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PERIODIC BOUNDARY VALUE PROBLEMS FOR THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH DELAY

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We study the periodic boundary value problem

$$\dddot{x}(t) + f(\dot{x}(t))\ddot{x}(t) + g(t, \dot{x}(t-\tau)) + h(x(t)) = p(t)$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

under some resonant conditions on the asymptotic behaviour of $x^{-1} g(t, x)$ for $|x| \to \infty$. The uniqueness of periodic solutions is also examined.

1. INTRODUCTION

In this paper we study the periodic boundary value problem

$$\left\{ \begin{array}{l} \dddot{x} + f(\dot{x})\ddot{x} + g(t, \dot{x}(t-\tau)) + h(x) = p(t) \\ x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0 \end{array} \right\} \qquad \ldots (1.1)$$

with fixed delay $\tau \in [0, 2\pi)$, where $f : R \to R$ is continuous, $P : [0, 2\pi] \to R$ and $g : [0, 2\pi] \times R \to R$ are 2π-periodic in t and g satisfies certain Caratheodory conditions. The unknown function $x : [0, 2\pi] \to R$ is defined for $0 < t \leq \tau$ by $x(t - \tau) = [2\pi - (t - \tau)]$. We are specifically concerned with the existence of periodic solutions of eqn. (1.1) under some resonant conditions.

The differential equations

$$\dddot{x} + a\ddot{x} + f(x)\dot{x} + g(t, x(t-\tau)) = p(t)$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi), \; \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

in which $a \neq 0$ is a constant and

PROOF : We consider a solution of the form $x(t) = e^{\lambda t}$ where $\lambda = in$ with $i^2 = -1$.

Then Lemma 2.1 will follow if

$$\psi(n, \tau) = -n^2 + b \cos n\tau \neq 0 \qquad \dots (2.3)$$

for all $n \geq 1$ and $\tau \in [0, 2\pi)$.

By (2.2) we get

$$\psi(n, \tau) \leq -n^2 + b < 0.$$

Therefore $\psi(n, \tau) \neq 0$ and the result follows. If $x \in L^1_{2\pi}$ we shall write

$$\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, dt, \quad \tilde{x}(t) = x(t) - \bar{x}$$

so that

$$\int_0^{2\pi} \tilde{x}(t)\, dt = 0.$$

Our next result concerns the delay equation

$$\dddot{x} + a\ddot{x} + b(t)\dot{x}(t-\tau) + cx = 0 \qquad \dots (2.4)$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

where a, c are constants and $b \in L^2_{2\pi}$.

Theorem 2.1 — Let $c \neq 0$. Suppose that $b(t)$ satisfies

$$0 < b(t) < 1, \; t \in [0, 2\pi]. \qquad \dots (2.5)$$

Then for arbitrary a eqn. (2.4) admits in $W^{3,2}_{2\pi}$ only the trivial solution.

PROOF : If x is a possible solution of (2.4) then since

$$\frac{1}{2\pi} \int_0^{2\pi} -\dot{\tilde{x}}(a\ddot{x} + cx)\, dt = 0$$

as can be easily verified, we have from (2.5) that

$$0 = \frac{1}{2\pi} \int_0^{2\pi} -\dot{\tilde{x}}(\dddot{x} + a\ddot{x} + b(t)\dot{x}(t-\tau) + cx)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\ddot{\tilde{x}}^2 - b(t)\dot{\tilde{x}}\dot{x}(t-\tau))\, dt.$$

PROOF : Integrating by parts and using the identity

$$-ab = \frac{[a-b]^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

and noting that

$$\frac{1}{2\pi}\int_0^{2\pi} -\dot{\tilde{x}}(t)\,(a\ddot{x} + cx))\,dt = 0$$

we get

$$\frac{1}{2\pi}\int_{00}^{2\pi} -\tilde{x}\,(\dddot{x} + V(t)\,\dot{x}\,(t-\tau))\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\ddot{\tilde{x}}^2(t) - V(t)\,\dot{\tilde{x}}^2(t))\,dt$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} (\ddot{\tilde{x}}^2(t) - b(t)\,\dot{\tilde{x}}^2)\,dt - \frac{\varepsilon}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t)\,dt$$

$$\geq \delta\,|\dot{\tilde{x}}|^2_{H^1_{2\pi}} - \varepsilon\,|\dot{\tilde{x}}|^2_2$$

$$\geq \delta\,|\ddot{\tilde{x}}|^2_2 - \varepsilon\,|\ddot{\tilde{x}}|^2_2$$

$$= (\delta - \varepsilon)\,|\ddot{x}|^2_2 .$$

We shall next consider the non-linear delay equation

$$\dddot{x} + f(\dot{x})\,\ddot{x} + g(t, \dot{x}\,(t, \tau)) + h(x) = p(t) \qquad \text{... (3.1)}$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}\,(2\pi) = \ddot{x}\,(0) - \ddot{x}\,(2\pi) = 0$$

where $f, h : R \to R$ are continuous functions and $g : [0, 2\pi] \times R \to R$ is such that $g(\cdot\ x)$ is measurable on $[0, 2\pi]$ for each $x \in R$ and $g(t, \cdot)$ is continuous on R for almost each $t \in [0, 2\pi]$.

We assume moreover that for each $r > 0$ there exists $\gamma_r \in L^2_{2\pi}$ such that $|g(t, y)| \leq \gamma_r(t)$ for a.e $t \in [0, 2\pi]$ and all $x \in [-r, r]$ such a g is said to satisfy Caratheodory's conditions.

Theorem 3.1 — Let g be a Caratheodory's function with respect to the space $L^2_{2\pi}$ such that

(i) There exists $s > 0$ such that

$$xg(t, x) \geq 0 \text{ for } |x| \geq s$$

Define as in Mawhin and Ward[5]

$$L : \operatorname{dom}L \subset X \to Z,\ x \to \ddot{x}$$

$$F : X \to Z,\ x \to f(\dot{x})\ddot{x}$$

$$G : X \to Z,\ x \to \tilde{\gamma}(t, \dot{x}(t-\tau))\,\dot{x}(t-\tau)$$

$$H : X \to Z,\ x \to h(x)$$

$$A : X \to Z,\ x \to b(t)\,\dot{x}(t-\tau)$$

$$G_0 : X \to Z,\ x \to g_0(t, \dot{x}(t-\tau)).$$

The proof of the theorem will follow from Theorem 4.5 of Mawhin[7] if we show that the possible solutions of the equation

$$Lx + \lambda Fx + (1-\lambda)Ax + \lambda Gx + \lambda G_0x + (1-\lambda)cx + \lambda Hx = \lambda p(t) \quad \text{... (3.5)}$$

where $c > 0$ are *a priori* bounded independently of $\lambda \in [0, 1]$.

For $\lambda = 0$ we get the equation

$$\ddot{x} + b(t)\,\dot{x}(t-\tau) + cx = 0$$

which by theorem (2.1) has only the trivial solution.

Observe that

$$0 \le (1-\lambda)\,b(t) + \lambda\,\tilde{\gamma}(t, \dot{x}(t-\tau)) \le b(t) + \frac{\delta}{2}.$$

Hence by Lemma 3.1 we get

$$\frac{1}{2\pi}\int_0^{2\pi} -\dddot{x}(t)\,\{\ddot{x} + \lambda f(\dot{x})\,\ddot{x} + [(1-\lambda)\,b(t) + \lambda\tilde{\gamma}(t, \dot{x}(t-\tau))]$$

$$\dot{x}(t-\tau) + (1-\lambda)\,cx\}$$

$$\ge \frac{\delta}{2}\,|\ddot{x}|_2^2.$$

Thus

$$0 = \frac{1}{2\pi}\int_0^{2\pi} -\dddot{x}(t)\,\{\ddot{x} + \lambda f(\dot{x})\,\ddot{x} + [(1-\lambda)\,b(t)$$

$$+ \lambda\tilde{\gamma}(t, \dot{x}(t-\tau))]\,\dot{x}(t-\tau) + (1-\lambda)\,cx$$

$$+ \lambda\,g_0(t, \dot{x}(t-\tau)) + \lambda h(x) - \lambda p(t)\}\ dt$$

From (3.8) we obtain

$$x(t) = x(t^*) + \frac{1}{2\pi} \int_{t^*}^{2\pi} \dot{x}(s)\, ds.$$

Hence

$$|x|_\infty \le \beta_6 + |\dot{x}|_\infty \le \beta_6 + \beta_2 = \beta_7 \qquad \text{... (3.9)}$$

for some $\beta_7 > 0$.

From eqn. (3.5) and by continuity of h we obtain

$$|\dddot{x}|_1 \le \beta_8 \quad \text{for some } \beta_8 > 0. \qquad \text{... (4.0)}$$

Now since $\dot{x}(0) = \dot{x}(2\pi)$, there exists $t_0 \in (0, 2\pi)$ such that $\ddot{x}(t_0) = 0$. Hence

$$\ddot{x}(t) = \ddot{x}(t_0) + \int_{t_0}^{2\pi} \dddot{x}(s)\, ds.$$

Therefore

$$|\ddot{x}|_\infty \le \beta_9 \text{ for some } \beta_9 > 0.$$

Hence

$$|x|_{c^2} = |x|_\infty + |\dot{x}|_\infty + |\ddot{x}|_\infty \le \beta_7 + \beta_2 + \beta_9 = \beta_{10}.$$

Choosing $\rho > \beta_{10} > 0$ we obtain the required *a priori* bound in $c^2\,[0, 2\pi]$ independently of x and λ.

4. Uniqueness Result

If in (1.1) $f(\ddot{x}) = a$, $h(x) = d$ where a and d are constants, then we have the following uniqueness result.

Theorem 4.1 — Let a and d be constants with $d > 0$. Suppose g is a Caratheodory function satisfying

$$0 \le \frac{g(t, \dot{x}_1) - g(t, \dot{x}_2)}{(\dot{x}_1 - \dot{x}_2)} \le b(t)$$

for all $\dot{x}_1, \dot{x}_2 \in R$, $\dot{x}_1 \ne \dot{x}_2$, where $b(t) \in L^2_{2\pi}$ is scuh that $0 < b(t) < 1$. Then for all arbitrary constant a and every $\tau \in [0, 2\pi)$ the boundary value problem

$$\dddot{x} + a\ddot{x} + g(t, \dot{x}(t - \tau)) + dx = p(t) \qquad \text{... (4.1)}$$

$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

has at most one solution.