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The h-Integrability and the Weak Laws of Large Numbers for Arrays

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Abstract: *In this paper, the concept of weak laws of large numbers for arrays (WLLNFA) is studied, and a new notion of uniform integrability referred to as h-integrability is introduced as a condition for WLLNFA in obtaining the main results.*

Keywords: *Weak Laws of Large Numbers, Uniform Integrability, Boundedness, h-Integrability and Convergence.*

1. Introduction

The law of large numbers (LLN) as a theorem describes the result of performing the same experiment a large number of times. According to this, the average of the results obtained from a large number of trials should be close to the expected value, and have a tendency to become closer as more trials are performed. The LLN ensures stable long-term results for the averages of random events. It only applies when a large number of observations are considered. While the weak law of large numbers (WLLN) states that the sample average converges in probability towards the expected value.

The concept of uniform integrability has been a core aspect in applied probability with regard to the theory of convergence of weak law of large numbers. Chandra [1] obtains the weak law of large numbers under a new condition known as the Cesa´ro uniform integrability, which is weaker than uniform integrability. Cabrera [2] studies the weak convergence of weighted sums of random variables and introduces the condition of uniform integrability concerning the weights, which is weaker than uniform integrability; this leads to a special case known as Cesa´ro uniform integrability.

On the condition of $\{a_{ni}\}$-uniform integrability, Cabrera [2] obtains the weak law of large numbers for weighted sums of pairwise independent random variables. Sung [3] introduces the concept of Cesa´ro type uniform integrability with exponent r. Chandra and Goswami [4] introduce the concept of Cesa´ro α-integrability (α > 0), and show that Cesa´ro α-integrability for any α > 0 is weaker than Cesa´ro uniform integrability.

Cabrera and Volodin [5] introduce the notion of *h*-integrability for an array of random variables concerning an array of constant weights, and prove that this concept is weaker than Cesa´ro uniform integrability, $\{a_{ni}\}$- uniform integrability and Cesa´ro α-integrability, and also show that *h*-integrability concerning the weights is sufficient for the weak law of large numbers to hold for weighted sums of an array of random variables, when these random variables are subject to some special kind of rowwise dependence.

Sung et al [6] remarked that the main idea of notions of $\{a_{ni}\}$-uniform integrability introduced in Cabrera [2] and *h*-integrability with respect to the array of constants $\{a_{ni}\}$ introduced in Cabrera and Volodin [5] is to deal with weighted sums of random variables. Sung et al in [7] introduce a new concept of integrability which deals with usual normed sums of random variables. Adeosun and Edeki [8], on a survey of uniform integrability of sequences of random variables, noted some new conditions required for such. In this paper, we study and present the basic theorem regarding uniform integrability, in terms of weak laws of large numbers for arrays.

2. Preliminary Definitions and Lemmas

In this section, the needed technical definitions and lemmas for the main results will be discussed accordingly, with their corresponding proof, if need be.

Definition 2.1 An array of random variables $\{X_{ni}, 1 \le i \le k < \infty,\ n \ge 1\}$ is said to be Cesa´ ro uniform integrability if

$$\lim_{a\to\infty} \sup_{n\ge1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}|X_{ni}| I(|X_{ni}| > a) = 0$$

where $\{k_n, n \ge 1\}$ is a sequence of positive integers such that $k_n \to \infty$ as n → ∞.

Definition 2.2 Let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of random variables and $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ an array of constants with $\sum_{i=u_n}^{v_n} |a_{ni}| \le C$ for all $n \in \mathbb{N}$ and some constant $C > 0$. The array $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is $\{a_{ni}\}$-*uniform integrable* if

$$\lim_{a\to\infty} \sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}| \mathbb{E}|X_{ni}| I(|X_{ni}| > a) = 0$$

$\{a_{ni}\}$-uniform integrability tends to Cesaro uniform integrability when $a_{ni} = K_n^{-1}$, $1 \le j \le K_n$, $n \ge 1$

Definition 2.3 Let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}\}$ be an array of random variables and r >0. The array $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is said to be *Cesa´ro type uniformly integrable with exponent* r if

$$\sup_{n\ge1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbb{E}|X_{ni}|^r < \infty \quad \text{and} \quad \lim_{a\to\infty} \sup_{n\ge1} \sum_{i=u_n}^{v_n} \mathbb{E}|X_{ni}|^r I(|X_{ni}|^r > a) = 0$$

We note that the conditions of Cesa´ro uniform integrability and Cesa´ro type uniformly integrable with exponent r are equivalent when $u_n = 1$, $v_n = k_n, n \ge 1$, and $r = 1$. Sung [3] obtains the weak law of large numbers for an array $\{X_{ni}\}$ satisfying Cesa´ro type uniform integrability with exponent r for some $0 < r < 2$.

Definition 2.4 For any $\alpha > 0$, a sequence $\{X_n, n \ge 1\}$ of random variables is said to be *Cesa´ro α-integrable* if

$$\sup_{n\ge1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}|X_i| < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}|X_i| I(|X_i| > i^\alpha) = 0$$

Definition 2.5 Let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of random variables and $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ an array of constants with $\sum_{i=u_n}^{v_n} |a_{ni}| \le C$ for all $n \in \mathbb{N}$ and some constant $C > 0$. Let moreover $\{h(n), n \ge 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as n ↑ ∞. The array $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is said to be *h*-integrable with respect to the array of constants $\{a_{ni}\}$ if

$$\sup_{n\geq 1}\sum_{i=u_n}^{v_n}|a_{ni}|\mathbb{E}|X_{ni}|<\infty \quad \text{and} \quad \lim_{n\to\infty}\sum_{i=u_n}^{v_n}|a_{ni}|\mathbb{E}|X_{ni}|I(|X_{ni}|>h(n))=0$$

Definition 2.6 Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $r > 0$. Let moreover, $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h(n)\uparrow \infty$ as n $\uparrow \infty$. The array $\{X_{ni}\}$ is said to be *h-integrable with exponent r* if

$$\sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{ni}|^r<\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{ni}|^r I(|X_{ni}|^r>h(n))=0$$

Lemma 2.1 [6] If the array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ satisfies the condition of Cesa´ro type uniform integrability with exponent *r* > 0, then it satisfies the condition of *h*-integrability with exponent *r*.

Proof: We note that the first condition of the Cesa´ro type uniform integrability with exponent r and the first condition of the h-integrability with exponent r are the same. Hence, it suffices to show that the second condition of Cesa´ro type uniform integrability with exponent r implies the second condition of h-integrability with exponent r. If $\{X_{ni}\}$ satisfies the Cesa´ro type uniform integrability with exponent r, then there exist A > 0 such that $\sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{ni}|^r I(|X_{ni}|^r>a)<\epsilon$ if $a>A$. Since $h(m)\uparrow\infty$ as $m\to\infty$, $\exists\, M$ such that $h(m)>A$ if $m>M$. For $m>M$,

$$\frac{1}{k_n}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{mi}|^r I(|X_{mi}|^r>h(m)) \leq \sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}\mathbb{E}\,|X_{ni}|^r I(|X_{ni}|^r>h(m))<\epsilon\,.$$

Hence the second condition of h-integrability with exponent r is satisfied ⊡

Note 2.1. As noted by [6], the concept of h-integrability with exponent r is strictly weaker than the concept of Cesa´ro type uniform integrability with exponent r, i.e., there exists an array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ which is h-integrability with exponent r, but not Cesa´ro type uniform integrability with exponent r.

Remark 2.1 In considering an array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and setting $\mathcal{F}_{nj} = \sigma\{X_{ni}, u_n \leq i \leq j\}$, $u_n \leq i \leq v_n, n \geq 1$, and $\mathcal{F}_{n,u_n-1} = \{\emptyset,\ \Omega\}$, $n \geq 1$, authors like [10], [9], [7], [5], [3] established weak laws of large numbers. Gut [9] proved that, for some $0<r<2$,

$\frac{\sum_{i=1}^{k_n}(X_{ni}-a_{ni})}{k_n^{1/r}} \to 0$ in L^r if $\{|X_{ni}|^r, 1 \leq i \leq k_n, n \geq 1\}$ is an array of Cesa´ro uniformly integrable random variables, where $a_{ni}=0$ if $0<r<1$ and $a_{ni}=\mathbb{E}(X_{ni}|\,\mathcal{F}_{n,i-1})$ if $1\leq r\leq 2$.

Lemma 2.2[3] Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of h-integrability with exponent r for some $r>0$, $k_n\to\infty$, $h(n)\uparrow\infty$, and $\frac{h(n)}{k_n}\to 0$. Then the following statements hold.

(i) $\sum_{i=u_n}^{v_n}\mathbb{E}|X_{ni}|^{\alpha} I(|X_{ni}|^r>k_n)=o\left(k_n^{\alpha/r}\right)$ if $0<\alpha\leq r$,

(ii) $\sum_{i=u_n}^{v_n}\mathbb{E}|X_{ni}|^{\beta} I(|X_{ni}|^r\leq k_n)=o\left(k_n^{\beta/r}\right)$ if $r<\beta$.

3. UNIFORM LAWS OF LARGE NUMBERS AND CONVERGENCE

This section deals with uniform laws of large numbers as condition for convergence of a sample mean of random variables on a compact space, and the general WLLNFA.

3.1 Uniform Laws of Large Numbers (ULLN) and a Compact Euclidean Space

Let $\mathcal{X}$ and Θ be two Euclidean sets with a Cartesian product $\mathcal{X} \times \Theta$. Define on $\mathcal{X} \times \Theta$ a real-valued function $h(x,\phi)$ such that $h(\cdot,\phi)$ is Lebesgue measurable for every $\phi \in \Theta$. Then for any fixed ϕ, the sequence $h(X_i,\phi)$ will be a sequence of independent and identically distributed (iid) random variables on $\mathcal{X}$ with its sample mean converges in probability to $\mathbb{E}[h(X,\phi)]$. A uniform weak law of large numbers therefore defines a set of conditions on which

$$\mathrm{Sup}_{\phi\in\Theta}\left[\frac{1}{n}\sum_{i=1}^{n} h(X_i,\phi) - \mathbb{E}h(X_i,\phi)\right] \xrightarrow{p} 0$$

Theorem 3.1: If (a) Θ is compact, (b) $h(X,\phi)$ is continuous at each $\phi \in \Theta$ with probability one, (c) $|h(X,\phi)| \le H(X_i)$, and (d) $\mathbb{E}[H(X_i)] < \infty$. Then:

$$\mathrm{Sup}_{\phi\in\Theta}\left[\frac{1}{n}\sum_{i=1}^{n} h(X_i,\phi) - \mathbb{E}h(X_i,\phi)\right] \xrightarrow{p} 0$$

Proof: Let $\Delta_\gamma(X_i,\phi_0) = \sup_{\phi\in B(\phi_0,\gamma)} h(X_i,\phi) - \inf_{\phi\in B(\phi_0,\gamma)} h(X_i,\phi)$. So
$\mathbb{E}\Delta_\gamma(X_i,\phi_0) \downarrow 0$ as $\gamma \downarrow 0$ since

(i) $\Delta_\gamma(X_i,\phi_0) \downarrow 0$ a.s. by condition (b).

(ii) $\Delta_\gamma(X_i,\phi_0) \le 2\sup_{\phi\in\Theta}|h(X_i,\phi)| \le 2H(X_i)$ by condition (c), and condition (d).

Hence, for all $\phi \in \Theta$ and $\varepsilon > 0$, there exists $\gamma_\varepsilon(\phi)$ such that $\mathbb{E}[\Delta_{\gamma_\varepsilon}(X_i,\phi)] < \varepsilon$.

Obviously, the whole parameter space $\boldsymbol{\Theta}$, is covered by $\{B(\phi,\gamma_\varepsilon(\phi)) : \phi \in \Theta\}$. So since, $\boldsymbol{\Theta}$ is compact, we can find a finite subcover, such that $\boldsymbol{\Theta}$ is covered by $\cup_{k=1}^{K} B(\phi_k,\gamma_\varepsilon(\phi_k))$.

We recall that

$$\begin{aligned}
&\mathrm{Sup}_{\phi\in\Theta}\left[\frac{1}{n}\sum_{i=1}^{n} h(X_i,\phi) - \mathbb{E}h(X_i,\phi)\right] \\
&\quad = \max_k \sup_{\phi\in B(\phi_k,\gamma_\varepsilon(\phi_k))}\left[\sum_{i=1}^{n} h(X_i,\phi) - \mathbb{E}h(X_i,\phi)\right] \\
&\quad \le \max_k \left[n^{-1}\sum_{i=1}^{n} \sup_{\phi\in B(\phi_k,\gamma_\varepsilon(\phi_k))} h(X_i,\phi) - \mathbb{E}\inf_{\phi\in B(\phi_k,\gamma_\varepsilon(\phi_k))} h(X_i,\phi)\right] \\
&\quad = o_p(1) + \max_k \left[\mathbb{E}\sup_{\phi\in B(\phi_k,\gamma_\varepsilon(\phi_k))} h(X_i,\phi) - \mathbb{E}\inf_{\phi\in B(\phi_k,\gamma_\varepsilon(\phi_k))} h(X_i,\phi)\right] \\
&\quad = o_p(1) + \max_k \mathbb{E}\Delta_{\gamma_\varepsilon(\phi_k)}(X_i,\phi_k) \\
&\quad \le o_p(1) + \varepsilon,
\end{aligned}$$

Note. *The first equality holds by the WLLN, since* $\mathbb{E}\left|\sup_{\phi\in B(\phi_k,\gamma_\varepsilon(\phi_k))} h(X_i,\phi)\right| \le \mathbb{E}H(X_i) < \infty$, *and the last inequality follows the definition of* $\gamma_\varepsilon(\phi_k)$, *with* ε *arbitrarily chosen.*
Whence,

$$\inf_{\phi\in\Theta}\left[\frac{1}{n}\sum_{i=1}^{n} h(X_i,\phi) - \mathbb{E}h(X_i,\phi)\right] + \varepsilon \ge o_p(1).$$

Remark 3.1

- The uniform law of large numbers states the condition under which the convergence happens uniformly.

- Uniform strong law of large numbers defines such set of conditions if the convergence satisfies almost surely instead of in probability.

Theorem 3.2 Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of h-integrability with exponent $0 < r < 2$ random variables, $k_n \to \infty$, $h(n) \uparrow \infty$ and $h(n)/k_n \to \infty$. Then $\frac{\sum_{i=u_n}^{v_n}(X_{ni}-a_{ni})}{k_n{}^{1/r}} \to 0$ in L^r and, hence, in probability as $n \to \infty$, where $a_{ni} = 0$ if $0 < r < 1$ and $a_{ni} = \mathbb{E}(X_{ni}| \mathcal{F}_{n,i-1})$ if $1 \leq r < 2$.

Proof: Let $X_{ni}{}' = X_{ni}I(|X_{ni}|^r \leq k_n)$ and $X_{ni}{}'' = X_{ni}I(|X_{ni}|^r > k_n)$. First we consider the case $0 < r < 1$. Since $a_{ni} = 0$, it follows that

$$\sum_{i=u_n}^{v_n}(X_{ni} - a_{ni}) = \sum_{i=u_n}^{v_n} X_{ni}{}' + \sum_{i=u_n}^{v_n} X_{ni}{}'' .$$

By the C_r-inequality, Jensen's inequality, and Lemma 2.2 with $\alpha = r$ and $\beta = 1$, we have

$$\begin{aligned}\mathbb{E}\left|\sum_{i=u_n}^{v_n}(X_{ni} - a_{ni})\right|^r &\leq \mathbb{E}\left|\sum_{i=u_n}^{v_n} X_{ni}{}'\right|^r + \mathbb{E}\left|\sum_{i=u_n}^{v_n} X_{ni}{}''\right|^r \\ &\leq \left(\mathbb{E}\left|\sum_{i=u_n}^{v_n} X_{ni}{}'\right|\right)^r + \mathbb{E}\left|\sum_{i=u_n}^{v_n} X_{ni}{}''\right|^r \\ &\leq \left(\sum_{i=u_n}^{v_n} \mathbb{E}|X_{ni}{}'|\right)^r + \sum_{i=u_n}^{v_n} \mathbb{E}|X_{ni}{}''| = o(k_n),\end{aligned}$$

which completes the proof for the case $0 < r < 1$.

Now considering the case of $1 \leq r < 2$, it is observed that:

$$\begin{aligned}\sum_{i=u_n}^{v_n}(X_{ni} - a_{ni}) = &\sum_{i=u_n}^{v_n}\left(X_{ni}{}' - \mathbb{E}(X_{ni}{}'| \mathcal{F}_{n,i-1})\right) \\ &+ \sum_{i=u_n}^{v_n}\left(X_{ni}{}'' - \mathbb{E}(X_{ni}{}''| \mathcal{F}_{n,i-1})\right).\end{aligned}$$

Hence, using the C_r-inequality, Burkholder's and Davis' inequalities [12]; [13] for $1 < r < 2$ and, $r = 1$ respectively), Jensen's inequality, and Lemma 2.2 (i.e. Lemma 1, Sung [3]) with $\alpha = r$ and $\beta = 2$, we obtain

$$\begin{aligned}\mathbb{E}\left|\sum_{i=u_n}^{v_n}(X_{ni} - a_{ni})\right|^r &\leq 2^{r-1}\left\{\mathbb{E}\left|\sum_{i=u_n}^{v_n}\left(X_{ni}{}' - \mathbb{E}(X_{ni}{}'| \mathcal{F}_{n,i-1})\right)\right|^r \right. \\ &\quad \left. + \mathbb{E}\left|\sum_{i=u_n}^{v_n}\left(X_{ni}{}'' - \mathbb{E}(X_{ni}{}''| \mathcal{F}_{n,i-1})\right)\right|^r\right\} \\ &\leq 2^{r-1}C_r\left\{\mathbb{E}\left|\sum_{i=u_n}^{v_n}\left(X_{ni}{}' - \mathbb{E}(X_{ni}{}'| \mathcal{F}_{n,i-1})\right)^2\right|^{\frac{r}{2}}\right. \\ &\quad \left. + \mathbb{E}\left|\sum_{i=u_n}^{v_n}\left(X_{ni}{}'' - \mathbb{E}(X_{ni}{}''| \mathcal{F}_{n,i-1})\right)^2\right|^{\frac{r}{2}}\right\} \\ &\leq 2^{r-1}C_r\left\{\left(\sum_{i=u_n}^{v_n} \mathbb{E}\left(X_{ni}{}' - \mathbb{E}(X_{ni}{}'| \mathcal{F}_{n,i-1})\right)^2\right)^{\frac{r}{2}}\right. . \\ &\quad \left. + \sum_{i=u_n}^{v_n} \mathbb{E}|X_{ni}{}'' - \mathbb{E}(X_{ni}{}''| \mathcal{F}_{n,i-1})|^r\right\} \\ &\leq 2^{r-1}C_r\left\{+ \sum_{i=u_n}^{v_n} \mathbb{E}|X_{ni}{}'' - \mathbb{E}(X_{ni}{}''| \mathcal{F}_{n,i-1})|^r\right\} \\ &\leq 2^{r-1}C_r\left\{\left(\sum_{i=u_n}^{v_n} \mathbb{E}|X_{ni}{}'|^2\right)^{\frac{r}{2}} + 2^r\sum_{i=u_n}^{v_n} |X_{ni}{}''|^r\right\}\end{aligned}$$

$$= o(k_n)$$

where C_r is a constant depending only on r. Hence, the proof of the case $1 \leq r < 2$ is done ⊡

Corollary 3.1 Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of random variables satisfying the Cesa´ro type uniform integrability with exponent 0 < r < 2 and $k_n \to \infty$. Then

$\frac{\sum_{i=u_n}^{v_n}(X_{ni}-a_{ni})}{k_n{}^{1/r}} \to 0$ in L^r and, hence, in probability as n → ∞, where a_{ni} = 0 if $0 < r < 1$ and $a_{ni} = \mathbb{E}(X_{ni}|\ \mathcal{F}_{n,i-1})$ if $1 \leq r < 2$.

Proof: By lemma2.1 [7] the condition of Cesa´ro type uniform integrability with exponent *r* implies the condition of *h*-integrability with exponent r, and so the result follows from Thrm 3.2. ⊡

4. Concluding Remarks

In conclusion, weak laws of large numbers for the array of dependent random variables satisfying the condition of h-integrability with exponent r is obtained. The work extends and sharpens previous results with regard to applied probability, in terms of theorems and conditions for boundedness, and convergence of laws of large numbers with applications to stationary data.

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