous and strongly accretive operators with $R(I - T_1)$ bounded, where I is the identity mapping on E. Let p denote the unique common solution to the equation $T_i x = f$, $(i = 1, 2, 3)$. For a given $f \in E$, define the operator $H_i : E \to E$ by $H_i x = f + x - T_i x$, $(i = 1, 2, 3)$. For any $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \beta_n - e_n)x_n + \alpha_n H_1 y_n + \beta_n H_1 z_n + e_n u_n, & n \geq 0, \\ y_n = (1 - a_n - b_n - e'_n)x_n + a_n H_2 z_n + b_n H_2 x_n + e'_n v_n, \\ z_n = (1 - c_n - e''_n)x_n + c_n H_3 x_n + e''_n w_n, \end{cases} \tag{2.11}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{e_n\}_{n=0}^{\infty}$, $\{e'_n\}_{n=0}^{\infty}$, $\{e''_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying the conditions: $a_n, b_n, e'_n, \alpha_n, \beta_n, e_n \longrightarrow 0$ as $n \to \infty$, $\alpha_n + \beta_n + e_n < 1$, $a_n + b_n + e'_n < 1$, $c_n + e''_n < 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ are bounded sequences in E. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common solution to $T_i x = f$ $(i = 1, 2, 3)$.

Proof. Clearly, if p is the unique common solution to the equation $T_i x = f$ $(i = 1, 2, 3)$, it follows that p is the unique common fixed point of H_1, H_2 and H_3. Using the fact that T_1, T_2 and T_3 are all srtongly accretive operators, then H_1, H_2 and H_3 are all strongly pseudocontractive with constant $k = \max\{k_1, k_2, k_3\}$ where $k_1, k_2, k_3 \in (0, 1)$ are strongly pseudocontractive constants for H_1, H_2 and H_3 respectively. Since T_i $(i = 1, 2, 3)$ is uniformly continuous with $R(I - T_1)$ bounded, this implies that H_i $(i = 1, 2, 3)$ is uniformly continuous with $R(H_1)$ bounded. Hence, Theorem 2.4 follows from Theorem 2.1. $\square$

Remark 2.5 Theorem 2.4 improves and extends Theorem 2.4 of Mogbademu and Olaleru [11] and Theorem 2.2 of Xue and Fan [25] which in turn is a correction of Rafiq [20].

Example 2.6 Let $E = (-\infty, +\infty)$ with the usual norm and let $D = [0, +\infty)$. We define $T_1 : D \to D$ by $T_1 x := \frac{x}{2(1+x)}$ for each $x \in D$. Hence, $F(T_1) = \{0\}$, $R(T_1) = [0, \frac{1}{2})$ and T_1 is a uniformly continuous and strongly pseudocontractive mapping. Define $T_2 : D \to D$ by $T_2 x := \frac{x}{4}$ for all $x \in D$. Hence, $F(T_2) = \{0\}$ and T_2 is a uniformly continuous and strongly pseudocontractive mapping. Define $T_3 : D \to D$ by $T_3 x := \frac{\sin^4 x}{4}$ for each $x \in D$. Then $F(T_3) = \{0\}$ and T_3 is a uniformly continuous and strongly pseudocontractive mapping. Set $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{(n+1)+(n+1)^2}$, $e_n = \frac{1}{(n+1)^2}$, $a_n = \frac{1}{4(n+1)^{\frac{1}{2}}}$, $b_n = \frac{1}{2(n+1)^2+(n+1)}$, $e'_n = \frac{1}{(n+1)+(n+1)^2+(n+1)^3}$, $c_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $e''_n = \frac{1}{(n+1)^{\frac{1}{2}}+(n+1)^{\frac{1}{3}}}$, for all $n \geq 0$. Clearly, $F(T_1) \cap F(T_2) \cap F(T_3) = \{0\} = p \neq \emptyset$. For an arbitrary $x_0 \in D$, the sequence $\{x_n\}_{n=0}^{\infty} \subset D$ defined by (1.11) converges strongly to the common fixed point of T_1, T_2 and T_3 which is $\{0\}$, satisfying Theorem 2.1. This means that Theorem 2.1 is applicable.

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