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EXISTENCE THEOREMS FOR A THIRD ORDER THREE POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. In this paper we present some results concerning the existence of solutions for the third order three point boundary value problems of the form

$$x'''(t) = f(t, x, x', x'')$$

$$x'(1) = x''(0) = 0, \;\; x(1) = ax(\eta),$$

where $\eta \in (0, 1)$, $f[0, 1] \times R^3 \to R^3$ is continuous and $a \in R$, with $a \neq 1$.

Introduction

Multipoint boundary value problem for second order differential equations have recently been the focus of study by several authors (see [5], [6], [10], [11]). However there are relatively few papers dealing with the study of third order multipoint boundary value problems. Multipoint boundary value problems arise from various sources. For instance, in solving linear partial differential equations by the method of separation of variables, one comes across differential equations containing several parameters with auxiliary conditions that the solutions satisfy a boundary condition at several points (see [4]).

In this paper we present some results concerning the existence of solutions for the third order three point boundary value problems of the form

$$x'''(t) = f(t, x, x', x'') \tag{1.1}$$

$$x'(1) = x''(0) = 0, \;\; x(1) = ax(\eta) \tag{1.2}$$

where $\eta \in (0, 1)$. $f[0, 1] \times R^3 \to R^3$ is continuous and $a \in R$, with $a \neq 1$.

In a recent paper [8] we obtained existence results for the above problem with $a = 1$ using a continuation theorem based on Mawhin's coincidence degree. Similarly Gupta and V Lakshmikan than [7] obtained existence and uniqueness results for the third order three point boundary value problem of the form.

$$x'''(t) = f(t, x, x', x'') - e(t) \tag{1.3}$$

$$x(0) = x(\eta) = x(1) = 0 \tag{1.4}$$

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(c) $zf(t,x,y,z) \leq (|z|^2+1)(D(t,x,y)+a(t))$ where $D(t,x,y)$ is bounded on bounded sets and $\alpha \in L^1[0,1]$. Then for $a \leq 0$ the boundary value problem (1.1)-(1.2) has at least one solution in $C^2[0,1]$ provided $M < \frac{\Pi^3}{16\sqrt{4+\Pi^2}}$

Proof: Since $a \neq 1$, L is one-to-one mapping. Let $K = L^{-1}$ so that $KN : X \to X$ is compact by the Arzela theorem. From the Leray-Schauder degree theory, existence of solution will follow if we can prove that the set of all possible solutions of the family of equations

$$x'''(t) = \lambda f(t,x,x',x'') \tag{2.2}$$

$$x'(1) = x''(0) = 0,\ x(1) = ax(\eta) \tag{2.3}$$

is bounded in $C^2[0,1]$ by a constant independent of $\lambda \in [0,1]$. To very this, suppose x is a solution of (2.2) - (2.3), so that $x \in D(L)$. The relation

$$x'(1) = x''(0) = 0 \quad \text{yields}$$

$$\int_0^1 x'x''dt = \int_0^1 |x''^2|dt$$

Therefore,

$$\begin{aligned}\int |x''|^2 dt &= -\lambda \int_0^1 x'g(t,x,x',x'')dt - \lambda \int_0^1 x'h(t,x,x',x'')dt \\ &\leq \int_0^1 |x'|h(t,x,x',x'')dt\end{aligned}$$

using the Cauchy inequality $|ab| \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ for $\epsilon > 0$.
we have

$$\int_0^1 |x'|\left|h(t,x,x',x'')dt \leq \frac{\epsilon}{2}\int_0^1 |x'|^2 dt + \frac{1}{2\epsilon}\int_0^1\right|h(t,x,x',x'')dt$$

From condition (b), we obtained the estimate

$$|h(t,x,y,z)^2 \leq 4M^2\{|x|^2 + |y|^2 + |z|^{2\beta}\}$$

Therefore, from Holder's inequality we get

$$|x''|_2^2 - \frac{\epsilon}{2}|\,|x'|_2^2 \leq \frac{2M^2}{\epsilon}\left\{|x|_2^2 + |x'|_2^2 + |x''|_2^{2\beta}\right\} \tag{2.4}$$

Since $x'(1) = 0$, we obtain from Lemma that

$$\frac{1}{2}|x''|_2^2 + \left(\frac{\pi^2}{8} - \frac{\epsilon}{2}\right)|x'|_2^2 \leq \frac{2M^2}{\epsilon}\left\{|x|_2^2 + |x'|_2^2 + |x''|_2^{2\beta}\right\}$$

Further, since $a \leq 0$, $x(1)$ and $x(\eta)$ have opposite signs. Therefore, there exist $\xi \in (\eta, 1)$ such that $x(\xi) = 0$. Hence for each $t \in [0,1]$, we gave

$$\left|x\right|_2^2 \leq \frac{4}{\pi^2}\left|x'\right|_2^2 \tag{2.5}$$

Theorem 2.2

Suppose the assumptions of theorems 2.1 hold. Then for $1 \neq a > 0$, the boundary value problem (1.1)-(1.2) has at least one solution in $C^2[0,1]$ provided

$$M < \min\left(\frac{1}{2B}, \frac{\pi^2}{16}\right) \quad \text{where} \quad B = \left(\frac{1-\eta}{|a-1|} + \frac{\sqrt{2}}{\pi}\right)^2.$$

Proof: As in the proof of Theorem 2.1, it suffices to verify that the set of all possible solutions of the family of equations

$$x'''(t) = \lambda f(t, x, x', x'') \tag{2.12}$$

$$x'(1) = x''(0) = 0, \quad x(1) = ax(\eta) \tag{2.13}$$

is bounded in $C^2[0,1]$ by a constant independent of $\lambda \in [0,1]$. By the mean value theorem there exist $\xi \in (\eta, 1)$ such that

$$x(\eta) = \frac{1-\eta}{a-1} x'(\xi) \qquad \text{(see [7] Lemma 2.2)}$$

Hence for $t \in [0,1]$, we have

$$\begin{aligned} x(t) &= x(\eta) + \int_\eta^1 x'(s)ds = \frac{1-\eta}{a-1} x'(\xi) + \int_\eta^1 x'(s)ds \\ &= \frac{1-\eta}{a-1} \int_0^\xi (x''(s)ds + \int_\eta^1 x'(s)ds \end{aligned} \tag{2.14}$$

Therefore

$$|x|_2 \leq \left(\frac{1-\eta}{|a-1|} + \frac{\sqrt{2}}{\pi}\right) |x''|_2 = \sqrt{B}|x''|_2 \tag{2.15}$$

where $B = \left(\frac{1-\eta}{|a-1|} + \frac{\sqrt{2}}{\pi}\right)^2$.

Proceeding as in the proof of Theorem 2.1, we substitute (2.15) in (2.4) to get

$$\left(\frac{1}{2} - \frac{2M^2 B}{\epsilon}\right) |x''|_2^2 + \left(\frac{\pi^2}{8} - \frac{\pi}{2} - \frac{2M^2}{\epsilon}\right) |x'|_2^2 \leq \frac{2M^2}{\epsilon} |x''|_2^{2\beta} \tag{2.16}$$

Thus, there exist constants C_1 andC_2 such that

$$|x''|_2 < C_1 \text{ provided } M < \sqrt{\frac{\epsilon}{3B}} \tag{2.17}$$

and

$$|x'|_2 < C_2 \text{ provided } \frac{\pi^2}{8} > \frac{\eta}{2} + \frac{2M^2}{\epsilon} \tag{2.18}$$

The choice $\epsilon = 2M$ minimizes the right hand sides of (2.18) and the minimum value is $2M$. Hence (2.17) and (2.18) will hold simultaneously provided

$$M < \min\left(\frac{1}{2B}, \frac{\pi^2}{16}\right)$$

Since $x'(1) = 0$, we derive that

$$|x'|_\infty \leq |x''|_2 < C_2 \tag{2.19}$$

Further, since $x'(1) = 0$, we get

$$|x|_\infty < \left[1 + \frac{1-\eta}{|a-1|}\right] |x'|_\infty < \left[1 + \frac{1-\eta}{|a-1|}\right] C_5 = C_6$$

The remaining part of the proof is completed in Theorem 2.1.

Corollary 2.3: The results of Theorem 2.3 and Theorem 2.4 remain valid if assumption (d) is replaced by either of the following

(d_1) $\quad |h(t, x, y, z)| \leq M\{|x| + |y|^q + |z|^r\}$ for $0 \leq q,\ r < 1$

provided $M < \frac{\pi^3}{32}$.

(d_2) $\quad |h(t, x, y, z)| \leq M\{|x|^p + |y| + |z|^r\}$ for $0 \leq p,\ r < 1$

provided $M < \frac{\pi^3}{16}$.

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