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ON A BLOCK INTEGRATOR FOR THE SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT: *An efficient one step Adam type implicit block numerical algorithm developed by simultaneous employment of interpolation and collocation techniques is proposed in this paper. Non mesh points were introduced to upgrade the order of consistency and improve the rate of convergence of the method. Further analysis revealed a wide interval of absolute stability. The method is implemented as a block to improve efficiency and reduce computational cost. Comparison of the new method with previous methods in terms of absolute errors of approximation established an improvement over those methods.*

Keywords: *Block, algorithm, collocation, interpolation, convergence, mesh*

2010 AMS Subject Classification: 65D30, 65L05, 65L06, 65L12.

1. INTRODUCTION

Differential equations usually arise from the mathematical modeling of real life phenomena such as in dynamical systems. For example, in chemical reactor dynamics, predator-prey problems, atmospheric fluid dynamics, biological dynamics, finance, electrical circuits and mechanical systems, etc. Very often, closed form solutions can not be obtained for these equations hence, the need for numerical approximations. However, the development of accurate numerical approximations is not usually an easy task and furthermore, there is usually a trade off between accuracy, stability and efficiency. Some of the several numerical methods proposed include the Runge-Kutta type methods [7, 9, and 10], the Adam type methods [1, 2, 3, 4, 5, and 6] and the backward differentiation formulae [11, 12].

The aim in this paper is to develop a stable, accurate and efficient method for the solution of initial value problems (IVP) of first order ordinary differential equations (ODEs) of the form:

$$F\left(x, y, y'\right) = 0, \quad \mathrm{y}(\mathrm{t}_0) = \gamma;\ t \in [\mathrm{t}_0, \mathrm{T}], \tag{1}$$

where $(\mathrm{T} \le \infty) \in \mathbb{R}$.

Conventionally, higher order ODEs are sometimes reduced to a system of first order ODEs before their solutions are sought. Many times problems in this class are mildly stiff or outright stiff problems. By stiff problems, we mean problems that correspond to physical processes whose components have disparate time scales or whose time scale is small compared to interval over which it is studied. For example, in chemical kinetics where reactions take place at different speeds. The motivation for this paper is therefore, to develop an accurate numerical algorithm for the solution of initial value problems of first order ODEs of any nature. In what follows, we shall assume standard conditions for the existence and uniqueness of the solutions of (1). Some of the other methods proposed for the solution of (1) can be seen in [1, 11, 13, and 12]. In the sections that follow, we shall describe the method of derivation, then an analysis of the properties of the method is presented. Furthermore, numerical experiments are conducted on sample problems and comparisons of the numerical results, with respect to the absolute error of approximation, with other proposed methods are presented in tables.

2. DERIVATION OF THE METHOD

Let $(t)_h = \{t_n : n = 0, \ldots, N\}$, with a fixed step size h, be the time discretization of the interval $[t_0, T]$ such that $t_{n+1} = t_n + h,\ n = 0, \ldots, N-1$, defined as a sequence $t_0 < t_1 < \cdots < t_{n-1} < t_N = T$ of mesh points. Following [3], consecutive mesh points, say $\{(t_n, t_{n+1}) : n = 0, 1, \ldots, N-1\}$, are further partitioned into q so called non mesh points given by $t_{n+\mu_u}, \mu_u = \frac{u}{q+1};\ u = 1, 2, \ldots, q$.

Let

$$P_m(x) = \sum_{i=0}^{m} a_i x^i \tag{2}$$

be a power series polynomial completely determined by m+1 unknown parameters a_i, i = 0,1,..., m. Now, interpolate (2), Stormer-Cowell style, at the non mesh points, $t_{n+\mu_q}$ and collocate (2) at all points, $t_{n+j},\ j = 0(\mu_u)1, \forall u$. These procedures together give rise to a system of m+1 equations:

$$\sum_{i=0}^{m} a_i x_{n+\mu_s}^i = y_{n+\mu_s},\ s = q \tag{3a}$$

$$\sum_{i=0}^{m} i(i-1) a_i x_{n+r}^i = f_{n+r},\ r = 0, \mu_u, 1 \tag{3b}$$

where $m = (s + r) - 1$ and s, r represent the interpolation and collocation points respectively. Equations (3a) and (3b) together satisfy the matrix equation:

$$AX = B \tag{4}$$

where A is an $(m+1)\times(m+1)$ coefficient matrix, (note that the coefficients here are the points, $t_{n+j},\ j = 0(\mu_u)1, \forall u$), obtained from the interpolation and collocation equations; X and B are column matrices of the unknown parameters to be solved for, that is, $a_j,\ j = 0(\mu_u)1$. The solutions so obtained when substituted back into (2) give rise to an algebraic equation from where a continuous implicit hybrid multistep method is obtained, after some algebraic manipulations as follows:

$$y_{n+j} = \alpha_{\mu_q}(z) y_{n+\mu_q} + h \sum_j \beta_j(z) f_{n+j} \tag{5}$$

where $j = 0(\mu_u)1, \forall u$. The coefficients $\alpha_{\mu_q}(z)$ and $\beta_j(z)$ are continuous for all values of $z \in [0, 1]$ and are obtained by evaluating the transformation:

$$z = \frac{t - t_n}{h} \tag{6}$$

In what follows, for reasons of accuracy, stability and convergence desired, we make $q = 7$, that is a choice of seven non mesh points. Hence, the coefficients $\alpha_{\mu_q}(z)$ and $\beta_j(z), \forall\, j$ were obtained as follows:

$$\alpha_{\frac{7}{8}}(z) = 1 \tag{7}$$

$$\begin{aligned}\beta_0(z) = {} & \frac{13102}{2835}z^9 - \frac{8192}{35}z^8 + \frac{53248}{105}z^7 \\ & - \frac{3072}{5}z^6 + \frac{34208}{75}z^5 - \frac{1068}{5}z^4 \\ & + \frac{59062}{945}z^3 - \frac{761}{70}z^2 + z + \frac{149527}{4147200}\end{aligned} \tag{8}$$

$$\begin{aligned}\beta_{\frac{1}{8}}(z) = {} & -\frac{1048576}{2835}z^9 + \frac{16384}{9}z^8 - \frac{1196032}{315}z^7 \\ & + \frac{117760}{27}z^6 - \frac{673792}{225}z^5 + \frac{11168}{9}z^4 \\ & - \frac{30784}{105}z^3 + 32z^2 - \frac{408317}{2073600}\end{aligned} \tag{9}$$

$$\begin{aligned}\beta_{\frac{1}{4}}(z) = {} & \frac{524288}{405}z^9 - \frac{278528}{45}z^8 + \frac{3915776}{315}z^7 \\ & - \frac{366592}{27}z^6 + \frac{1956992}{225}z^5 - \frac{146824}{45}z^4 \\ & + \frac{3312}{5}z^3 - 56z^2 - \frac{24353}{2073600}\end{aligned} \tag{10}$$

$$\begin{aligned}\beta_{\frac{3}{8}}(z) = {} & -\frac{1048576}{405}z^9 + \frac{180224}{15}z^8 - \frac{2441216}{105}z^7 \\ & + \frac{72704}{3}z^6 - \frac{1097728}{75}z^5 + \frac{25504}{5}z^4 \\ & - \frac{128192}{135}z^3 + \frac{224}{3}z^2 - \frac{542969}{2073600}\end{aligned} \tag{11}$$

$$\begin{aligned}\beta_{\frac{1}{2}}(z) = {} & \frac{262144}{81}z^9 - \frac{131072}{9}z^8 + \frac{1712128}{63}z^7 \\ & - \frac{733184}{27}z^6 + \frac{703552}{45}z^5 - \frac{46624}{9}z^4 \\ & + \frac{2764}{3}z^3 - 70z^2 - \frac{343}{25920}\end{aligned} \tag{12}$$

$$\beta_{\frac{5}{8}}(z) = -\frac{1048576}{405}z^9 + \frac{507904}{45}z^8 - \frac{6406144}{315}z^7 + \frac{2642944}{135}z^6 - \frac{2443264}{225}z^5 + \frac{156512}{45}z^4 - \frac{3008}{5}z^3 + \frac{224}{5}z^2 - \frac{368039}{2073600} \tag{13}$$

$$\beta_{\frac{3}{4}}(z) = \frac{524288}{405}z^9 - \frac{16384}{3}z^8 + \frac{999424}{105}z^7 - \frac{26624}{3}z^6 + \frac{358784}{75}z^5 - 1496z^4 + \frac{34288}{135}z^3 - \frac{56}{3}z^2 - \frac{261023}{2073600} \tag{14}$$

$$\beta_{\frac{7}{8}}(z) = -\frac{1048576}{2835}z^9 + \frac{475136}{315}z^8 - \frac{114688}{45}z^7 + \frac{62464}{27}z^6 - \frac{274432}{225}z^5 + \frac{16864}{45}z^4 - \frac{6592}{105}z^3 + \frac{32}{7}z^2 - \frac{111587}{2073600} \tag{15}$$

$$\beta_1(z) = -\frac{1048576}{2835}z^9 + \frac{106496}{63}z^8 - \frac{204800}{63}z^7 + \frac{461824}{135}z^6 - \frac{482176}{225}z^5 + \frac{36572}{45}z^4 - \frac{56884}{315}z^3 + \frac{1487}{70}z^2 - t + \frac{8183}{4147200} \tag{16}$$

Furthermore, evaluating (6) at the point $t = t_{n+j},\ j = 0, \mu_1, \ldots, \mu_{q-1}, 1$ respectively, gives the values $z = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$ and 1 respectively, yielding eight discrete schemes. To derive the block method, we obtained an additional scheme from differentiating (5) and evaluating (6) at $t = t_n$

The block method comprises of a set of eight independent solutions (both at the mesh and non mesh points) as follows:

$$y_{n+\frac{1}{8}} = y_n + \frac{h}{29030400}[-33953 f_{n+1} + 312874 f_{n+\frac{7}{8}} - 1291214 f_{n+\frac{3}{4}} + 3146338 f_{n+\frac{5}{8}} - 5033120 f_{n+\frac{1}{2}} + 5595358 f_{n+\frac{3}{8}} - 4604594 f_{n+\frac{1}{4}} + 4467094 f_{n+\frac{1}{8}} + 1070017 f_n] \tag{17}$$

$$
\begin{aligned}
y_{n+\frac{1}{4}} = y_n + \frac{h}{907200}[&-833 f_{n+1} + \\
&7624 f_{n+\frac{7}{8}} - 31154 f_{n+\frac{3}{4}} + \\
&74728 f_{n+\frac{5}{8}} - 116120 f_{n+\frac{1}{2}} + \\
&120088 f_{n+\frac{3}{8}} - 42494 f_{n+\frac{1}{4}} + \\
&182584 f_{n+\frac{1}{8}} + 32377 f_n]
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
y_{n+\frac{3}{8}} = y_n + \frac{h}{358400}[&-369 f_{n+1} + \\
&3402 f_{n+\frac{7}{8}} - 14062 f_{n+\frac{3}{4}} + \\
&34434 f_{n+\frac{5}{8}} - 56160 f_{n+\frac{1}{2}} + \\
&79934 f_{n+\frac{3}{8}} + 3438 f_{n+\frac{1}{4}} + \\
&70902 f_{n+\frac{1}{8}} + f_n]
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
y_{n+\frac{1}{2}} = y_n + \frac{h}{113400}[&-107 f_{n+1} + \\
&976 f_{n+\frac{7}{8}} - 3956 f_{n+\frac{3}{4}} + \\
&9232 f_{n+\frac{5}{8}} - 9080 f_{n+\frac{1}{2}} + \\
&32752 f_{n+\frac{3}{8}} + 244 f_{n+\frac{1}{4}} + \\
&22576 f_{n+\frac{1}{8}} + 4063 f_n]
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
y_{n+\frac{5}{8}} = y_n + \frac{h}{1161216}[&-1225 f_{n+1} + \\
&11450 f_{n+\frac{7}{8}} - 49150 f_{n+\frac{3}{4}} + \\
&170930 f_{n+\frac{5}{8}} - 4000 f_{n+\frac{1}{2}} + \\
&318350 f_{n+\frac{3}{8}} + 7550 f_{n+\frac{1}{4}} + \\
&230150 f_{n+\frac{1}{8}} + 41705 f_n]
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
y_{n+\frac{3}{4}} = y_n + \frac{h}{11200}[&-9 f_{n+1} + \\
&72 f_{n+\frac{7}{8}} + 158 f_{n+\frac{3}{4}} + 2664 f_{n+\frac{5}{8}} \\
&- 360 f_{n+\frac{1}{2}} + 3224 f_{n+\frac{3}{8}} + 18 f_{n+\frac{1}{4}} \\
&+ 2232 f_{n+\frac{1}{8}} + 401 f_n]
\end{aligned} \tag{22}
$$

$$y_{n+\frac{7}{8}} = y_n + \frac{h}{4147200}[-8183 f_{n+1} + 223174 f_{n+\frac{7}{8}} + 522046 f_{n+\frac{3}{4}} + 736078 f_{n+\frac{5}{8}} + 54880 f_{n+\frac{1}{2}} + 1085938 f_{n+\frac{3}{8}} + 48706 f_{n+\frac{1}{4}} + 816634 f_{n+\frac{1}{8}} + 149527 f_n] \tag{23}$$

$$y_{n+1} = y_n + \frac{h}{28350}[989 f_{n+1} + 5888 f_{n+\frac{7}{8}} - 928 f_{n+\frac{3}{4}} + 10496 f_{n+\frac{5}{8}} - 4540 f_{n+\frac{1}{2}} + 10496 f_{n+\frac{3}{8}} - 928 f_{n+\frac{1}{4}} + 5888 f_{n+\frac{1}{8}} + 989 f_n] \tag{24}$$

Which in compact form can be written as:

$$\bar{A}Y_m = \bar{B}y_m + h^{\gamma}\left[\bar{C}F\left(y_m\right) + \bar{D}F\left(Y_m\right)\right] \tag{25}$$

where $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D}$ are coefficients matrices associated respectively with $Y_m = (y_{n+\mu_u}, y_{n+1},, \; y'_{n+\mu_u}, y'_{n+1})^T$, $y_m = \left(y_m, y'_m\right)^T$ $F\left(Y_m\right) = \left(f_{n+\mu_u}, f_{n+1}\right)^T$, $F\left(y_m\right) = \left(f_n\right)$ and γ =1, coincides with the order of (1).

3. ANALYSIS OF THE METHOD

In this section, the order of the consistency, local error term, convergence and stability of the method (25) are determined.

3.1. Order and Local Error Term

Following [3] and [10], the linear difference operator associated with the block method is defined as:

$$\mathfrak{L}\left[y(x), h\right] = \bar{A}Y_m - \bar{B}y_m - h[\bar{C}F\left(y_m\right) - \bar{D}F\left(Y_m\right) \tag{26}$$

Now, expanding (26) in Taylor series about x_n and collecting terms in powers of h yields the following:

$$\mathfrak{L}\left[y\left(x_n,\right), h\right] = \bar{c}_0 y\left(x_n\right) + \bar{c}_1 h y^{(1)}\left(x_n\right) + \bar{c}_2 h^2 y^{(2)}\left(x_n\right) + \cdots + \bar{c}_p h^p y^{(p)}\left(x_n\right) + \bar{c}_{p+1} h^{p+1} y^{(p+1)}\left(x_n\right) \tag{27}$$

where $y^{(p)}(x_n)$ represents the *pth* derivative of y with respect to x_n and the $\bar{c}_i, i = 0, \ldots, p, p+1, \ldots$ are vector coefficients.

Definition 3.1

1. *The block method (25) and the associated linear difference operator (26) have order equal to p if* $\bar{c}_0 = \bar{c}_1 = \cdots = \bar{c}_p = 0$ *and* $\bar{c}_{p+1} \neq 0$.

2. *The term* $\bar{c}_{p+1}$ *is called the error term and implies that the local truncation error of (25) is given by the term:*

$$T_n = \bar{c}_{p+1} h^{p+1} y^{(p+1)}(x_n) + 0\left(h^{p+2}\right) \tag{28}$$

From our computation, the block method (25) is of uniform order $p = 9$ with local error term given as $\bar{c}_{p+1} = (7.3505\times10^{-12},\ 5.9871\times10^{-12}, 6.4964\times10^{-12}\ 6.1760\times10^{-12};\ 6.4964\times10^{-12};\ 5.9871\times10^{-12}; 7.3505\times10^{-12};\ 0)^T$

3.2. Stability of the Method

The region of stability and zero stability of the method (25) is investigated in this section.

Consider the scalar test equation:

$$y' = \lambda y, \quad \lambda < 0 \tag{29}$$

Furthermore, let $\bar{h} = \lambda h$ then, applying (29) to (25) , (see [12]), gives the stability polynomial

$$\begin{aligned}\Pi\left(z,\bar{h}\right) = {} & -\frac{\bar{h}^8 z^7}{150994944} - \frac{761\bar{h}^7 z^7}{2642411520} - \\ & \frac{29531\bar{h}^6 z^7}{3963617280} - \frac{89\bar{h}^5 z^7}{655360} - \frac{1069\bar{h}^4 z^7}{589824} \\ & -\frac{9\bar{h}^3 z^7}{512} - \frac{91\bar{h}^2 z^7}{768} - \frac{\bar{h} z^7}{2} - z^7 + \\ & \frac{\bar{h}^8 z^8}{150994944} - \frac{761\bar{h}^7 z^8}{2642411520} - \\ & \frac{29531\bar{h}^6 z^8}{3963617280} - \frac{89\bar{h}^5 z^8}{655360} - \frac{1069\bar{h}^4 z^8}{589824} \\ & -\frac{9\bar{h}^3 z^8}{512} - \frac{91\bar{h}^2 z^8}{768} - \frac{\bar{h} z^8}{2} - z^8 \\ = {} & 0\end{aligned} \tag{30}$$

From (30), it can be established that the method is zero stable since as $h \to 0$ the equation reduces to the first characteristic polynomial of (25) as follows:

$$z^8 - z^7 = 0 \tag{31}$$

with $z = 0$ or 1 satisfying the root conditions for zero stability, [10].

From our computation, the method is found to have an interval of absolute stability of (–12000,0).

3.3. Consistency and Convergence

It can be deduced that the block method (25) is consistent from the fact that the order of the method is greater than one.

Convergence for the method is considered in the light of the result by [8]. That is, our claim for convergence follows from the fact that the method (25) is consistent and zero stable.

4. IMPLEMENTATION

Implementation of the block formula (25) is according to [5]. A single application of the block formula generates simultaneously, approximate solutions at the step points t_n, t_{n+1} and all the non mesh points $t_{n+\mu_u}, u=1,\ldots,q$. The procedure is a block by block procedure where initial conditions are obtained explicitly at $t_{n+1}, n=0,1,...,N-1$ using the computed values y_{n+1}. The starting values for subsequent blocks are then computed from the previous block for the implementation of the method over the subintervals: $[t_0,t_1], [t_1,t_2],..., [t_{N-1},t_N]$.

5. TEST PROBLEMS

In this section, the proposed block method is tested on the following sample stiff problems:

Problem 5.1

$$y' = xy,\ y(0)=1$$
$$Exact\ Solution:\ y(x) = e^{\frac{1}{2}x^2}$$
$$Source: [1]$$

Problem 5.2

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$$
$$y\left(\frac{\pi}{4}\right) = 3\sqrt{2},\ \frac{\pi}{4} \le x \le \frac{\pi}{2}$$
$$Exact\ Solution: y(x) = -\frac{1}{2}\cos(x)\sin(2x) - \sin(x) + 7\cos(x)$$
$$Source: [13]$$

Problem 5.3

$$y' = -5(y-x)+1,\ y(0)=1$$
$$Exact\ \ Solution: y(x) = e^{-5x} + x$$
$$Source: [11]$$

Problem 5.4

$$y' = -20y + 24, \quad y(0) = 0, 0 \le x \le 1.0$$

$$Exact\ Solution : y(x) = \frac{6}{5} - \frac{6}{5}e^{-20x}$$

$$Source : [12]$$

6. NUMERICAL RESULTS

In the following are tables showing the exact solutions, the approximate solutions, the maximum error recorded for the proposed method and maximum errors obtained from the previous methods. The computation was carried out for each problem with a prescribed step size in order to make for a good basis of comparison with previous methods by other authors. The results reported in Tables 1, 2, 3 and 4 respectively, for each of the four test problems and their absolute errors of approximation obtained from the proposed method are compared respectively, with results obtained for the same problems from hybrid linear multistep formulae ([1] and [13]) and backward differentiation formulae ([11] and [12]). Note in particular, that although in [12] Problem 5.4 is solved using the variable step size method with prescribed tolerances based on a block backward differentiation formula, the performance of the new method using a fixed step size of h^{-2} suggest a better accuracy over the results obtained in [12] for tolerance values of 10^{-2}, 10^{-4} and 10^{-6} respectively.

7. CONCLUSION

An accurate and efficient one step block numerical algorithm of order p = 9 has been developed in this paper for the solutions of initial value problems of first order ordinary differential equations. From the analysis of the method, it has been ascertained that the block numerical algorithm is consistent, locally stable with very low error constants and converges. Furthermore, the method is absolutely stable with a very wide interval of absolute stability. The mode of implementation makes the method computationally inexpensive. Therefore, we recommend the one step hybrid block numerical algorithm for the solutions of initial value problems of firrst order ordinary differential equations.

Table 1
Absolute Errors in the New Method Compared to Errors in [1] for Problem 5.1

X	*ExactResult*	*Computed Result*	*Error inmethod (25)*	*Error in [1]*
0.1	1.0050125	1.0050125	8.4377(-15)	4.0345(-13)
0.2	1.0202013	1.0202013	3.9524(-14)	9.2947(-13)
0.3	1.0460279	1.0460279	9.6589(-14)	1.6266(-12)
0.4	1.0832871	1.0832871	1.8407(-13)	2.6270(-12)
0.5	1.1331485	1.1331485	3.0931(-13)	4.1049(-12)
0.6	1.1972174	1.1972174	4.8295(-13)	6.3136(-12)
0.7	1.2776213	1.2776213	7.1920(-13)	9.6096(-12)
0.8	1.3771278	1.3771278	1.0387(-12)	1.4527(-11)
0.9	1.4993025	1.4993025	1.4695(-12)	2.1842(-11)
1.0	1.6487213	1.6487213	2.0508(-12)	3.2709(-11)

Table 2
Absolute Errors in the New Method Compared to Errors in [13] for Problem 5.2

X	*Exact Result*	*Computed Result*	*Error in method (25)*	*Error in [13]*
$\frac{7\pi}{25}$	3.7511752	3.7511752	1.26852(-16)	2.61568(-13)
$\frac{37\pi}{100}$	1.9982136	1.9982136	1.86700(-16)	3.27738(-13)
$\frac{23\pi}{50}$	-0.0540845	-0.0540845	3.21412(-16)	2.96670(-13)
$\frac{47\pi}{100}$	-0.2905834	-0.2905834	3.11545 (-16)	2.73614(-13)
$\frac{12\pi}{25}$	-0.5273457	-0.5273457	2.75062(-16)	2.16604(-13)
$\frac{49\pi}{100}$	-0.7639572	-0.7639572	3.51010(-16)	1.93289(-13)
$\frac{\pi}{2}$	-1.0000003	-1.0000003	9.63645(-13)	1.07637(-03)

Table 3
Absolute Errors in the New Method Compared to Errors in [11] for Problem 5.3

X	*Exact Result*	*Computed Result*	*Error in Method (25)*
0.01	0.21752310	0.21752319	9.4061712(-08)
0.02	0.39561595	0.39561612	1.6453692(-07)
0.03	0.54142605	0.54142625	2.0480460(-07)
0.04	0.66080526	0.66080548	2.2350043(-07)
0.05	0.75854469	0.75854492	2.2712190(-07)
0.06	0.83856697	0.83856719	2.2051972(-07)
0.07	0.90408367	0.90408388	2.0728138(-07)
0.08	0.95772421	0.95772440	1.9002966(-07)
0.09	0.10016414	0.10016415	1.7065458(-07)
0.10	0.10375977	0.10375978	1.5049279(-07)

Table 4
Absolute Errors in the New Method for Problem 5.4

X	*Exact Result*	*Computed Result*	*Error in method (25)*	*Error in [11]*
0.01	0.96122943	0.96122943	8.10(-15)	5.03(-10)
0.02	0.92483742	0.92483742	1.67(-14)	3.90(-11)
0.03	0.89070798	0.89070798	2.42(-14)	4.27(-10)
0.04	0.85873076	0.85873076	3.10(-14)	8.00(-11)
0.05	0.82880079	0.82880079	3.71(-14)	7.40(-11)
0.06	0.80081822	0.80081822	4.25(-14)	3.17(-10)
0.07	0.77468809	0.77468809	4.73(-14)	2.80(-10)
0.08	0.75032005	0.75032005	5.15(-14)	3.70(-11)
0.09	0.72762816	0.72762816	5.54(-14)	3.77(-10)
0.10	0.70653066	0.70653066	5.86(-14)	2.86(-10)

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