Nonlocal effects in two-dimensional conductivity

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Abstract

The paper deals with the asymptotic behaviour as $\varepsilon \to 0$ of a two-dimensional conduction problem whose matrix-valued conductivity a_ε is ε-periodic and not uniformly bounded with respect to ε. We prove that only under the assumptions of equi-coerciveness and L^1-boundedness of the sequence a_ε, the limit problem is a conduction problem of same nature. This new result points out a fundamental difference between the two-dimensional conductivity and the three-dimensional one. Indeed, under the same assumptions of periodicity, equi-coerciveness and L^1-boundedness, it is known that the high-conductivity regions can induce nonlocal effects in three (or greater) dimensions.

1. Introduction

In the paper we are interested in the limit behaviour as $\varepsilon \to 0$ of the two-dimensional conduction problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega \\ \qquad\qquad u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

in a bounded open set Ω of $\mathbb{R}^2$ and for a given f in $H^{-1}(\Omega)$. For each $\varepsilon > 0$, the conductivity a_ε is a symmetric positive definite matrix-valued function in $L^\infty(\Omega; \mathbb{R}^{2\times 2})$ which is ε-periodic, i.e. $a_\varepsilon(x) = A_\varepsilon(\frac{x}{\varepsilon})$ with $A_\varepsilon(y_1+1, y_2) = A_\varepsilon(y_1, y_2+1) = A_\varepsilon(y)$ for a.e. $y \in \mathbb{R}^2$. The sequence a_ε is assumed to be equi-coercive in Ω (i.e. there exists $\alpha > 0$ such that $a_\varepsilon \geqq \alpha I$ a.e. in Ω) and bounded in $L^1(\Omega; \mathbb{R}^{2\times 2})$, but not bounded in $L^\infty(\Omega; \mathbb{R}^{2\times 2})$.

The question we ask is can the high-conductivity regions induce nonlocal effects in the limit problem? In three (or greater) dimensions the answer is known to be positive. Indeed, FENCHENKO and KHRUSLOV [13] (see also [15]) first obtained nonlocal effects from microstructures a_ε with high-conductivity regions. The model example, which was extended by BELLIEUD and BOUCHITTÉ [2] to a nonlinear framework, consists of a medium reinforced by one-directional and high-conductivity fibres. More precisely, in a cylinder $\Omega := \omega \times (0, 1)$ the fibres form an ε-periodic lattice of x_3-directional cylinders of radius $\varepsilon\, r_\varepsilon$ such that $\gamma\, \varepsilon^2\, |\ln r_\varepsilon| = 1$ with $\gamma > 0$, their conductivity is equal to $\kappa\, r_\varepsilon^{-2}$ with $\kappa > 0$, and they are embedded in a medium of conductivity equal to 1. Then, the solution u_ε of the conduction problem (1.1) weakly converges in $H_0^1(\Omega)$ to the solution u_0 of the nonlocal homogenized equation

$$\begin{cases} -\Delta u_0 + 2\pi\gamma \left(u_0 - \displaystyle\int_0^1 u_0(x_1, x_2, t)\, \theta_{\gamma,\kappa}(t, x_3)\, dt \right) = f & \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the kernel $\theta_{\gamma,\kappa}$ can be explicitely computed (see [2] for details). The nonlocal term in (1.2) is due to the diffusion along the fibres combined with their capacitary effect.

These works were also extended by [8] and [6] in conduction, as well as by [19] and [3] in elasticity. More generally, MOSCO [16] proved that the energy associated with (1.1) converges to a quadratic form according to the BEURLING & DENY [5] representation formula. In some sense CAMAR-EDDINE and SEPPECHER [10] closed the topic not only in three-dimensional conduction by proving that any nonlocal effect can be attained by a suitable conductivity sequence, but also in three-dimensional elasticity [11] by proving a remarkable closure result.

On the other hand, in any dimension, various conditions on the conductivity sequence a_ε prevent the appearance of nonlocal effects. Firstly, SPAGNOLO [20] with the G-convergence theory, then MURAT and TARTAR [21,18] with the H-convergence theory, proved that the equi-coerciveness combined with the equi-boundedness of the sequence a_ε (without periodicity restriction) implies a compactness result for the sequence of problems (1.1). BUTTAZZO and DAL MASO [9] (see also [12]) extended this compactness result to any sequence of isotropic conductivities $a_\varepsilon = \alpha_\varepsilon\, I$ such that α_ε is bounded and equi-integrable in $L^1(\Omega)$. More recently, BRIANE [7] proved that for any ε-periodic conductivity $a_\varepsilon(x) := A_\varepsilon(\frac{x}{\varepsilon})$ with A_ε bounded in $L^1(Y)$, $Y := (0, 1)^d$, the estimate of the weighted Poincaré–Wirtinger inequality

$$\forall\, V \in H^1(Y), \quad \int_Y A_\varepsilon \left(V - \fint_Y V \right)^2 dy \leqq C(\varepsilon) \int_Y A_\varepsilon \nabla V \cdot \nabla V\, dy, \quad \text{with} \quad \varepsilon^2\, C(\varepsilon) \to 0, \tag{1.3}$$

also leads to a classical limit of problem (1.1). However, the opposite behaviour $\varepsilon^2 C(\varepsilon) \nrightarrow 0$ can imply nonlocal effects in three dimensions.

In contrast with these previous works, the present paper points out the gap between the second and third (or greater) dimension regarding the appearance of nonlocal effects in conductivity. The main result of the paper (see Theorem 1) claims that any sequence of ε-periodic conductivities a_ε, which is equi-coercive and bounded only in $L^1(\Omega; \mathbb{R}^{2\times 2})$, cannot induce nonlocal effects in dimension two.

The proof is based on a Poincaré–Wirtinger type inequality and a div-curl type lemma. These two auxiliary results are specific to dimension two and allow us to apply the method of the oscillating test functions of Tartar [21], which implies a classical limit behaviour of the conduction problem (1.1).

On the one hand, the Poincaré–Wirtinger inequality (see Proposition 2) reads as, in the ε-periodic case,

$$\forall\, V \in H^1(Y), \quad \int_Y \left(V - \fint_Y V \right)^2 dy \leqq C \int_Y \tilde{A}_\varepsilon \nabla V \cdot \nabla V \, dy \qquad (1.4)$$
$$\text{where} \quad \tilde{A}_\varepsilon := \frac{A_\varepsilon}{\det A_\varepsilon}.$$

Inequality (1.4) can be regarded as the conjugate of inequality (1.3). However, contrary to (1.3) the constant C of the Poincaré–Wirtinger inequality (1.4) is independent of ε and thus cannot be blown up.

On the other hand, the div-curl result (see Proposition 3) is an extension of the classical div-curl lemma of MURAT & TARTAR [17], for any sequence ξ_ε with compact divergence in $H^{-1}(\Omega)$, and such that $a_\varepsilon^{-1/2}\xi_\varepsilon$ (but not ξ_ε) is bounded in $L^2(\Omega; \mathbb{R}^2)$. The ingredients of this weak div-curl lemma is the representation of a divergence-free function by a stream function and the approximation of this stream function by a piecewise-constant function, based on the Poincaré–Wirtinger inequality (1.4). At this level and contrary to the third (or greater) dimension, an estimate on the two-dimensional curl of the stream function yields an estimate on its whole gradient. A three-dimensional counter-example (see Example 1) clarifies the two-dimensional character of the div-curl result.

The paper is organised as follows. In the first section we state the main result of the paper. The second section is devoted to the proofs and is divided into three parts. The first part deals with the Poincaré–Wirtinger inequality (3.2), the second one with the div-curl result, and the third one with the proof of Theorem 1.

2. Statement of the result

In the following:

(i) $|\cdot|$ denotes the euclidian norm in $\mathbb{R}^2$ as well as its subordinate matrix-norm:

$$|A| := \max_{x \in \mathbb{R}^2 \setminus \{0\}} \frac{|Ax|}{|x|} = \sqrt{\rho(AA^T)} \quad \text{for } A \in \mathbb{R}^{2\times 2},$$

where ρ is the spectral radius and A^T the transpose of the matrix A. Note that $|A| = \rho(A)$ if A is symmetric, which will be the case in the sequel;

(ii) I denotes the unit matrix of $\mathbb{R}^{2\times 2}$ and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;

(iii) Y denotes the unit square $(0, 1)^2$ of $\mathbb{R}^2$;

(iv) $L^p_\#(Y)$ (resp. $H^1_\#(Y)$) denotes the set of the Y-periodic functions which belong to $L^p_{\text{loc}}(\mathbb{R}^2)$ (resp. $H^1_{\text{loc}}(\mathbb{R}^2)$);

(v) Ω denotes a bounded open subset of $\mathbb{R}^2$; and

(vi) $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support on Ω.

Let A_ε, for $\varepsilon > 0$, be a sequence of symmetric positive definite matrix-valued fiunctions and Y-periodic matrix-valued functions in $L^\infty_\#(Y)$. We assume that there exist two positive constants α, β such that

$$\forall\, \varepsilon > 0, \quad A_\varepsilon \geqq \alpha\, I \quad \text{a.e. in } \mathbb{R}^2, \tag{2.1}$$

$$\forall\, \varepsilon > 0, \quad \int_Y |A_\varepsilon|\, dy \leqq \beta. \tag{2.2}$$

Therefore, the sequence A_ε is equi-coercive by (2.1) but only bounded in $L^1(Y)$ due to (2.2).

For each $\lambda \in \mathbb{R}^2$, let X^λ_ε be the unique function in $H^1_\#(Y)$ with zero average value, solution of the equation

$$\operatorname{div}(A_\varepsilon \nabla W^\lambda_\varepsilon) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^2), \quad \text{where} \quad W^\lambda_\varepsilon(y) := \lambda \cdot y - X^\lambda_\varepsilon(y), \tag{2.3}$$

and let A^*_ε be the constant matrix defined by

$$A^*_\varepsilon \lambda := \int_Y A_\varepsilon \nabla W^\lambda_\varepsilon\, dy, \tag{2.4}$$

which satisfies the equality

$$A^*_\varepsilon \lambda \cdot \lambda = \int_Y A_\varepsilon \nabla W^\lambda_\varepsilon \cdot \nabla W^\lambda_\varepsilon\, dy. \tag{2.5}$$

For a fixed $\varepsilon > 0$, A^*_ε is the homogenized matrix induced by the oscillating sequence $A_\varepsilon(\frac{x}{\delta})$ as δ tends to 0 (see e.g. [1] or [4]).

We easily deduce from the equi-coerciveness (2.1), the boundedness (2.2) and from (2.5), that the sequence W^λ_ε satisfies the bound

$$\|\nabla W^\lambda_\varepsilon\|_{L^2(Y)} \leqq \sqrt{\frac{\beta}{\alpha}}\, |\lambda|, \tag{2.6}$$

and that A^*_ε satisfies the estimates

$$A^*_\varepsilon \geqq \alpha\, I \quad \text{and} \quad |A^*_\varepsilon| \leqq \beta. \tag{2.7}$$

Taking into account (2.7) we can assume that (up to a subsequence)

$$A^*_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} A^*_0, \tag{2.8}$$

where A^*_0 is a symmetric positive definite matrix.

The main result of the paper is the following:

Theorem 1. *Assume that conditions* (2.1) *and* (2.2) *hold true. Then, the solution u_ε of the conduction problem* (1.1) *with conductivity $a_\varepsilon(x) = A_\varepsilon(\frac{x}{\varepsilon})$, weakly converges in $H^1_0(\Omega)$ to the solution u_0 of the conduction problem with the constant conductivity A^*_0 defined by* (2.8) *and* (2.4).

Remark 1. The result of Theorem 1 implies that the equi-coerciveness constraint (2.1) combined with the one of L^1-boundedness (2.2) prevents the appearance of nonlocal effects in dimension two.

Convergence (2.8), rather than the more restrictive condition (2.2), seems to be the natural assumption to obtain the previous homogenization result. We did not succeed in proving Theorem 1 by only assuming (2.8) together with the equi-coerciveness (2.1). Indeed, our approach, through the Propositions 1 and 2 is essentially based on the boundedness (2.2). However, this condition is sufficiently general to point out the difference between the second and third dimension and the appearance of nonlocal effects in strong conductivity.

3. Proof of the result

The first section is devoted to a Poincaré–Wirtinger inequality and the second one to a div-curl lemma. We prove these two auxiliary results for a class of microstructures satisfying a kind of uniform L^1-boundedness (see Definition 1), which contains any ε-periodic and L^1-bounded conductivity. The third section deals with the proof of Theorem 1 in the case of ε-periodic microstructures.

3.1. A Poincaré–Wirtinger inequality

Definition 1. A sequence b_ε, for $\varepsilon > 0$, of nonnegative measurable functions on Ω is said to be ω-bounded in $L^1(\Omega)$ if there exists a positive function $\omega : (0, +\infty) \longrightarrow (0, +\infty)$ with zero limit at 0, satisfying

$$\begin{aligned}&\forall\,\delta > 0,\ \exists\,\varepsilon_0 > 0 \quad \text{such that}\\ &\forall\,\varepsilon \in (0, \varepsilon_0),\ \forall\, Q \text{ square of } \Omega \text{ with } |Q| \geqq \delta, \quad \int_Q b_\varepsilon\, dx \leqq \omega(|Q|),\end{aligned} \tag{3.1}$$

where $|Q|$ denotes the Lebesgue measure of Q.

Proposition 1. *Let b_ε be the sequence defined on Ω by $b_\varepsilon(x) := B_\varepsilon(\frac{x}{\varepsilon})$, where B_ε is a Y-periodic positive sequence bounded in $L^1(Y)$. Then, b_ε is ω-bounded in $L^1(\Omega)$.*

Proof. Let Q be a square of Ω with $|Q| \geqq \varepsilon^2$. The square Q is included in a minimal square Q_ε composed of a number $N_\varepsilon \leqq 9\,\varepsilon^{-2}\,|Q|$, of cells of the type $\varepsilon(k + Y)$, $k \in \mathbb{Z}^2$. The εY-periodicity of b_ε implies that

$$\int_Q b_\varepsilon\, dx \leqq \int_{Q_\varepsilon} b_\varepsilon\, dx = N_\varepsilon\, \varepsilon^2 \int_Y B_\varepsilon\, dy \leqq 9 \left(\sup_{\varepsilon > 0} \|B_\varepsilon\|_{L^1(Y)} \right) |Q|.$$

Therefore, the sequence b_ε is ω-bounded in $L^1(\Omega)$ with $\omega(t) := c\,t$, where c is a positive constant.

With Definition 1 we have the following result:

Proposition 2. *Let a_ε, for $\varepsilon > 0$, be a sequence of symmetric positive definite matrix-valued functions with a_ε and a_ε^{-1} in $L^\infty(\Omega; \mathbb{R}^{2\times 2})$, such that the sequence $|a_\varepsilon|$ is ω-bounded in $L^1(\Omega)$. Then, there exists a positive constant C such that, for any $\delta > 0$ and any $\varepsilon > 0$ small enough, each square $Q \subset \Omega$, with $|Q| \geqq \delta$, satisfies the Poincaré–Wirtinger inequality*

$$\forall\, v \in H^1(Q), \qquad \int_Q \left(v - \fint_Q v \right)^2 dx \leqq C\,\omega(|Q|) \int_Q \tilde{a}_\varepsilon \nabla v \cdot \nabla v\, dx, \quad \text{where} \quad \tilde{a}_\varepsilon := \frac{a_\varepsilon}{\det a_\varepsilon}. \tag{3.2}$$

Remark 2. Inequality (3.2) is weighted by the matrix-valued $\tilde{a}_\varepsilon$ but, in contrast with (1.3), with a constant which is independent of δ and ε provided that ε is small enough with respect to δ. This constant also tends to 0 with the measure of Q. This result is strongly linked to dimension two as shown in the following proof.

Proof of Proposition 2. Let $\delta > 0$ and let Q be a square of Ω of side $h \geqq \sqrt{\delta}$. Let $v \in H^1(Q)$ with $\int_Q v = 0$ and let $V \in H^1(Y)$ be defined by $v(x) := V(\frac{x - x_h}{h})$, where $Q = x_h + hY$. By the change of variable $y := \frac{x - x_h}{h}$, and using the embedding of $W^{1,1}(Y)$ into $L^2(Y)$ (which is specific to the second dimension) combined with the classical Poincaré–Wirtinger inequality in $W^{1,1}(Y)$, we have

$$\int_Q v^2\, dx = h^2 \int_Y V^2\, dy \leqq C\, h^2 \left(\int_Y |\nabla V|\, dy \right)^2 = C \left(\int_Q |\nabla v|\, dx \right)^2,$$

where C is a positive constant. Moreover, if $\lambda_\varepsilon \leqq \mu_\varepsilon := |a_\varepsilon|$ are the eigenvalues of a_ε, then the eigenvalues of $\tilde{a}_\varepsilon$ are $\mu_\varepsilon^{-1} \leqq \lambda_\varepsilon^{-1}$, the ones of $\tilde{a}_\varepsilon{}^{-1/2}$ are thus $\sqrt{\lambda_\varepsilon} \leqq \sqrt{\mu_\varepsilon}$, from which $|\tilde{a}_\varepsilon{}^{-1/2}| = \sqrt{\mu_\varepsilon} = |a_\varepsilon|^{1/2}$. Then, combining the equality $|\tilde{a}_\varepsilon{}^{-1/2}| = |a_\varepsilon|^{1/2}$ and the inequality $|\nabla v| \leqq |\tilde{a}_\varepsilon{}^{-1/2}|\, |\tilde{a}_\varepsilon{}^{1/2} \nabla v|$ with the Cauchy–Schwarz inequality yields

$$\left(\int_Q |\nabla v|\, dx \right)^2 \leqq \int_Q |a_\varepsilon|\, dx \int_Q \tilde{a}_\varepsilon \nabla v \cdot \nabla v\, dx.$$

Therefore, we obtain the estimate

$$\int_Q v^2\, dx \leqq C \int_Q |a_\varepsilon|\, dx \int_Q \tilde{a}_\varepsilon \nabla v \cdot \nabla v\, dx,$$

which combined with the ω-boundedness (3.1) of $|a_\varepsilon|$ implies the desired inequality (3.2), provided that ε is small enough.

3.2. A div-curl result

In this section we extend the classical div-curl lemma of Murat & Tartar [17] to sequences which are not bounded in $L^2(\Omega; \mathbb{R}^2)$:

Proposition 3. *Let a_ε, for $\varepsilon > 0$, be a sequence of symmetric positive definite matrix-valued functions with $a_\varepsilon \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$, such that for a given $\alpha > 0$, $a_\varepsilon \geqq \alpha I$ a.e. in Ω and the sequence $|a_\varepsilon|$ is ω-bounded in $L^1(\Omega)$. Let ξ_ε be a sequence in $L^2(\Omega; \mathbb{R}^2)$ and let v_ε be a sequence in $H^1(\Omega; \mathbb{R}^2)$ which satisfy the following assumptions:*

$$\int_\Omega a_\varepsilon^{-1} \xi_\varepsilon \cdot \xi_\varepsilon \, dx \leqq c, \tag{3.3}$$

$$\int_\Omega |\nabla v_\varepsilon|^2 \, dx \leqq c, \tag{3.4}$$

where c is a positive constant,

$$\operatorname{div} \xi_\varepsilon \text{ is compact in } H^{-1}(\Omega), \tag{3.5}$$

and

$$\nabla v_\varepsilon \rightharpoonup 0 \ \text{ weakly in } L^2(\Omega; \mathbb{R}^2) \quad \text{or} \quad \xi_\varepsilon \rightharpoonup 0 \ \text{ weakly} * \text{ in } \mathcal{M}(\Omega; \mathbb{R}^2) \tag{3.6}$$

in the weak $*$ sense of the Radon measures on Ω.
Then, the following convergence in the sense of distributions holds true

$$\xi_\varepsilon \cdot \nabla v_\varepsilon \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.7}$$

The following example shows that the previous div-curl result does not hold in dimension three.

Example 1. With reference to the model example presented in the Introduction. Let $\Omega := (0, 1)^3$, let $\omega_\varepsilon \subset \Omega$ be the ε-periodic lattice of x_3-parallel cylinders of axis $x_1 = k_1 \varepsilon, x_2 = k_2 \varepsilon$, for $k_1, k_2 \in \mathbb{N}$, and of radius $\varepsilon r_\varepsilon$, and let a_ε be the ε-periodic isotropic conductivity defined by

$$a_\varepsilon(x) := \begin{cases} \frac{\kappa}{r_\varepsilon^2} I_3 & \text{if } x \in \omega_\varepsilon \\ I_3 & \text{if } x \in \Omega \setminus \omega_\varepsilon, \end{cases} \qquad \text{where} \quad r_\varepsilon := \exp\left(\frac{-1}{\gamma\, \varepsilon^2}\right) \quad \text{and} \quad \kappa, \gamma > 0.$$

Let u_ε be the solution in $H_0^1(\Omega)$ of $-\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f$, where f is a given function in $L^2(\Omega)$. For a fixed $R_0 \in (0, \frac{1}{2})$, let v_ε be the ε-periodic function defined in Ω by $v_\varepsilon(x) = V_\varepsilon(\frac{x}{\varepsilon})$, where V_ε is the continuous periodic function of period $\left(-\frac{1}{2}, \frac{1}{2}\right)^3$, independent of y_3, and defined on its period by

$$V_\varepsilon(y) := \begin{cases} 0 & \text{if } r \leqq r_\varepsilon \\ \frac{\ln r - \ln r_\varepsilon}{\ln R_0 - \ln r_\varepsilon} & \text{if } r_\varepsilon < r < R_0, \\ 1 & \text{if } r \geqq R_0 \end{cases} \qquad \text{where} \quad r := \sqrt{y_1^2 + y_2^2}.$$

It can be checked that the sequences $\xi_\varepsilon := a_\varepsilon \nabla u_\varepsilon$ and v_ε satisfy the assumptions (3.3)–(3.6) of Proposition 3. In particular, $\nabla v_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega; \mathbb{R}^3)$ since $v_\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$. Moreover, it can be proven (see e.g. [8]) that

$$\xi_\varepsilon \cdot \nabla v_\varepsilon = \nabla u_\varepsilon \cdot \nabla v_\varepsilon \rightharpoonup 2\pi\gamma\, (u_0 - v_0) \quad \text{in } \mathcal{D}'(\Omega), \tag{3.8}$$

where u_0 is the weak limit of u_ε in $H_0^1(\Omega)$ and v_0 is the weak $*$ limit of $\frac{1_{\omega_\varepsilon}}{\pi r_\varepsilon^2}\, u_\varepsilon$ in the Radon measures sense on Ω. The functions u_0, v_0 are the solutions of the coupled system

$$\begin{cases} -\Delta u_0 + 2\pi\gamma\,(u_0 - v_0) = f \ \text{ in } \Omega \\ -\kappa\,\frac{\partial^2 v_0}{\partial x_3^2} + 2\gamma\,(v_0 - u_0) = 0 \ \text{ in } \Omega \\ u_0(x) = 0 \ \text{ if } x \in \partial\Omega \\ v_0(x',0) = v_0(x',1) = 0 \ \text{ if } x' = (x_1,x_2) \in (0,1)^2, \end{cases} \tag{3.9}$$

which is equivalent to the nonlocal problem (1.2). We easily deduce from (3.9) that $u_0 - v_0$ is nonzero if f is a nonzero function. Therefore, in this case convergence (3.8) contradicts the result (3.7) of Proposition 3.

Proof of Proposition 3. We have to prove that, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \xi_\varepsilon \cdot \nabla v_\varepsilon\, \varphi\, dx \ \underset{\varepsilon\to 0}{\longrightarrow}\ 0.$$

By using a partition of the unity we may assume that the support of the test function φ is included in an open square Q with $\overline{Q} \subset \Omega$.

The proof is divided in three steps. In the first step we replace the sequence ξ_ε by a divergence-free one $J\nabla\tilde{u}_\varepsilon$, where $\tilde{u}_\varepsilon$ is a stream function. In the second step we approach $\tilde{u}_\varepsilon$ by a piecewise-constant function $\bar{u}_\varepsilon$. In the third step we prove that the sequence $\tilde{u}_\varepsilon \nabla v_\varepsilon$ converges to 0 in the sense of distributions.

Step 1. Introduction of a stream function. First note that there exists a positive constant c_Q such that

$$\int_Q |\xi_\varepsilon|\, dx \leqq c_Q. \tag{3.10}$$

Indeed, the Cauchy–Schwarz inequality combined with the ω-boundedness (3.1) of $|a_\varepsilon|$ and estimate (3.3), implies that for any ε small enough,

$$\begin{aligned} \int_Q |\xi_\varepsilon|\, dx \leqq \int_\Omega |a_\varepsilon^{\frac12}|\, |a_\varepsilon^{-\frac12}\xi_\varepsilon|\, dx &\leqq \left(\int_Q |a_\varepsilon|\, dx\right)^{\frac12} \left(\int_Q a_\varepsilon^{-1}\xi_\varepsilon\cdot\xi_\varepsilon\, dx\right)^{\frac12} \\ &\leqq \sqrt{\omega(|Q|)} \left(\int_\Omega a_\varepsilon^{-1}\xi_\varepsilon\cdot\xi_\varepsilon\, dx\right)^{\frac12} \leqq c_Q, \end{aligned}$$

from which we get the desired estimate (3.10).

Let u_ε be the solution in $H_0^1(\Omega)$ of the equation $\Delta u_\varepsilon = \operatorname{div}\xi_\varepsilon$ in $\mathcal{D}'(\Omega)$. Since the function $\xi_\varepsilon - \nabla u_\varepsilon$ is divergence-free in Q, there exists a stream function $\tilde{u}_\varepsilon \in H^1(Q)$ (see e.g. [14], page 22) such that

$$\xi_\varepsilon = \nabla u_\varepsilon + J\nabla\tilde{u}_\varepsilon \quad \text{with} \quad \int_Q \tilde{u}_\varepsilon\, dx = 0. \tag{3.11}$$

Due to the compactness (3.5), the sequence u_ε strongly converges in $H_0^1(\Omega)$ to some function u_0. According to (3.6) we have the two following alternatives:

(i) If ξ_ε weakly $*$ converges to 0 in $\mathcal{M}(\Omega; \mathbb{R}^2)$, then $\operatorname{div} \xi_\varepsilon = \Delta u_\varepsilon$ converges to $0 = \Delta u_0$ in $\mathcal{D}'(\Omega)$, from which $u_0 = 0$ and ∇u_ε strongly converges to 0 in $L^2(\Omega; \mathbb{R}^2)$. Therefore, the sequence $\nabla u_\varepsilon \cdot \nabla v_\varepsilon$ strongly converges to 0 in $L^1(\Omega)$, which implies

$$\int_Q \xi_\varepsilon \cdot \nabla v_\varepsilon \, \varphi \, dx - \int_Q J \nabla \tilde{u}_\varepsilon \cdot \nabla v_\varepsilon \, \varphi \, dx \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{3.12}$$

(ii) Otherwise, ∇v_ε weakly converges to 0 in $L^2(\Omega; \mathbb{R}^2)$, then the strong convergence of ∇u_ε in $L^2(\Omega; \mathbb{R}^2)$ implies that $\nabla u_\varepsilon \cdot \nabla v_\varepsilon$ weakly converges to 0 in $L^1(\Omega)$. Therefore, limit (3.12) still holds true.

Moreover, since $J^T = -J$ and $J \nabla v_\varepsilon$ is divergence-free, integrating by parts yields

$$\begin{aligned} \int_Q J \nabla \tilde{u}_\varepsilon \cdot \nabla v_\varepsilon \, \varphi \, dx &= -\int_Q \nabla(\varphi \tilde{u}_\varepsilon) \cdot J \nabla v_\varepsilon \, dx + \int_Q \tilde{u}_\varepsilon \nabla \varphi \cdot J \nabla v_\varepsilon \, dx \\ &= \int_Q \tilde{u}_\varepsilon \nabla \varphi \cdot J \nabla v_\varepsilon \, dx. \end{aligned}$$

Therefore, to prove the div-curl convergence (3.7) it is sufficient to prove that the sequence $\tilde{u}_\varepsilon \nabla v_\varepsilon$ converges to 0 in $\mathcal{D}'(Q; \mathbb{R}^2)$. Note that the sequence $\tilde{u}_\varepsilon$ is only bounded in $W^{1,1}(Q)$ due to (3.11) and (3.10), which does not imply its strong convergence in $L^2(Q)$ since the embedding of $W^{1,1}(Q)$ into $L^2(Q)$ is not compact in the second dimension. The next step provides an alternative based on the approximation of $\tilde{u}_\varepsilon$ by a piecewise-constant function combined with the Poincaré–Wirtinger inequality (3.2).

Step 2. Approximation of $\tilde{u}_\varepsilon$ by a piecewise-constant function. Let $\Phi \in \mathcal{D}(Q; \mathbb{R}^2)$. For a fixed $h > 0$ small enough, let $(Q_k)_{k \in K_h}$ be a finite covering of the support of Φ by the squares $Q_k := h(k + Y) \subset Q$, for $k \in K_h \subset \mathbb{Z}^2$. By Proposition 2 there exists $\omega_h > 0$ which tends to 0 as $h \to 0$ such that, for any $\varepsilon > 0$ small enough (it is sufficient that $\varepsilon \leqq h$ by the proof of Proposition 1) and any $k \in K_h$,

$$\int_{Q_k} \left(\tilde{u}_\varepsilon - \fint_{Q_k} \tilde{u}_\varepsilon \right)^2 dx \leqq \omega_h \int_{Q_k} \tilde{a}_\varepsilon \nabla \tilde{u}_\varepsilon \cdot \nabla \tilde{u}_\varepsilon \, dx.$$

Moreover, the equalities $\nabla \tilde{u}_\varepsilon = J\,(\nabla u_\varepsilon - \xi_\varepsilon)$ and $a_\varepsilon^{-1} = J^T \tilde{a}_\varepsilon J$ imply that

$$\int_{Q_k} \tilde{a}_\varepsilon \nabla \tilde{u}_\varepsilon \cdot \nabla \tilde{u}_\varepsilon \, dx = \int_{Q_k} a_\varepsilon^{-1} (\xi_\varepsilon - \nabla u_\varepsilon) \cdot (\xi_\varepsilon - \nabla u_\varepsilon) \, dx,$$

from which the Cauchy–Schwarz inequality combined with $a_\varepsilon \geqq \alpha\, I$, yields

$$\int_{Q_k} \left(\tilde{u}_\varepsilon - \fint_Q \tilde{u}_\varepsilon \right)^2 dx \leqq 2\,\omega_h \int_{Q_k} \left(a_\varepsilon^{-1} \xi_\varepsilon \cdot \xi_\varepsilon + \alpha^{-1} |\nabla u_\varepsilon|^2 \right) dx. \tag{3.13}$$

On the other hand, let $\bar{u}_\varepsilon$ be the piecewise-constant function defined from the function $\tilde{u}_\varepsilon$ and the covering $(Q_k)_{k \in K_h}$ by

$$\bar{u}_\varepsilon := \sum_{k \in K_h} \left(\fint_{Q_k} \tilde{u}_\varepsilon \right) 1_{Q_k}, \tag{3.14}$$

where 1_{Q_k} denotes the characteristic function of the set Q_k. Then, summing the inequalities (3.13) over $k \in K_h$, yields

$$\int_Q |\Phi|^2 \, (\tilde{u}_\varepsilon - \bar{u}_\varepsilon)^2 \, dx \leqq 2 \, \|\Phi\|^2_{L^\infty(Q)} \, \omega_h \int_Q \left(a_\varepsilon^{-1} \xi_\varepsilon \cdot \xi_\varepsilon + \alpha^{-1} |\nabla u_\varepsilon|^2 \right) dx.$$

Thus, the estimate (3.3) and the boundedness of ∇u_ε in $L^2(\Omega; \mathbb{R}^2)$ (which is strongly convergent) imply that

$$\int_Q |\Phi|^2 \, (\tilde{u}_\varepsilon - \bar{u}_\varepsilon)^2 \, dx \leqq c_\Phi \, \omega_h.$$

Finally, by the Cauchy–Schwarz inequality combined with the boundedness of ∇v_ε in $L^2(\Omega; \mathbb{R}^2)$, we obtain that for any $\varepsilon > 0$ small enough,

$$\left| \int_Q \Phi \cdot \nabla v_\varepsilon \, (\tilde{u}_\varepsilon - \bar{u}_\varepsilon) \, dx \right| \leqq c_\Phi \, \sqrt{\omega_h}, \tag{3.15}$$

where $c_\Phi > 0$ is independent of h and ε and $\omega_h \to 0$ as $h \to 0$.

The third step of the proof deals with the convergence of $\bar{u}_\varepsilon \nabla v_\varepsilon$. The convergence of $\tilde{u}_\varepsilon \nabla v_\varepsilon$ then follows thanks to the previous step.

Step 3. Convergence of $\bar{u}_\varepsilon \nabla v_\varepsilon$ and $\tilde{u}_\varepsilon \nabla v_\varepsilon$. Let us fix $h > 0$. The sequence $\nabla \tilde{u}_\varepsilon$ is bounded in $L^1(Q; \mathbb{R}^2)$ by its definition (3.11) and estimate (3.10). Then, since $\int_Q \tilde{u}_\varepsilon = 0$ the sequence $\tilde{u}_\varepsilon$ is bounded in $W^{1,1}(Q)$ by the classical Poincaré–Wirtinger inequality, and thus in $L^2(Q)$ by the embedding of $W^{1,1}(Q)$ into $L^2(Q)$. Therefore, the sequence $\tilde{u}_\varepsilon$ weakly converges (up to a subsequence) in $L^2(Q)$ to some function $\tilde{u}_0$, from which the sequence $\bar{u}_\varepsilon$ defined by (3.14) strongly converges in $L^\infty(Q)$ to the function

$$\bar{u}_0 := \sum_{k \in K_h} \left(\fint_{Q_k} \tilde{u}_0 \right) 1_{Q_k}.$$

Moreover, by (3.4) and by the regularity of Q, the sequence ∇v_ε weakly converges (up to a subsequence) in $L^2(Q; \mathbb{R}^2)$ to ∇v_0 with $v_0 \in H^1(Q)$, whence

$$\int_Q \bar{u}_\varepsilon \nabla v_\varepsilon \cdot \Phi \, dx \;\underset{\varepsilon \to 0}{\longrightarrow}\; \int_Q \bar{u}_0 \nabla v_0 \cdot \Phi \, dx. \tag{3.16}$$

According to (3.6) we have the two following alternatives:

(i) If ξ_ε weakly $*$ converges to 0 in $\mathcal{M}(\Omega; \mathbb{R}^2)$, so does $\nabla \tilde{u}_\varepsilon$ by (3.11). Then, $\nabla \tilde{u}_0 = 0$ in $\mathcal{D}'(Q)$, which implies $\tilde{u}_0 = 0$ since $\int_Q \tilde{u}_0 = 0$. The right-hand side of (3.16) is thus equal to 0.

(ii) Otherwise, ∇v_ε weakly converges to 0 in $L^2(Q; \mathbb{R}^2)$ and the right-hand side of (3.16) is still equal to 0.

Therefore, for any $h > 0$ and for the whole sequence ε, we obtain

$$\int_Q \bar{u}_\varepsilon \nabla v_\varepsilon \cdot \Phi \, dx \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

The previous limit combined with the uniform (with respect to ε) estimate (3.15) yields

$$\int_Q \tilde{u}_\varepsilon \nabla v_\varepsilon \cdot \Phi \, dx \underset{\varepsilon \to 0}{\longrightarrow} 0 \quad \text{for any } \Phi \in \mathcal{D}(Q; \mathbb{R}^2),$$

which concludes the proof.

3.3. Proof of Theorem 1

We will apply the method of the oscillating test functions of TARTAR [21] by using the div-curl result of Proposition 3. To this end we consider for a fixed $\lambda \in \mathbb{R}^2$, the oscillating function $w_\varepsilon^\lambda(x) := \varepsilon W_\varepsilon^\lambda(\frac{x}{\varepsilon})$ for $x \in \Omega$, where W_ε^λ is defined by (2.3). We will determine the limit in the sense of distributions of the sequence $a_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon^\lambda = a_\varepsilon \nabla w_\varepsilon^\lambda \cdot \nabla u_\varepsilon$.

First note that, in virtue of Proposition 1 and the boundedness (2.2) of A_ε, the sequence $|a_\varepsilon|$ is ω-bounded in $L^1(\Omega)$.

Step 1. Limit of $a_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon^\lambda$. Set $\xi_\varepsilon := a_\varepsilon \nabla u_\varepsilon$ and $v_\varepsilon(x) := w_\varepsilon^\lambda(x) - \lambda \cdot x$, for $x \in \Omega$. By the classical Poincaré inequality in $H_0^1(\Omega)$ and the equi-coerciveness $a_\varepsilon \geqq \alpha I$, we have

$$\begin{aligned}\int_\Omega \xi_\varepsilon \cdot \nabla u_\varepsilon \, dx = \langle f, u_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &\leqq c \, \|f\|_{H^{-1}(\Omega)} \, \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\leqq \frac{c}{\sqrt{\alpha}} \, \|f\|_{H^{-1}(\Omega)} \left(\int_\Omega \xi_\varepsilon \cdot \nabla u_\varepsilon \, dx \right)^{\frac{1}{2}},\end{aligned}$$

from which $\xi_\varepsilon \cdot \nabla u_\varepsilon = a_\varepsilon^{-1} \xi_\varepsilon \cdot \xi_\varepsilon$ is bounded in $L^1(\Omega)$ and estimate (3.3) holds true. The sequence ∇v_ε satisfies estimate (3.4) since $\nabla W_\varepsilon^\lambda$ is bounded in $L^2_\#(Y; \mathbb{R}^2)$ by (2.6). The equality $-\operatorname{div} \xi_\varepsilon = f$ implies (3.5). Moreover, successively using the Y-periodicity of the zero average value function X_ε^λ and the Poincaré–Wirtinger inequality in $H^1_\#(Y)$, yields

$$\|w_\varepsilon^\lambda - \lambda \cdot x\|_{L^2(\Omega)} \leqq c \, \varepsilon \, \|X_\varepsilon^\lambda\|_{L^2(Y)} \leqq c' \, \varepsilon \, \|\nabla X_\varepsilon^\lambda\|_{L^2(Y)} = O(\varepsilon) \quad \text{by (2.6)},$$

from which $\nabla v_\varepsilon = \nabla w_\varepsilon^\lambda - \lambda$ weakly converges to 0 in $L^2(\Omega; \mathbb{R}^2)$, which implies (3.6).

Therefore, the convergence (3.7) of Proposition 3 yields (up to a subsequence)

$$a_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon^\lambda = \xi_\varepsilon \cdot \lambda + \xi_\varepsilon \cdot \nabla v_\varepsilon \rightharpoonup \xi_0 \cdot \lambda \quad \text{in } \mathcal{D}'(\Omega), \tag{3.17}$$

where ξ_0 is the weak $*$ limit of $a_\varepsilon \nabla u_\varepsilon$ in $\mathcal{M}(\Omega; \mathbb{R}^2)$.

Step 2. Limit of $a_\varepsilon \nabla w_\varepsilon^\lambda \cdot \nabla u_\varepsilon$. Set $\xi_\varepsilon := a_\varepsilon \nabla w_\varepsilon^\lambda - A_\varepsilon^* \lambda$, where A_ε^* is the matrix defined by (2.3)–(2.4), and $v_\varepsilon := u_\varepsilon$.

Thanks to the Y-periodicity of $A_\varepsilon \nabla W_\varepsilon^\lambda \cdot \nabla W_\varepsilon^\lambda$ and estimate (2.6), the sequence ξ_ε satisfies the bound (3.3). This combined with the bound (2.2) satisfied by A_ε, implies that ξ_ε is also bounded in $L^1(\Omega; \mathbb{R}^2)$ (see the proof of (3.10)). The sequence v_ε clearly satisfies (3.4). Moreover, the compactness (3.5) holds true since $\operatorname{div} \xi_\varepsilon = 0$ by rescaling (2.3). Thus, it remains to prove condition (3.6).

The function $\xi_\varepsilon(x)$ reads as $\Sigma_\varepsilon(\frac{x}{\varepsilon})$, where Σ_ε is Y-periodic with zero average value and bounded in $L^1(Y; \mathbb{R}^2)$. Let $\Phi \in \mathcal{D}(\Omega; \mathbb{R}^2)$ and let Φ_ε be a piecewise-constant function with compact support in Ω, constant in each square $\varepsilon(k + Y)$, for $k \in \mathbb{Z}^2$, and such that $\|\Phi - \Phi_\varepsilon\|_{L^\infty(\Omega)} = o(1)$. Since the Y-periodicity of Σ_ε implies that

$$\int_{\varepsilon(k+Y)} \Sigma_\varepsilon\left(\frac{x}{\varepsilon}\right) dx = \varepsilon^2 \int_Y \Sigma_\varepsilon(y)\, dy = 0,$$

and since ξ_ε is bounded in $L^1(\Omega; \mathbb{R}^2)$, we have

$$\int_\Omega \xi_\varepsilon \cdot \Phi\, dx = \int_{\mathbb{R}^2} \Sigma_\varepsilon\left(\frac{x}{\varepsilon}\right) \cdot \Phi_\varepsilon(x)\, dx + o(1) = 0 + o(1) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Therefore, the sequence ξ_ε weakly $*$ converges to 0 in $\mathcal{D}'(\Omega; \mathbb{R}^2)$ and is bounded in $L^1(\Omega; \mathbb{R}^2)$, which implies (3.6).

By applying Proposition 3 and convergence (2.8) we thus obtain

$$a_\varepsilon \nabla w_\varepsilon^\lambda \cdot \nabla u_\varepsilon = A_\varepsilon^* \lambda \cdot \nabla u_\varepsilon + \xi_\varepsilon \cdot \nabla v_\varepsilon \rightharpoonup A_0^* \lambda \cdot \nabla u_0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.18}$$

where u_0 is the weak limit (up to a subsequence) of u_ε in $H_0^1(\Omega)$.

Step 3. Conclusion. The limits (3.17) and (3.18) imply the equality $\xi_0 \cdot \lambda = A_0^* \lambda \cdot \nabla u_0$ in $\mathcal{D}'(\Omega)$, for any $\lambda \in \mathbb{R}^2$, from which $\xi_0 = A_0^* \nabla u_0$. Since the sequence $-\operatorname{div} \xi_\varepsilon = f$ converges to $-\operatorname{div} \xi_0$ in $\mathcal{D}'(\Omega)$, we thus obtain the equation $-\operatorname{div}\left(A_0^* \nabla u_0\right) = f$ in $\mathcal{D}'(\Omega)$. Theorem 1 is now proved.

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