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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A FOURTH ORDER FOUR POINT BOUNDARY VALUE PROBLEM

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1. Introduction

The purpose of this paper is to provide existence and uniqueness results for the fourth order four point boundary value problem

$$x^{(4)} = f(t, x, x, \ddot{x}, x^{(3)}) + e(t) \tag{1.1}$$

$$x(0) = x(\eta_1) = x(\eta_2) = x(1) = 0 \tag{1.2}$$

where $f: [0,1] \times \mathbf{R}^4 \to \mathbf{R}$ is a given function satisfying Caratheodory's conditions, $e: [0,1] \to \mathbf{R}$ is a function in $L^1[0,1]$ and η_1, η_2 are in $(0,1)$ with $0 < \eta_1 < \eta_2 < 1$. Note, for example, that $\dot{x} = \frac{dx}{dt}$ and $x^{(k)} = \frac{d^k x}{dt^k}$. Various fourth order boundary value problems are used to model deformation of elastic beams which have found applications in structures such as aircrafts, buildings, ships and bridges. Some of these equations have been extensively studied in recent years (see e.g. **[2]**, **[4]**, **[11]** and the references therein).

Our study is motivated by the recent results of C. P. Gupta and V. Lakshmikanthan **[3]** for a third order three point boundary value problem.

We shall use topological degree **[7]** and Wirtinger type inequalities to obtain the necessary a-priori estimates.

2. Existence results

For convenience we gather together some of the inequalities we shall use in the forthcoming sections.

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where

$$B = \frac{\eta_1}{6\eta_2(\eta_2-1)(\eta_1-\eta_2)}\int_0^{\eta_2}(\eta_2-s)^3 e(s)\,ds - \frac{\eta_2}{6\eta_1(\eta_1-1)(\eta_1-\eta_2)}$$
$$\times \int_0^{\eta_1}(\eta_1-s)^3 e(s)\,ds - \frac{\eta_1\eta_2}{6(\eta_2-1)(\eta_1-1)}\int_0^1(1-s)^3 e(s)\,ds$$
$$C = \frac{\eta_2+\eta_1}{6(\eta_2-1)(\eta_1-1)}\int_0^1(1-s)^3 e(s)\,ds + \frac{1+\eta_2}{6\eta_1(\eta_1-1)(\eta_1-\eta_2)}$$
$$\times \int_0^{\eta_1}(\eta_1-s)^3 e(s)\,ds - \frac{1+\eta_1}{6\eta_2(\eta_2-1)(\eta_1-\eta_2)}\int_0^{\eta_2}(\eta_2-s)^3 e(s)\,ds$$
$$D = \frac{1}{6\eta_2(\eta_2-1)(\eta_1-\eta_2)}\int_0^{\eta_2}(\eta_2-s)^3 e(s)\,ds - \frac{1}{6(\eta_1-1)(\eta_2-1)}$$
$$\times \int_0^1(1-s)^3 e(s)\,ds - \frac{1}{6\eta_1(\eta_1-1)(\eta_1-\eta_2)}\int_0^{\eta_1}(\eta_1-s)^3 e(s)\,ds.$$

THEOREM 2.1. *Let $f\colon[0,1]\times\mathbf{R}^4\to\mathbf{R}$ be a Caratheodory's function; that is,*

(i) *$f(\cdot,x)$ is measurable for each $x\in\mathbf{R}^4$*

(ii) *$f(t,\cdot)$ is continuous for a.e. $t\in[0,1]$*

(iii) *For every $s>0$ there exists $h_s\in L^1[0,1]$ such that $|f(t,x)|\le h_s$ for a.e. $t\in[0,1]$ where $\|x\|\le s$.*

Assume that

(1) *there exists a,b,c,d in $\mathbf{R}$ and $\alpha(t)\in L^1[0,1]$ such that*

$$\ddot{x}f(t,x,\dot{x},\ddot{x},x^{(3)}) \ge a\ddot{x}^2 + b|\ddot{x}|\,|\dot{x}| + c|\ddot{x}|\,|x^{(3)}| + d|x|\,|\ddot{x}| + \alpha(t)|\ddot{x}| \tag{2.4}$$

(2) *there exist functions $p(t),q(t),r(t),s(t)$ in $L^2[0,1]$ and a function $T(t)\in L^1[0,1]$ such that*

$$|f(t,x,\dot{x},\ddot{x},x^{(3)})| \le p(t)|x| + q(t)|\dot{x}| + r(t)|\ddot{x}| + s(t)|x^{(3)}| + T(t) \tag{2.5}$$

for a.e. $t\in[0,1]$ and all $(x.\dot{x},\ddot{x},x^{(3)})\in\mathbf{R}^4$.

Let η_1,η_2 in $(0,1)$ be given with $0<\eta_1<\eta_2<1$.

Then for every given $e\in L^1[0,1]$ the four point bounday value problem

$$x^{(4)} = f(t,x,\dot{x},\ddot{x},x^{(3)}) - e(t)$$
$$x(0) = x(\eta_1) = x(\eta_2) = x(1) = 0$$

has at least one solution in $C^3[0,1]$ provided

$$\frac{8}{\pi^3}\|p\|_2 + \frac{8}{\pi^3}\|q\|_2 + \frac{4}{\pi}\|r\|_2 + 2\|s\|_2 + |a|\frac{4}{\pi^2} + |b|\frac{8}{\pi^3} + |c|\frac{2}{\pi} + |d|\frac{8}{\pi^4} < 1 \tag{2.6}$$

From equation (2.7) and condition (2.5) we get

$$
\begin{aligned}
\|x^{(4)}\|_1 &\leq \lambda \|f(t, x, \dot{x}, \ddot{x}, x^{(3)})\|_1 + \|e\|_1 \\
&\leq \|p(t)|x| + q(t)|\dot{x}| + r(t)|\ddot{x}| + s(t)|x^{(3)}| \\
&\quad + T(t)\|_1 + \|e\|_1 \\
&\leq (\|p\|_2 \frac{4}{\pi^3} + \|q\|_2 \frac{4}{\pi^2} + \|r\|_2 \frac{2}{\pi} + \|s\|_2)\|x^{(3)}\|_2 \\
&\quad + \|T\|_1 + \|e\|_1 \qquad (2.11)
\end{aligned}
$$

Using (2.11) in (2.10) we obtain

$$
\begin{aligned}
0 &\geq \|x^{(3)}\|_2^2 - 2\|x^{(3)}\|_1 \left(\|p\|_2 \frac{4}{\pi^3} + \|q\|_2 \frac{4}{\pi^2} + \|r\|_2 \frac{2}{\pi} + \|s\|_2 \right) \|x^{(3)}\|_2 \\
&\quad - 2\|x^{(3)}\|_1 (\|T\|_1 + \|e\|_1) - \left(|a| \frac{4}{\pi^2} + |b| \frac{8}{\pi^3} + |c| \frac{2}{\pi} \right. \\
&\quad \left. + |d| \frac{8}{\pi^4} \right) \|x^{(3)}\|_2^2 - (\|\alpha\|_1 + \|e\|_1)\|x^{(3)}\|_2 \\
&\geq \|x^{(3)}\|_2^2 - \left(\|p\|_2 \frac{8}{\pi^3} + \|q\|_2 \frac{8}{\pi^2} + \|r\|_2 \frac{4}{\pi} + 2\|s\|_2 \right) \|x^{(3)}\|_2^2 \\
&\quad - \left(|a| \frac{4}{\pi^2} + |b| \frac{8}{\pi^3} + |c| \frac{2}{\pi} + |d| \frac{8}{\pi^4} \right) \|x^{(3)}\|_2^2 \\
&\quad - (\|\alpha\|_1 + 2\|T\|_1 + 3\|e\|_1)\|x^{(3)}\|_2^2 \\
&\geq \left[1 - \left(\frac{8}{\pi^3} \|p\|_2 + \frac{8}{\pi^2} \|q\|_2 + \frac{4}{\pi} \|r\|_2 + 2\|s\|_2 + |a| \frac{4}{\pi^2} \right. \right. \\
&\quad \left. \left. + |b| \frac{8}{\pi^3} + |c| \frac{2}{\pi} + |d| \frac{8}{\pi^4} \right) \right] \|x^{(3)}\|_2^2 \\
&\quad - (2\|T\|_1 + 3\|e\|_1 + \|\alpha\|_1)\|x^{(3)}\|_2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(2.12) \quad \|x^{(3)}\|_2 \\
&\leq \frac{2\|T\|_1 + 3\|e\|_1 + \|\alpha\|_1}{1 - \left(\frac{8}{\pi^3}\|p\|_2 + \frac{8}{\pi^2}\|q\|_2 + \frac{4}{\pi}\|r\|_2 + 2\|s\|_2 + |a|\frac{4}{\pi^2} + |b|\frac{8}{\pi^3} + |c|\frac{2}{\pi} + |d|\frac{8}{\pi^4} \right)} \\
&\equiv \beta_1
\end{aligned}
$$

for some $\beta_1 > 0$.

Now

$$x^{(3)}(t) = \int_{t_0}^{t} x^{(4)}(s)\, ds.$$

Hence $\|x^{(3)}\|_\infty \leq \|x^{(4)}\|_1$. It follows from (2.11) and (2.12) that there exists $\beta_2 > 0$ such that

$$\|x^{(3)}\|_\infty \leq \beta_2. \qquad (2.13)$$

(2.18) $$|f(t,x,\dot{x},\ddot{x},x^{(3)})| \leq p(t)|x| + q(t)|\dot{x}| + r(t)|\ddot{x}| + s(t)|x^{(3)}| + T(t)$$

for a.e. $t \in [0,1]$ *and all* $(x,\dot{x},\ddot{x},x^{(3)}) \in \mathbf{R}^4$. *Let* η_1, η_2 *in* $(0,1)$ *be given with* $0 < \eta_1 < \eta_2 < 1$.

Then for every $e(t)$ *in* $L^1[0,1]$ *the boudary value problem (1.1)-(1.2) has at least one solution in* $C^3[0,1]$ *provided condition (2.6) holds.*

PROOF. Multiply (2.7) by $\ddot{x}$ and integrating over $[0,1]$ we obtain

$$\begin{aligned} 0 &= -\int_0^1 x^{(4)}\ddot{x}\,dt + \lambda\int_0^1 f(t,x,\dot{x},\ddot{x},x^{(3)})\ddot{x}\,dt + \lambda\int_0^1 e(t)\ddot{x}\,dt \\ &= \int_0^1 (x^{(3)})^2\,dt - x^{(3)}(1)\ddot{x}(1) + x^{(3)}(0)\ddot{x}(0) \\ &\quad + \lambda\int_0^1 f(t,x,\dot{x},\ddot{x},x^{(3)})\ddot{x}\,dt + \lambda\int_0^1 e(t)\ddot{x}\,dt \\ &\geq \|x^{(3)}\|_2^2 - |x^{(3)}(1)|\,|\ddot{x}(1)| - |x^{(3)}(0)|\,|\ddot{x}(0)| + \lambda\int_0^1 \big[a\ddot{x}^2 \\ &\quad + b|\ddot{x}|\,|\dot{x}| + c|\ddot{x}|\,|x^{(3)}| + d|x|\,|\ddot{x}| + \alpha(t)\,|\ddot{x}| + s(t)\,|x| \\ &\quad + \beta(t)\,|\dot{x}| + \delta(t)\,|x^{(3)}|\big] + \lambda\int_0^1 e(t)\ddot{x}\,dt \\ &\geq \|x^{(3)}\|_2^2 - 2\|x^{(3)}\|_1\|x^{(4)}\|_1 - \left(|a|\frac{4}{\pi^2} + |b|\frac{8}{\pi^3} + |c|\frac{2}{\pi}\right. \\ &\quad \left. + |d|\frac{8}{\pi^4}\right)\|x^{(3)}\|_2^2 - \left(\|\alpha\|_1 + \|s\|_1\frac{4}{\pi^2} + \|\beta\|_1\frac{2}{\pi}\right. \\ &\quad \left. + \|\delta\|_2\right)\|x^{(3)}\|_2 - \|e\|_1\|x^{(3)}\|_2 \end{aligned} \tag{2.19}$$

From equation (2.7) and condition (2.18) we get

$$\begin{aligned} \|x^{(4)}\|_1 \leq \|p\|_2\|x\|_2 + \|q\|_2\|\dot{x}\|_2 + \|r\|_2\|\ddot{x}\|_2 + \|s\|_2\|x^{(3)}\|_2 \\ + \|T\|_1 + \|e\|_1 \end{aligned} \tag{2.20}$$

Using (2.20) in (2.19) we get

$$\begin{aligned} 0 \geq \|x^{(3)}\|_2^2 &- \left(\|p\|_2\frac{8}{\pi^3} + \|q\|_2\frac{8}{\pi^2} + \|r\|_2\frac{4}{\pi} + 2\|s\| + |a|\frac{4}{\pi^2}\right. \\ &\left. + |b|\frac{8}{\pi^3} + |c|\frac{2}{\pi} + |d|\frac{8}{\pi^4}\right)\|x^{(3)}\|_2^2 - (\|T\|_1 + \|e\|_1)\|x^{(3)}\|_2 \\ &- \left(\|\alpha\|_1 + \|s\|_1\frac{4}{\pi^2} + \|\beta\|_1\frac{2}{\pi} + \|\delta\|_2\right)\|x^{(3)}\|_2\,, \end{aligned}$$