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ON THE NON-RESONANT OSCILIATION OF A FOURTH ORDER PERIODIC BOUNDARY VALUE PROBLEM WITH DELAY

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ABSTRACT. We use coincidence degree arguments to prove the existence and uniqueness of periodic solutions of the equation

$$x^{iv}(t) + a\dddot{x}(t) + b\ddot{x}(t) + g(t, \dot{x}(t-\tau)) + dx = p(t)$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3.$$

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1. INTRODUCTION

In a recent paper [5] we proved the existence of periodic solution of the fourth order delay equation of the form

$$x^{iv}(t) + a\dddot{x}(t) + b\ddot{x}(t) + c\dot{x}(t) + g(t, x(t-\tau)) = p(t)$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3.$$

where a, b, c are constants, g is a Caratheodory's function $p(t) \in L^1_{2\pi}$ and $\tau \in [0, 2\pi)$ is a fixed time delay.

In this paper we shall investigate the existence and uniqueness of 2π - periodic solution for the fourth order periodic boundary value problem with delay of the form.

$$x^{iv}(t) + a\dddot{x}(t) + b\ddot{x}(t) + g(t, \dot{x}(t-\tau)) + dx = p(t)$$
$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3. \tag{1.1}$$

where a, b and d are constants g is a Caratheodory's function $p(t) \in L^1_{2\pi}$ and $\tau \in [0, 2\pi)$ is a fixed time delay. The unknown function x :

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$[0, 2\pi] \to R$ is defined for $0 \le t \le \tau$ by $x(t-\tau) = x(2\pi-(t-\tau))$. It is pertinent to note that some problems in biological or physiological systems can be modeled by fourth order differential equations with time delay. For instance, the oscillatory movements of muscles that occur from the interaction of a muscle with its load [7]. Other applications can be found in [3] and references therein.

In section 2 of this paper we shall consider the problem of non-existence of non-trivial 2π periodic solutions of some linear analogues of (1.1). In section 3 we shall prove that under suitable conditions on the constants a, b, d and on the asymptotic behaviour of the ratio $\dfrac{g(t,y)}{ay}$ the equation (1.1) possesses at least one 2π-periodic solution for each $p(t) \in L^1_{2\pi}$. The techniques of proof uses coincidence degree theory [6] and the apriori estimates are obtained by adapting the methods established in [4].

In section 4 we shall obtain uniqueness results. In what follows we shall use the following notations and definitions. Let R denote the real time and I the interval $[0, 2\pi]$. The following spaces will be used. $L^k_{2\pi} = L^k(I, R)$ are the usual Lebesque spaces, $1 \le k < \infty$ with $x \in L^k_{2\pi}$, 2π-periodic

$$H^k_{2\pi} = H^k(I,R)\begin{cases} x : I \to R, \quad x, \dot{x}, \ldots, x^{k-1} \quad \text{are absolutely} \\ \text{continuous and } \; x^k \in L^2_{2\pi} \\ \\ x^{(i)}(0) = x^{(i)}(2\pi) \quad i = 0, 1, 2, 3, \ldots, k-1 \end{cases}$$

with norm

$$\|x\|^2_{H^k_{2\pi}} = \left(\frac{1}{2\pi}\int_0^{2\pi} x(t)dt\right)^2 + \frac{1}{2\pi}\sum_{i=1}^{k}\int_0^{2\pi} |x^{(i)}(t)|^2$$

and

$W^{k,1}_{2\pi} = \{x : I \to R, x, \dot{x}, \ldots, x^{k-1}$ are absolutely continuous,
$x^k \in L^2_{2\pi}$
and $x^{(i)}(0) = x^{(i)}(2\pi), \;\; i = 0, 1, 2, 3, \ldots, k-1\}$
with norm

$$\|x\|^2_{W^{k,1}_{2\pi}} = \frac{1}{2\pi}\sum_{i=0}^{k}\int_0^{2\pi} |x^{(i)}(t)|dt.$$

A function $x \in W^{4,1}_{2\pi}$ is a solution of (1.1) if it satisfies (1.1) almost everywhere on R.

For such a solution we set

$x = \bar{x} + \tilde{x}$ where

$$\bar{x}(t) = a_0 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt) \tag{1.2}$$

$$\tilde{x}(t) = \sum_{k=n+1}^{\infty}(a_k \cos kt + b_k \sin kt) \tag{1.3}$$

2. The Linear Case

We consider in this section the problem of non-existence of non-trivial periodic solution for some linear analogue of (1.1). We shall consider the linear equation.

$$\begin{aligned} &x^{iv} + a\dddot{x}(t) + b\ddot{x}(t) + c(t)\dot{x}(t-\tau) + dx = 0 \\ &x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0,1,2,3 \end{aligned} \tag{2.1}$$

where a, b, d are constants and $c(t) \in L^1_{2\pi}$.
We have the following results.

Theorem 2.1

Let $n \geq 1$ be an integer and let the following conditions be satisfied.

(i) $a \neq 0$
(ii) $b > n^2$
(iii) $0 < d < n$
(iv) $n^2 \leq a^{-1}c(t) \leq (n+1)^2$ holds uniformly a.e. in $t \in [0, 2\pi]$ with strict inequalities $n^2 < a^{-1}c(t), a^{-1}c(t) < (n+1)^2$ holding on subsets of $[0, 2\pi]$ of positive measure.

Suppose that there exists constant $\delta > 0$ with $\delta > |a|^{-1}$ then the boundary value problem (2.1) has no non-trivial periodic solution in $W^{4,1}_{2\pi}$.

Proof.

We set $\Gamma(t) = a^{-1}c(t)$ and rewrite (2.1) in the form

$$a^{-1}[x^{iv}(t) + b\ddot{x}(t) + dx(t)] + \dddot{x} + \Gamma(t)\dot{x}(t-\tau) = 0 \tag{2.2}$$

Let $x = \bar{x} + \tilde{x} \in H^3_{2\pi}$ be any solution of (2.2). Then on multiplying (2.2) by $\dot{\bar{x}}(t-\tau) - \dot{\tilde{x}}(t)$ and integrating over I, we obtain $I_1 + I_2 = 0$, where

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi}(\dot{\ddot{x}}(t-\tau)-\dot{\ddot{x}}(t))a^{-1}[x^{iv}+b\ddot{x}+dx]dx \equiv I_1. \\ &+\frac{1}{2\pi}\int_0^{2\pi}(\dot{\ddot{x}}(t-\tau)-\dot{\ddot{x}}(t))\{\dddot{x}+\Gamma(t)\dot{x}(t-\tau)\}dt \equiv I_2 \\ &\equiv I_1+I_2 \end{aligned}$$

To estimate I_1 we observe from definition (1.2) and (1.3) that

$$\frac{1}{2\pi}\int_0^{2\pi}\dot{\ddot{x}}(t-\tau)x^{iv}(t)dt = \sum_{k=1}^{n}k^5(a_k^2+b_k^2)\sin k\tau$$

$$\frac{1}{2\pi}\int_0^{2\pi}\dot{\ddot{x}}(t-\tau)\ddot{x}(t)dt = -\sum_{k=1}^{n}k^3(a_k^2+b_k^2)\sin k\tau$$

$$\frac{1}{2\pi}\int_0^{2\pi}\dot{\ddot{x}}(t-\tau)\dddot{x}(t) = -\sum_{k=1}^{n}k^4(a_k^2+b_k^2)\cos k\tau$$

$$\frac{1}{2\pi}\int_0^{2\pi}\dot{\ddot{x}}(t-\tau)xdt = -\sum_{k=1}^{n}(a_k^2+b_k^2)\sin k\tau$$

Thus,

$$I_1 = a^{-1}\sum_{k=1}^{n}[k^5-bk^3-dk][a_k^2+b_k^2k]\sin k\tau$$

$$|I_1| \le |a^{-1}|\sum_{k=1}^{n}\{|k^5-bk^3+dk|[a_k^2+b_k^2]\}$$

from conditions (ii) and (iii) we get

$$\begin{aligned} |I_1| &\le |a^{-1}|\sum_{k=1}^{n}k^2(a_k^2+b_k^2) = \frac{|a|^{-1}}{2\pi}\int_0^{2\pi}\dot{\ddot{x}}^2(t)dt \\ &\le |a^{-1}||\ddot{x}|_2^2 \\ &\le |a|^{-1}|\dot{x}|^2_{H^1_{2\pi}} \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{2\pi}\int_0^{2\pi}(\dot{\ddot{x}}(t-\tau)-\dot{\ddot{x}}(t))(\dddot{x}+\Gamma(t)\dot{x}(t-\tau))dt \\ &= -\sum_{k=1}^{n}k^4(a_k^2+b_k^2)\cos k\tau + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\dot{\ddot{x}}^2(t-\tau))dt \end{aligned}$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\ddot{x}(t-\tau)\ddot{x}(t-\tau)dt + \frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2 dt$$
$$- \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\ddot{x}(t)\dot{x}(t-\tau)dt$$
$$\geq -\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2 dt - \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\ddot{x}^2(t-\tau)dt$$
$$+ \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\ddot{x}(t-\tau)\ddot{x}(t-\tau)dt + \frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t)dt$$
$$- \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\ddot{x}(t)\ddot{x}(t-\tau)dt$$

using

$$-ab = \frac{(a-b)^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

we get

$$-\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\ddot{x}^2(t-\tau)dt$$
$$+\tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \Gamma(t)\ddot{x}^2(t-\tau)\ddot{x}(t-\tau)dt$$

$$+\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}[\ddot{x}(t-\tau) + \dddot{x}(t-\tau) - \ddot{x}(t)]^2 dt$$

$$+\frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}[\ddot{x}^2(t-\tau) - \ddot{x}^2(t-\tau) - 2\ddot{x}(t-\tau)\ddot{x}(t-\tau) - \ddot{x}^2(t)]dt$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}\ddot{x}^2(t-\tau)dt$$
$$+\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}[-\ddot{x}^2(t-\tau) - \ddot{x}^2(t)]dt$$
$$-\frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}\ddot{x}^2(t-\tau)dt + \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}[\ddot{x}(t-\tau) + \ddot{x}(t-\tau) - \ddot{x}(t)]^2 dt$$
$$= \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} [\dddot{x}^2(t-\tau) - \Gamma(t)\ddot{x}^2(t-\tau)]dt\right) + \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t-\tau)\right)$$
$$+\frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} [\Gamma(t)\ddot{x}^2(t-\tau) - \dddot{x}^2(t-\tau)]dt\right)$$
$$+\frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}[\ddot{x}(t-\tau) + \ddot{x}(t-\tau) - \ddot{x}(t)]^2 dt$$

$$-\tfrac{1}{2}\left(\tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \dddot{x}^2(t-\tau)dt\right)$$

Since $\Gamma(t) \geq n^2$ for a.e., t $\in [0, 2\pi]$ the last two terms imply that

$$-\frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} \ddddot{x}^2(t-\tau)dt\right)$$
$$+\tfrac{1}{2}\left(\tfrac{1}{2\pi}\int_0^{2\pi} \Gamma(t)[\dddot{x}(t-\tau) + (\dddot{x}(t-\tau) - \dddot{x})]^2\right) dt$$

$$\geq -\frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} \ddddot{x}^2(t-\tau)dt\right) + \frac{n^2}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t-\tau)dt\right)$$

$$+\frac{n^2}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}(t-\tau)\dddot{x}(t)\right)^2 dt$$

$$+\frac{n^2}{2\pi}\int_0^{2\pi} \dddot{x}(t-\tau)\dddot{x}(t-\tau)dt - \frac{n^2}{2\pi}\int_0^{2\pi} \dddot{x}(t-\tau)\dddot{x}(t)dt \geq 0$$

Since the last two terms are zero by orthogonality of $\dddot{x}$ and $\dddot{x}$ and the sum of the first two terms is non-negative by Parseval's equality. It follows that

$$I_2 \geq \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} [\ddddot{x}^2(t-\tau) - \Gamma(t)\dddot{x}^2(t-\tau)]\right) dt$$
$$+\frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} [\Gamma(t)\dddot{x}^2(t-\tau) - \ddddot{x}^2(t-\tau)]dt\right) \geq \delta|\dot{x}|^2_{H^1_{2\pi}}$$

by Lemma (2.2) and (2.3) of [4]. Therefore,

$$\begin{aligned} 0 &= I_1 + I_2 \geq \delta|\dot{x}|^2_{H^2_{2\pi}} - |a|^{-1}\delta|\dot{x}|^2_{H^1_{2\pi}} \\ &= (\delta - |a|^{-1})\delta|\dot{x}|^2_{H^1_{2\pi}} \end{aligned}$$

Since $\delta > |a|^{-1}$ we conclude that $\dot{x} = 0$ and hence $x =$ constant. It is clear that $x =$ constant cannot be a solution of (2.1) since $d \neq 0$.
Therefore $x = 0$.

3. The Non-Linear Case

We shall consider here the non-linear boundary value problem of the form.

$$\begin{aligned} &x^{iv} + a\dddot{x} + b\ddot{x} + g(t, \dot{x}(t-\tau)) + dx = p(t) \\ &x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3. \end{aligned} \tag{3.1}$$

where a, b, d are constants and $p(t) \in L^1_{2\pi}$, $g : I \times R \to R$ is such that $g(t + 2\pi, x) = g(t, x)$ and is a Caratheodory function with respect to $L^1_{2\pi}$, that is

(i) $g(\cdot, x)$ is measurable on I for each $x \in R$
(ii) $g(t)$ is continuous on R for a.e. $t \in I$
(iii) for each $r > 0$ there exists $Y_r \in L^1_{2\pi}$ such that

$$|g(t, x| \leq Y_r(t) \tag{3.2}$$

For almost everywhere $t \in I$ and all $x \in R$ such that $|x| \leq r$.
We have the following Lemma.

Lemma 3.1

Let all the conditions of theorem 2.1 be satisfied. Assume that $\alpha, \beta \in L^1_{2\pi}$ satisfy the following conditions:

$$n^2 \leq a^{-1}a(t) \leq a^{-1}\beta(t) \leq (n+1)^2$$

For a.e. $t \in [0, 2\pi]$ where $n \geq 1$ is an integer and $n^2 \leq a^{-1}\alpha(t)$, $\alpha^{-1}\beta(t) < (n+1)^2$ on subsets of $[0, 2\pi]$ of positive measure. Suppose that there exists constant $\varepsilon > 0$, and $\delta_0 > 0$ with

$$a^{-1}\alpha(t) - \varepsilon \leq a^{-1}c(t) \leq a^{-1}\beta(t) + \varepsilon \tag{3.3}$$

Then

$$\int_0^{2\pi} |x^{iv} + a\,\dddot{x} + b\ddot{x} + c(t)\dot{x}(t-\tau) + dx|dt \geq \delta_0 |x|_{H^{4,1}_{2\pi}} \tag{3.4}$$

Proof.

The proof follows the same procedure as in Lemma 3.1 of [5].
We shall now prove the following existence result for equation (3.1).

Theorem 3.1

Let a, b, d be constants such that

(i) $a \neq 0$
(ii) $b > n^2$
(iii) $0 < d < n$

and let g be a Caratheodory's function such that the inequalities

$$n^2 \leq \frac{\alpha(t)}{\alpha} \leq \lim_{|y|\to\infty} \inf \frac{g(t,y)}{ay} \leq \lim_{|y|\to\infty} \sup \frac{g(t,y)}{ay} \leq \frac{\beta(t)}{a} \leq (n+1)^2 \tag{3.5}$$

hold uniformly for a.e. $t \in I$, where $n \geq 1$ is an integer, $\alpha, \beta \in L^1_{2\pi}$ and the strict inequalities $n^2 < a^{-1}c(t)$, $a^{-1}c(t) < (n+1)^2$ hold on subsets of I of positive measure. Suppose that there exists $\delta > 0$ such that $\delta > |a|^{-1}$ then the boundary value problem (3.1) has at least one solution in $W^{4,1}_{2\pi}$.

Proof.

Let $\varepsilon > 0$ be associated to α, β in Lemma 3.1 then by (3.5) there exists a constant $r = r(\varepsilon)$ such that

$$\frac{\alpha(t)}{a} - \varepsilon \leq \frac{g(t,y)}{ay} \leq \frac{\beta(t)}{a} + \varepsilon$$

for a.e. $t \in I$ and all $y \in R$ with $|y| \geq r$.
Define a function

$$\tilde{Y} . R \to R \tag{3.6}$$

by

$$\tilde{Y}(t,y) = \begin{cases} y^{-1}g(t,y) & \text{if } |y| \geq r \\ yr^{-1}g(t,r) + (1 - yr^{-1})\beta(t), & 0 \leq y < r \\ yr^{-2}g(t,r) + (1 + yr^{-1})\beta(t), & -r < y \leq 0 \end{cases} \tag{3.7}$$

Hence,

$$\frac{\alpha(t)}{a} - \varepsilon \leq \frac{\tilde{Y}(t,y)}{ay} \leq \frac{\beta(t)}{a} + \varepsilon \tag{3.8}$$

For a.e. $t \in [0, 2\pi\}$ and all $y \in R$ with $|y| \geq r$.
Define $\tilde{g}$ and ϕ by $\tilde{g}(t,x) = \tilde{Y}(t,x)x$, $\phi(t,x) = g(t,x) - \tilde{g}(t,x)$ and observe that both $\tilde{g}$ and ϕ are caratheodory's functions.
Hence there exists $Y_r \in L^1_{2\pi}$ such that

$$|\phi(t,x)| \leq Y_r(t) \tag{3.9}$$

for a.e. $t \in I$ and all $x \in R$ where $Y_r = Y_r(\alpha, \beta)$.
Thus equation (3.1) is equivalent to

$$x^{iv} + a\,\dddot{x} + b\ddot{x} + \tilde{Y}(t, \dot{x}(t-\tau))\dot{x}(t-\tau) + \phi(t, \dot{x}(t-\tau)) + dx = p(t)$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0,1,2,3 \tag{3.10}$$

To apply coincidence degree theory [6] to (3.1) written in the form (3.10) we set

$$X = W^{4,1}_{2\pi}, \; Z = L^1_{2\pi},$$

To complete the proof we choose ρ such that $\rho > \delta_0^{-1}(|Y_r|_{L^1_{2\pi}} + |P|_{L^1_{2\pi}})$ □

4. Uniqueness Result

In this section we shall establish a uniqueness result for equation (3.1).

Theorem 4.1

Let all the conditions of theorem 3.1 hold with g satisfying

$$n^2 \le \frac{\alpha(t)}{a} \le \frac{g(t,\dot{x}) - g(t,\dot{y}))}{a(\dot{x}-\dot{y})} \le \frac{\beta(t)}{a} \le (n+1)^2 \tag{4.1}$$

for a.e. $t \in [0, 2\pi]$ and all $\dot{x} \neq \dot{y} \in R$ with α and β as in theorem 3.1.

Then problem (3.1) has a unique solution for each $P \in L^1_{2\pi}$

Proof.

Since condition (4.1) implies (3.5), theorem 3.1 ensures the existence of at least one solution.

Now let x and y be solutions of (3.1). Then by setting $v = x - y$, v is a solution of the problem

$$v^{iv} + a\,\dddot{v} + b\ddot{v} + g(t, \dot{v} + \dot{y} - g(t, \dot{y}) + dv = 0 \tag{4.2}$$

Define $f : I \times R \to R$ by

$$f(t) = \begin{cases} \dot{v}^{-1}[g(t, \dot{v} + \dot{y}) - g(t, \dot{y})], & \text{if } \dot{v} \neq 0 \\ \alpha(t), & \text{if } \dot{v} = 0 \end{cases}$$

Then (4.2) can be written in the form

$$v^{iv} + a\,\dddot{v} + b\ddot{v} + f(t)\dot{v} + dv = 0 \tag{4.3}$$

with

$$\frac{\alpha(t)}{a} \le \frac{f(t)}{a} \le \frac{\beta(t)}{a}$$

for a.e. $t \in I$ and all $\dot{v} \in R$.

If $\dot{v} = 0$ on every subset of $[0, 2\pi]$ of positive measure then $v =$ constant $= 0$.

Since $d \neq 0$. Hence $x = y$.

Suppose on the other hand that $\dot{v}(t) \neq 0$ on a certain subset of $[0, 2\pi]$ of positive measure, then using the arguments of theorem 2.1 we arrive that $v = 0$ and hence $x = y$. □