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THREE STEPS HYBRID BLOCK METHOD FOR THE SOLUTION OF GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

Block method is adopted in this paper for the direct solution of second order ordinary differential equations. The method is derived by collocation and interpolation of power series approximate solution to give a continuous hybrid linear multistep method which is implemented in block method to derive the independent solution at selected grid points. The properties of the derived scheme were investigated and found to be zero-stable, consistent and convergent. The efficiency of the derived method was tested and found to compare favourably with the existing methods.

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1. Introduction

In this paper, we consider approximate techniques for the solution of second-order initial value problems of the form

$$y'' = f(x, y, y'), \quad y(a) = y_a, \quad y'(a) = y'_a, \tag{1}$$

where a is the initial point, y_a and y'_a are the solutions at the initial point a, f is assumed to be continuous within the interval of integration and satisfies the existence and uniqueness conditions. Awoyemi et al. [1] had given the theorem for the existence and uniqueness of (1). Scholars have discussed method of reduction of higher order ordinary differential equations to systems of first order ordinary differential equation to increase the dimension of resulting equation by the order of the differential equations, this invariably involves more human and computer efforts, Vigo-Aguiar and Ramos [2] reported that the method of reduction does not utilize additional information associated with specific ordinary differential equations such as the oscillatory nature of the solution. Bun and Vasil'Yev [3] also reported that another disadvantage of method of reduction is the fact that resulting system of first order equation can not be solved explicitly with respect to the derivatives of the highest order. Conclusively, method of reduction is not efficient and unstable for general purpose.

Continuous linear multistep method for the direct solution of higher order ordinary differential equations has been proposed by scholars. Awoyemi [4], Kayode and Adeyeye [5], Adesanya et al. [6], Adee et al. [7], Yusuf and Onumanyi [8] proposed implicit continuous linear mutistep method which was implemented in predictor corrector mode, using Taylor series approximation to supply the starting values. This method was found to be costly to implement and the derived predictors are in reducing order of accuracy, which has an adverse effect on the results generated.

Block method which is more cost effective and does not require starting values has been proposed by scholars for the solution of initial value problems. Jator and Li [9] proposed a family of linear multistep method using methods of collocation and interpolation of power series approximate solution for the solution of two point boundary value problems. Adesanya et al. [10] proposed a block method through interpolation and collocation to solve second order initial value problem. This method gives better approximation and is cost effective.

In this paper, we propose that a hybrid block method is implemented as a simultaneous integrator for the solution of general second order ordinary differential equations.

2. Methodology

We propose an approximate solution to (1) in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j, \tag{2}$$

where r and s are the number of interpolation and collocation points, respectively, a_j''s are constant parameters to be determined, x is the polynomial basis function of degree j.

Substituting the second derivative of (2) into (1) gives

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1)a_j x^j. \tag{3}$$

In this paper, we consider a grid with step length of three with constant step length (h), where $h = x_{n+i} - x_n$, $i = 0, 1, 2, 3$ and off step points at $\frac{1}{2}$, $\frac{3}{2}$ and $\frac{5}{2}$.

Interpolating (2) at x_{n+r}, $r = 1, 2$, and collocating (3), at all grid and off grid points gives

$$\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 & 42x_{n+\frac{1}{2}}^5 & 56x_{n+\frac{1}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x_{n+\frac{3}{2}}^2 & 20x_{n+\frac{3}{2}}^3 & 30x_{n+\frac{3}{2}}^4 & 42x_{n+\frac{3}{2}}^5 & 56x_{n+\frac{3}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{5}{2}} & 12x_{n+\frac{5}{2}}^2 & 20x_{n+\frac{5}{2}}^3 & 30x_{n+\frac{5}{2}}^4 & 42x_{n+\frac{5}{2}}^5 & 56x_{n+\frac{5}{2}}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix}. \tag{4}$$

Solving (4) for a_j's using Guassian elimination method and substituting into (2) gives a linear multistep in the form

$$y(x) = \sum_{j=1}^{2} \alpha_j y_{n+j} + h^2 \left[\sum_{j=0}^{3} \beta_j f_{n+j} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} \right], \tag{5}$$

where $y_{n+j} = y(x_n + jh)$, $f_{n+j} = y''((x_n + jh, y(x_n + jh), y'(x_n + jh))$,

$$\alpha_1 = -t + 2,$$

$$\alpha_2 = t - 1,$$

$$\beta_0 = \frac{1}{7560}(12t^8 - 168t^7 + 980t^6 - 3087t^5 + 5684t^4 - 6174t^3$$

$$+ 3780t^2 - 1149t + 122),$$

$$\beta_{\frac{1}{2}} = -\frac{1}{630}(6t^8 - 80t^7 + 434t^6 - 1218t^5 + 1827t^4 - 1260t^3$$

$$+ 461t - 170),$$

$$\beta_1 = \frac{1}{2520}(60t^8 - 760t^7 + 3836t^6 - 9681t^5 + 12285t^4$$

$$- 6300t^3 - 512t + 1072),$$

$$\beta_{\frac{3}{2}} = -\frac{1}{945}(30t^8 - 360t^7 + 1694t^6 - 3906t^5 + 4445t^4$$

$$- 2100t^3 + 459t - 262),$$

$$\beta_2 = \frac{1}{2520}(60t^8 - 680t^7 + 2996t^6 - 6447t^5 + 6930t^4$$

$$- 3150t^3 + 269t + 22),$$

$$\beta_{\frac{5}{2}} = -\frac{1}{630}(6t^8 - 64t^7 + 266t^6 - 546t^5 + 567t^4$$

$$- 252t^3 + 25t - 2),$$

$$\beta_3 = \frac{1}{7560}(12t^8 - 120t^7 + 476t^6 - 945t^5 + 959t^4$$

$$- 420t^3 + 42t - 4),$$

$$t = \frac{x - x_n}{h}.$$

Solving (5) for the independent solution gives a continuous hybrid block formula of the form

$$y(x) = \sum_{m=0}^{1} \frac{(jh)^m}{m!} y_n^{(m)} + h^2 \left[\sum_{j=0}^{3} \sigma_j f_{n+j} + \sigma_{\frac{1}{2}} f_{n+\frac{1}{2}} + \sigma_{\frac{3}{2}} f_{n+\frac{3}{2}} + \sigma_{\frac{5}{2}} f_{n+\frac{5}{2}} \right], \qquad (6)$$

where

$$\sigma_0 = \frac{1}{7560}(12t^8 - 168t^7 - 3087t^5 + 5684t^4 - 6174t^3 + 3780t^2 + 5880t),$$

$$\sigma_{\frac{1}{2}} = -\frac{1}{630}(6t^8 - 80t^7 - 1218t^5 + 1827t^4 - 1260t^3 + 2604t),$$

$$\sigma_{\frac{3}{2}} = -\frac{1}{945}(30t^8 - 360t^7 - 3906t^5 + 4445t^4 - 2100t^3 + 10164t),$$

$$\sigma_2 = \frac{1}{2520}(60t^8 - 680t^7 - 6447t^5 + 6930t^4 - 3150t^3 + 17976t),$$

$$\sigma_{\frac{5}{2}} = -\frac{1}{630}(6t^8 - 64t^7 - 546t^5 + 567t^4 - 252t^3 + 1596t),$$

$$\sigma_3 = \frac{1}{7560}(12t^8 - 12t^7 - 945t^5 + 959t^4 - 420t^3 + 2856t).$$

Evaluating (6) at $t = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ gives a discrete hybrid block formula of the form

$$A^{(0)}Y_m^{(n)} = h^{\lambda}By_{m-1}^{(n)} + h^{\mu-\lambda}CF(Y_m), \quad n = 0, 1, \tag{7}$$

where

$$Y_m^{(n)} = \left[y_{n+\frac{1}{2}} \quad y_{n+1} \quad y_{n+\frac{3}{2}} \quad y_{n+2} \quad y_{n+\frac{5}{2}} \quad y_{n+3} \quad y'_{n+\frac{1}{2}} \right.$$
$$\left. y'_{n+1} \quad y'_{n+\frac{3}{2}} \quad y'_{n+2} \quad y'_{n+\frac{5}{2}} \quad y'_{n+3} \right]^T,$$

$$Y_{m-1}^{(n)} = \left[y_{n-\frac{1}{2}} \quad y_{n-1} \quad y_{n-\frac{3}{2}} \quad y_{n-2} \quad y_{n-\frac{5}{2}} \quad y_n \quad y'_{n-\frac{1}{2}} \right.$$
$$\left. y'_{n-1} \quad y'_{n-\frac{3}{2}} \quad y'_{n-2} \quad y'_{n-\frac{5}{2}} \quad y'_n \right]^T,$$

$$F(Y_m) = \left[f_n \quad f_{n+\frac{1}{2}} \quad f_{n+1} \quad f_{n+\frac{3}{2}} \quad f_{n+2} \quad f_{n+\frac{5}{2}} \quad f_{n+3} \right]^T,$$

n is the order of the derivatives, $\lambda \in \Re$ represents the power of A and μ is the order of the problem (see Awoyemi et al. [1] and Anake et al. [11] for details),

$A = 12 \times 12$ identity matrix,

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{28549}{483840} & \frac{275}{2304} & \frac{-5717}{53760} & \frac{10621}{120960} & \frac{-7703}{161280} & \frac{403}{26880} & \frac{-199}{96768} \\ \frac{1027}{7560} & \frac{97}{210} & \frac{-2}{9} & \frac{197}{945} & \frac{-97}{840} & \frac{23}{630} & \frac{-19}{3780} \\ \frac{759}{7560} & \frac{1485}{1792} & \frac{-2403}{17920} & \frac{45}{128} & \frac{-3267}{17920} & \frac{513}{8960} & \frac{-141}{17920} \\ \frac{272}{945} & \frac{376}{315} & \frac{-2}{105} & \frac{656}{945} & \frac{-2}{9} & \frac{8}{105} & \frac{-2}{189} \\ \frac{35225}{96768} & \frac{8385}{5376} & \frac{3125}{32256} & \frac{25625}{24192} & \frac{-625}{10752} & \frac{275}{2304} & \frac{-1375}{96768} \\ \frac{123}{280} & \frac{27}{14} & \frac{27}{140} & \frac{51}{35} & \frac{27}{280} & \frac{27}{280} & 0 \\ \frac{19087}{120960} & \frac{2713}{5040} & \frac{-15487}{40320} & \frac{293}{945} & \frac{-6737}{40320} & \frac{263}{5040} & \frac{-863}{120960} \\ \frac{1139}{7560} & \frac{47}{63} & \frac{11}{2520} & \frac{166}{845} & \frac{-269}{2520} & \frac{11}{315} & \frac{-27}{7560} \\ \frac{137}{896} & \frac{81}{112} & \frac{1161}{4480} & \frac{17}{35} & \frac{-729}{4480} & \frac{27}{560} & \frac{-29}{4480} \\ \frac{143}{945} & \frac{232}{315} & \frac{64}{315} & \frac{752}{945} & \frac{29}{315} & \frac{8}{315} & \frac{-4}{945} \\ \frac{3735}{24192} & \frac{725}{1008} & \frac{2125}{8064} & \frac{125}{189} & \frac{3875}{8064} & \frac{235}{1008} & \frac{-275}{24192} \\ \frac{41}{280} & \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & \frac{41}{280} \end{bmatrix}.$$

3. Analysis of the Block Method

3.1. Order of the block

Let the linear operator $\mathcal{L}\{y(x) : h\}$ associated with the discrete block method

(8) be defined as

$$\mathcal{L}\{y(x):h\} = A^{(0)}Y_m^{(n)} - h^{\lambda}By_{m-1}^{(n)} - h^{\mu-\lambda}CF(Y_m). \tag{8}$$

Expanding Y_m and $F(Y_m)$ in Taylor series gives

$$\mathcal{L}\{y(x):h\} = C_0 y(x) + C_1 y'(x)$$

$$+ \cdots + C_p h^p y^p(x) + C_{p+1}h^{p+1}y^{p+1}(x) + C_{p+2}h^{p+2}y^{p+2}(x).$$

The linear operator $\mathcal{L}$ and associated block formula are said to be of *order p* if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0$, $C_{p+2} \neq 0$, C_{p+2} is called the *error constant* and implies that the truncation error is given by

$$t_{n+k} = C_{p+2}h^{p+2}y^{p+2}(x) + O(h^{p+3}). \tag{9}$$

Hence the block (10) has order 7 with error constant

$$\left[\frac{6031}{464486400}\ \frac{233}{7257600}\ \frac{9}{179200}\ \frac{31}{453600}\ \frac{1625}{18579456}\ \frac{9}{89600}\ \frac{275}{6193152}\right.$$

$$\left.\frac{1}{30240}\ \frac{9}{229376}\ \frac{1}{30240}\ \frac{275}{6193152}\ \frac{-9}{716800}\right].$$

3.2. Zero stability

The block (7) is said to be *zero stable*, if the roots $z_s = 1, 2, \ldots, N$ of the characteristic polynomial $\rho(z) = \det(zA - E)$ satisfy $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover as $h^{\mu} \to 0$, $\rho(z) = z^{r-\mu}(\lambda - 1)^{\mu}$, where μ is the order of differential equations $r = \dim(A^{(0)})$ (see Awoyemi et al. [1] and Adesanya et al. [10] for details).

For our method $\rho(z) = z^{10}(\lambda - 1)^2$, hence our method is zero stable.

3.3. Convergence

A method is said to be *convergent*, if it is zero stable and has order $p \geq 1$.

From the above, our method is convergent.

3.4. Stability interval of the method

The block (7) is said to be *absolutely stable* within a given interval, if for a given

h, all roots z_s of the characteristics polynomial $\pi(z, h) = \rho(z) + h^2\sigma(z) = 0$ satisfy $|z| < 1$, $s = 1, 2, \ldots, n$, where $\bar{h} = \lambda^2 h^2$ and $\lambda = \frac{\partial f}{\partial y}$.

We adopted boundary locus method to determine the stability interval of our block. Substituting $y'' = -\lambda^2 y$ into the block (7). Substituting $e^{i\theta} = \cos\theta + i\sin\theta$ and after evaluation at interval [0, 180] gives the stability interval as [0, 74566].

4. Numerical Examples

Problem 1. Consider the non linear initial value problem (I.V.P.)

$$y'' - x(y') = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}, \quad h = 0.01.$$

Exact solution: $y(x) = 1 + \frac{1}{2}\text{In}(\frac{2+x}{2-x})$.

This problem was solved by Adesanya et al. [10] where a method implemented in block method of order seven was proposed with $h = 0.01$. Adesanya et al. [12] also solved this problem where a method implemented in block predictor-block corrector was proposed. The result is shown in Table 1.

Table 1. Showing result of Problem 1

x	Exact result	Computed result	Error	ERAB	ERABP
0.1	1.05004172927849	1.050041729278490	1.1102(– 15)	9.992(– 15)	1.3300(– 13)
0.2	1.10033534773107	1.100335347731056	1.9318(– 14)	8.149(– 14)	9.9498(– 13)
0.3	1.15114053593646	1.151140435936355	1.1169(– 13)	4.700(– 13)	3.4396(– 12)
0.4	1.20273255405408	1.202732554053672	4.1034(– 13)	1.637(– 12)	8.6803(– 12)
0.5	1.25241281188299	1.255412811881847	1.1482(– 12)	4.664(– 12)	1.8373(– 11)
0.6	1.30951960420311	1.309519604200349	2.7629(– 12)	1.116(– 11)	3.5087(– 11)
0.7	1.36544375427139	1.36544754265240	6.1566(– 12)	2.501(– 11)	6.3313(– 11)
0.8	1.43264893019360	1.423648930180650	1.2952(– 11)	5.215(– 11)	1.1013(– 10)
0.9	1.48470027859405	1.484700278567621	2.6431(– 11)	1.076(– 11)	1.8818(– 10)
1.0	1.54930614433405	1.549306144280011	5.4045(– 11)	2.170(– 10)	3.2211(– 10)

Note.

ERAB $\rightarrow$ Error in Adesanya et al. [10].

ERABP $\rightarrow$ Error in Adesanya et al. [12].

ERROR $\rightarrow |\text{Exact result} - \text{Computed result}|$.

Problem 2. We consider the non linear initial value problem

$$y'' + \left(\frac{6}{x}\right) y' + \left(\frac{4}{x^2}\right) y = 0, \quad y(1) = 1, \quad y'(1) = 1, \quad h = 0.01.$$

Exact solution: $y(x) = \dfrac{5x^3 - 2}{3x^4}$.

This problem was solved by Adesanya et al. [10] where a block method of order seven was proposed. We compared our result with their result as shown in Table 2.

Table 2. Showing result of Problem 2

x	Exact result	Computed result	Error	ERA
1.0031	1.00307652585769	1.00307652587696	2.2204(– 16)	1.332(– 15)
1.0062	1.00605750308351	1.006057503083516	2.2204(– 16)	8.215(– 15)
1.0094	1.008944995088837	1.008944995088838	6.6613(– 16)	1.909(– 14)
1.0125	1.01174101816798	1.011741018167990	1.7764(– 15)	3.508(– 14)
1.0156	1.01444754268641	1.01444754268415	1.7764(– 15)	5.484(– 14)
1.0187	1.01706649423567	1.017066494235674	1.7764(– 15)	7.904(– 14)
1.0219	1.01959975475628	1.019599754756289	1.9984(– 15)	1.063(– 13)
1.0250	1.02204916362943	1.022049163629432	1.5543(– 15)	1.374(– 13)
1.0281	1.02441651873840	1.024416518738403	1.1102(– 15)	1.723(– 13)
1.0312	1.02670357750080	1.026703577500806	1.1102(– 15)	2.100(– 13)

5. Conclusion

We have proposed a three steps hybrid block method of order seven in this paper. The method was found to be consistent, zero stable and convergence. The stability interval was investigated. The efficiency of our method was tested on some numerical examples and found to compete favourably with the existing methods as shown in Tables 1 and 2.

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