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AN A(α)-STABLE METHOD FOR SOLVING INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

An A(α)-stable implicit one step hybrid method for the numerical approximation of solutions of initial value problems of general second order ordinary differential equations is proposed. The method is developed by interpolation and collocation of a power series approximate solution and implemented as simultaneous integrators via block method. The stability and convergence of the methods are determined. Numerical experiments are conducted on sample problems and the absolute error estimates of the results are presented.

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1. Introduction

Due to sophistication in computing, mathematical modeling of real life systems throw up complex mathematical equations which pose the challenge of obtaining close form solutions. Mostly, these complex mathematical equations are either in the form of partial differential equations (PDE) or ordinary differential equations (ODE). Of interest to us however, is the latter, where the models pose initial value problems (IVP).

For these kinds of problems, developing efficient and accurate numerical methods has increasingly been of much interest to researchers in the area of numerical methods and analysis over the years. Indeed, through a variety of approaches, several numerical methods have been proposed; ranging from the single step Runge-Kutta type methods [11, 17], through Adam type multistep methods [4, 5, 9, 18, 20] to the now very popular block methods [1, 2, 3, 8, 13, 19]. Besides these methods, are their hybrid variants [6, 7, 10, 12, 14, 16]. These methods respectively, have their setbacks which impacts on their efficiency and accuracy. Therefore, the overriding objective in developing new methods has always been to improve on the efficiency and convergence with the ultimate aim of reducing the error of approximation.

Thus, it is our intention in this paper to develop a more efficient and accurate implicit one step hybrid method for the direct solution of general second order IVP of ODE of the form:

$$\begin{cases} y'' = f(x, y, y'), \quad x \in [a, b] \\ y(a) = \zeta_0, \\ y'(a) = \zeta_1. \end{cases} \tag{1}$$

This class of problems often arises in areas such as control theory, chemical kinetics, circuit theory, mechanics and biology. Unique solutions have been shown to exist for problems of this class in [21].

The layout of this paper is as follows: the next section describes the derivation of the proposed numerical method, this is followed by the analysis of the method for stability and convergence in section three. In section four, the associated block formulation for the implementation of the method is presented, this is followed by numerical experiments on some selected problems in section five. Finally, conclusion is given in section six and references thereafter.

2. Derivation of the Method

In this section, a continuous representation of an implicit one-step hybrid method is derived.

Let $\pi_N : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = n$ be a partition of the integration interval $[a, b]$, into N subintervals, $[x_j, x_{j+1}]$, with constant step size given by $h = x_{j+1} - x_j$; $j = 0, 1, ..., N - 1$. Also, let the basis polynomial be a power series polynomial of the form

$$Y(x) = \sum_{i=0}^{m} a_i x^i \tag{2}$$

completely determined by $m + 1$ unknown parameters a_i, $i = 0, 1, 2, ..., m$.

Introducing n offstep points, $\mu_u = \frac{u}{n+1}$, $u = 1, 2, ..., n - 1, n$, in the one step structure, (see [6]), a continuous implicit one step hybrid method is obtained. This is accomplished, by interpolating (2) at the points $x_{j+\mu_{(n-1)}}$ and $x_{j+\mu_n}$ in a Stormer-Cowell fashion, (see [13]), and collocating (1) at the points x_{j+i}, $i = 0(\mu_u)1$. A combination of these procedures give rise to a system of $m + 1$ equations of degree at most m in the form:

$$\sum_{i=0}^{m} a_i x_{j+\mu_s}^i = y_{j+\mu_s}, \quad s = n - 1, n \tag{3a}$$

$$\sum_{i=0}^{m} i(i-1) a_i x_{j+r}^{i-2} = f_{j+r}, \; r = 0, \mu_u, 1, \tag{3b}$$

where s and r represent the interpolation and collocation points respectively.

The system of equations (3) is solved for the values of the unknown parameters a_i, $i = 0, 1, \ldots, m$ which are then substituted into (2). By using the transformation, $t = \dfrac{x - x_{j+\mu_n}}{h}$ in the resulting algebraic system, we obtained the proposed continuous implicit one-step hybrid method and its first derivative in the form:

$$y(t) = \alpha_{\mu_n}(t)y_{j+\mu_n} + \alpha_{\mu_{(n-1)}}(t)y_{j+\mu_{(n-1)}} + h^2\left[\sum_{i=0}^{1}\beta_i(t)f_{j+i} + \sum_{u=1}^{n}\beta_{\mu_u}(t)f_{j+\mu_u}\right], \tag{4a}$$

$$y'(t) = \alpha'_{\mu_n}(t)y_{j+\mu_n} + \alpha'_{\mu_{(n-1)}}(t)y_{j+\mu_{(n-1)}} + h\left[\sum_{i=0}^{1}\beta'_i(t)f_{j+i} + \sum_{u=1}^{n}\beta'_{\mu_u}(t)f_{j+\mu_u}\right], \tag{4b}$$

where for arbitrary $d \in \mathbb{R}$, $\alpha_d(t)$ and $\beta_d(t)$ are continuous coefficients in t, $y_{j+i} = y(x_{j+i})$ is the numerical approximation of the analytical solution at the point $x_{j+i} = x_j + ih$ and $f_{j+i} = f(x_{j+i}, y_{j+i}, y'_{j+i})$.

Obtaining values of t by evaluating (4a) $x = x_{j+i}$, $i = 0, \mu_u, 1$; $u = 1, 2, \ldots, n-2$ and (4b) at $x = x_{j+i}$, $i = 0, \mu_u, 1$; $u = 1, 2, \ldots, n$, respectively, the desired discrete numerical methods and their first derivatives are obtained.

In particular, if we set $n = 6$, that is, if six offstep points are introduced between x_j and x_{j+1}. Then, a power series approximate solution (2) of degree $m = 9$ yields a system of equations, each completely determined by the coefficients a_j, $j = 0, 1, \ldots, 9$. Following the procedure described earlier we obtained the continuous implicit one step hybrid methods:

$$y(t) = \alpha_{\frac{5}{7}}(t)\, y_{j+\frac{5}{7}} + \alpha_{\frac{6}{7}}(t)\, y_{j+\frac{6}{7}}$$

$$+ h^2 \left[\sum_{i=0}^{1} \beta_i(t)\, f_{j+i} + \sum_{u=1}^{6} \beta_{\mu_u}(t)\, f_{j+\mu_u} \right], \tag{5a}$$

$$y'(t) = \alpha'_{\frac{5}{7}}(t)\, y_{j+\frac{5}{7}} + \alpha'_{\frac{6}{7}}(t)\, y_{j+\frac{6}{7}}$$

$$+ h \left[\sum_{i=0}^{1} \beta'_i(t)\, f_{j+i} + \sum_{u=1}^{6} \beta'_{\mu_u}(t)\, f_{j+\mu_u} \right]. \tag{5b}$$

Obtaining values for t by evaluating (5a) at the points $x = x_{j+1}$, $i = 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, 1$, yields specific discrete methods expressed in terms of their coefficients in Table 1. In a similar manner, when (5b) is evaluated at $x = x_{j+i}$, $i = 0\left(\frac{1}{7}\right)1$ for values of t, specific derivative methods expressed by their coefficients in Table 2 are obtained.

Table 1. The coefficients of the method (5a) for $t = -\frac{6}{7}, -\frac{5}{7}, -\frac{4}{7}, -\frac{3}{7}, -\frac{2}{7}, \frac{1}{7}$

i	$\alpha_{\frac{5}{7}}(t)$	$\alpha_{\frac{6}{7}}(t)$	$\beta_0(t)$	$\beta_{\frac{1}{7}}(t)$	$\beta_{\frac{2}{7}}(t)$	$\beta_{\frac{3}{7}}(t)$	$\beta_{\frac{4}{7}}(t)$	$\beta_{\frac{5}{7}}(t)$	$\beta_{\frac{6}{7}}(t)$	$\beta_1(t)$
0	6	-5	$\frac{3683}{2963520}$	$\frac{8333}{370440}$	$\frac{35771}{987840}$	$\frac{20063}{296352}$	$\frac{45089}{592704}$	$\frac{23221}{246960}$	$\frac{25511}{2963520}$	$\frac{-349}{1481760}$
$\frac{1}{7}$	5	-4	$\frac{-1}{11760}$	$\frac{2927}{1481760}$	$\frac{304}{15435}$	$\frac{4187}{98784}$	$\frac{8825}{148176}$	$\frac{36307}{493920}$	$\frac{899}{123480}$	$\frac{-349}{1481760}$
$\frac{2}{7}$	4	-3	$\frac{-31}{1481760}$	$\frac{23}{185220}$	$\frac{521}{493920}$	$\frac{16297}{740880}$	$\frac{58039}{1481760}$	$\frac{1355}{24696}$	$\frac{1609}{296352}$	$\frac{-127}{740880}$
$\frac{3}{7}$	3	-2	$\frac{-31}{1481760}$	$\frac{527}{2963520}$	$\frac{-11}{15435}$	$\frac{9109}{2963520}$	$\frac{28477}{1481760}$	$\frac{35507}{987840}$	$\frac{2713}{740880}$	$\frac{-349}{2963520}$
$\frac{4}{7}$	2	-1	$\frac{-31}{2963520}$	$\frac{31}{370440}$	$\frac{-31}{109760}$	$\frac{649}{1481760}$	$\frac{4343}{2963520}$	$\frac{463}{27440}$	$\frac{1129}{592704}$	$\frac{-19}{296352}$
1	-1	2	$\frac{19}{296352}$	$\frac{-517}{987840}$	$\frac{29}{15435}$	$\frac{-11477}{2963520}$	$\frac{811}{164640}$	$\frac{-2099}{987840}$	$\frac{13831}{740880}$	$\frac{275}{197568}$

Table 2. The coefficients for the method (5b) for $t = -\frac{6}{7},\ -\frac{5}{7},\ -\frac{4}{7},\ -\frac{3}{7},$ $-\frac{2}{7},\ -\frac{1}{7},\ 0,\ \frac{1}{7}$

i	$\alpha_{\frac{5}{7}}(t)$	$\alpha_{\frac{6}{7}}(t)$	$\beta_0(t)$	$\beta_{\frac{1}{7}}(t)$	$\beta_{\frac{2}{7}}(t)$	$\beta_{\frac{3}{7}}(t)$	$\beta_{\frac{4}{7}}(t)$	$\beta_{\frac{5}{7}}(t)$	$\beta_{\frac{6}{7}}(t)$	$\beta_1(t)$
0	-7	7	$\frac{-534223}{12700800}$	$\frac{-346873}{1587600}$	$\frac{-147751}{4233600}$	$\frac{-331003}{1270080}$	$\frac{-19987}{362880}$	$\frac{-182201}{1058400}$	$\frac{-18691}{12700800}$	$\frac{-6031}{6350400}$
$\frac{1}{7}$	-7	7	$\frac{8881}{6350400}$	$\frac{-677249}{12700800}$	$\frac{-5912}{33075}$	$\frac{-41801}{362880}$	$\frac{-40555}{254016}$	$\frac{-521309}{4233600}$	$\frac{-47239}{3175200}$	$\frac{8563}{12700800}$
$\frac{2}{7}$	-7	7	$\frac{-409}{1814400}$	$\frac{4967}{1587600}$	$\frac{-35713}{604800}$	$\frac{-213499}{1270080}$	$\frac{-324901}{2540160}$	$\frac{-144761}{1058400}$	$\frac{-143971}{12700800}$	$\frac{1649}{6350400}$
$\frac{3}{7}$	-7	7	$\frac{1201}{6350400}$	$\frac{-3287}{1814400}$	$\frac{298}{33075}$	$\frac{-181919}{2540160}$	$\frac{-192791}{1270080}$	$\frac{-542909}{4233600}$	$\frac{-42439}{3175200}$	$\frac{6163}{12700800}$
$\frac{4}{7}$	-7	7	$\frac{-463}{12700800}$	$\frac{647}{1587600}$	$\frac{-9511}{4233600}$	$\frac{2305}{254016}$	$\frac{-180613}{2540160}$	$\frac{-147641}{1058400}$	$\frac{-141571}{12700800}$	$\frac{1649}{6350400}$
$\frac{5}{7}$	-7	7	$\frac{1201}{6350400}$	$\frac{-20609}{12700800}$	$\frac{208}{33075}$	$\frac{-37631}{2540160}$	$\frac{32233}{1270080}$	$\frac{-302429}{4233600}$	$\frac{-7297}{453600}$	$\frac{8563}{12700800}$
$\frac{6}{7}$	-7	7	$\frac{-409}{1814400}$	$\frac{3047}{1587600}$	$\frac{-31111}{4233600}$	$\frac{21509}{1270080}$	$\frac{-13985}{508032}$	$\frac{7297}{151200}$	$\frac{512669}{12700800}$	$\frac{-6031}{6350400}$
1	-7	7	$\frac{8881}{6350400}$	$\frac{-145889}{12700800}$	$\frac{1378}{33075}$	$\frac{-222623}{2540160}$	$\frac{21391}{181440}$	$\frac{-404669}{4233600}$	$\frac{652601}{3175200}$	$\frac{539923}{12700800}$

In particular, the main method and its first derivative are obtained as follows:

$$y_{j+1} = 2y_{j+\frac{6}{7}} - y_{j+\frac{5}{7}} + \frac{h^2}{2963520}\Big[4125 f_{j+1} + 55324 f_{j+\frac{6}{7}} - 6297 f_{j+\frac{5}{7}}$$

$$+ 14598 f_{j+\frac{4}{7}} - 11477 f_{j+\frac{3}{7}} + 5568 f_{j+\frac{2}{7}} - 1551 f_{j+\frac{1}{7}} + 190 f_n\Big], \quad (6)$$

$$y'_{j+1} = 7y_{j+\frac{6}{7}} - 7y_{j+\frac{5}{7}} + \frac{h}{12700800}\Big[539923 f_{j+1} + 2610404 f_{j+\frac{6}{7}}$$

$$- 1214007 f_{j+\frac{5}{7}} + 1497370 f_{j+\frac{4}{7}} - 1113115 f_{j+\frac{3}{7}}$$

$$+ 5291252 f_{j+\frac{2}{7}} - 145889 f_{j+\frac{1}{7}} + 17762 f_n\Big]. \quad (7)$$

3. Analysis of the Methods

In this section, the order, local error constant, zero stability, consistency, convergence and absolute stability and A(α)-stability of the method (6) is determined.

3.1. Order and error constant

To obtain the order and error constants for the new methods, rewrite (6) in the form of the linear difference operator

$$\mathcal{L}[y(x);\ h] = y(x_j + h) - \alpha_{\mu_{n-1}} y(x_j + \mu_{n-1}h) - \alpha_{\mu_n} y(x_j + \mu_n h)$$

$$- h^2\left[\sum_{i=0}^{1} \beta_i y''(x_j + ih) + \sum_{u=1}^{n} \beta_{\mu_u} y''(x_j + \mu_u h)\right], \qquad (8)$$

where $y(x) \in C^d[a, b]$ is an arbitrary test function. Then expand $y(x_j + ih)$ and $y''(x_j + ih)$, $i = 0, \mu_u, 1$ for all i respectively in Taylor series about x_j and collect terms in powers of y such that (8) becomes:

$$\mathcal{L}[y(x);\ h] = C_0 y(x) + C_1 h y^{(1)}(x) + C_2 h^2 y^{(2)}(x) + \cdots + C_p h^p y^{(p)}(x)$$

$$+ C_{p+1} h^{p+1} y^{(p+1)}(x) + \cdots, \qquad (9)$$

where the constant coefficients C_q, $q = 0, 1, 2, \ldots$ are defined as follows:

$$C_0 = \sum_{i=0}^{k} \alpha_i$$

$$C_1 = \sum_{i=0}^{k} i\alpha_i$$

$$\vdots$$

$$C_q = \frac{1}{q!}\left[\sum_{i=1}^{k} i^q \alpha_i - q(q-1)\left(\sum_{i=1}^{k} i^{q-2}\beta_i + \sum_{i=1}^{k} \mu_i^{(q-2)}\beta_{\mu_i}\right)\right].$$

Definition 3.1. 1. The difference operator $\mathcal{L}$ and the associated method is said to be of *order p* if $C_0 = C_1 = \cdots = C_p = 0$ and $C_{p+2} \neq 0$.

2. The term C_{p+2} is called the *error constant* and it implies that the

local truncation error (l.t.e) is defined by:

$$l.t.e. = C_{p+2}h^{p+2}y^{p+2} + O(h^{p+3}). \tag{10}$$

We have established from our computation that method (6) has order $p = 8$ and error constants $C_{p+2} = -9.5888 \times 10^{-12}$.

3.2. Zero stability, consistency and convergence

Definition 3.2. The first and second characteristic polynomials of the algorithm (6) are defined respectively as

$$\rho(z) = \sum_{i=0}^{k} \alpha_i z^i, \tag{11a}$$

$$\sigma(z) = \sum_{i=0}^{k} \beta_i z^i, \tag{11b}$$

where z is the principal root, $\alpha_k \neq 0$ and $\alpha_0^2 + \beta_0^2 \neq 0$.

Definition 3.3. The method (6) is said to be *zero stable* as $h \to 0$ if no root of (11a), $\rho(z)$ has modulus greater than one, and if every root of modulus one has multiplicity not greater than one.

For our method (6), we obtained (11a) as follows;

$$\rho(z) = z - 2z^{\frac{6}{7}} + z^{\frac{5}{7}}. \tag{12}$$

Clearly, the conditions in Definition 3.3 are satisfied hence, the algorithm is zero stable. The consistency of the method is established by the fact that the order of the algorithm is greater than one, (see [17]).

Following [15], convergence is established by the zero stability and consistency of method (6).

3.3. Stability

Absolute stability for the algorithm is determined by means of the boundary locus method. Consider the stability polynomial

$$\Pi(z, \bar{h}) = \rho(z) - \bar{h}\sigma(z) = 0, \tag{13}$$

where $\bar{h} = h^2\omega^2$ and $\omega = \frac{df}{dy}$ are assumed constant.

The stability polynomial (13) is obtained by applying the continuous implicit one step hybrid methods (6) to the scalar test problem;

$$y'' = -\omega^2 y. \tag{14}$$

The following definitions shall guide our conclusions.

Definition 3.4 (Absolute stability)**.** The algorithm, (6) is said to be *absolutely stable* if for a given $\bar{h}$ all the roots z_ϕ of (13) satisfy $|z_\phi| < 1$, $\phi = 1, 2, ..., (r-1)$.

Definition 3.5 (Region of absolute stability)**.** The region $\mathcal{R}$ of the complex $\bar{h}$-plane such that the roots of the polynomial $\Pi(z, \bar{h})$ lie within the unit circle whenever $\bar{h}$ lies in the interior of the region is called the *region of absolute stability*.

Definition 3.6 (A(α)-stability)**.** A linear multistep method is *A*(α)-*stable*, $\alpha \in \left(0, \frac{\pi}{2}\right)$ if the region of absolute stability includes the infinite wedge

$$S_\alpha = \{\bar{h} : |\pi - \arg(\bar{h})| < \alpha\}. \tag{15}$$

We established from our computation that method (6), is absolutely stable and indeed A(α)-stable. The A(α)-stability property is shown in Figure 1.

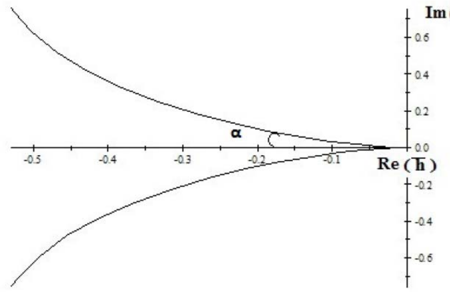

Figure 1. A(α)-stability of method (6).

4. Implementation

The methods obtained from Tables 1 and 2 are combined to form a block method given in vector notation by

$$\bar{A}Y_m = \bar{E}y_m + h^{\gamma-\lambda}[\bar{D}F(y_m) + \bar{B}F(y_m)], \tag{16}$$

where $\bar{A}$ is a square identity matrix of order 14; $\bar{E}, \bar{D}, \bar{B}$ are constant coefficient matrices, $Y_m = (y_{j+\mu_u}, y_{j+1}, y'_{j+\mu_u}, y'_{j+1})^T$, $y(y_j, y'_j)^T$, $\bar{F}(Y_m) = (f_{j+\mu_u}, f_{j+i})^T$, $F(y_m) = (f_j)$, λ is the power of derivative in (4) and γ is the order of the problem.

The constant coefficient matrices are obtained as follows:

$$\bar{E} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$$\bar{D} = \left[\frac{416173}{88905600}\ \frac{14939}{1389150}\ \frac{18399}{1097600}\ \frac{15824}{694575}\ \frac{102425}{3556224}\ \frac{597}{17150}\ \frac{10597}{259200}\right.$$

$$\left.\frac{751}{17280}\ \frac{41}{980}\ \frac{265}{6272}\ \frac{278}{6615}\ \frac{275}{6272}\ \frac{41}{980}\ \frac{751}{17280}\right]^T$$

$$\bar{B} = \begin{bmatrix} \frac{33953}{3175200} & \frac{27821}{694575} & \frac{39141}{548800} & \frac{71152}{694575} & \frac{59375}{444528} & \frac{1413}{8575} & \frac{25333}{129600} & \frac{139849}{846720} & \frac{1466}{6615} & \frac{1359}{6272} & \frac{1448}{6615} & \frac{36725}{169344} & \frac{54}{245} & \frac{3577}{17280} \\ \frac{-341699}{29635200} & \frac{-17}{675} & \frac{-24111}{1097600} & \frac{-3832}{231525} & \frac{-13375}{1185408} & \frac{-54}{8575} & \frac{49}{86400} & \frac{-4511}{31360} & \frac{-71}{2940} & \frac{1377}{31360} & \frac{8}{245} & \frac{775}{18816} & \frac{27}{980} & \frac{49}{640} \\ \frac{105943}{8890560} & \frac{799}{27783} & \frac{369}{7840} & \frac{11344}{138915} & \frac{210625}{1778112} & \frac{267}{1715} & \frac{245}{1296} & \frac{123133}{846720} & \frac{68}{735} & \frac{5927}{31360} & \frac{1784}{6615} & \frac{4625}{18816} & \frac{68}{245} & \frac{2989}{17280} \\ \frac{-153761}{17781120} & \frac{-5881}{277830} & \frac{-7299}{219520} & \frac{-856}{19845} & \frac{-130625}{3556224} & \frac{-99}{3430} & \frac{-833}{51840} & \frac{-88547}{846720} & \frac{-1927}{26460} & \frac{-3033}{31360} & \frac{609239}{12700800} & \frac{13625}{169344} & \frac{27}{980} & \frac{2989}{17280} \\ \frac{943}{231525} & \frac{2321}{231525} & \frac{8613}{548800} & \frac{4912}{231525} & \frac{25}{864} & \frac{459}{8575} & \frac{3283}{43200} & \frac{1537}{31360} & \frac{26}{735} & \frac{1377}{31360} & \frac{8}{245} & \frac{1895}{18816} & \frac{54}{245} & \frac{49}{640} \\ \frac{-99359}{88905600} & \frac{-1916}{694575} & \frac{-4737}{1097600} & \frac{-4072}{694575} & \frac{-26875}{3556224} & \frac{-9}{1225} & \frac{2989}{259200} & \frac{-11351}{846720} & \frac{-29}{2940} & \frac{-373}{31360} & \frac{-64}{6615} & \frac{-275}{18816} & \frac{41}{980} & \frac{3577}{17280} \\ \frac{6031}{44452800} & \frac{233}{694575} & \frac{9}{17150} & \frac{496}{694575} & \frac{1625}{1778112} & \frac{9}{8575} & \frac{167}{4800} & \frac{275}{169344} & \frac{8}{6615} & \frac{9}{6272} & \frac{8}{6615} & \frac{275}{169344} & 0 & \frac{751}{17280} \end{bmatrix}^T$$

The block formulation for the implementation of these schemes is according to [8]. A single application of the revised block formula generates simultaneously, approximate solutions and first derivative solutions at the step points x_j, x_{j+1} and all the offstep points: $x_{j+\mu_u}$, $u = 1, \ldots, n.$ The procedure is a block by block procedure where initial conditions are obtained explicitly at x_{j+1}, $j = 0, 1, \ldots, N-1$ using the computed values y_{j+1}. The starting values for subsequent block is then computed from the previous block for the implementation of the method over the subintervals: $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$.

5. Numerical Experiments

In this section, numerical experiments are performed using some sample problems to test the efficiency and accuracy of the hybrid methods. The results are compared with results obtained from existing methods in Tables 3, 4 and 5 respectively, using in each case a fixed step size $h = 0.01$.

Problem 5.1.

$$y'' - x(y')^2 = 0,\ y(0) = 1,\ y'(0) = \frac{1}{2}$$

Theoretical solution:

$$y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right).$$

Table 3. Comparison of absolute errors and CPU time between method (6), [3] and [8] for Problem 5.1

X	Exact Result	Computed Result	Error in Method (6)	Error in [3]	Error in [8]	Time(s) for [3]	Time(s) for [8]	Time(s) for method(6)
0.1	1.0500417	1.0500417	0.0000(00)	4.8627(-14)	6.5501(-11)	0.0312	0.0156	0.0624
0.2	1.1003353	1.1003353	6.6661(-16)	2.1604(-13)	5.4803(-10)	0.0624	0.0468	0.0780
0.3	1.1511404	1.1511404	1.5543(-15)	5.2557(-13)	1.9256(-09)	0.1092	0.1092	0.0936
0.4	1.2027325	1.2027325	2.8866(-15)	1.0254(-12)	4.8029(-09)	0.1404	0.1716	0.1092
0.5	1.2554128	1.2554128	4.8850(-15)	1.8032(-12)	1.0006(-08)	0.2653	0.1872	0.1404
0.6	1.3095196	1.3095196	7.1054(-15)	3.0078(-12)	1.8727(-08)	0.2964	0.2028	0.1872
0.7	1.3654437	1.3654437	1.2657(-14)	4.8991(-12)	3.2746(-08)	0.3744	0.2340	0.2496
0.8	1.4236489	1.4236489	2.3315(-14)	7.9460(-12)	5.3969(-08)	0.4524	0.2496	0.2808
0.9	1.4847003	1.4847003	4.1522(-14)	1.3702(-11)	8.8004(-08)	0.5460	0.3432	0.2808
1.0	1.5493061	1.5493061	7.4829(-14)	2.1885(-11)	1.4353(-07)	0.6552	0.4748	0.2964

Problem 5.2.

$$y'' - \frac{(y')^2}{2y} + 2y = 0,\ y\left(\frac{\pi}{6}\right) = \frac{1}{4},\ y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Theoretical solution:

$$y = \sin^2 x.$$

Table 4. Comparison of absolute errors and CPU time between method (6) and [1] for Problem 5.2

X	Exact Result	Computed Result	Error in Method (6)	Error in [1]	Time(s) for [1]	Time(s) for method (6)
1.1035988	0.7971525280	0.7971525281	3.5274(-11)	1.8811(-10)	0.3900	0.0936
1.2035988	0.8711181442	0.8711181442	2.9418(-11)	2.4539(-10)	0.4368	0.1092
1.3035988	0.9302884505	0.9302884505	2.2389(-11)	3.0306(-10)	0.4680	0.1092
1.4035988	0.9723045137	0.9723045137	1.4467(-11)	3.5819(-10)	0.4992	0.1404
1.5035988	0.9954912857	0.9954912857	5.9693(-12)	4.0838(-10)	0.5460	0.1560
1.6035988	0.9989243831	0.9989243831	2.7670(-12)	4.5128(-10)	0.5772	0.1560
1.7035988	0.9824669392	0.9824669392	1.1393(-11)	4.8473(-10)	0.6552	0.1560
1.8035988	0.9467750604	0.9467750604	1.9564(-11)	5.0696(-10)	0.6864	0.1716
1.9035988	0.8932716692	0.8932716692	2.6956(-11)	5.1697(-10)	0.7020	0.2028
2.0035988	0.8240897771	0.8240897770	3.3274(-11)	5.1381(-10)	0.7644	0.2184

Problem 5.3.

$$y'' + \psi y = 0,\ y(0) = 1,\ y'(0) = 2,\ \psi = 2$$

Theoretical solution:

$$y(x) = \cos 2x + \sin 2x.$$

Table 5. Comparison of absolute errors and CPU time between method (6) and [2] for Problem 5.3

X	Exact Result	Computed Result	Error in Method (6)	Error in [2]	Time(s) for [2]	Time(s) for method (6)
0.01	1.0197987	1.0197987	7.3274720(-15)	9.5379(-13)	0.0000	0.0000
0.02	1.0391894	1.0391894	6.6613381(-15)	2.1846(-12)	0.0156	0.0000
0.03	1.0581645	1.0581645	6.2172489(-15)	3.6890(-12)	0.0312	0.0000
0.04	1.0767164	1.0767164	5.5511151(-15)	7.1798(-12)	0.0312	0.0156
0.05	1.0948376	1.0948376	5.1070259(-15)	1.0965(-11)	0.0624	0.0156
0.06	1.1125208	1.1125208	4.2188475(-15)	1.5016(-11)	0.0624	0.0312
0.07	1.1297591	1.1297591	3.7747583(-15)	2.1162(-11)	0.0780	0.0624
0.08	1.1465455	1.1465455	3.5527137(-15)	2.7600(-11)	0.0780	0.0624
0.09	1.1628733	1.1628733	3.1086245(-15)	3.4333(-11)	0.0862	0.0780
0.10	1.1787359	1.1787359	2.8865799(-15)	4.3238(-11)	0.0862	0.0780

6. Conclusion

An A(α)-stable continuous implicit one step hybrid method which is both efficient, accurate and economical has been developed in this paper. It has been established that the order $p = 8.$ method obtained converges very fast for fixed step sizes as shown in the time it takes to obtain solutions at the respective grid points. It is worth noting that apart from serving as starting values, the simultaneous block solutions can themselves be used as integrators. It is also evident that derivative solutions can be obtained at individual grid points as well. Numerical experiments performed on sample problems yielded the results reported in Tables 3, 4 and 5 respectively. In view of the comparison made with solutions obtained from block method [8], block predictor-corrector method [2] and block hybrid predictor-corrector method [3], we observed that method (6) gave better result; yielding very low error of approximation and used lesser CPU time (in seconds) than these methods. The method developed is recommended for the direct solution of higher order initial value problems of ordinary differential equations, even for stiff problems.

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