A lenticular version of a von Neumann inequality

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By

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Abstract. We generalize to lens-shaped domains the classical von Neumann inequality for the disk.

1. Introduction. We will say that L is a convex lens-shaped domain of the complex plane, with vertices σ and σ', if

- either there exist two disks

$$D_1 := \{z \in \mathbb{C};\, |z-\alpha_1| < r_1\} \text{ and } D_2 := \{z \in \mathbb{C};\, |z-\alpha_2| < r_2\}$$

  such that $L = D_1 \cap D_2$, $\sigma \neq \sigma'$ and $\{\sigma, \sigma'\} = \partial D_1 \cap \partial D_2$,
- or there exist a disk and a half-plane

$$D_1 := \{z \in \mathbb{C};\, |z-\alpha_1| < r_1\} \text{ and } \Pi_2 := \{z \in \mathbb{C};\, \text{Re } e^{i\theta}(z-\sigma) < 0\}$$

  such that $L = D_1 \cap \Pi_2$, $\sigma \neq \sigma'$ and $\{\sigma, \sigma'\} = \partial D_1 \cap \partial \Pi_2$.

We will denote by $2\alpha \in]0, \pi]$ the angle of the lens L at the vertices. We will consider also as a lens the limit case where $L = D_1 = D_2$ is a disk. Then, any point of the boundary may be considered as a vertex and $\alpha = \frac{\pi}{2}$.

Now let us consider a bounded linear operator $A \in \mathcal{B}(H)$ on a complex Hilbert space H. We will say that the operator A is of the lenticular L-type if we have

- $\|A-\alpha_1 I\| \leqq r_1$ and $\|A-\alpha_2 I\| \leqq r_2$, if $L = D_1 \cap D_2$,
- $\|A-\alpha_1 I\| \leqq r_1$ and $\text{Re } e^{i\theta}((A-\sigma)v, v) \leqq 0$, $\forall\, v \in H$, if $L = D_1 \cap \Pi_2$.

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In this paper, the norm used for a linear operator on a Hilbert space H (as well as for a matrix) is always the operator norm induced by the hilbertian structure.

The aim of this paper is to prove the following result.

Theorem 1. *Let L be a convex lens-shaped domain of the complex plane with angle 2α. There exists a best constant $C(\alpha) \in \mathbb{R}$ such that the inequality*

$$\|p(A)\| \leqq C(\alpha) \sup_{z\in L} |p(z)|, \tag{1}$$

holds for all polynomials $p : \mathbb{C} \to \mathbb{C}$, for all linear operators $A \in \mathcal{B}(H)$ of L-type and for all Hilbert spaces H. Furthermore this constant, which is only depending on the angle α, is a continuous decreasing function of $\alpha \in (0, \frac{\pi}{2}]$, and we have the estimate

$$\frac{\pi}{2\alpha} \sin\alpha \leqq C(\alpha) \leqq \min\left(2 + 2/\sqrt{3}, \frac{\pi - \alpha}{\alpha}\right). \tag{2}$$

Note that for $\alpha = \frac{\pi}{2}$, which corresponds to the case where L is a disk, we have $C(\frac{\pi}{2}) = 1$, and we recover a famous von Neumann inequality [4]. Except for this value $\frac{\pi}{2}$, we do not know the exact values of $C(\alpha)$. A small improvement

$$C(\alpha) \leqq \frac{\pi - \alpha}{\pi}\left(2 - \frac{2}{\pi} \log\tan\left(\frac{\alpha\pi}{4(\pi - \alpha)}\right)\right)$$

of the upper bound in (2) can be deduced from Theorem 4.2 in [1].

Theorem 1 can be generalized in several directions. For instance, by Mergelyan's Theorem, the inequality (1) remains valid if instead of polynomials we take p holomorphic in L and continuous in $\overline{L}$. The theorem is also valid in a completely bounded form. More precisely, if we consider now polynomial functions P with matrix values: $\mathbb{C} \in z \mapsto P(z) = (p_{ij}(z)) \in \mathbb{C}^{n,n}$, then there exists a continuous decreasing function $C_{cb}(\alpha)$ (which satisfies the bounds given for $C(\alpha)$) such that the inequality

$$\|P(A)\| \leqq C_{cb}(\alpha) \sup_{z\in L} \|P(z)\|$$

holds for all polynomials P with matrix values, for all linear operators $A \in \mathcal{B}(H)$ of type L and for all Hilbert spaces H. The adverb *completely* points out the fact that the inequality holds independently of the size n of the matrices. We do not know if $C(\alpha) = C_{cb}(\alpha)$ or not.

We should mention that a preliminary version of this theorem, in the particular case where L has a straight face, has been implicitly used in [2] to study the convergence of the GMRES method.

2. The proof. Our proof of Theorem 1 is heavily based on the result of the paper [3], that we recall now. Let S_α be a convex sector of the complex plane with angle 2α. An operator $B \in \mathcal{B}(H)$ is said S_α-accretive iff $(Bv, v) \in \overline{S_\alpha}$, for all $v \in H$ satisfying $\|v\| = 1$. The result proved in [3] is

there exists a best constant $C_\alpha \in \mathbb{R}$ such that the inequality

$$\|r(B)\| \leqq C_\alpha \sup_{z \in S_\alpha} |r(z)|, \tag{3}$$

holds for all rational functions bounded in S_α and for all S_α-accretive operators B. Furthermore this constant C_α (which only depends of α) is a continuous and decreasing function of α and it satisfies the estimates

$$\frac{\pi}{2\alpha} \sin\alpha \leqq C_\alpha \leqq \min\left(2 + 2/\sqrt{3}, \frac{\pi - \alpha}{\alpha}\right).$$

Therefore it is sufficient to prove that $C_\alpha = C(\alpha)$ for getting the theorem.

We turn now to the proof of this equality. Without loss of generality, we can assume that the vertices of L are $\sigma = 0$ and $\sigma' = 1$, and that $\operatorname{Im} \alpha_1 < 0$. We introduce the rational function $g(z) := \frac{z}{z-1}$. It is easily seen that g is an involution and that g realizes a bijection of the disk $D_j := \{z \in \mathbb{C};\, |z - \alpha_j| < |\alpha_j|\}$ onto the half-plane $P_j := \{z \in \mathbb{C};\, \operatorname{Re} \bar{\alpha}_j z < 0\}$. In the case where the lens has a straight face $L = D_1 \cap \Pi_2$ with $\Pi_2 := \{z \in \mathbb{C};\, \operatorname{Re} iz < 0\}$, we remark also that g realizes a bijection of the half-plane Π_2 onto the half-plane $P_2 := \{z \in \mathbb{C};\, \operatorname{Re} iz > 0\}$. Therefore g is a bijection of the lens L onto the sector $S_\alpha = P_1 \cap P_2$. Note that the sector and the lens have the same angle 2α and that $1 \notin S_\alpha$.

Let us consider now a linear operator A such that 1 does not belong to its spectrum $\sigma(A)$, and we set $B = g(A) = A(A - I)^{-1}$. It is easily seen that $(B - I)(A - I) = I$, thus $1 \notin \sigma(B)$, and $A = g(B)$.

Using that $\operatorname{Re} \alpha_j = \frac{1}{2}$, we remark by setting $v = (A - I)w$ that

$$\begin{aligned}
&|\alpha_j|^2\, \|w\|^2 - \|(A - \alpha_j I)w\|^2 \geqq 0, \forall\, w \in H,\\
&\iff \|Aw\|^2 - 2 \operatorname{Re} \bar{\alpha}_j (Aw, w) \leqq 0, \forall\, w \in H,\\
&\iff 2 \operatorname{Re} \bar{\alpha}_j (Aw, (A - I)w) \leqq 0, \forall\, w \in H,\\
&\iff \operatorname{Re} \bar{\alpha}_j (Bv, v) \leqq 0, \forall\, v \in H.
\end{aligned}$$

In the case where L has a straight face, we also remark that

$$\begin{aligned}
&\operatorname{Im}(Aw, w) \geqq 0, \forall\, w \in H,\\
&\iff \operatorname{Im}(Aw, (A - I)w) \leqq 0, \forall\, w \in H,\\
&\iff \operatorname{Im}(Bv, v) \leqq 0, \forall\, v \in H.
\end{aligned}$$

Therefore if the linear operator A is of L-type, then the operator B is S_α-accretive. Conversely if B is S_α-accretive then $1 \notin \sigma(B)$ (since $1 \notin S_\alpha$) and $A = g(B)$ is of L-type.

Let us consider now a polynomial p and set $r(z) = p(g(z))$, then we have $p(A) = r(B)$ and $\sup_{z \in S_\alpha} |r(z)| = \sup_{\zeta \in L} |p(\zeta)|$. We deduce from (3) that

$$\|p(A)\| \leqq C_\alpha \sup_{\zeta \in L} |p(\zeta)|.$$

Note that, if $1 \in \sigma(A)$, then for $0 < \varepsilon < 1$, the operator $A_\varepsilon := (1-\varepsilon)A$ is of L-type and $1 \notin \sigma(A_\varepsilon)$, which shows that the previous inequality is still valid by using a limit argument. Therefore we have $C(\alpha) \leqq C_\alpha$.

Conversely if we consider a rational function r bounded in S_α, $p(z) = r(g(z))$ is a rational function bounded in L. Note that p is then a uniform limit in L of a sequence of polynomial functions, therefore the estimate (1) is still valid. We deduce that

$$\|r(B)\| \leqq C(\alpha) \sup_{z \in S_\alpha} |r(z)|,$$

which implies $C(\alpha) \geqq C_\alpha$, and thus finally $C(\alpha) = C_\alpha$.

The proofs would be the same for the completely bounded form of our estimates.

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