A Third Order Three-Point Boundary Value Problem at Resonance

S.A. Iyase,
C/O National Mathematical Centre,
P.M.B. 118, Abuja.

1 Introduction

Multipoint boundary value problems for second order differential equations have recently been the focus of study by several authors (see [4],[7],[8]). However, there are relatively few papers dealing with the study of third order multipoint boundary value problems. Multipoint boundary problems arise from various sources. For instance in solving linear partial differential equations by the method of separation of variables, one comes across differential equations containing several parameters with auxiliary condition that the solutions satisfy a boundary condition at several points (see [5]).
In this paper we present some results concerning the existence of solutions for the third order three point boundary value problem of the form

$$x'''(t) = f(t, x, x', x'') \tag{1.1}$$

$$x'(1) = x''(0) = 0, \quad x(1) = x(\eta) \tag{1.2}$$

where $\eta \in\rightarrow (0,1); f : [0,1] \times \mathbb{R}^3 \in \mathbb{R}$ is continuous. Our method of proof consists of imposing a decomposition condition on f of the form

$$f(t, x, y, z) = g(t, x, y, z) + h(t, x, y, z).$$

We shall employ the coincidence degree theory of Mawhim to obtain our existence results.

2 Existence Results

Let X denote the Banach space C^2 [0,1] and Z denote the Banach space L^1 [0,1]. We define the linear mapping

$$L : D(L) \subset X \rightarrow Z \quad \text{by setting}$$

$$D(L) = \{x \in W^{3,1}(0,1) : x'(1) = x''(0) = 0, \quad x(1) = x(\eta)\}$$

and for

$$x \in D(L), Lx = x'''$$

Let $N : X \rightarrow Z$ be the nonlinear mapping defined by

$$(Nx)(t) = f(t, x, x', X''), t \in [0,1]$$

Then the boundary value problem (1.1) - (1.2) can be put in the abstract form

$$Lx = Nx \tag{2.1}$$

We shall prove the following theorem.

0.0.1 Theorem 2.1

Assume that $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and has the decomposition

$$f(t,x,y,z) = g(t,x,y,z) + h(t,x,y,z)$$

such that

(i) $xf(t,x,y,z) >$ for a.e.t. $\in [0.1]$ and $(x,y,z) \in \mathbb{R}^3$

(ii) $yg(t,x,y,z) \geq 0$

(iii) $| h(t,x,y,z) | \leq M\{| x | + | y | + | z |^{\beta}\}$ for $0 \leq \beta < 1$

(iv) $zf(t,x,y,z) \leq (| z |^2 + 1)(D(t,x,y) + \alpha(y))$ where $D(t,x,y)$ is bounded on bounded sets and $\alpha \in L^1[0,1]$.

Then the boundary value problem (1.1) - (1.2) has at least one solution in $C^2[0,1]$ provided

$$M < \frac{\pi^3}{16\sqrt{4+\pi^2}}$$

In the proof of Theorem 2.1 we shall need the following continuation theorem based on Mawhin's coincidence degree.

Theorem: Let Ω be a bounded open set in X and suppose that the following conditions hold

(1) $Lx \neq \lambda Nx$ for any $(x,\lambda) \in (domL \cap \partial\Omega) \times (0,1)$

(2) $QNx \neq 0$ for $x \in$ ker $L \cap \partial\Omega$

(3) The Brouwer degree deg $_{\beta}(JQN)_{\text{ker}L,}$ $\Omega \cap \text{ker}L, 0) \neq 0$ where $J : imQ \to$ ker$L$ is some isomorphism. Then there exists $x \in \bar{\Omega} \cap$ dom$L$ such that

$$Lx = Nx$$

0.1 Proof of Theorem 2.1

Let L be defined as above. Then ker $L = \{x \in X : x$ is a constant mapping $\} \simeq \mathbb{R}$

$$imL = \{z \in Z : \int_{\eta}^{1}\int_{t}^{1}\int_{o}^{\tau} z(s)dsd\tau dt = 0\}$$

the latter is closed in Z and of co-dimension 1. Thus L is Fredholm operator of index zero. Therefore from the results of linear functional analysis there exist continuous projections

$$p : X \to \text{ker } L \quad \text{and} \quad Q : Z \to Z_1$$

which we define by

$$(px)(t) = x(0)$$

and

$$Qz = \frac{6}{(\eta+2)(\eta-1)^2}\int_\eta^1\int_t^1\int_o^\tau z(s)dsd\tau dt, z \in imL$$

so that

$$X = \ker L \oplus \ker p, \quad z = imL \oplus imQ$$

and

$$L_p = L\mid_{D(L)\cap \ker p}$$

The operator $k = L_p^{-1} : im : L \to D(L) \cap \ker p$ is the linear operator defined by

$$(ky)(t) = \frac{1}{2}\int_o^t (t-s)^2 y(s)ds \tag{2.2}$$

By the Arzela-Ascoli theorem it can be shown that k is compact. Hence N is L-compact.
We shall prove that the conditions of the theorem are satisfied. To do this, we shall show that for $\lambda \in (0,1)$, the set of solutions of the family of equations

$$x''' = \lambda f(t, x, x' x'') \tag{2.3}$$

$$x'(1) = x''(0) = 0, \quad x(1) = x(\eta) \tag{2.4}$$

is apriori bounded and then construct Ω accordingly.
Let $x \in C^2[0,1]$ satisfy (2.3)-(2.4). From condition (i) we derive that if $x(t) > 0$ then $f(t,x,y,z) > 0$ and if $x(t) < 0$ then $f(t,x,y,z) < 0$. Since $x(1) = x(\eta)$, there exist $\xi \in (\eta, 1)$ such that $x'(\xi) = 0$ and from $x'(1) = x'(\xi) = 0$ there exist $t_1 \in (\xi, 1)$ such that $x''(t_1) = 0$. Hence if $x(t) > 0$ then

$$0 = \int_o^{t_1} x'''(s)ds = \lambda\int_o^{t_1} f(t, x(s), x'(s), x''(s)ds > 0$$

a contradiction. If $x(t) < 0$ we derive a similar contradiction. Hence there exist $t_o \in (0, t_1)$ such that $x(t_o) = 0$. Hence for each $t \in [0,1]$ we have

$$\mid x \mid_2^2 \leq \frac{4}{\pi^2} \mid x' \mid_2^2 \tag{2.5}$$

Multiplying (2.3) by $x'(t)$ and using the relation

$$x'(1) = x''(0) = 0$$

yields

$$\int_o^1 x' x''' dt = -\int_o^1 \mid x'' \mid^2 dt$$

Therefore

$$\begin{aligned} \textstyle\int_o^1 \mid x'' \mid^2 dt &= -\lambda \textstyle\int_o^1 x' g(t,x,x',x'')dt - \lambda\int_o^1 x' h(t,x,x',x'')dt \\ &\leq \textstyle\int_o^1 \mid x' \mid\mid h(t,x,x',x'')dt. \end{aligned}$$

Using the Cauchy inequality $| ab |\leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ for $\epsilon > 0$, we have

$$\int_o^1 | x^{'} || h(t,x,x^{'},x^{''}) | \, dt \leq \frac{\epsilon}{2}\int_o^1 | x^{'} |^2 \, dt + \frac{1}{2\epsilon}\int_o^1 | h(t,x,x^{'},x^{''}) |^2 \, dt$$

From condition (iii), we obtain the estimate

$$| h(t,x,y,z) |^2 \leq 4M^2 | \{ | x |^2 + | y |^2 + | z |^{2\beta} \}$$

Therefore from the Holder's inequality, we get

$$| x^{''} |_2^2 - \frac{\epsilon}{2} | x^{'} |_2^2 \leq \frac{2M^2}{\epsilon}\{ | x |_2^2 + | x^{'} |_2^2 + | x^{''} |_2^{2\beta} \}$$

Since $x^{'}(1) = 0$, we obtain

$$\frac{1}{2} | x^{''} |_2^2 + \left(\frac{\pi^2}{8} - \frac{\epsilon}{2}\right) | x^{'} |_2^2 \leq \frac{2M^2}{\epsilon}\{ | x |_2^2 + | x^{'} |_2^2 + | x^{''} |_2^{2\beta} \} \qquad (2.6)$$

Using (2.5) in (2.6), we get

$$\frac{1}{2} | x^{''} |_2^2 + \left(\frac{\pi^2}{8} - \frac{\epsilon}{2} - \frac{8M^2}{\epsilon\pi^2} - \frac{2M^2}{\epsilon}\right) | x^{'} |_2^2 \leq \frac{2M^2}{\epsilon} | x^{''} |_2^{2\beta}$$

Since $0 \leq \beta < 1$ we infer the existence of a constant M_1 such that

$$| x^{'} |_2 < | x^{''} |_2 < M_1, \qquad (2.7)$$

provided

$$\frac{\pi^4}{8} > \frac{8M^2}{\epsilon} + \frac{2\pi M^2}{\epsilon} + \frac{\epsilon\pi^2}{2} \qquad (2.8)$$

The choice $\epsilon = 2M\sqrt{4+\pi^2}$ minimizes the right hand side of (2.8) and the minimum values is $2M\pi\sqrt{4+\pi^2}$. Therefore (2.8) holds provided $M < \frac{\pi^3}{16\sqrt{4+\pi^2}}$.

Furthermore, since $x(t_o) = x^{'}(1) = 0$ for $t_o \in (\eta, 1)$ we get from (2.7) that

$$| x |_\infty < | x^{'} |_\infty < M_2 \qquad (2.9)$$

for some constant $M_2 > 0$.

From condition (iv) of Theorem 1.1, we obtain

$$\frac{x^{''}x^{'''}}{| x^{''} |^2 + 1} \leq D(t,x,x^{'}) + \alpha(t). \qquad (2.10)$$

Integrating (2.10) from 0 to t we get

$$\int_o^t \frac{x^{''}(s)x^{'''}(s)}{| x^{''}(s) |^2 + 1} ds = \left[\frac{1}{2}\log_e (| x^{''}(s) |^2 + 1)\right]_o^t < D + | \alpha |_1 = N, \qquad (2.11)$$

where the constant D depends only on M_2. Furthermore since $x^{''}(0) = 0$ we get from (2.11) that

$$| x^{''} |_\infty < e^N = M_3. \qquad (2.12)$$

Let

$$\|x\| = \max(|x|_\infty, |x'|_\infty, |x''|_\infty) < \max(M_2, M_3) = R.$$

It follows that $\|x\| < R$.
We take $\Omega = \{x \in X : \|x\| < R\}$, then if $x \in D(L) \cap \partial\Omega$ then $Lx \neq \lambda Nx, 0 < \lambda < 1$.
If $x \in \ker L \cap \partial\Omega$ then $x = \pm R$.
Now if $x = R$ we derive from condition (i) that

$$QNx = \frac{6}{(\eta+2)(\eta-1)^2} \int_\eta^1 \int_t^1 \int_o^\tau f(S, R, 0, 0) > 0$$

and if $x = -R$ we get

$$QNx < 0.$$

Thus $QNx \neq 0$ for $x \in \ker L \cap \partial\Omega$, verifying condition (2) of the theorem. It is easily verified that

$$H(\mu, x) = \mu x + (1-\mu)QNx, \quad 0 \leq \mu < 1$$

is a homotopy from the identity I to QN on $\bar{\Omega}$ and is such that $H(\mu, x) \neq 0$ on [0,1] $\times(\partial\Omega \cap \ker L)$. Hence taking J in condition (3) of theorem to be the identity we get.

$$\deg_B [QN|_{\ker L}, \Omega \cap \ker L, 0] = \det_B[I, \Omega \cap \ker L, O] = 1$$

This completes the of proof our theorem.

Remark 2.1 The results of Theorem 2.1 still hold if condition (i) is replaced by the condition

$$xf(t, x, y, z) < 0.$$

Remark 2.2 The results of Theorem 2.1 remains valid if assumption (iii) is replaced by any of the following assumptions.

1. $| h(t, x, y, z) | \leq M(| x | + | y |^\beta + | z |)$ for $0 \leq \beta < 1$ provided $M < \frac{\pi}{8\sqrt{16+\pi^4}}$
2. $| h(t, x, y) | \leq M(| x |^\beta + | y | + | z |)$ for $0 \leq \beta < 1$ provided $M < \frac{1}{\pi}$
3. $| h(t, x, y, z) | \leq M\{| x |^r + | y |^\beta + | z |^\gamma\}$ for $0 \leq \gamma, \beta, r < 1$ for some constant M.
4. $| h(t, x, y, z) | \leq M\{| x |^\beta + | y |^r + | z |\}, 0 \leq \beta, r < 1$ for some constant M.

References

1. **Aftabizadeh, A.R., Gupta, C.P. and Jian; Ming Xu**, *Existence uniqueness theorems for three point boundary value problems.* SIAM J. Math. Anal. 20, 716 - 726 (1989).

2. *Constantin. A. A note on a boundary value problem.* Nonlinear Analysis 27(1), 13 - 16 (1996).

3. **Dymand, McKean, H.P.** *Fourier series and Integrals.* Academic Press, New York.

4. **Feng. W. and Webb, R.L.** *Solvability of m-point boundary value problem with nonlinear growth.* Journal of Mathematical Analysis and Applications 21, 467 - 480.

5. **Gregus, M., Neumann, F. and Arscott, F.M.** *Three points boundary value problem for differential equations.* J. London Math. Soc. 3, 429 - 436 (1971).

6. **Gaines, R.E. and Mawhin, J.** *Coincidence degree and Nonlinear differential equations.* Lecture Notes. Math 568 Berlin (1977).

7. **Granas, R. Guenther, B. and Lee, W.** *Some general existence principles in the caratheodory theory of nonlinear differential systems*, J. Math. Pures et Appl. 70, 153 - 196 (1991).

8. **Gupta, C.P., Ntouyas, K. and Tsamatos, P.** *Solvability of an m-point boundary value problem for second order ordinary differential equations.* J. Math. Anal. Appl. 189, 575 - 584 (1995).

9. **Gupta, C.P. and Lakshmikantham, V.** *Existence and uniqueness theorems for a third order three point boundary value problem.* Nonlinear Analysis 16 (11), 949-957 (1991).

10. **Mawhin, J.** *Topological degree methods in Nonlinear boundary value problems, NSF-CBMS Reg. Conf. Ser. Math. 40 Americ.* Math. Soc. Providence RI (1979).

11. **Schmitt, K.** *Periodic solutions of a forced nonlinear oscillator involving a one sided restoring force.* Arch. Math. Vol. 31, 70 - 73 (1978).