Analytical Solution of the Extended Schwartz and Moon (2000, 2001) Growth Option Model

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Abstract- **In the Schwartz and Moon growth option model and other extended versions of the model, the procedure of the valuation process is to use the discrete version of the continuous-time model and then use the Monte Carlo simulation for the valuation. We are of the opinion that if the equation for valuing the company is well posed and necessary adjustments made, our derived equation could be solved by a stochastic partial differential equation.**

Key words: Ito Lemma, Stochastic Equation, Stochastic Partial Differential Equation, Monte Carlo.

Introduction

We considered an extended version of [7], Internet company valuation model, and a special case of [2].

Assumptions of the model in [7] which applies to [2]:

- Revenue, growth rate of revenue and variable cost are stochastic and is a Brownian motion process.
- There are no correlation between Brownian motion processes of revenues, growth rate of revenues and variable costs.
- Only revenues have a risk premium associated with them.
- Growth rate of revenue and volatilities of revenue and variable cost is a mean – reverting process to a long –term average drift.
- In the long run revenue will become stable.
- In the long run, when the company has no more growth, volatility of growth rate of revenue is assumed to be zero.
- The company can have future equity and debt financing when its cash available become negative.
- The interest rate is assumed to be static

- After the initial period, the capital spending is assumed to be a fixed percentage of revenue.

- Depreciation in period t is assumed to be a fixed portion of the accumulated property and equipment at period t_{n-1}.

In the [7], [8] model and in other literatures, [1], [3], [6] etc where the model has been used, a discrete version of the continuous-time process is used to simulate the value of a company. The said value of the company can be expressed as a function of several variables

$$V = V(R, \mu, \gamma, L, X, P, t) \qquad (1)$$

where,

R = revenues

$\mu = growth\ rate\ of\ revenues$

$\gamma = \mathrm{var}\,iable \quad \cos ts$

L = loss carry forward

X = cash balances

P = property and equipment

t = time

We are of the opinion that if equation (1) is well posed and necessary adjustments made, the resultant equation as in equation (26) can be solved by a Stochastic Partial Differential Equation.

Mathematical Formulation of the extended case:

According to [2], the main sources of revenue for banks come from interest spread between loans and deposits. Assume a bank loan at time t is given by L_t . We assume that the dynamics of these loans are given by the stochastic differential equation:

$$\frac{dL_t}{L_t} = \mu_t^L dt + \sigma_t^L dW_1^L \qquad (2)$$

where,

$L_t = Loan\ in\ period\ t$

$\mu_t^L = growth\ rate\ in\ loans$

$\sigma_t^L = volatility\ in\ loans$

$dW_1^L = a\ Wienene\ process$

Meanwhile, the volatility of loan, σ_t^L, is assumed to converge to a normal level in the long – term, and can be defined as:

$$d\sigma_t^L = k_1(\overline{\sigma^L} - \sigma_t^L)dt \qquad (3)$$

where,

$k_1 = mean\ reversion\ coefficien\ t$

$\overline{\sigma^L} = long - term\ average\ volatility\ of\ loans$

$\sigma_t^L = initial\ volatility\ of\ loans$

Similarly, assume the bank deposit at time t is given by D_t. We also assume that the deposit are given by the stochastic differential equation:

$$\frac{dD_t}{D_t} = \mu_t^D dt + \sigma_t^D dW_2^D \qquad (4)$$

where,

$D_t = Deposit\ in\ period\ t$

$\mu_t^D = growth\ rate\ in\ deposits$

$\sigma_t^D = volatility\ in\ deposits$

$dW_2^D = a\ Wiener\ process$

Meanwhile, the volatility of deposit σ_t^D, is assumed to also converge to a normal level in the long – term, and can be defined as:

$$d\sigma_t^D = k_2(\overline{\sigma^D} - \sigma_t^D)dt \qquad (5)$$

where,

$k_2 = mean\ reversion\ coefficien\ t$

$\overline{\sigma^D} = long - term\ average\ volatility\ of\ deposits$

$\sigma_t^D = initial\ volatility\ of\ deposits$

Growth rate of Loans:

The growth rate of loans in equation (2) can be expressed as the following stochastic differential equation

$$d\mu_t^L = \kappa_3 \left(\overline{\mu_L} - \mu_t^L \right)dt + \eta_t^L dW_3^L \qquad (6)$$

where,

$\mu^L = initial\ growth\ rate\ in\ loan$

$k_3 = mean\ reversion\ coefficien\ t$

$\overline{\mu} = long\ term\ growth\ rate\ in\ loan$

$\eta_t^L = volatility\ of\ growth\ rate\ in\ loan$

$dW_3^L = a\ Wiener\ process$

In the long run, the unanticipated volatility of growth rate in loan is assumed to converge to zero, which means that equation (6) can be written as follows:

$$d\eta_t^L = -k_4\eta_t^L dt \qquad (7)$$

where,

$\eta_t^L = volatility\ of\ growth\ rate\ of\ loans$

$k_4 = mean\ reversion\ coefficien\,t$

Growth rate of Deposits

Similarly, the growth rate of deposits in equation (4) can be expressed as the following stochastic differential equation:

$$d\mu_t^D = \kappa_5 \left(\overline{\mu}_D - \mu_t^D\right)dt + \eta_t^D dW_4^D \qquad (8)$$

where,

$\mu^D = initial\ growth\ rate\ in\ deposit$

$k_4 = mean\ reversion\ coefficien\,t$

$\overline{\mu^D} = long\ term\ growth\ rate\ in\ deposit$

$\eta_t^D = volatility\ of\ growth\ rate$

$dW_4^D = a\ Wiener\ process$

In the long run, the unanticipated growth rate in deposit is assumed to converge to zero, which means that equation (8) can be written as follows:

$$d\eta_t^D = -k_5\eta_t^D dt \qquad (9)$$

where,

$\eta_t^D = volatility\ of\ growth\ rate\ in\ deposit$

$k_5 = mean\ reversion\ coefficien\,t.$

In equations (3), (5), (6), (7), (8) and (9), the mean reversion coefficient (k) influences the process of initial growth rate converging to the long − term growth rate of loans and deposits. The half − life of the deviation is $\dfrac{\ln(2)}{k}$, which means that any deviations in the growth rate are expected to be halved in the time period.

Interest Spread

The interest spreads relates to changes in the interest rate on loan and interest rate on deposits. Although there are different kinds of deposit rate, we make use of the average for a proxy. The loans and deposits rates follow two stochastic processes. We define loans interest as a spread S_t above the average deposits rate r_t, and they both follow the square-root process of CIR (1985):

$$dr_t = a(b - r_t)dt + \sigma_t \sqrt{r_t}dW_5^r \qquad (10)$$

$$dS_t = a^s(b^s - S_t)dt + \sigma_t^s \sqrt{S}dW_6^S \qquad (11)$$

where,

$a\ and\ a^s = the\ reversion\ speed\ parameters$

b and b^s = the values towards the $interest$
$rate\ revert\ over\ time$

σ_t and σ_t^s = the $standard\ deviation$

dW_5^r and dW_6^s = Wiener processes

Net Income

We define the net interest income

$$R_t = L_t \times (r_t + S_t) - D_t \times r_t \qquad (12)$$

where,

$R_t = net\ interest\ income$

$L_t = loan\ in\ period\ t$

$r_t = deposit\ rate$

$S_t = interest\ spread$

$D_t = deposit\ in\ period\ t$

In defining all the parameters about net interest income, there are seven standard Wiener processes, $dW_1^L, dW_2^\mu, dW_3^D, dW_4^\mu, dW_5^r, dW_6^S dW_7^\gamma$ and each Wiener process is instantaneously correlated with each other as follows:

$$dW_1^L dW_3^L = \varphi_{Lg^L} dt \qquad (13.1)$$

$$dW_1^L dW_2^D = \varphi_{LD}\, dt \qquad (13.2)$$

$$dW_1^L dW_4^D = \varphi_{Lg^D} dt \qquad (13.3)$$

$$dW_1^L dW_5^r = \varphi_{Lr} dt \qquad (13.4)$$

$$dW_1^L dW_6^S = \varphi_{LS}\, dt \qquad (13.5)$$

$$dW_1^L dW_7^\gamma = \varphi_{L\gamma} dt \qquad (13.6)$$

$$dW_3^L dW_2^D = \varphi_{Dg^L} dt \qquad (13.7)$$

$$dW_3^L dW_4^D = \varphi_{g^L g^D} dt \qquad (13.8)$$

$$dW_3^L dW_5^r = \varphi_{g^L r}\, dt \qquad (13.9)$$

$$dW_3^L dW_6^S = \varphi_{g^L S}\, dt \qquad (13.10)$$

$$dW_3^L dW_7^\gamma = \varphi_{g^L \gamma} dt \qquad (13.11)$$

$$dW_2^D dW_4^D = \varphi_{Dg^D} dt \qquad (13.12)$$

$$dW_2^D dW_5^r = \varphi_{Dr} dt \qquad (13.13)$$

$$dW_2^D dW_6^S = \varphi_{DS} dt \qquad (13.14)$$

$$dW_2^D dW_7^\gamma = \varphi_{D\gamma} dt \qquad (13.15)$$

$$dW_4^D dW_5^r = \varphi_{g^D r}\, dt \qquad (13.16)$$

$$dW_4^D dW_6^S = \varphi_{g^D S}\, dt \qquad (13.17)$$

$$dW_4^D dW_7^\gamma = \varphi_{g^D \gamma}\, dt \qquad (13.18)$$

$$dW_5^r dW_6^S = \varphi_{rs} dt \qquad (13.19)$$

$$dW_5^r \, dW_7^\gamma = \varphi_{r\gamma} dt \qquad\qquad (13.20)$$

$$dW_6^S \, dW_7^\gamma = \varphi_{S\gamma} dt \qquad\qquad (13.21)$$

where $\varphi_{L_g L}$ is the correlation between loans and loan growth rate; φ_{LD} is the correlation between loans and deposits; $\varphi_{L_g D}$ is the correlation between loans and deposit growth rate; φ_{Lr} is the correlation between loans and deposits rates; φ_{LS} is the correlation between loans and interest spread; $\varphi_{L\gamma}$ is the correlation between loans and variable cost; $\varphi_{D_g L}$ is the correlation between deposits and loans growth rate; $\varphi_{{}_g L_g D}$ is the correlation between loan growth rate and deposit growth rate; $\varphi_{{}_g L_r}$ is the correlation between loan growth rate and deposit rate; $\varphi_{{}_g L_S}$ is the correlation between loan growth rate and interest spread; $\varphi_{{}_g L_\gamma}$ is the correlation between growth in loan and variable cost; $\varphi_{D_g D}$ is the correlation between deposit and deposit growth rate; φ_{Dr} is the correlation between deposit and deposit rate; φ_{DS} is the correlation between deposit and interest spread; $\varphi_{D\gamma}$ is the correlation between deposit and variable cost; $\varphi_{{}_g D_r}$ is the correlation between deposit growth rate and deposit rate, $\varphi_{{}_g D_S}$ is the correlation between deposit growth rate and interest spread; $\varphi_{{}_g D_\gamma}$ is the correlation between deposit growth rate and variable cost; φ_{rs} is the correlation between the deposit rate and the interest spread; $\varphi_{r\gamma}$ is the correlation between deposit rate and variable cost and $\varphi_{s\gamma}$ is the correlation between interest spread and variable cost.

Variable costs

The total cost of bank will consist of two components, the variable cost and fixed cost. The variable cost can be defined as the non – operating loan, which is a percentage of the total amount of loans. The fixed cost is the building and operating equipment cost, it is assumed to be constant.

The total cost of bank can therefore be presented in the following equation

$$C_t = \gamma_t L_t + F \qquad\qquad (14)$$

where,

$$C_t = total \; \cos t$$

$$\gamma_t = \mathrm{var} iable \; \cos t \; at \; time \; t$$

$L_t = loan \ at \ time \ t$

$F = fixed \ \cos t$

In Schwartz and Moon (2000, 2001) models, the cost function should be stochastic to reflect the uncertainty of future technology advancement and competition. Since only the variable cost would allow the stochastic process, equation (14) can be written as

$$d\gamma_t = k_6(\bar{\gamma} - \gamma_t)dt + \xi_t dW_7^\gamma \qquad (15)$$

where,

$\gamma = initial \ \ var iable \ \cos t$

$\bar{\gamma} = long \ term \ average \ of \ var iable \ \cos t$

$\xi_t = volatility \ \ of \ var iable \ \cos t$

$k_6 = mean \ reversion \ coeffiecie nt$

$dW_7^\gamma = a \ Wiener \ process$

The mean reversion coefficient k_6 is similar to the one described earlier on, it describe the rate at which the variable cost are expected to converge to its long – term average. The volatility of variable costs is also assumed to converge to a normal level in the long run. Equation (15) can be expressed as follows

$$d\xi_t = k_7(\bar{\xi} - \xi_t)dt \qquad (16)$$

where,

$\xi_t = int ial \ \ volatility \ \ of \ var iable \ \cos ts$

$k_7 = mean \ reversion \ coeficient$

$\bar{\xi} = long - term \ average \ volatility \ of \ var iable \ \cos ts$

Loss carry - forward

With the bank revenue and cost known, we define the after –tax net income as Y_t given by:

$$Y_t = (R_t - C_t - Dep_t) \times (1 - \tau_t), \qquad (17)$$

where

$Y_t = after \ tax \ net \ income$

$R_t = net \ int erest \ income$

$C_t = total \ \cos t$

$Dep = depreciati on$

$\tau_t = corporate \ tax \ rate$

The company only has to pay taxes where there is no accumulated loss carry –forward (F_t).

The condition can be defined as:

$$dF_t = -Y_t dt \qquad \qquad if \quad F_t > 0$$
$$dF_t = Max(-Y_t dt, 0) \qquad if \quad F_t = 0 \quad (18)$$

Accumulated property, plant and equipment

There are two elements that affect accumulated property, plant and equipment: the rate of capital expenditures and corresponding rate of depreciation. Therefore, the accumulated property, plant and equipment can be expressed as,

$$dP_t = Capx_t - dep_t dt \qquad\qquad (19)$$
$$Capx_t = CX_{t'} \qquad\qquad for \quad t \le t'$$
$$Dep_t = DR \times P_t$$

where,

P = accumulated property plant and equipment

Capx = capital expenditures

Dep = depreciation

CX = planned capital expenditures

DR = percentage of depreciation

t = initial period of time

The planned capital expenditures are assumed to be fixed at the beginning, after a certain period of time, it will become a percentage of planned capital expenditures. Depreciation is also assumed to be certain percentage of accumulated property, plant and equipment.

Amount of cash available

The total amount of cash available for the company can be defined as:

$$X_t = (R_t + Y_t + Dep_t - Capx_t)dt \qquad (20)$$

where,

X = cash balance

R = net interest income

Y = after tax net income

Dep = Depreciation

Capx = capital expenditure

To simplify the model, [7] and [8] assumed a zero dividend policy at the initial stage. In other words, all profits generated will be retained and grow at risk free rate of interest. After the initial volatile start up period, when the company is now sustainable, then it can distribute all its accumulated profits to its shareholders.

The value of the Company

The objective of the model is to determine the value of the firm at the current time. According to standard theory this value is obtained by discounting the expected value of the firm at time horizon under the risk neutral measure (the equivalent martingale measure) at the risk free rate of interest. After defining all the variables in the model, the value of the company at time T, $T \in [0, T]$ has two components. First, the cash balance outstanding and the second the value of the firm as a going concern, the value which is assumed to be a multiple M of the Earning Before Interest, Taxes, Depreciation and Amortization (EBITDA):

$$V_0 = E_Q[(X_T + M(R_T - C_T))e^{-rT} \qquad (21)$$

where,

$V_0 = value\ of\ the\ company\ at\ present\ time$

$X_T = outs\tan ding\ cash\ balanch\ at\ time\ T$

$M = multiplier$

$C_T = \cos t\ at\ time\ T$

$r = risk\ free\ rate$

$R_T = net\ \text{int}erest\ income\ at\ time\ T$

$e^{-rT} = continuously\ compounded\ discount\ factor$

$E_Q = equivalent\ martingale\ measure$

In this model, the value of a bank is a function of the state variables (loans, expected growth in loans, deposit, expected growth in deposits, variable cost, interest rates, interest spread, loss – carry forward, accumulated property, plant and equipment, cash balances) and time. This can be written as

$$V \equiv V\left(L, \mu^L, D, \mu^D, \gamma, r, S, X, Y, P, t\right) \qquad (22)$$

Applying Ito's Lemma to (22) yields an expression for bank value dynamics:

$$\begin{aligned}
dV_t &= \frac{\partial V_t}{\partial L} dL + \frac{\partial V_t}{\partial \mu^L} d\mu^L + \frac{\partial V_t}{\partial D} dD + \frac{\partial V_t}{\partial \mu^D} d\mu^D + \frac{\partial V_t}{\partial \gamma} d\gamma + \frac{\partial V_t}{\partial r} dr + \frac{\partial V_t}{\partial S} dS + \frac{\partial V_t}{\partial X} dX + \frac{\partial V_t}{\partial Y} dY + \frac{\partial V}{\partial P} dP + \\
&\quad \frac{\partial V_t}{\partial t} dt + \frac{1}{2}\frac{\partial^2 V_t}{\partial L^2} dL^2 + \frac{1}{2}\frac{\partial^2 V_t}{\partial \mu^{L^2}} d\mu^{L^2} + \frac{1}{2}\frac{\partial^2 V_t}{\partial D^2} dD^2 + \frac{1}{2}\frac{\partial^2 V_t}{\partial \mu^{D^2}} d\mu^{D^2} + \frac{1}{2}\frac{\partial^2 V_t}{\partial \gamma^2} d\gamma^2 + \frac{1}{2}\frac{\partial^2 V_t}{\partial r^2} dr^2 + \frac{1}{2}\frac{\partial^2 V_t}{\partial S^2} dS^2 + \\
&\quad \frac{\partial V_t \partial V_t}{\partial L \partial \mu^L} dL d\mu^L + \frac{\partial V_t \partial V_t}{\partial L \partial D} dL dD + \frac{\partial V_t \partial V_t}{\partial L \partial \mu^D} dL d\mu^D + \frac{\partial V_t \partial V_t}{\partial L \partial \gamma} dL d\gamma + \frac{\partial V_t \partial V_t}{\partial L \partial r} dL dr + \frac{\partial V_t \partial V_t}{\partial L \partial S} dL dS + \\
&\quad \frac{\partial V_t \partial V_t}{\partial \mu^L \partial D} d\mu^L dD + \frac{\partial V_t \partial V_t}{\partial \mu^L \partial \mu^D} d\mu^L d\mu^D + \frac{\partial V_t \partial V_t}{\partial \mu^L \partial \gamma} d\mu^L d\gamma + \frac{\partial V_t \partial V_t}{\partial \mu^L \partial r} d\mu^L dr + \frac{\partial V_t \partial V_t}{\partial \mu^L \partial S} d\mu^L dS + \frac{\partial V_t \partial V_t}{\partial D \partial \mu^D} dD d\mu^D \\
&\quad + \frac{\partial V_t \partial V_t}{\partial D \partial \gamma} dD d\gamma + \frac{\partial V_t \partial V_t}{\partial D \partial r} dD dr + \frac{\partial V_t \partial V_t}{\partial D \partial S} dD dS + \frac{\partial V_t \partial V_t}{\partial \mu^D \partial \gamma} d\mu^D d\gamma + \frac{\partial V_t \partial V_t}{\partial \mu^D \partial r} d\mu^D dr + \frac{\partial V_t \partial V_t}{\partial \mu^D \partial S} d\mu^D dS + \\
&\quad \frac{\partial V_t \partial V_t}{\partial \gamma \partial r} d\gamma dr + \frac{\partial V_t \partial V_t}{\partial \gamma \partial S} d\gamma dS + \frac{\partial V_t \partial V_t}{\partial r \partial S} dr dS. \bullet \qquad\qquad \bullet(23)
\end{aligned}$$

Now, applying equations (2, 4, 6, 8, 10, 11, 15, 17, 19 and 20) and the correlated Wiener processes, equations (13.1 – 13.21) to equation (23), we have that

$$dV_t = \frac{\partial V_t}{\partial L}\left[L\mu^L dt + L\sigma^L dW_1^L\right] + \frac{\partial V_t}{\partial \mu^L}\left[K_s\left(\overline{\mu}_L - \mu\right)dt + \gamma^L dW_3^L\right] + \frac{\partial V_t}{\partial D}\left[D\mu^D dt + D\sigma^D dW_2^D\right] +$$

(equation 24, continued)

$$\bullet(24)$$

adopt the discrete time approximation of the model and apply the Monte Carlos simulation in solving the problem.

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Rearranging equation (24) we have,

$$dV_t = \left\{ \ldots \right\}dt + \frac{\partial V_t}{\partial L}L\sigma_L dW_1^L + \frac{\partial V_t}{\partial D}D\sigma_D dW_2^D + \frac{\partial V_t}{\partial \mu_L}\gamma_L dW_3^L + \frac{\partial V_t}{\partial \mu_D}\gamma_D dW_4^D + \frac{\partial V_t}{\partial r}\sigma\sqrt{r_t}dW_5^r +$$

$$\frac{\partial V_t}{\partial S}\sigma_x\sqrt{S_t}dW_6^s + \frac{\partial V_t}{\partial \gamma}\xi_t dW_7^\gamma \qquad (25)$$

Conclusion

Taking the integral of both sides of equation (25) and with the necessary adjustments, gives the value of the company. It is when these adjustments cannot be made that we

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