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Abstract

Let $(X, \mathfrak{X}, \mu, \tau)$ be an ergodic dynamical system and φ be a measurable map from X to a locally compact second countable group G with left Haar measure m_G. We consider the map τ_φ defined on $X \times G$ by $\tau_\varphi : (x, g) \to (\tau x, \varphi(x)g)$ and the cocycle $(\varphi_n)_{n\in\mathbb{Z}}$ generated by φ.

Using a characterization of the ergodic invariant measures for τ_φ ([Ra06]), we give the form of the ergodic decomposition of $\mu(dx) \otimes m_G(dg)$ or more generally of the τ_φ-invariant measures $\mu_\chi(dx) \otimes \chi(g)m_G(dg)$, where $\mu_\chi(dx)$ is $\chi \circ \varphi$-conformal for an exponential χ on G.

Contents

1 Introduction

We consider a dynamical system $(X, \mathfrak{X}, \mu, \tau)$, where $(X, \mathfrak{X})$ is a standard Borel space, μ a σ-finite measure on $\mathfrak{X}$ and τ an invertible measurable transformation on X such that μ is *quasi-invariant* and *ergodic* for the action of τ.

Let G be a locally compact second countable (lcsc) group. We denote by $\mathfrak{B}_G$ the σ-algebra of its Borel sets, $m_G(dg)$ (or simply dg) a left Haar measure on G, e its identity element.

Let φ be a measurable function on X taking its values in G and τ_φ the map on $X \times G$ (*skew product*) defined by

$$\tau_\varphi : (x, g) \to (\tau x, \varphi(x)g). \tag{1}$$

The corresponding G-valued cocycle $(\varphi_n)_{n\in\mathbb{Z}}$ over (X, μ, τ) (noted also (φ, τ)) is

$$\varphi_n(x) = \begin{cases} \varphi(\tau^{n-1}x) \cdots \varphi(x), & \text{for } n > 0, \\ e, & \text{for } n = 0, \\ \varphi(\tau^n x)^{-1} \cdots \varphi(\tau^{-1}x)^{-1}, & \text{for } n < 0. \end{cases}$$

If μ is τ-invariant, the map τ_φ leaves invariant the product measure $\lambda_1 := \mu \otimes m_G$. The cocycle (φ_n) can be seen as a *stationary walk* in G over the dynamical system (X, μ, τ).

More generally, let χ be an exponential on G, i.e. a continuous map from G to $]0,+\infty[$ such that: $\forall g,g' \in G,\ \chi(g\,g') = \chi(g)\,\chi(g')$. If μ_χ is a $\chi \circ \varphi$ conformal σ-finite measure on X, i.e. such that

$$(\tau \mu_\chi)(dx) = \chi(\varphi(\tau^{-1}x))\ \mu_\chi(dx), \tag{2}$$

then the measure $\lambda_\chi(dx, dg) := \mu_\chi(dx) \otimes \chi(g) m_G(dg)$ (sometimes called Maharam measure) is a σ-finite measure on $X \times G$ which is τ_φ-invariant.

The study of cocycles was the subject of many papers since K. Schmidt ([Sc77]), J. Feldman and C.C. Moore ([FeMo77]). There has been recently a new interest for the invariant measures for skew products (cf. [ANSS02], [Sa04], [LeSa07]).

Our main goal is to give the precise form of the ergodic decomposition (for the skew product τ_φ) of the measures λ_χ on $X \times G$. We give in the first section the statement of the results on this ergodic decomposition, then some consequences in terms of regularity, boundness and essential values of the cocycle $(\varphi_n)_{n\in\mathbb{Z}}$. The following sections are devoted to the proof of the main results. We also discuss a conjugacy equation for the closed subgroups of G which arises in the ergodic decomposition. In the appendix, we recall and specify some results on ergodic decompositions and regular conditional probabilities.

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2 Statement of the main results

2.1 Ergodic decomposition

Before we state the main results, we recall some facts about a topology on the set $\mathcal{F}(G)$ of closed subsets of G and give some notations.

• **A topology on** $\mathcal{F}(G)$

Let G be a lcsc group. We equip the set $\mathcal{F}(G)$ of closed subsets of G with the so-called *Chabauty's topology*. In this topology the open sets are defined by

$$U(\mathcal{O}, C) = \{S \in \mathcal{F}(G) : \forall U\ \in \mathcal{O},\ S \cap U \neq \emptyset \ \text{ and } \ S \cap C = \emptyset\},$$

where $\mathcal{O}$ is a finite family of open sets of G and C is a compact subset of G.

It can be shown that a sequence (F_n) of closed subsets of G converges to a closed subset F in Chabauty's topology if and only if the two following properties are satisfied :

(i) Let $\xi : \mathbb{N} \mapsto \mathbb{N}$ be an increasing sequence and let $(g_n)_{n\in\mathbb{N}}$ be a sequence such that $g_n \in F_{\xi(n)}$ for every $n \geq 0$. If $(g_n)_{n\in\mathbb{N}}$ converges to $g \in G$, then the limit g is in F.

(ii) Each $g \in F$ is the limit of a sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n \in F_n$ for every $n \geq 0$.

The Borel structure associated to this topology is generated by the sets $\{S \in \mathcal{F}(G) : S \subseteq F\}$ where $F \in \mathcal{F}(G)$. The lcsc group G is metrizable. We denote by d a metric on G which defines the topology of G. For any dense sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G, the family of continuous functions $\{d(g_n, \cdot), n \in \mathbb{N}\}$ separates the points of $\mathcal{F}(G)$ (see [AuMo66], Ch. II section 2).

- **Notations**

Notations 2.1.1 For a locally compact second countable group H, we denote by $m_H(d\gamma)$ (or simply $d\gamma$) a *left Haar measure* on the Borel sets of H, by δ_u the Dirac measure at a point $u \in H$. The identity element is denoted by e.

If ρ_1 and ρ_2 are positive measures on the Borel sets of H, we denote by $\rho_1 * \rho_2$ their convolution (i.e. the image of the product measure $\rho_1 \otimes \rho_2$ by the map $(g, g') \in H \times H \longrightarrow g\, g' \in H$).

As in the introduction we consider a measurable map φ from X to G and τ_φ the skew-product defined by (1). Let λ be a τ_φ-quasi-invariant positive measure on $X \times G$. We denote by $\mathfrak{J}$ or $\mathfrak{J}_\varphi$ the σ-algebra of τ_φ-*invariant subsets*. We are interested in the $\mathfrak{X} \times \mathfrak{B}_G$-measurable functions on $X \times G$ which are invariant by the map τ_φ.

The following remark is useful. If f is τ_φ-invariant λ-a.e., then there is a τ_φ-invariant function g such that $f = g$ λ-a.e.. Therefore it is enough to consider functions which are *everywhere* τ_φ-*invariant*.

Recall that two G-valued cocycles (φ, τ) and (ψ, τ) over the dynamical system (X, μ, τ) are μ-*cohomologous*, if there is a measurable map $u : X \to G$ such that

$$\varphi(x) = u(\tau x)\, \psi(x)\, (u(x))^{-1} \text{ for } \mu - a.e.\ x. \tag{3}$$

The function u in (3) is called *transfer function*. We write $\varphi \overset{(u,\mu)}{\sim} \psi$ when (3) is satisfied. A cocycle (φ, τ) is a μ-*coboundary* if it is μ-*cohomologous* to the constant function $\psi \equiv e$.

Notations 2.1.2 *In what follows, we consider a τ_φ-invariant measure λ_χ of the form $\lambda_\chi = \mu_\chi \otimes (\chi m_G)$, where χ is an exponential on G, μ_χ is a σ-finite measure which is $\chi \circ \varphi$-conformal and τ-ergodic on X. When $\chi \equiv 1$, the measure μ_χ is τ-invariant.*

Once and for all we choose a measurable positive function h on $X \times G$ such that

$$\int_{X\times G} h(x, g)\ \mu_\chi(dx)\ \chi(g)\, m_G(dg) = 1.$$

The existence of h results from the facts that μ_χ is σ-finite on X and that G is a lcsc group.

Let P^h be a regular conditional probability with respect to the probability measure $h\,\lambda_\chi$ and the σ-algebra $\mathfrak{J}$ of τ_φ-invariant subsets (i.e. P^h is a transition probability on $X \times G$

such that, for every nonnegative measurable function f on $X \times G$, $P^h f$ is a version of the conditional expectation $\mathbb{E}_{h\lambda_\chi}[f|\mathfrak{J}]$).

We define a positive kernel M^h on $X \times G$ by

$$\forall (x,g) \in X \times G,\ \ M^h f(x,g) = P^h(f/h)(x,g),$$

for any measurable nonnegative function f on $X \times G$.

If we replace h by an other density h', we have $M^{h'}((x,g),.) = P^h(h/h')(x,g)\, M^h((x,g),.)$. For λ_χ-a.e. $(x,g) \in X \times G$, the positive measure $M^h((x,g),.)$ on $X \times G$ is τ_φ-invariant ergodic. (See the appendix)

• Statement of the main result

The formula $\mathbb{E}_{h\lambda_\chi}[\,\cdot\,] = \mathbb{E}_{h\lambda_\chi}[\mathbb{E}_{h\lambda_\chi}[\,\cdot\,|\mathfrak{J}]]$ can be written

$$\lambda_\chi(dy,dt) = \int_{X\times G} M^h((x,g),(dy,dt))h(x,g)\,\lambda_\chi(dx,dg),$$

which represents a decomposition of λ_χ in τ_φ-ergodic components. Our goal is to give a precise description of these ergodic components. This is the content of the following theorem:

Theorem 2.1.3 *(Ergodic decomposition of λ_χ)*

1) There exist:

- *a family $(\mu_x)_{x\in X}$ of σ-finite τ-quasi-invariant measures on X defining a σ-finite positive kernel from $(X,\mathfrak{X})$ to $(X,\mathfrak{X})$ (i.e. for every $x \in X$, μ_x is a σ-finite positive measure on $\mathfrak{X}$ and for every $A \in \mathfrak{X}$ the map $x \to \mu_x(A) \in [0,+\infty]$ is $\mathfrak{X}$-measurable),*
- *a family $(H_x)_{x\in X}$ of closed amenable subgroups of G such that the map $x \to H_x$ from X to $\mathcal{F}(G)$ is measurable,*
- *a measurable map $\eta : X \times G \mapsto \mathbb{R}_+^*$ such that, for each $x \in X$, $\chi_x(\cdot) := \eta(x,\cdot)$ defines an exponential on H_x,*
- *a measurable map $u : X \times X \mapsto G$ (for $x \in X$, we set $u_x(.) = u(x,.)$)*

satisfying for μ_χ-a.e. $x \in X$ and every $g \in G$ the following properties (4) to (10):

$$H_{\tau x} = \varphi(x) H_x\ \varphi(x)^{-1}, \tag{4}$$

$$\psi(y) := u_x(\tau y)^{-1}\,\varphi(y)\ u_x(y) \in H_x,\ \ \textit{for}\ \mu_x\text{–a.e.}\,y, \tag{5}$$
$$\tau\mu_x(dy) = \chi_x(\psi(\tau^{-1}y))\ \mu_x(dy), \tag{6}$$
$$\chi_x(\gamma) = \chi_{\tau x}(\varphi(x)\,\gamma\,(\varphi(x))^{-1}),\ \ \forall \gamma \in H_x, \tag{7}$$
$$\zeta_x(y) := (u_x(y))^{-1}\,u_{\tau x}(y)\,\varphi(x) \in H_x,\ \ \textit{for}\ \mu_x\text{–a.e.}\,y, \tag{8}$$
$$\mu_{\tau x}(dy) = c(x)\,\chi_x(\zeta_x(y))\,\mu_x(dy),\ \ \textit{for a positive constant}\ c(x). \tag{9}$$

and

$$M^h f(x,g) = \frac{\int_X (\int_{H_x} f(y, u_x(y)\,\gamma\, g)\ \chi_x(\gamma)\, m_{H_x}(d\gamma)\ \mu_x(dy)}{\int_X (\int_{H_x} h(y, u_x(y)\,\gamma\, g)\ \chi_x(\gamma)\, m_{H_x}(d\gamma)\ \mu_x(dy)}. \tag{10}$$

If we take for m_{H_x}, $x \in X$, the unique left Haar measure on H_x such that

$$\int_{H_x \cap \{d(e,\cdot)\leq 1\}} \chi_x(\gamma)\, m_{H_x}(d\gamma) = 1,$$

then $K(x, dt) := m_{H_x}(dt)$ is a positive kernel from $(X, \mathfrak{X})$ to $(G, \mathfrak{B}_G)$.

An ergodic decomposition of the measure $\lambda_\chi = \mu_\chi \otimes (\chi\, m_G)$ is given by

$$\lambda_\chi(dy, dt) = \int_{X\times G} M^h((x,g),(dy,dt))\ h(x,g)\ \lambda_\chi(dx,dg). \tag{11}$$

For every nonnegative measurable λ_χ-a.e. τ_φ-invariant function f, we have, λ_χ-a.e., $f = P^h f$ (the last function being τ_φ-invariant according to the definition of a regular conditional probability).

2) When there exist a fixed closed subgroup H of G and a measurable map $a : X \to G$ such that $H_x = a(x)H(a(x))^{-1}$ for μ_χ-a.e. $x \in X$ (which is the case when G is a nilpotent connected Lie group (Theorem 5.1.1)), the ergodic measures can be written, with $\tilde\chi_x(\gamma) := \chi_x(a_x \gamma a_x^{-1})$,

$$M^h f(x,g) = \frac{\int_X (\int_H f(y, u_x(y)\, a(x)\,\gamma\,(a(x))^{-1}\, g)\ \tilde\chi_x(\gamma)\, d\gamma)\ \mu_x(dy)}{\int_X (\int_H h(y, u_x(y)\, a(x)\,\gamma\,(a(x))^{-1}\, g)\ \tilde\chi_x(\gamma)\, d\gamma)\ \mu_x(dy)}. \tag{12}$$

3) When G is abelian, the subgroups H_x are equal to a fixed closed subgroup H of G, the exponentials χ_x are equal to the exponential χ and the ergodic measures are given by

$$M^h f(x,g) = \frac{\int_X (\int_H f(y, u_x(y)\,\gamma\, g)\ \chi(\gamma)\, d\gamma)\ \mu_x(dy)}{\int_X (\int_H h(y, u_x(y)\,\gamma\, g)\ \chi(\gamma)\, d\gamma)\ \mu_x(dy)}. \tag{13}$$

The proof of Theorem 2.1.3 will be given in section 3.

2.2 Notion of regularity for a cocycle

• Regularity

Definitions 2.2.1 We say that the cocycle defined by φ is μ_χ-*regular*, if there exist a closed subgroup H of G and a measurable map $u : X \to G$ such that the cocycle $\psi := (u \circ \tau)^{-1}\, \varphi\, u$ takes μ_χ-a.e. its values in H and $\tau_\psi : (x, h) \to (\tau x, \psi(x)h)$ is ergodic for the product measure $\mu_\chi \otimes (\chi\, m_H)$.

The measure $(\chi \circ u)\ \mu_\chi \otimes \chi\, m_H$ is τ_ψ-invariant. In the regular case we have a "good" ergodic decomposition of $\mu_\chi \otimes (\chi dg)$ and the subgroups H_x of Theorem 2.1.3 are conjugate to H: $H_x = u(x)\, H\, u(x)^{-1}$.

Theorem 2.2.2 - *1) For $x_0 \in X$, the set $\{x \in X : \mu_x \sim \mu_{x_0}\}$ is measurable and has zero or full μ_χ-measure.*

2) Assume that the cocycle (φ, τ) is μ_χ-regular. Then every measurable τ_φ-invariant function f can be written $f(x,g) = F_f((u(x))^{-1} g)$, $\mu_\chi \otimes m_G$-a.e., where F_f is a left H-invariant function on G. The ergodic components of λ_χ (see (10)) can be written:

$$M^h f(x,g) = \frac{\int_X (\int_H f(y, u(y)\, \gamma\, (u(x))^{-1} g)\ \chi(\gamma)\, d\gamma)\ \chi(u(y))\ \mu_\chi(dy)}{\int_X (\int_H h(y, u(y)\, \gamma\, (u(x))^{-1} g)\ \chi(\gamma)\, d\gamma)\ \chi(u(y))\ \mu_\chi(dy)}.$$

In other words we have $H_x = u(x)\, H\, u(x)^{-1}$ and $\chi_x(\gamma) = \chi(u(x)\, \gamma\, u(x)^{-1})$. We can take $u_x(y) = u(y)\, u(x)^{-1}$ and $\mu_x(dy) = \chi(u(y))\, \mu_\chi(dy)$.

3) Assume that the cocycle (φ, τ) is not μ_χ-regular. Then for μ_χ-a.e. x, the measures μ_x of the ergodic decomposition of $\mu_\chi \otimes (\chi\, m_G)$ are singular with respect to the measure μ_χ. There are uncountably many of them pairwise mutually singular. If G is abelian and μ_χ is finite, then, for μ_χ-a.e. $x \in X$, the measure μ_x is infinite.

The proof of Theorem 2.2.2 will be given in section 4.

Examples of nonregular cocycle over rotations were given by Lemańczyk in [Le95]. In 5.2, Remark 5.2.2, we give an example of a nonregular cocycle over a rotation which is the difference $1_{[0,\beta]} - 1_{[0,\beta]}(. + r)$, for some β and r on circle.

- **Boundness**

In the proposition below, we discuss the boundness of the map u and of the cocycle (φ_n). The notations are those of Theorem 2.1.3.

As the group G is lcsc, we can write $G = \bigcup_n U_n$ for an increasing sequence of open sets such that $K_n = \overline{U_n}$ is compact. Consequently $G = \bigcup_{n \in \mathbb{N}} K_n$ and for any compact subset K of G there exists $n \in \mathbb{N}$ such that $K \subset K_n$.

Lemma 2.2.3 *1) Let u be the measurable map from $X \times X$ to G defined in Theorem 2.1.3. For any compact subset K of G we define the following subset of X:*

$$X_K = \{x \in X : u_x(y)\, H_x \subset K\, H_x,\ \text{for } \mu_x\text{-a.e. } y \in X\} = \{x \in X : \mathrm{Supp}\,(u_x(\mu_x)) \subset K\, H_x\}.$$

Then X_K is measurable and $x \in X_K \Rightarrow \tau x \in X_{K\,(\varphi(x))^{-1}}$.

The set $\bigcup_{n \in \mathbb{N}} X_{K_n}$ is a τ-invariant measurable subset of X and (ergodicity of μ_χ) has zero or full μ_χ-measure.

2) If there exists a compact subset K of G such that $\mu_\chi(X_K) > 0$, then $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has full μ_χ-measure. In this case, we can replace the measurable map u by another measurable map u satisfying, for any $n \in \mathbb{N}$,

$$\text{for } \mu_\chi\text{-a.e. } x \in X_n = X_{K_n} \setminus X_{K_{n-1}},\ u_x(y) \in K_n,\ \text{ for } \mu_x\text{-a.e. } y \in X. \tag{14}$$

3) In particular, the set $\{x : G/H_x \text{ is compact}\}$ is measurable and has zero or full measure. If this set has full measure, we are in the above situation.

Proof of Lemma 2.2.3

1) If K is a fixed compact set in G, the map $F \to K.F$ from the set $\mathcal{F}(G)$ of closed subsets of G into itself is continuous. Since $x \to H_x$ is measurable, the map $x \to K.H_x$ is measurable. In section 3, we will see that the map $(x, y) \in X \times X \mapsto u(x, y)\, H_x \in \mathcal{F}(G)$ is measurable. We also know that, for any $g \in G$, the map $F \in \mathcal{F}(G) \mapsto d(g, F) \in \mathbb{R}_+$ is continuous. It follows that the set $\{(x, y) \in X \times X : d(g, KH_x) \leq d(g, u(x, y)H_x)\}$ is measurable. Let $(g_n)_{n\in\mathbb{N}}$ be a dense sequence in G. Then we have

$$X_K = \{x \in X : \forall n \in \mathbb{N},\ \nu_{(x,e)}(\{y \in X : d(g_n, K\, H_x) \leq d(g_n, u(x, y)\, H_x)\}) = 1\}.$$

It shows that X_K is measurable.

From the formulas (4) and (8) of Theorem 2.1.3, we obtain $x \in X_K \Rightarrow \tau x \in X_{K\,(\varphi(x))^{-1}}$. Since for any compact subset K of G, there exists $n \in \mathbb{N}$ such that $K \subset K_n$, we deduce that the measurable set $\bigcup_{n\in\mathbb{N}} X_{K_n}$ is τ-invariant and (ergodicity of μ_χ) has zero or full measure.

2) If $\mu_\chi(X_K) > 0$ for some compact subset K of G, then the same argument shows that $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has full μ_χ-measure.

The last assertion follows from the construction of u (cf. Lemma 7.1.1).

3) We have

$$\{x \in X : G/H_x \text{ is compact}\} = \bigcup_{n\in\mathbb{N}} \{x \in X : K_n.H_x = G\};$$

which shows that the set is measurable. By the conjugacy relation (4) this set is τ-invariant and (ergodicity of μ_χ) has zero or full μ_χ-measure. In the last case, for μ_χ-a.e. $x \in X$, we have $\bigcup_{n\in\mathbb{N}} K_n H_x = G$, which implies that the set $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has a full μ_χ-measure. □

Proposition 2.2.4 *1) Assume that the measure μ_χ in the basis is a finite measure and that there exists a compact subset K of G such that $\mu_\chi(X_K) > 0$. Then when the map u satisfies the boundness condition (14), the measures μ_x are finite, for μ_χ-a.e. $x \in X$.*

2) Assume G abelian and there exists a compact subset K of G such that $\mu_\chi(X_K) > 0$. Then the cocycle is regular.

3) Assume G abelian, μ_χ finite and τ conservative for μ_χ. If $\mu_\chi(\{x \in X : \tilde\mu_x(X) < +\infty\}) > 0$, where

$$\tilde\mu_x(dy) := (\chi(u_x(y)))^{-1}\ \mu_x(dy), \tag{15}$$

then the cocycle is regular.

4) Assume that τ is conservative for μ_χ. If the cocycle (φ_n) is μ_χ-bounded (i.e. there exists a compact subset K of G such that, for μ_χ-a.e. $x \in X$, $\forall n \geq 0$, $\varphi_n(x) \in K$), then H_x is a compact subgroup of G and the cocycle is cohomologous with a bounded transfer function to a cocycle taking its values in a compact subgroup of G.

Proof Let r be a positive continuous function on G such that $\int_G r(t)\,\chi(t)\,dt = 1$. For any compact subset K of G, we set $r_K(g) := \min_{u\in K} r(ug) > 0$. For all measurable nonnegative functions f on X, we have (cf. 34),

$$\begin{aligned}
M^h(f\otimes r)(x,g) &= c(x,g)\int_X f(y)\Big(\int_{H_x} r(u_x(y)\,\gamma\,g)\ \chi_x(\gamma)\,m_{H_x}(d\gamma)\Big)\ \mu_x(dy)\\
&\geq c(x,g)\int_X f(y)\Big(\int_{H_x} 1_K(u_x(y))\,r(u_x(y)\,\gamma\,g)\ \chi_x(\gamma)\,m_{H_x}(d\gamma)\Big)\ \mu_x(dy)\\
&= c(x,g)\Big(\int_{H_x} r_K(\gamma\,g)\ \chi_x(\gamma)\,m_{H_x}(d\gamma)\Big)\ \int_X f(y)1_K(u_x(y))\,\mu_x(dy)
\end{aligned}$$

and therefore

$$\mu_\chi(f) = \lambda_\chi(f\otimes r) \geq \int_X \Psi_K(x)\ \Big(\int_X f(y)1_K(u_x(y))\,\mu_x(dy)\Big)\ \mu_\chi(dx), \tag{16}$$

where $\Psi_K(x) := \int_G c(x,g)\,(\int_{H_x} r_K(\gamma\,g)\,\chi_x(\gamma)\,m_{H_x}(d\gamma))\,h(x,g)\,\chi(g)\,dg > 0$.

1) Under the assumptions of the first assertion, we have from (14) and (16), for each $n \in \mathbb{N}$,

$$\mu_\chi(f) \geq \int_{X_n} \Psi_{K_n}(x)\,\mu_x(f)\ \mu_\chi(dx)$$

and taking $f = 1_X$, we obtain that $\mu_x(X) < +\infty$, for μ_χ-a.e. $x \in X_n$, hence for μ_χ-a.e. $x \in X$ since $\cup_n X_n$ has full measure in X.

2) In this section 2) and in section 3), we assume that G is abelian. With the notations of Theorem 2.1.3, the exponentials in (13) do not depend on x and the measures μ_x satisfy $(\tau\mu_x)(dy) = \chi(\psi(\tau^{-1}y))\,\mu_x(dy)$. One easily sees that the measures $\tilde\mu_x(dy)$ defined by (15) satisfy, as the measure μ_χ, the conformal property

$$\tau\tilde\mu_x(dy) = \chi(\varphi(\tau^{-1}y))\ \tilde\mu_x(dy). \tag{17}$$

By (16) we have, for any $n \in \mathbb{N}$,

$$\mu_\chi(f) \geq \int_{X_n} \Phi_{K_n}(x)\,\tilde\mu_x(f)\ \mu_\chi(dx), \tag{18}$$

where $\Phi_{K_n}(x) = \Psi_{K_n}(x) \inf_{u \in K_n} \chi(u)$.

This implies that, for any $B \in \mathfrak{X}$, $B \subset X_n$, there exists a nonnegative measurable function ξ_B on X such that

$$\int 1_B(x) \Phi_{K_n}(x)\ \tilde{\mu}_x(dy)\ \mu_\chi(dx) = \xi_B(y)\ \mu_\chi(dy).$$

From the conformal property (17), it follows that $\xi_B \circ \tau^{-1} = \xi_B$, μ_χ-a.e.. As μ_χ is τ-ergodic, ξ_B is μ_χ-a.e. equal to a constant $\nu(B)$. The map $B \to \nu(B)$ defines a positive measure ν on $(X_n, X_n \cap \mathfrak{X})$ absolutely continuous with respect to the measure μ_χ. Therefore there exists a measurable nonnegative function ξ on X such that:

$$\int 1_B(x) \Phi_{K_n}(x)\ \tilde{\mu}_x(dy)\ \mu_\chi(dx) = \nu(B)\ \mu_\chi(dy) = (\int 1_B(x) \xi(x)\, \mu_\chi(dx))\ \mu_\chi(dy)$$

and, for μ_χ-a.e. $x \in X_n$,

$$\xi(x)\ \mu_\chi(dy) = \Phi_{K_n}(x)\ \tilde{\mu}_x(dy).$$

As $\bigcup_{n \in \mathbb{N}} X_n$ is of full measure, by gluing the Φ_{K_n}, we obtain a function Φ such that, for μ_χ-a.e. $x \in X$,

$$\xi(x)\ \mu_\chi(dy) = \Phi(x)\ \tilde{\mu}_x(dy).$$

This shows the regularity of the cocycle.

3) We set $X_0 = \{x \in X : \tilde{\mu}_x(X) < +\infty\}$. For $x \in X_0$, we denote by $\hat{\mu}_x$ the probability $\tilde{\mu}_x / \tilde{\mu}_x(X)$. From (16), for any compact subset K of G, we have

$$\mu_\chi(f) \geq \int_{X_0} \Phi_K(x)\ \Big(\int_X f(y) 1_K(u_x(y))\, \hat{\mu}_x(dy) \Big)\ \mu_\chi(dx); \tag{19}$$

where $\Phi_K(x) := \Psi_K(x) \inf_{u \in K} \chi(u)\, \tilde{\mu}_x(X)$.

Let h_1 be a positive bounded measurable function on X. We know that τ is conservative; i.e. $\mu_\chi(\{\sum_{k \geq 0} h_1 \circ \tau^k < +\infty\}) = 0$. From (19), it follows that, for μ_χ-a.e. $x \in X_0$,

$$\forall n \in \mathbb{N},\ \hat{\mu}_x(\{\sum_{k \geq 0} h_1 \circ \tau^k < +\infty\} \cap \{u_x \in K_n\}) = 0.$$

When n increases towards $+\infty$, from the monotone convergence theorem, we obtain, for μ_χ-a.e. $x \in X$,

$$\hat{\mu}_x(\{\sum_{k \geq 0} h_1 \circ \tau^k < +\infty\}) = 0.$$

Since h_1 is bounded and thus $\hat{\mu}_x$ integrable, for $x \in X_0$, we deduce that, for μ_χ-a.e. $x \in X_0$, τ is conservative for $\hat{\mu}_x$. Replacing X_0 by $X_0 \cap \{x \in X : \hat{\mu}_x(\{\sum_{k \geq 0} h_1 \circ \tau^k < +\infty\}) = 0\}$ we can assume that, for any $x \in X_0$, τ is conservative for $\hat{\mu}_x$.

From (19), there exists a measurable $[0,1]$-valued function ξ_K such that

$$\xi_K(y)\,\mu_\chi(dy) = \int_{X_0} \Phi_K(x)\,1_K(u_x(y))\,\hat{\mu}_x(dy)\ \mu_\chi(dx) \leq \int_{X_0} \Phi_K(x)\,\hat{\mu}_x(dy)\ \mu_\chi(dx).$$

Consequently, there exists a measurable $[0,1]$-valued function ψ_K such that

$$\xi_K(y)\,\mu_\chi(dy) = \psi_K(y) \int_{X_0} \Phi_K(x)\ \hat{\mu}_x(dy)\ \mu_\chi(dx).$$

Since the measure μ_χ and the measures $\hat{\mu}_x$ satisfy the same conformal property (cf. (17)), we have

$$\sum_{k=0}^{n-1} T^k \xi_K(y)\ \mu_\chi(dy) = \int_{X_0} \Phi_K(x)\ \sum_{k=0}^{n-1} T^k \psi_K(y)\ \hat{\mu}_x(dy)\ \mu_\chi(dx)$$

where T is the operator defined by

$$Tf(y) = f \circ \tau^{-1}(y)\ \ \chi(\varphi(\tau^{-1}y)).$$

As τ is conservative for μ_χ and for $\hat{\mu}_x$, $x \in X_0$, by Hurewicz's ergodic theorem, for any bounded measurable function f on X, the sequence of functions

$$\Big(\sum_{k=0}^{n-1} T^k f \,/\, \sum_{k=0}^{n-1} T^k 1 \Big)_{n \in \mathbb{N}}$$

converges μ_χ-a.e. to $\mu_\chi(f)$ and converges $\hat{\mu}_x$-a.e. to $\hat{\mu}_x(f)$, for $x \in X_0$. As the sequence of functions is bounded and the measures are finite, these convergences also hold in $\mathbb{L}^1$-norm.

Therefore, for any bounded measurable function f,

$$\int_X f(y)\,\frac{\sum_{k=0}^{n-1} T^k \xi_K(y)}{\sum_{k=0}^{n-1} T^k 1(y)}\ \mu_\chi(dy) \underset{n\to+\infty}{\longrightarrow} \mu_\chi(\xi_K)\,\mu_\chi(f)$$

and for μ_χ-a.e. $x \in X$

$$\alpha_n(x) = \int_X f(y)\,\frac{\sum_{k=0}^{n-1} T^k \psi_K(y)}{\sum_{k=0}^{n-1} T^k 1(y)}\ \hat{\mu}_x(dy) \underset{n\to+\infty}{\longrightarrow} \hat{\mu}_x(\psi_K)\,\hat{\mu}_x(f).$$

The inequality (19) shows that Φ_K is μ_χ-integrable. Moreover the sequence of functions (α_n) is bounded. By the dominated convergence theorem, it follows that

$$\int_{X_0} \Phi_K(x)\,\alpha_n(x)\,\mu_\chi(dx) \underset{n\to+\infty}{\longrightarrow} \int_{X_0} \Phi_K(x)\,\hat{\mu}_x(\psi_K)\,\hat{\mu}_x(f)\ \mu_\chi(dx).$$

We deduce,

$$\mu_\chi(dy) = \int_{X_0} \widehat{\Phi}_K(x)\ \hat{\mu}_x(dy)\ \mu_\chi(dx),$$

where $\widehat{\Phi}_K(x) = \Phi_K(x)\,\hat{\mu}_x(\psi_K)/\mu_\chi(\xi_K)$.

Now, as above, for any $B \in \mathfrak{X}$, $B \subset X_0$, there exists a nonnegative measurable function ξ_B such that

$$\xi_B(y)\ \mu_\chi(dy) = \int_B \hat{\Phi}_K(x)\ \hat{\mu}_x(dy)\ \mu_\chi(dx).$$

From the conformal property (17), it follows that $\xi_B \circ \tau^{-1} = \xi_B$, μ_χ-a.e.. With the same argument as in 2), since μ_χ is τ-ergodic, ξ_B is μ_χ-a.e. equal to $\nu(B)$, where ν if a positive measure on $(X_n, X_n \cap \mathfrak{X})$ absolutely continuous with respect to the measure μ_χ. Therefore there exists a measurable nonnegative function ξ on X such that:

$$\int 1_B(x)\hat{\Phi}_K(x)\ \hat{\mu}_x(dy)\ \mu_\chi(dx) = \nu(B)\ \mu_\chi(dy) = (\int_B \xi(x)\mu_\chi(dx))\ \mu_\chi(dy)$$

and, for μ_χ-a.e. $x \in X_0$,

$$\xi(x)\ \mu_\chi(dy) = \hat{\Phi}_K(x)\ \hat{\mu}_x(dy).$$

This shows the regularity of the cocycle.

4) Now let us assume that τ is conservative for μ_χ and that there exists a compact subset K such that, for μ_χ-a.e. $x \in X$, $\varphi_n(x) \in K$, for every $n \in \mathbb{N}$.

For any nonnegative measurable function on X with $\mu_\chi(f) \in]0, +\infty[$, we have

$$\sum_{n\geq 0} f(\tau^n x) 1_K(\varphi_n(x)) = \sum_{n\geq 0} f(\tau^n x) = +\infty, \mu_\chi\text{-a.e.}.$$

Hence τ_φ is conservative for λ_χ. We deduce that, for $x \in X_0$, where X_0 is a set of full μ_χ-measure, and any $g \in G$, τ_φ is conservative for $M^h((x, g), \cdot)$.

We take $x \in X_0$. Let $s \in \text{Supp}\,(u_x(\mu_x))$ and $t \in H_x$. Then, for any neighborhoods V and W of s and t, for μ_x-a.e. $y \in X$, $\sum_{n\geq 0} 1_V(\tau^n y)\, 1_W(\varphi_n(y)) = +\infty$. From the inclusion

$$u_x(\tau^n y)\,\psi_n(y) = \varphi_n(y)\,u_x(y) \subset K\,u_x(y), \text{ for } \mu_x\text{-a.e. } y \in X,$$

it follows that, for μ_χ-a.e. $x \in X$ and for μ_x-a.e. $y \in X$, $s\,t \in K\,u_x(y)$. Taking a fixed s and a dense sequence (t_n) in H_x, we obtain that, for μ_χ-a.e. $x \in X$ and for μ_x-a.e. $y \in X$, $\forall n \geq 0$, $t_n \in s^{-1}\,K\,u_x(y)$. Therefore $H_x \subset s^{-1}\,K\,u_x(y)$ is a compact subgroup of G and, with a similar argument, for μ_χ-a.e. $x \in X$ and for μ_x-a.e. $y \in X$, $\text{Supp}\,(u_x(\mu_x)) \subset K\,u_x(y)\,H_x$. This implies that, for μ_χ-a.e. $x \in X$, there exists a compact subset K_x of G such that $\text{Supp}\,(u_x(\mu_x)) \subset K_x\,H_x$. Since any compact subset K of G satisfies $K \subset K_n$, for n large enough, we deduce that $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has full μ_χ-measure. So we can assume that u satisfies the boundness condition (14) (cf. Lemma 2.2.3).

By (16) we have, for any $n \in \mathbb{N}$,

$$\mu_\chi(f) \geq \int_{X_n} \Psi_{K_n}(x)\,\mu_x(f)\ \mu_\chi(dx). \tag{20}$$

This implies that, there exists a $[0,1]$-valued measurable function ξ such that

$$\int_{X_n} \Psi_{K_n}(x)\ \mu_x(dy)\ \mu_\chi(dx) = \xi(y)\ \mu_\chi(dy).$$

Observe that for any $x \in X$, the exponential χ_x on the compact group is trivial and consequently the measures μ_x, $x \in X$, are τ-invariant.

From the conformal property (17), it follows that $\xi \circ \tau^{-1} d\tau\mu_\chi / d\mu_\chi = \xi$, μ_χ-a.e.. This shows that the measure $\xi\,\mu_\chi$ is τ-invariant. Moreover $\{\xi > 0\}$ is μ_χ-a.e. τ-invariant and therefore has full μ_χ-measure.

For any $B \in \mathfrak{x}$, $B \subset X_n$, there exists a $[0,1]$-valued measurable function ξ_B such that

$$\int_B \Psi_{K_n}(x)\ \mu_x(dy)\ \mu_\chi(dx) = \xi_B(y)\,\xi(y)\,\mu_\chi(dy). \tag{21}$$

From the conformal property (17), it follows that $\xi_B \circ \tau^{-1} = \xi_B$, μ_χ-a.e.. As in 2) and 3), ξ_B is μ_χ-a.e. equal to $\nu(B)$, where ν if a positive measure on $(X_n, X_n \cap \mathfrak{x})$ absolutely continuous with respect to the measure μ_χ. Therefore there exists a measurable nonnegative function ψ on X such that:

$$\int_B \Psi_{K_n}(x)\ \mu_x(dy)\ \mu_\chi(dx) = \nu(B)\,\xi(y)\ \mu_\chi(dy) = (\int_B \psi(x)\,\mu_\chi(dx))\,\xi(y)\ \mu_\chi(dy)$$

and, for μ_χ-a.e. $x \in X_n$,

$$\psi(x)\,\xi(y)\ \mu_\chi(dy) = \Psi_{K_n}(x)\ \mu_x(dy).$$

This shows the regularity of the cocycle. hence the last assertion of 4). □

Remark If G is a compact group, then it is well known that every G-valued cocycle φ is regular and therefore cohomologous to a cocycle ψ taking its values in a compact subgroup K of G such that $\mu \otimes m_K$ is ergodic for τ_ψ. (cf. [PaPo97], [Pa97] for the regularity of the cohomology when G is compact and the cocycle φ is Hölderian over a subshift of finite type.)

See also [AW00] for results under the assumption of tightness for the cocycle (φ_n).

2.3 Essential values and periods of invariant functions

The notion of essential values was introduced by K. Schmidt [Sc77], J. Feldman and C.C. Moore [FeMo77]. See also [Sc79], [Sc81], [Aa97]. The results in this section, excepted Proposition 2.3.6, are not new, at least when μ is τ-invariant. For the sake of completeness, we will give proofs. Remark that we are here in the more general case of a quasi-invariant measure.

Definitions 2.3.1 : Let μ be a τ-*quasi-invariant conservative* measure on X. An element $a \in G \cup \{\infty\}$ is an *essential value* of the cocycle (φ, τ) (with respect to μ) if, for every neighborhood V of a, for every subset B such that $\mu(B) > 0$, there is $n \in \mathbb{Z}$ such that

$$\mu(B \cap \tau^{-n} B \cap \{x : \varphi_n(x) \in V\}) > 0.$$

We denote by $\overline{\mathcal{E}}(\varphi)$ the set of essential values of the cocycle (φ, τ) and by $\mathcal{E}(\varphi) = \overline{\mathcal{E}}(\varphi) \cap G$ the *set of finite essential values.*

Let B be a measurable set of positive μ-measure. Let τ_B be the induced transformation on B and $\varphi^B(x) := \varphi_{n(x)}(x)$, where $n(x) = n_B(x) := \inf\{j \geq 1 : \tau^j x \in B\}$, for $x \in B$. The "induced" cocycle is given, for $n \geq 1$, by $\varphi_n^B(x) := \varphi^B(x)\, \varphi^B(\tau_B x) \cdots \varphi^B(\tau_B^{n-1} x)$.

Equivalently to the definition 2.3.1, an element $a \in G \cup \{\infty\}$ is an essential value of the cocycle (φ, τ) if and only if, for every subset B such that $\mu(B) > 0$, for any neighborhood V of a, $\mu(\{x : \varphi_n^B(x) \in V\}) > 0$ for some $n \in \mathbb{Z}$.

Proposition 2.3.2 *Assume that τ is conservative for μ_χ. If $\infty \notin \overline{\mathcal{E}}(\varphi)$, φ is cohomologous to a cocycle taking its values in a compact subgroup of G. When G is abelian we have $\overline{\mathcal{E}}(\varphi) = \{e\}$ if and only if φ is a coboundary.*

<u>Proof</u> If $\infty \notin \overline{\mathcal{E}}(\varphi)$, then there is B with $\mu_\chi(B) > 0$ such that $(\varphi_n^B)_{n \in \mathbb{Z}}$ is a bounded sequence. This implies that φ^B is τ_B-cohomologous to a cocycle taking values in a compact subgroup of G (cf. Proposition 2.2.4), i.e. there are measurable maps ζ^B from B to G and ψ^B from B to a compact subgroup of G such that

$$\varphi^B = \zeta^B \circ \tau_B\ \psi^B\ (\zeta^B)^{-1}. \tag{22}$$

By ergodicity and conservativity of (X, μ_χ, τ), for μ_χ-a.e. $y \in X$ there are an unique $x \in B$ and an integer k, $0 \leq k < n_B(x)$, such that $y = \tau^k x$. We define ζ on X by taking, for $y = \tau^k x$, $0 \leq k < n_B(x)$.

$$\zeta(y) = \varphi_k(x)\, \zeta^B(x)\ (\psi(y))^{-1},$$

with $\psi(y) = e$, if $k < n_B(x) - 1$, and $\psi(y) = \psi^B(x)$, for $k = n_B(x) - 1$.

For $0 \leq k < n_B(x) - 1$, the cocycle relation is clearly satisfied by construction. For $k = n_B(x) - 1$, it results from the cocycle relation (22) for the induced cocycle.

Now we consider the abelian case. Let us show that, when $\overline{\mathcal{E}}(\varphi) = \{e\}$ then φ is a coboundary. From the first assertion we know that the cocycle is cohomologous to a cocycle ψ taking values in a compact subgroup K of G. The set of essential values is the same for ϕ and ψ (see below). As τ_ψ is ergodic conservative and $\overline{\mathcal{E}}(\psi) = \{e\}$, one has $K = \{e\}$. □

We consider now, as in Theorem 2.1.3, a measure λ_χ.

Notation 2.3.3 Let $\mathcal{P}(\varphi)$ be the closed subgroup of G of *left periods* of the τ_φ-invariant measurable functions (i.e. the subgroup of elements $\gamma \in G$ such that, for every τ_φ-invariant function f, $f(x, \gamma\, g) = f(x, g)$, for λ_χ-a.e $(x, g) \in X \times G$).

Remark that we should write $\mathcal{P}(\varphi, \mu_\chi)$, since $\mathcal{P}(\varphi)$ and $\overline{\mathcal{E}}(\varphi)$ depend on the measure μ_χ. We will show that $\mathcal{P}(\varphi) = \mathcal{E}(\varphi)$ by using the following lemma from [ArNgOs].

Let (Y, ρ) be a complete separable metric space with a continuous action $(g, y) \to g.y$ of a group G on it. Let f be a measurable map from X to Y. Given a G-valued cocycle φ, we say that f is (φ, τ)-invariant if $f(\tau x) = \varphi(x).f(x)$, μ-a.e..

Lemma 2.3.4 *([ArNgOs]) If f is (φ, τ)-invariant, then $a.f(x) = f(x)$ $\mu -$ a.e., $\forall a \in \mathcal{E}(\varphi)$.*

Proof (Y, ρ) being a separable metric space, the set

$$X_f := \{x \in X : \mu(\{x' \in X : \rho(f(x'), f(x)) < \varepsilon\}) > 0, \text{ for every } \varepsilon > 0\}$$

has full μ-measure since it contains $f^{-1}(supp f(\mu))$. Let $x \in X_f$ and $a \in \mathcal{E}(\varphi)$. Let $\varepsilon > 0$ be arbitrary. Then the subset $E_x = \{x' : \rho(f(x'), f(x)) < \varepsilon\}$ has positive μ-measure. Since $a \in \mathcal{E}(\varphi)$, for every $\varepsilon_1 > 0$ there exist $x_1 \in E_x$ and $n \in \mathbb{Z}$ such that $\tau^n x_1 \in E_x$ and $d(a, \varphi_n(x_1)) < \varepsilon_1$, where d is a distance on G. By the invariance of f we have

$$\rho(a.f(x), f(x)) \leq \rho(af(x), af(x_1)) + \rho(af(x_1), \varphi_n(x_1).f(x_1)) + \rho(f(\tau^n x_1), f(x)).$$

Since ε and ε_1 are arbitrary and the action of G is continuous, we get $\rho(af(x), f(x)) = 0$. □

Proposition 2.3.5 $\mathcal{E}(\varphi) = \mathcal{P}(\varphi)$

Proof 1) If $a \notin \mathcal{E}(\varphi)$, there are a subset A, with $\mu(A) > 0$, and a neighborhood V of e such that

$$A \cap \tau^{-n} A \cap \{\varphi_n \in aVV^{-1}\} = \emptyset, \forall n \in \mathbb{Z}.$$

This implies that a is not a period of the τ_φ-invariant set $B = \cup_{n \in \mathbb{Z}} \tau_\varphi^n (A \times V)$.

2) Let h be a strictly positive function on G such that $\int h(g) m_G(dg) = 1$. We apply Lemma 2.3.4 to the G-space Y of real measurable functions on G, with the metric defined by $\rho(f_1, f_2) = \int_X \inf(|f_1 - f_2|, 1)\, h\, dm_G$. A function on $X \times G$ can be viewed as a function on X taking its values in Y. By Lemma 2.3.4, if a function f on $X \times G$ is τ_φ-invariant, then every element of $\mathcal{E}(\varphi)$ is a period for f. □

The proposition shows that $\mathcal{E}(\varphi) = G$ if and only if λ_χ is ergodic for τ_φ.

With the notations of Theorem 2.1.3, we have:

Proposition 2.3.6 *An element γ in G belongs to $\mathcal{P}(\varphi)$ if and only if γ belongs to H_x, for μ_χ-a.e. $x \in X$.*

In the abelian case, $\mathcal{P}(\varphi)$ (and therefore $\mathcal{E}(\varphi)$) coincides with the subgroup H.

Proof For $(x, g) \in X \times G$, we set (cf. (34))

$$c(x,g) = \Big(\int_X (\int_{H_x} h(y, u_x(y)\gamma g) \chi_x(\gamma) d\gamma) \mu_x(dy) \Big)^{-1}.$$

According to Theorem 2.1.3, we have

$$\gamma \in \mathcal{P}(\varphi) \Leftrightarrow M^h((x,\gamma g),\cdot) = M^h((x,g),\cdot) \text{ for } \lambda_\chi\text{-a.e. } (x,g) \in X \times G.$$

For λ_χ-a.e. $(x, g) \in X \times G$, the right member is equivalent to

$$c(x,g)\ \mu_x(dy)\ \delta_{u_x(y)} * (\chi_x m_{H_x}) * \delta_g = c(x,\gamma g)\ \mu_x(dy)\ \delta_{u_x(y)} * (\chi_x m_{H_x}) * \delta_{\gamma g},$$

that is, for μ_x-a.e. $y \in X$,

$$c(x,g)\ \delta_{u_x(y)} * (\chi_x m_{H_x}) * \delta_\gamma = c(x,\gamma g)\ \delta_{u_x(y)} * (\chi_x m_{H_x}).$$

The equality of the supports of these measures implies $H_x\gamma = H_x$, for μ_χ-a.e. $x \in X$. Hence the result. □

Abelian groups

If φ and ψ are two cohomologous cocycles, $\varphi \overset{(u,\mu)}{\sim} \psi$, then f is τ_φ-invariant if and only if $\tilde{f}$ is τ_ψ-invariant, where $\tilde{f}(x, g) = f(x, u(x)g)$.

If G is abelian, this implies that $\mathcal{P}(\varphi) = \mathcal{P}(\psi)$, so that two cohomologous cocycles have the same set of essential values. This is false in the nonabelian case (cf. [ArNgOs]).

When G is abelian, the cocycle $\tilde{\varphi} := \varphi \bmod \mathcal{E}(\varphi)$ satisfies $\mathcal{E}(\tilde{\varphi}) = \{0\}$. If $\overline{\mathcal{E}}(\tilde{\varphi}) = \{0\}$, then by 2.3.2, φ is μ_χ-cohomologous to a cocycle taking its values in $\mathcal{E}(\varphi)$. Therefore the regularity of the cocycle is equivalent to $\overline{\mathcal{E}}(\tilde{\varphi}) = \{0\}$. This last property, for an invariant measure, corresponds to the definition of regularity given by K. Schmidt for a cocycle (defined for a group action) taking its values in an abelian group.

If $G/\mathcal{E}(\varphi)$ is compact, then $\overline{\mathcal{E}}(\tilde{\varphi}) = \{0\}$ and φ is regular. In particular this is the case when $G = \mathbb{R}$ and $\mathcal{E}(\varphi) \neq \{0\}$.

Remark that if φ is cohomologous to φ_1 and to φ_2, two functions with values respectively in closed subgroups whose intersection is reduced to the identity element e of G, then $\mathcal{E}(\varphi) = \{e\}$.

For instance, if φ is a $\mathbb{Z}$-valued cocycle such that there is $s \notin \mathbb{Q}$ for which the multiplicative equation $e^{2\pi i s\varphi} = \psi/\psi \circ \tau$ has a measurable solution ψ, then either φ is a coboundary or the cocycle φ is not regular. We will use this remark to give an example of nonregular cocycle in section 5.

3 Proof of Theorem 2.1.3

3.1 Characterization of the τ_φ-invariant ergodic measures

The key tool in the proof of Theorem 2.1.3 is the following result:

Theorem 3.1.1 *([Ra06]) Let λ be a τ_φ-invariant ergodic measure of the form $\lambda(dy, dg) = \mu(dy)N(y, dg)$, where μ is a probability measure on X and N a positive Radon kernel (i.e. such that, for every $y \in X$, $N(y, dg)$ is a positive Radon measure on the Borel sets of G and, for every Borel set B of G, the map $y \to N(y, B)$ is measurable).*

Then there exist a closed subgroup H of G and a measurable map u from X to G such that:

- $\varphi_u(y) := (u(\tau y))^{-1}\, \varphi(y)\, u(y) \in H$ for μ-a.e. $y \in X$;

- the measure $\tilde{\lambda}$ image of λ by the map $(y, g) \to (y, (u(y))^{-1}\, g)$ is a τ_{φ_u}-invariant ergodic measure with support $X \times H$ and has the form :

$$\tilde{\lambda}(dy, dh) = \tilde{\mu}(dy)\chi(h)\, dh, \tag{23}$$

where χ is an exponential on H and $\tilde{\mu}$ a positive σ-finite measure, equivalent to μ such that

$$\tau\tilde{\mu}(dy) = \chi(\varphi_u(\tau^{-1}y))\, \tilde{\mu}(dy). \tag{24}$$

If $H = G$, $u(y) \equiv e$, $\lambda(dy, dg) = \tilde{\mu}(dy)\ \chi(g)\, dg$, $\tau\tilde{\mu}(dy) = \chi(\varphi(\tau^{-1}y))\, \tilde{\mu}(dy)$.

3.2 Ergodic decomposition of λ_χ

• Abstract ergodic decomposition

Let h be a positive measurable function on $X \times G$ such that $\lambda_\chi(h) = 1$ (cf. 2.1.2). We apply to the Borel standard space $(X \times G, \mathfrak{X} \times \mathfrak{B}_G)$ and to the probability measure $h\, \lambda_\chi$ the results of the appendix.

We denote by P^h a regular conditional probability with respect to $h\, \lambda_\chi$ and the σ-algebra of τ_φ-invariant sets $\mathfrak{J}$, by M^h the positive kernel on $X \times G$ defined, for any measurable nonnegative function f on $X \times G$ by :

$$\forall (x, g) \in X \times G,\ M^h f(x, g) = P^h(f/h)(x, g).$$

We have

$$\lambda_\chi(dy, dt) = \int_{X \times G} M^h((x, g), (dy, dt))\ h(x, g)\, \lambda_\chi(dx, dg). \tag{25}$$

For λ_χ-a.e. $(x,g) \in X \times G$, the probability measure $P^h((x,g),\cdot)$ is τ_φ-ergodic (Theorem 7.4.5) (i.e. $\forall A \in \mathfrak{J}, P^h((x,g),A) = 0$ or 1). Moreover, according to (49), Lemma 7.2.1, we have

$$\tau_\varphi P^h((x,g),(dy,dt)) = \frac{h \circ \tau_\varphi^{-1}(y,t)}{h(y,t)} \, P^h((x,g),(dy,dt)), \tag{26}$$

which is equivalent to

$$\tau_\varphi M^h((x,g),(dy,dt)) = M^h((x,g),(dy,dt)). \tag{27}$$

We write

$$P^h((x,g),(dy,dt)) = \rho((x,g),dy) \; Q((x,g,y),dt),$$

where ρ is a transition probability from $(X \times G, \mathfrak{X} \otimes \mathfrak{B}_G)$ to $(X, \mathfrak{X})$ and Q a transition probability from $(X \times G \times X, \mathfrak{X} \otimes \mathfrak{B}_G \otimes \mathfrak{X})$ to $(G, \mathfrak{B})$.

We introduce also the notations :

$$\nu_{(x,g)}(dy) := \rho((x,g),dy) \quad \text{and} \quad N_{(x,g)}(y,dt) := Q((x,g,y),dt).$$

Let $(x,g) \in X \times G$. The probability $\nu_{(x,g)}$ is uniquely determined by, for any $A \in \mathfrak{X}, \nu_{(x,g)}(A) = P^h((x,g), A \times G)$. The family of probabilities $\{N_{(x,g)}(y,\cdot) : y \in X\}$ is determined up to a set of $\nu_{(x,g)}$-measure zero. If we consider on the probability space $(X \times G, \mathfrak{X} \times \mathfrak{B}_G, P^h((x,g),\cdot))$ the projections U and V on X and G, $\nu_{(x,g)}$ is the law of U and $N_{(x,g)}$ is a version of the conditional law of V with respect to U.
The kernel M^h can then be written:

$$M^h((x,g),(dy,dt)) = \rho((x,g),dy) \; \tilde{Q}((x,g,y),dt) = \nu_{(x,g)}(dy) \; \tilde{N}_{(x,g)}(y,dt), \tag{28}$$

where $\tilde{Q}((x,g,y),dt) = \tilde{N}_{(x,g)}(y,dt) = h(y,t)^{-1} N_{(x,g)}(y,dt)$ is a positive kernel from $(X \times G \times X, \mathfrak{X} \times \mathfrak{B}_G \times \mathfrak{X})$ to $(G, \mathfrak{B}_G)$.

Let f be a measurable positive μ_χ-integrable function on X and K be a compact subset of G. We know that

$$\begin{aligned}
&\int_{X \times G} \left[\int_X f(y) \; \tilde{N}_{(x,g)}(y,K) \; \nu_{(x,g)}(dy) \right] h(x,g) \lambda_\chi(dx,dg) \\
&= \int_{X \times G} f(x) 1_K(g) \lambda_\chi(dx,dg) < +\infty.
\end{aligned}$$

Therefore, for λ_χ-a.e. (x,g), we have, for $\nu_{(x,g)}$-a.e. y, $\tilde{N}_{(x,g)}(y,K) < +\infty$.

Let $(K_n)_{n \geq 0}$ be the sequence of compact subsets of G such that $\bigcup_{n \in \mathbb{N}} K_n = G$. For λ_χ-a.e. (x,g), we have, for $\nu_{(x,g)}$-a.e. y, $\forall n \geq 0$, $\tilde{N}_{(x,g)}(y,K_n) < +\infty$, i.e. $\tilde{N}_{(x,g)}(y,\cdot)$ is a Radon measure on G.

After a modification of P^h on a set of λ_χ-measure zero followed, for any $(x,g) \in X \times G$, by a modification of the family of positives measures $\tilde{N}_{(x,g)}(y,\cdot) : y \in X\}$ on a set of $\nu_{(x,g)}$-measure zero, we can assume that:

For every $(x,g) \in X \times G$, the positive measure $M^h((x,g),\cdot)$ is τ_φ-invariant ergodic and, for every $y \in X$, $\tilde{N}_{(x,g)}(y,\cdot)$ is a Radon measure on G.

• **Explicit form of the ergodic decomposition**

According to Theorem 3.1.1, the τ_φ-invariant ergodic measure $M^h((x,g),\,\cdot\,)$ can be written, up to a multiplicative constant,

$$M^h((x,g),(dy,d\gamma)) = \tilde{\mu}_{(x,g)}(dy) \times [\delta_{v_{(x,g)}(y)} * \Big(\chi_{(x,g)}(\gamma)\, m_{H_{(x,g)}}(d\gamma)\Big)], \tag{29}$$

where $H_{(x,g)}$ is a closed subgroup of G, $\chi_{(x,g)}$ an exponential on $H_{(x,g)}$, $v_{(x,g)}$ a measurable map from X to G and $\tilde{\mu}_{(x,g)}$ a positive σ-finite measure on X, equivalent to the probability $\nu_{(x,g)}$, such that

$$\tau_\varphi(\tilde{\mu}_{(x,g)})(dy) = \chi(\varphi_{v_{(x,g)}}(\tau^{-1}y))\ \tilde{\mu}_{(x,g)}(dy), \tag{30}$$

where

$$\varphi_{v_{(x,g)}}(y) := (v_{(x,g)}(\tau y))^{-1}\, \varphi(y)\ v_{(x,g)}(y) \in H_{(x,g)},\ \ for\ \tilde{\mu}_{(x,g)} - a.e.\ y \in X. \tag{31}$$

For $t \in G$ and f defined on $X \times G$, let $R_t(f)(x,g) := f(x, g\,t)$. From Lemma 7.2.1 it follows that, for every $t \in G$, for every nonnegative measurable function f on $X \times G$ and for λ_χ-a.e. $(x,g) \in X \times G$,

$$M^h(R_t(f))(x,g) = P^h(R_th/h)(x,g)\ M^h(f)(x,g\,t). \tag{32}$$

Let $c_{(x,g),t}$ be defined by

$$c_{(x,g),t} = P^h(R_th/h)(x,g). \tag{33}$$

From (32), we have :

$$\begin{aligned} &\tilde{\mu}_{(x,g)}(dy) \times [\delta_{v_{(x,g)}}(y) * \Big(\chi_{(x,g)}(\gamma)\, m_{H_{(x,g)}}(d\gamma)\Big) * \delta_t] \\ = \ \ & c_{(x,g),t}\ \tilde{\mu}_{(x,gt)}(dy) \times [\delta_{v_{(x,gt)}}(y) * \Big(\chi_{(x,gt)}(\gamma)\, m_{H_{(x,gt)}}(d\gamma)\Big)]. \end{aligned}$$

Using Fubini's theorem and the separability of the σ-algebra $\mathfrak{X} \times \mathfrak{B}_G$, it follows that, for λ_χ-a.e. $(x,g) \in X \times G$ and for m_G-a.e. $t \in G$,

$$R_t\Big(M^h((x,g),\,\cdot\,)\Big) = P^h(R_th/h)(x,g)\ M^h((x,g\,t),\,\cdot\,)$$

and therefore

$$R_{g^{-1}}\Big(M^h((x,g),\,\cdot\,)\Big) = P^h(R_th/h)(x,g)\ R_{(g\,t)^{-1}}\Big(M^h((x,g\,t),\,\cdot\,)\Big).$$

This implies that, for λ_χ-a.e. $(x,g) \in X \times G$, the measure $M^h((x,g),(dy,dt))$ is equal, up to a multiplicative positive constant $c(x,g)$, to a fixed measure which has the form:

$$\tilde{\mu}_x(dy)\ [\delta_{v_x(y)} * (\chi_x\, \tilde{m}_{H_x}) * \delta_g](dt),$$

where $\tilde{m}_{H_x}$ is a left Haar measure on H_x (we will latter change $\tilde{m}_{H_x}$ into m_{H_x} by multiplying it by a factor).

Now, for λ_χ-a.e. $(x,g) \in X \times G$, $P^h(1)(x,g) = M^h(h)(x,g) = 1$. Therefore

$$(c(x,g))^{-1} = \int_X \Big(\int_{H_x} h(y, v_x(y)\,\gamma\, g)\ \chi_x(\gamma)\, \tilde{m}_{H_x}(d\gamma) \Big)\ \tilde{\mu}_x(dy) \tag{34}$$

and, for λ_χ-a.e. $(x,g) \in X \times G$ and every measurable nonnegative function f on $X \times G$,

$$M^h(f)(x,g) = \frac{\int_X \Big(\int_{H_x} f(y, v_x(y)\,\gamma\, g)\ \chi_x(\gamma)\, \tilde{m}_{H_x}(d\gamma) \Big)\ \tilde{\mu}_x(dy)}{\int_X \Big(\int_{H_x} h(y, v_x(y)\,\gamma\, g)\ \chi_x(\gamma)\, \tilde{m}_{H_x}(d\gamma) \Big)\ \tilde{\mu}_x(dy)}. \tag{35}$$

Now we carry out the suitable modifications in order to obtain the desired properties of measurability for the decomposition

- **Measurability**

We can explicit the decomposition of M^h given in (28). We have:

$$\nu_{(x,g)}(dy) = P^h((x,g), dy \times G) = c(x,g) \Big(\int_{H_x} h(y, v_x(y)\gamma g)\ \chi_x(\gamma)\, \tilde{m}_{H_x}(d\gamma) \Big)\ \tilde{\mu}_x(dy), \tag{36}$$

and

$$\tilde{N}_{(x,g)}(y,dt) = \Big(\int_{H_x} h(y, v_x(y)\gamma g)\ \chi_x(\gamma)\, \tilde{m}_{H_x}(d\gamma) \Big)^{-1}\ (\delta_{v_x(y)} * (\chi_x\, \tilde{m}_{H_x}) * \delta_g)(dt).$$

The closed set $v_x(y)H_x$ is the support $S(x,y)$ of the probability $Q((x,e,y),.) = N_{(x,e)}(y,\cdot)$ on G and H_x is the support of the probability measure $\widehat{Q}((x,e,y),.) * Q((x,e,y),.)$, where $\widehat{Q}((x,e,y),.)$ is the image of the positive measure $Q((x,e,y),.)$ by the transformation $t \mapsto t^{-1}$ of G. It follows that the maps $x \in X \mapsto H_x \in \mathcal{F}(G)$ and $(x,y) \in X \times X \mapsto v_x(y)H_x \in \mathcal{F}(G)$ are measurable. For instance, the last property follows from the fact that, for any closed subset F of G, we have

$$\{(x,y) \in X \times X : v_x(y)H_x \subset F\} = \{(x,e,y) : Q((x,e,y), F^c) = 0\}.$$

From Lemma 7.1.1 we can find a measurable map $u : X \times X \mapsto G$ such that, for any $(x,y) \in X \times X$, $u(x,y) \in S(x,y)$. Then, $v_x(y)H_x = u(x,y)H_x$ and, for any nonnegative measurable function f on $X \times G$,

$$\int_{H_x} f(y, v_x(y)\,\gamma\, g)\ \chi_x(\gamma)\tilde{m}_{H_x}(d\gamma) = \chi_x^{-1}((u(x,y))^{-1}\, v_x(y)) \int_{H_x} f(y, u(x,y)\,\gamma\, g)\ \chi_x(\gamma)\tilde{m}_{H_x}(d\gamma)$$

As
$$\delta_{(u(x,y))^{-1}\,v_x(y)} * (\chi_x \tilde{m}_{H_x}) = \chi_x^{-1}((u(x,y))^{-1}\,v_x(y))\,(\chi_x \tilde{m}_{H_x}),$$
the positive kernel $R((x,g,y),dt) = \delta_{(u(x,y))^{-1}} * \widetilde{N}_{x,g}(y,dt) * \delta_{g^{-1}}$ from $X \times X$ to G is equal to
$$\Big(\int_{H_x} h(y,u(x,y)\,\gamma\,g)\,\chi_x(\gamma)\,\tilde{m}_{H_x}(d\gamma)\Big)^{-1} \quad \chi_x(t) \quad \tilde{m}_{H_x}(dt).$$

Denoting by U the unit closed ball in G centered at e, we have
$$\Big(\int_{H_x} h(y,u(x,y)\,\gamma\,g)\,\chi_x(\gamma)\,\tilde{m}_{H_x}(d\gamma)\Big)^{-1} \int_{H_x\cap U} \chi_x(t)\,\tilde{m}_{H_x}(dt) = R((x,g,y),U) > 0$$
and, for any $\gamma \in H_x$,
$$\chi_x(\gamma) = \frac{R((x,e,y),\gamma\,U)}{R((x,e,y),U)},$$
which proves that there exists a measurable map $\eta : X \times G \mapsto \mathbb{R}_+^*$ such that, for μ_χ-a.e. $x \in X$, $\forall \gamma \in H_x$, $\chi_x(\gamma) = \eta(x,\gamma)$.

We also have
$$\frac{\tilde{m}_{H_x}(dt)}{\int_{H_x\cap U} \chi_x(\gamma)\,\tilde{m}_{H_x}(d\gamma)} = \frac{R((x,e,y),dt)}{R((x,e,y),t\,U)}$$
which shows that the left-hand member defines a positive kernel from X to G. We observe that the left-hand member is the unique left Haar measure denoted by m_{H_x} of H_x such that
$$\int_{H_x\cap U} \chi_x(\gamma)\,m_{H_x}(d\gamma) = 1.$$

Finally, we obtain
$$M^h((x,g),dy,dt) = R((x,g,y),U)\,\nu_{(x,g)}(dy)\,\Big(\delta_{u(x,y)} * (\chi_x\,m_{H_x}) * \delta_g\Big)(dt)$$
and
$$R((x,g,y),U)\,\nu_{(x,g)}(dy) = c(x,g)\,\chi_x((u(x,y))^{-1}\,v_x(y))\,(\int_{H_x\cap U} \chi_x(t)\,\tilde{m}_{H_x}(dt))\,\tilde{\mu}_x(dy).$$

We deduce that
$$\chi_x((u(x,y))^{-1}\,v_x(y))\;\tilde{\mu}_x(dy) = d(x)\,\mu_x(dy)$$
with
$$(d(x))^{-1} = c(x,e)\,(\int_{H_x\cap U} \chi_x(t)\,\tilde{m}_{H_x}(dt))$$
and
$$\mu_x(dy) = R((x,e,y),U)\,\nu_{(x,e)}(dy).$$

We observe that $(\mu_x(dy))_{x\in X}$ is a positive kernel on $(X,\mathfrak{X})$.

The formula (35) can be written

$$M^h(f)(x,g) = \frac{\int_X \Big(\int_{H_x} f(y, u(x,y)\,\gamma\, g)\ \chi_x(\gamma)\, m_{H_x}(d\gamma)\Big)\ \mu_x(dy)}{\int_X \Big(\int_{H_x} h(y, u(x,y)\,\gamma\, g)\ \chi_x(\gamma)\, m_{H_x}(d\gamma)\Big)\ \mu_x(dy)}. \qquad (37)$$

For every $(x,g) \in X \times G$, we choose the expression (37) for $M^h((x,g),\cdot)$.

• **Proof of the relations (4) to (9).**

The equality of measures $\tau_\varphi(M^h((x,g),(dy,dt)) = M^h((x,g),(dy,dt))$ is equivalent to

$$(\tau\mu_x)(dy)\ \Big(\delta_{\varphi(\tau^{-1}y)} * \delta_{u_x(\tau^{-1}y)} * (\chi_x\, m_{H_x}) * \delta_g)(dt) = \mu_x(dy)\ \Big(\delta_{u_x(y)} * (\chi_x\, m_{H_x}) * \delta_g\Big)(dt),$$

which leads to

$$\varphi(\tau^{-1}y)\, u_x(\tau^{-1}y)\, H_x = u_x(y) H_x, \text{ for } \mu_x\text{-a.e. } x \in X$$

and

$$(\tau\mu_x)(dy) = \chi_x\ ((u_x(y))^{-1}\, \varphi(\tau^{-1}y)\, u_x(\tau^{-1}y))\ \mu_x(dy);$$

hence the relations (5) and (6).

The equality $M^h((x,g),\cdot) = M^h(\tau_\varphi(x,g),\cdot)$ is equivalent to : $\nu_{(x,g)} = \nu_{\tau_\varphi(x,g)}$ and, for $\nu_{(x,g)}$-a.e. $y \in X$, $\tilde{N}_{(x,g)}(y,\cdot) = \tilde{N}_{\tau_\varphi(x,g)}(y,\cdot)$.

The equality $\tilde{N}_{(x,g)}(y,\cdot) = \tilde{N}_{\tau_\varphi(x,g)}(y,\cdot)$ is equivalent to the following conditions :

$$u_x(y) H_x = u_{\tau x}(y)\, H_{\tau x}\, \varphi(x)$$

(equality of the supports) which implies

$$\zeta_x(y) = (u_x(y))^{-1}\, u_{\tau x}(y)\, \varphi(x) \in H_x, \qquad (38)$$

$$H_{\tau x} = \varphi(x)\, H_x\, (\varphi(x))^{-1} \qquad (39)$$

and therefore

$$\chi_{\tau x}(\varphi(x)\, \zeta_x(y)\, (\varphi(x))^{-1})\ \delta_{u_{\tau x}}(y) * (\chi_{\tau x}\, m_{H_{\tau x}}) * \delta_{\varphi(x)} = \delta_{u_x(y)} * (\chi_{\tau x}(\varphi(x)\ \cdot\ (\varphi(x))^{-1})\ m_{H_x}$$

where $\widehat{m}_{H_x} = \delta_{\varphi(x)} * m_{H_{\tau x}} * \delta_{(\varphi(x))^{-1}}$ is a left Haar measure on H_x.

We write $\widehat{m}_{H_x} = d(x)\, m_{H_x}$ for a constant $d(x)$ depending on x and we obtain for any $\gamma \in H_x$,

$$\chi_x(\gamma) = \chi_{\tau x}(\varphi(x)\, \gamma\, (\varphi(x))^{-1})$$

and

$$\chi_x(\zeta_x(y)) \int_{H_{\tau x}} h(y, u_{\tau x}\gamma\varphi(x)g)\ \chi_{\tau x}(\gamma)\, d\gamma = d(x) \int_{H_x} h(y, u_x(y)\gamma g)\ \chi_x(\gamma)\, d\gamma.$$

Then the probability equality $\nu_{(x,g)} = \nu_{\tau_\varphi(x,g)}$ is equivalent to

$$\tilde{\mu}_{\tau x}(dy) = c(x)\, \chi_x(\zeta_x(y))\, \tilde{\mu}_x(dy)$$

for a constant $c(x)$ depending on x.

Hence the relations (4), (7), (8), (9).

The ergodicity of the cocycle φ_{u_x} on H_x over the σ-finite ergodic measure μ_x implies that H_x is amenable [Zi78].

The first assertion of Theorem 2.1.3 is proved.

- **Assertions 2) and 3) of Theorem 2.1.3**

a) We suppose that the subgroups H_x are conjugated to a fixed closed subgroup H (cf. Theorem 5.1.1 for the nilpotent connected Lie group case), i.e. there exists a measurable map $a : X \to G$ such that $H_x = a(x)H(a(x))^{-1}$.

Let $x \in X$. The element $a(x)$ is defined modulo the normalizer of H. The element $\psi(x) := a(\tau x)^{-1}\ \varphi(x)\ a(x)$ is in the normalizer of H and we have

$$(a(x))^{-1}\ (u_x(y))^{-1}\ u_{\tau x}(y)\ \varphi(x)\ a(x) \in H.$$

The ergodic components applied to a function f can be written

$$M^h f(x,g) = \frac{\int_X (\int_H f(y, u_x(y)\, a(x)\, \gamma\, a(x)^{-1} g)\ \chi_x(a(x)\, \gamma a(x)^{-1})\ d\gamma)\ \mu_x(dy)}{\int_X (\int_H h(y, u_x(y)\, a(x)\, \gamma\, a(x)^{-1} g)\ \chi_x(a(x)\gamma a(x)^{-1})\ d\gamma)\ \mu_x(dy)}. \qquad (40)$$

We have: $\chi_{\tau x}(a(\tau x)\, \gamma\, a(\tau x)^{-1}) = \chi_x(a(x)(\psi(x))^{-1}\gamma\psi(x)a(x)^{-1})$.

Setting $\tilde{\chi}_x(\gamma) := \chi_x(a(x)\, \gamma\, (a(x))^{-1})$, we have $\tilde{\chi}_{\tau x}(\gamma) = \tilde{\chi}_x((\psi(x))^{-1}\gamma\psi(x))$.

b) *Abelian groups*

If G is abelian, we have $H_{\tau(x)} = H_x$, for μ_χ-a.e. $x \in X$. Since the map $x \in X \to H_x \in \mathcal{F}(G)$ is measurable and the Chabauty's topology countably separates the points, there exists a closed subgroup H of G such that $H_x = H$, for μ_χ-a.e. $x \in X$.

For every $\gamma \in H$, we have $\lambda_\chi(R_\gamma(f)) = \chi^{-1}(\gamma)\, \lambda_\chi(f)$ and, for λ_χ-a.e. $(x,g) \in X \times G$, $M^h R_\gamma(f)(x,g) = \chi_x^{-1}(\gamma)\, M^h f(x,g)$. For $f = h$, it follows that,

$$\forall \gamma \in H,\ \chi(\gamma) = \int_{X\times G} \chi_x(\gamma)\, h(x,g)\, \lambda_\chi(dx, dg)$$

and therefore $\chi_x = \chi$, for μ_χ-a.e. $x \in X$.

The ergodic component of λ_χ applied to a function f can be written

$$M^h f(x,g) = \frac{\int_X (\int_H f(y, u_x(y)\, \gamma\, g)\ \chi(\gamma)\ d\gamma)\ \mu_x(dy)}{\int_X (\int_H h(y, u_x(y)\, \gamma\, g)\ \chi(\gamma)\ d\gamma)\ \mu_x(dy)}. \qquad (41)$$

This completes the proof of Theorem 2.1.3. $\square$

4 Proof of Theorem 2.2.2

4.1 Lemmas

For the proof of Theorem 2.2.2, we begin by a lemma which allows to compare the ergodic components.

Lemma 4.1.1 *1) Let φ be a cocycle with values in a closed subgroup H_1 of G and $\mu_1 \otimes m_{H_1}$ be a τ_φ-quasi-invariant positive measure. We suppose that the measure $\mu_1 \otimes m_{H_1}$ is τ_φ-ergodic and that the cocycle φ is μ_1-cohomologous to a cocycle ψ with values in a closed subgroup H_2 of G, with transfer function u.*

Then, there exists $g_0 \in G$ such that, for μ_1-a.e. $x \in X$,

$$u(x)\, H_2 = g_0\, H_2 \quad \textit{and} \quad H_1 \subset u(x)\, H_2\, (u(x))^{-1} = g_0\, H_2\, g_0^{-1}.$$

2) Assume in addition there exist a positive τ_ψ-quasi invariante positive measure $\mu_2 \otimes m_{H_2}$ with $\mu_2 \sim \mu_1$ which is τ_φ-ergodic. Then there exists $g_0 \in G$ such that

$$H_1\, u(x) = H_1\, g_0, u(x)\, H_2 = g_0\, H_2 \textit{ and } g_0^{-1}\, H_1\, g_0 = H_2.$$

3) Assume in addition that μ_1 [resp. μ_2] is $\chi_1 \circ \tau_\varphi$-conformal [resp. $\chi_2 \circ \tau_\psi$-conformal] for an exponential χ_1 on H_1 [resp. χ_2 on H_2]. Then
- for μ_1-a.e. $x \in X$ and every $\gamma \in H_1$, $\chi_1(\gamma) = \chi_2(g_0^{-1}\, \gamma\, g_0)$,
- for μ_1-a.e. $x \in X$, $\chi_1(u(x)\, g_0^{-1}) = \chi_2(g_0^{-1}\, u(x))$,
- up to a multiplicative constant, $\mu_2(dx) = \chi_1(u(x)\, g_0^{-1})\ \mu_1(dx)$.

The τ_φ-invariant ergodic measure $\mu_2 \otimes (\delta_{u(x)} * (\chi_2\, m_{H_2}))$ is equal to $\mu_1 \otimes ((\chi_1\, m_{H_1}) * \delta_{g_0})$, up to a multiplicative constant.

Proof 1) For every continuous left H_2-invariant function F on G and every $g \in G$ the function $f^g(x,t) = F((u(x))^{-1}\, t\, g)$ is τ_φ-invariant. This function is therefore $\mu_1 \otimes m_{H_1}$-a.e. constant. Applying Fubini's theorem and the continuity of F, it follows that, for μ_1-a.e. $x \in X$ and for any $g \in G$, the function $t \in H_1 \longrightarrow F((u(x))^{-1}\, t\, u(x)\, g)$ is constant and therefore equal to $F(g)$, its value for $t = e$. Consequently, $(u(x))^{-1}\, H_1\, u(x) \subset H_2$.

Since φ [resp. ψ] takes values in H_1 [resp. H_2], the above inclusion implies that, for μ_1-a.e. $x \in X$, $(u(\tau x))^{-1}\, u(x) \in H_2$. Therefore $u(\tau x)\, H_2 = u(x)\, H_2$. By ergodicity of (μ_1, τ), we deduce the existence of $g_0 \in G$ such that μ_1-a.e. $x \in X$, $u(x)\, H_2 = g_0\, H_2$.

2) The cocycle ψ is μ_2-cohomologous to the cocycle φ, via the map $x \in X \to (u(x))^{-1} \in G$. Then the second statement is a consequence of the first one.

3) Set $\mu_2 = \beta\, \mu_1$ where β is a positive function on X. From the conformity it follows that, for μ_1-a.e. $x \in X$,

$$\chi_2(\psi(x)) = \frac{\beta(x)}{\beta(\tau x)}\, \chi_1(\varphi(x)).$$

From part 2), this equality can be written

$$\frac{\chi_2((u(\tau x))^{-1}\, g_0)}{\chi_2((u(x)^{-1}\, g_0)}\, \chi_2((u(x))^{-1}\, \varphi(x)\, u(x)) = \frac{\beta(x)}{\beta(\tau x)} \chi_1(\varphi(x)).$$

For any $x \in X$, we consider the exponential $\widetilde{\chi}_x$ on H_1 and the function f on X, defined by :

$$\widetilde{\chi}_x(t) = \frac{\chi_2((u(x))^{-1}\, t\, u(x))}{\chi_1(t)} \quad \text{and} \quad f(x) = \beta(x)\, \chi_2((u(x))^{-1}\, g_0)$$

We observe that for any $t \in H_1$, $\widetilde{\chi}_{\tau x}(t) = \widetilde{\chi}_x(t)$ and the positive function $(x,t) \mapsto f(x)\, \chi_x(t)$ on $X \times H$ is τ_φ-invariant. It follows that this function is constant $\mu_1 \otimes m_{H_1}$-a.e. Hence: for μ_1-a.e. $x \in X$,

– for every $t \in H_1$, $\chi_2((u(x))^{-1}\, t\, u(x)) = \chi_1(t)$,

– and up to a multiplicative constant, the function $\beta(x)$ is equal to $\chi_2(g_0^{-1}\, u(x)) = \chi_1(u(x)\, g_0^{-1})$. $\square$

Corollary 4.1.2 *Let $\mu_x(dy)\otimes\Big(\delta_{u_x(y)}*(\chi_x\; m_{H_x})\Big)(dt)$ and $\mu_{x'}(dy)\otimes\Big(\delta_{u_{x'}(y)}*(\chi_{x'}\; m_{H_{x'}})\Big)(dt)$ two ergodic components of λ_χ. Then*
- either the measures μ_x and $\mu_{x'}$ on X are mutually singular;
- or there is $g_{x',x} \in G$ such that, for every $g \in G$,

$$\mu_{x'}(dy) \otimes \Big(\delta_{u_{x'}(y)} * (\chi_{x'}\; m_{H_{x'}})\Big) = \mu_x(dy) \otimes \Big(\delta_{u_x(y)} * (\chi_x\; m_{H_x})\Big) * \delta_{g_{x',x}}.$$

Hence $P^h((x,g),.) = P^h((x', g_{x',x}\, g),.)$.

Proof For a G-valued cocycle φ and a measurable map u from X to G, we denote by φ_u the cocycle $\varphi_u(y) := (u(\tau y))^{-1}\, \varphi(y)\, u(y), \forall y \in X$.

The values of the cocycles φ_{u_x} and $\varphi_{u_{x'}}$ are respectively in H_x and $H_{x'}$. The measures $\mu_{\chi_x} \otimes (\chi_x\, m_{H_x})$ and $\mu_{x'} \otimes (\chi_{x'}\, m_{H_{x'}})$ are respectively $\tau_{\varphi_{u_x}}$-invariant ergodic and $\tau_{\varphi_{u_{x'}}}$-invariant ergodic, and $\varphi_{u_{x'}} \overset{(u_x)^{-1}\, u_{x'}}{\sim} \varphi_{u_x}$.

The result follows from the previous lemma. $\square$

4.2 Proof of 2.2.2

1) Let $x_0 \in X$. From Corollary 4.1.2, for any $x \in X$, if the measure μ_x is equivalent to μ_{x_0} then there exists $g_x \in G$ such that $P^h((x,e),\cdot) = P^h((x_0,e),\cdot) * \delta_{g_x}$ and consequently,

with the notations of Subsection 3.2 (cf. (36)), we have $\nu_{(x,e)} = \nu_{(x_0,e)}$. Conversely, the equality $\nu_{(x,e)} = \nu_{(x_0,e)}$ implies the equivalence of the measures μ_x and μ_{x_0}.

The σ-algebra $\mathfrak{X} \times \mathfrak{B}(G)$ is separable, i.e. generated by a countable sub-algebra $\mathcal{A}$. We deduce the equality of the sets:

$$\{x \in X : \mu_x \sim \mu_{x_0}\} = \{x \in X : \nu_{(x,e)} = \nu_{(x_0,e)}\} = \{x \in X : \forall A \in \mathcal{A}, \nu_{(x,e)}(A) = \nu_{(x_0,e)}(A)\}$$

which proves that $\{x \in X : \mu_x \sim \mu_{x_0}\}$ is measurable. Since, for any $x \in X$, $\mu_x \sim \mu_{\tau x}$, this set is τ-invariant and therefore (ergodicity of μ_χ) has zero or full measure.

1) Assume that the cocycle is regular. Then every measurable τ_ψ-invariant function f is $\mu_\chi \otimes m_H$-a.e. constant. The function $F(g) := \|f(\cdot, \cdot\, g)\|_{\mathbb{L}^\infty(X \times H, \mu_\chi \otimes m_H)}$, is left H-invariant on G and we have, for every $g \in G$,

$$f(x, \gamma\, g) = F(g), \text{ for } \mu_\chi \otimes m_H\text{-a.e. } (x, \gamma) \in X \times H.$$

The first statement of 2) follows from the fact that f is a measurable τ_φ-invariant function if and only if the function $\tilde{f}(x, g) = f(x, u(x)\, g)$ is τ_ψ-invariant.

We consider the bijective map θ_u from $X \times G$ onto itself defined by: $\forall (x, g) \in X \times G$, $\theta_u(x, g) = (x, u(x)g)$. A measurable nonnegative function f on $X \times G$ is τ_φ-invariant if and only if $f \circ \theta_u$ is τ_ψ-invariant. If $\mathfrak{J} = \mathfrak{J}_\varphi$ is the σ-algebra of τ_φ-invariant subsets of $X \times G$ then $\theta_u\, \mathfrak{J}_\varphi$ is the σ-algebra $\mathfrak{J}_\psi$ of τ_ψ-invariant subsets of $X \times G$. From Lemma 7.2.1 we have, for any nonnegative measurable function f on $X \times G$ and for λ_χ-a.e. $(x, g) \in X \times G$,

$$\mathbb{E}_{h\,\lambda_\chi}[f|\mathfrak{J}_\varphi](x, g) = \frac{\mathbb{E}_{h\,\lambda_\chi}\left[f \circ \theta_u\, \frac{h \circ \theta_u}{h}\, \chi \circ u|\mathfrak{J}_\psi\right] \circ \theta_u(x, g)}{\mathbb{E}_{h\,\lambda_\chi}\left[\frac{h \circ \theta_u}{h}\, \chi \circ u|\mathfrak{J}_\psi\right] \circ \theta_u(x, g)}. \tag{42}$$

Any nonnegative measurable τ_ψ-invariant function is $\mu_\chi \otimes m_H$-a.e. constant. Hence, we have, for any nonnegative measurable function f and for λ_χ-a.e. $(x, g) \in X \times G$,

$$\mathbb{E}_{h\,\lambda_\chi}[f|\mathfrak{J}_\psi]x, g) = \frac{\int_{X \times H} f(y, \gamma\, g)\, h(y, \gamma\, g)\; d\gamma\, \mu_\chi(dy)}{\int_{X \times H} h(y, \gamma\, g)\; d\gamma\, \mu_\chi(dy)}.$$

From (42) it follows:

$$\begin{aligned} M^h f(x, g) &= \mathbb{E}_{h\,\lambda_\chi}[h\, f|\mathfrak{J}_\varphi](x, g) \\ &= \frac{\int_X \int_H f(y, u(y)\, \gamma\, (u(x))^{-1}\, g)\, \chi(u(y))\; d\gamma\, \mu_\chi(dy)}{\int_X \int_H h(y, u(y)\, \gamma\, (u(x))^{-1}\, g)\, \chi(u(y))\; d\gamma\, \mu_\chi(dy)}. \end{aligned}$$

3) If there exists some x such that $\mu_x \sim \mu_\chi$, then the reduction of the cocycle given by (8) is "global" μ_χ-a.e.: there exists a measurable function u and a closed subgroup H such that the cocycle is cohomologous to an ergodic cocycle with values in H and it is regular.

If there is a countable number of different equivalence classes among the measures $\mu_x, x \in X$, then by 1), for μ_χ-a.e. x, all the measures μ_x all equivalent and this equivalence class is that of μ_χ.

The last assertion of 3) follows from the assertion 3) of proposition 2.2.4. □

5 On the equation $H_{\tau x} = \varphi(x)\ H_x\ (\varphi(x))^{-1}$

In Theorem 2.1.3 we encounter a measurable family of subgroups H_x such that the following conjugacy equation holds:

$$H_{\tau x} = \varphi(x)\ H_x\ (\varphi(x))^{-1},\ \text{for } \mu_\chi-\text{a.e. } x \in X. \tag{43}$$

5.1 Nilpotent groups

When G is a nilpotent connected Lie group, the subgroups H_x are conjugate to a fixed subgroup H.

Theorem 5.1.1 *Assume G is a nilpotent connected Lie group. If (H_x) is a measurable family of subgroups such that (43) holds μ-a.e., where μ is a σ-finite measure which is quasi-invariant and ergodic for τ, then there is a fixed closed subgroup H and a measurable map $x \to a(x)$ from X into G such that for μ_χ-a.e. $x \in X$:*

$$H_x = a(x)\ H\ a(x)^{-1}.$$

Proof We equip the set $\mathcal{F}(G)$ of closed subsets of G with the Chabauty's topology (cf. section 2).

We know that the map $x \in X \longrightarrow H_x \in \mathcal{F}(G)$ is measurable. For any $F \in \mathcal{F}(G)$, we have

$$\{x \in X : \overline{\{gH_xg^{-1} : g \in G\}} \subset F\} = \{x \in X : H_x \subset \bigcap_{g \in G} g^{-1}Fg\}.$$

It follows that the map $x \in X \longrightarrow \overline{\{gH_xg^{-1} : g \in G\}} \in \mathcal{F}(G)$ is measurable. We denote by $\mathfrak{g}$ the Lie algebra of G and call ad the adjoint representation of $\mathfrak{g}$ (i.e. for any $(X, Y) \in \mathfrak{g}^2$, $\operatorname{ad} X\,(Y) = [X, Y]$). We denote by $\exp : \mathfrak{g} \longrightarrow G$ the exponential map and by Ad the adjoint representation of G on $\mathfrak{g}$. We have :

$$\begin{aligned} g\,\text{exp}X\,g^{-1} &= \exp(\text{Ad}g(X)),\ \forall g \in G, \forall X \in \mathfrak{g}, \\ \text{Ad}(\text{exp}Y) &= \text{Exp}(\text{ad}Y) = \sum_{k \in \mathbb{N}} \frac{(\text{ad}Y)^k}{k!},\ \forall Y \in \mathfrak{g}. \end{aligned}$$

First case We assume that G is a connected and simply connected nilpotent Lie group.

For μ-a.e. $x \in X$, we have:

$$\overline{\{gH_xg^{-1} : g \in G\}} = \overline{\{gH_{\tau x}g^{-1} : g \in G\}}.$$

Since the points of $\mathcal{F}(G)$ are separated by a countable family of continuous functions, there exists a closed subgroup H of G such that, for μ-a.e. $x \in X$,

$$\overline{\{gH_xg^{-1} : g \in G\}} = \overline{\{gHg^{-1} : g \in G\}}.$$

Now, from the proposition below, this equality implies that the two open dense subsets $\{gH_xg^{-1} : g \in G\}$ and $\{gHg^{-1} : g \in G\}$ of $\overline{\{gHg^{-1} : g \in G\}}$ are not disjoint. Therefore the two G-orbits coincide. Hence the result.

Second case We assume that G is a connected nilpotent Lie group.

Let $f : \tilde{G} \longrightarrow G$ be a group cover of G with $\tilde{G}$ connected and simply connected (see [Ho65], Ch. IV, Theorems 2.2 and 3.2).

If H is a closed subgroup of G then $\tilde{H} = f^{-1}(H)$ is a closed subgroup of $\tilde{G}$. Moreover the G-orbit of H is the image by f of the $\tilde{G}$-orbit of $\tilde{H}$. The theorem follows from the first case. $\square$

Proposition 5.1.2 *For any closed subgroup H of a connected simply connected nilpotent Lie group G, the G-orbit $\{gHg^{-1} : g \in G\}$ of H is open in its closure.*

Proof We know that the exponential map exp is an analytic diffeomorphism. We set $\Sigma = \{1, \ldots, \dim(\mathfrak{g})\}$. For any $p \in \Sigma$, we consider the exterior product $V_p = \bigwedge_p \mathfrak{g}$ and the corresponding projective space $\mathbf{P}(V_p)$. We denote by π_p the natural map from $V_p \setminus \{0\}$ onto $\mathbf{P}(V_p)$.

For each p-dimensional subspace $\mathfrak{v}$ of $\mathfrak{g}$ we associate the element $u_{\mathfrak{v}} = \pi_p(u_1 \wedge \cdots \wedge u_p)$ of $\mathbf{P}(V_p)$ where $(u_1, \ldots, u_p)$ is a linear basis of $\mathfrak{v}$. We denote by $\mathcal{D}_p$ the image in $\mathbf{P}(V_p)$ of the set of p-dimensional subspaces of $\mathfrak{g}$. We consider the disjoint union $\bigcup_{p\in\Sigma} \mathcal{D}_p$ equipped with the following topology. A sequence $(u_n)_{n\in\mathbb{N}}$ converges to x if the two following properties are satisfied:

(i) there exists $N \in \mathbb{N}$ and $r \in \Sigma$ such that, for $n \geq N$, $u_n \in \mathcal{D}_r$.

(ii) the sequence $(u_n)_{n\geq N}$ converges towards x on $\mathcal{D}_r$ for the usual induced topology of $\mathbf{P}(V_p)$.

One sees easily that a sequence $(\mathfrak{v}_n)$ of subspaces of $\mathfrak{g}$ converges in Chabauty's topology if and only if $(u_{\mathfrak{v}_n})_{n\in\mathbb{N}}$ converges in $\bigcup_{p\in\Sigma} \mathcal{D}_p$. Hence, the map $\mathfrak{v} \longrightarrow u_{\mathfrak{v}}$ is an homeomorphism from the set of non-trivial subspaces of $\mathfrak{g}$ onto $\bigcup_{p\in\Sigma} \mathcal{D}_p$.

Let H be a closed subgroup of G with Lie algebra $\mathfrak{h} = \exp^{-1}(H)$. The G-orbit $\{g\,H\,g^{-1} : g \in G\}$ of H is identified with the $\bigwedge_p \mathrm{Ad}G$-orbit of $u_{\mathfrak{h}}$. Now, for a connected simply connected nilpotent Lie group G, we know (see for example [BoSe64]) that this orbit is open in its closure. Hence the result. $\square$

5.2 A counter example

Let G be the semi-direct product of $\mathbb{R}$ and $\mathbb{C}^2$, with the composition law:

$$(t, z_1, z_2) * (t', z_1', z_2') = (t + t', z_1 + e^{2\pi i t} z_1', z_2 + e^{2\pi \theta i t} z_2'),$$

where θ is a fixed irrational.

The conjugate in G of $(0, z_1, z_2)$ by $a = (s, v_1, v_2)$ is:

$$(s, v_1, v_2)(0, z_1, z_2)(s, v_1, v_2)^{-1} = (0, e^{2\pi i s} z_1, e^{2\pi \theta i s} z_2). \tag{44}$$

Consider the dynamical system defined by an irrational rotation ($\tau : x \to x + \alpha \bmod 1$) on $X = \mathbb{R}/\mathbb{Z}$. Let $\Phi : X \to G$ be the cocycle defined by $\Phi(x) = (\varphi(x), 0, 0)$, where φ has its values in $\mathbb{Z}$.

Let $H_x := \{(0, v z_1, v e^{2\pi i \psi(x)} z_2), v \in \mathbb{R}\}$, where ψ is a function to be defined and z_1, z_2 are given natural real numbers. Consider the function $x \to H_x$ with values in the set of the closed subgroups of G. It satisfies the conjugacy relation:

$$H_{\tau x} = \Phi(x)\ H_x\ (\Phi(x))^{-1}, \tag{45}$$

if φ has integral values and satisfies

$$\theta\ \varphi(x) + \psi(x) = \psi(\tau x) \mod 1. \tag{46}$$

For every α whose partial quotients are not bounded, there are real numbers β and r for which the function

$$\varphi := 1_{[0,\beta]} - 1_{[0,\beta]}(. + r)$$

is not a coboundary and there are irrational values of s such that $e^{2\pi i s (1_{[0,\beta]} - 1_{[0,\beta]}(.+r))}$ is a multiplicative coboundary (cf. [Co07]).

If we take for θ one of these values of s and for ψ a function satisfying the multiplicative coboundary equation: $e^{2\pi i \theta \varphi} = e^{2\pi i (\psi \circ \tau - \psi)}$, we get (46).

Proposition 5.2.1 *For these choices of β, r, θ, ψ, there is no subgroup H such that the equation $H_x = a(x) H a(x)^{-1}$ has a measurable solution a.*

Proof Suppose that there are a fixed subgroup H and a measurable function $a : X \to G$ such that $H_x = a(x) H a(x)^{-1}$.

According to (44), this is equivalent to the existence of a function t defined on X such that the set

$$\{(0, v e^{2\pi i t(x)} z_1, v e^{2\pi i (\theta t(x) + \psi(x))} z_2), v \in \mathbb{R}\}$$

does not depend on x. This implies that t and $\psi + \theta t$ have a fixed value mod 1; therefore $\theta(\varphi(x) - t(x) + t(\tau x)) = \theta \varphi(x) + \psi(x) - \psi(\tau x) \mod 1 = 0$. As φ and $t - t \circ \tau$ have integral values and θ is irrational, it follows that $\varphi = t \circ \tau - t$, contrary to the fact that φ is not a coboundary. □

Remark 5.2.2 By the same arguments it can be shown that the cocycle $1_{[0,\beta]} - 1_{[0,\beta]}(.+r))$ is nonregular in the sense of Definition 2.2.1.

6 Comments

6.1 Remarks on transience/recurrence

The cocycle $(\varphi_n)_{n\in\mathbb{Z}}$ gives the position at time n of the "vertical" coordinate for the iterates τ_φ^n. If it is *recurrent* (i.e. if the stationary random walk (φ_n) returns infinitely often in any neighborhood of the identity element), the transformation τ_φ is conservative.

The ergodicity of the basis implies that the cocycle is either recurrent or transient. When φ has its values in $\mathbb{R}$ and is integrable, $(\varphi_n)_{n\in\mathbb{Z}}$ is recurrent if and only if $\mu(\varphi) = 0$.

For every amenable group G and every ergodic system (X, μ, τ), there is a measurable ergodic cocycle (φ, τ) over the system, taking its values in G, such that $(X \times G, \mu \otimes m_G, \tau_\varphi)$ is ergodic (cf. [He79], [GoSi85]). However, a problem is to construct explicitly recurrent cocycles generated by regular functions over particular dynamical systems and to find whether they are or not ergodic.

In the recurrent case, the transformation τ_φ is *conservative*: there is no wandering set E with a positive measure (wandering means that the images $(\tau_\varphi^{-k} E, k \in \mathbb{Z})$ are pairwise disjoint). This implies that every sub-invariant set is invariant $\mu \otimes m_G$-a.e..

Remark that the ergodic decomposition of a recurrent system gives recurrent systems. In particular the recurrence of the cocycle (φ_n) relatively to μ implies (with the notations of 2.1.3) that, for μ-a.e. x that the cocycle (φ_n) is recurrent relatively to the measure μ_x, which is infinite if the cocycle is not regular (cf. Theorem 2.2.2).

Assume now that the cocycle is transient. Let E be a wandering set E and $h > 0$ be a function on G such that $\int h \, dg = 1$. The series $\sum_{k\in\mathbb{Z}} h(\varphi_k(x)g)$ converges for $\mu \otimes m_G$ a.e. (x, g), according to:

$$\int_E \sum_{k\in\mathbb{Z}} h(\varphi_k(x)g) \, d\mu(x) \, dg = \int_E h(g) \, (\sum_{k\in\mathbb{Z}} 1_E(x, (\varphi_k(x))^{-1}g)) \, dg \, d\mu(x)$$

$$= \int_E h(g) \, (\sum_{k\in\mathbb{Z}} 1_E(\tau^{-k}x, (\varphi_k(\tau^{-k}x))^{-1}g)) \, d\mu(x) \, dg = \int h(g) \, dg = 1.$$

The function $\tilde{h}(x, g) := \sum_{k\in\mathbb{Z}} h(\varphi_k(x)g)$ is therefore τ_φ-invariant and finite for $\mu \otimes m_G$-a.e. (x, g).

The subgroups H_x defined in Theorem 2.1.3 are reduced to $\{e\}$ and the ergodic measures are given, up to a multiplicative factor, by $\lambda_{(x,g)}(f) = \sum_{k\in\mathbb{Z}} f(\tau^k x, \varphi_k(x)g)$.

The function φ is a coboundary with respect to the σ-finite measure $\tilde{\mu}_x(dy) := \sum_{k\in\mathbb{Z}} \delta_{\tau^k x}(dy)$ (we get $u_x(y) \, \varphi(y) \, (u_x(\tau y))^{-1} = e$ by setting $u_x(y) := \varphi_k(x)$ at the point $y = \tau^k x$). The ergodic decomposition of $\mu(dx) \times dg$ can be written:

$$\int_X \int_G f(x,g) d\mu(x)\ dg = \int_X \int_G \left[(\tilde{h}(x,g))^{-1} \sum_{k \in \mathbb{Z}} f(\tau^k x, \varphi_k(x) g) \right] \ h(g)\ d\mu(x)\ dg.$$

This shows that, in the transient case, there is no interesting information in the ergodic decomposition. Therefore it is suitable to have examples of recurrent cocycles (φ, τ). A family of such cocyles is provided when the basic system is a rotation on the circle and φ is a BV function with values in $\mathbb{R}^d$. There are also examples over rotations for cocycles taking their values in nilpotent groups (see [Gr05], [Co07]). For rotations, using BV-functions, one can construct conformal probability measures μ_χ for which the rotation is conservative. More precisely if φ is a BV-function on the circle with zero integral, χ is an exponential on $\mathbb{R}$ and τ an ergodic rotation, there is a unique probability measure μ_χ on the circle such that $d(\tau\mu_\chi)/d\mu_\chi = \chi \circ \varphi$ and the corresponding measure λ_χ on $X \times G$ is conservative due to Koksma's inequality (see for example [CoGu00]).

6.2 Extension to a group action

For simplicity, we have restricted the paper to the framework of a single transformation, but the domain of validity can be extended by taking more generally the action of a countable group Γ. This gives access to more examples of transient cocycle with a non trivial ergodic decomposition. Most of the results presented here when $\Gamma = \mathbb{Z}$ are still valid for the action of a countable group Γ.

Indeed we can use the result of Theorem 3.1.1, since one can easily extend it from the case of a single invertible transformation to an ergodic group action. Another important point is the ergodicity of the measures given by a regular conditional probability with respect to the σ-algebra of invariant sets. This point also remains valid (see Remark 7.3.3 at the end of 7.3).

7 Appendix

In this appendix, we recall a selection lemma and some results on the conditional expectation and the ergodic decomposition that were used in the previous sections.

7.1 A selection lemma

Let G be a lcsc group. Recall that the set $\mathcal{F}(G)$ of closed subsets of G is equipped by *Chabauty's topology*, for which the open sets are defined by

$$U(\mathcal{O}, C) = \{S \in \mathcal{F}(G) : \forall U \ \in \mathcal{O},\ S \cap U \neq \emptyset \ \text{ and } \ S \cap C = \emptyset\},$$

where $\mathcal{O}$ is a finite family of open sets of G and C is a compact subset of G.

The Borel structure associated to this topology is generated by the sets $\{S \in \mathcal{F}(G) : S \subseteq F\}$ where $F \in \mathcal{F}(G)$ (cf. 2.1). For the sake of completeness, we give a proof of a selection lemma (cf. the theorem of Kuratowski-Ryll-Nardzewski) that was used in section 3:

Lemma 7.1.1 *If $t \to F_t$ is a Borel map from a Borel space $(T, \mathcal{T})$ to $\mathcal{F}(G)$, then there exists a Borel map f from T to G such that $f(t) \in F_t$ for each $t \in T$.*

Proof 1) Let K be a compact set in G. Assume that $F_t \subset K$, for $t \in R \subset T$, where R is a Borel set in T.

For every $n \geq 1$, there exists a finite family $(K_{n,i}, i \in I_n)$ of compact sets such that $diameter(K_{n,i}) < \frac{1}{n}$ and $K \subset \cup_{j \in I_{n+1}} K_{n+1,j}$.

For a compact set C in G, the set $\{t : F_t \cap C \neq \emptyset\}$ is Borel (its complement is the union of the sets $\{t : F_t \subset G \setminus U_n\}$, where U_n is basis of open neighborhoods of C). Therefore, for every n and j, the set $\{t : F_t \cap K_{n,j} \neq \emptyset\}$ is Borel.

We define $i_n(t)$ by $i_n(t) = \inf\{j \in I_n : F_t \cap K_{n,j} \neq \emptyset\}$. The map $t \to K_{n,i_n(t)}$ is Borel.

Now we define the point $f(t)$ for $t \in R$ by

$$f(t) := \bigcap_{n \geq 1} K_{n,i_n(t)}.$$

From the condition on the diameters and the compactness of the sets, it follows that $f(t)$ is well defined for every $t \in T$.

We have to show that f is Borel, that is that the set $\{t \in T : f(t) \in C\}$ is a Borel set for any closed subset C in G. Let (O_k) be a decreasing sequence of open sets such that $O_{k+1} \subset \overline{O_{k+1}} \subset O_k$, for every k and $C = \bigcap_k O_k$. We have:

$$(f(t) \in C) \Leftrightarrow (\bigcap_{n \geq 1} K_{n,i_n(t)} \subset O_k, \forall k \geq 1) \Leftrightarrow (\bigcap_{n \geq 1} K_{n,i_n(t)} \subset \overline{O_k}, \forall k \geq 1).$$

As the set $\{t \in T : K_{n,i_n(t)} \subset \overline{O_k}\}$ for each k is Borel, the assertion follows.

2) Now we construct f on the whole space.

For any compact set K in G, the map $t \to F_t \cap K$ is Borel, since the map $(F, K) \to F \cap K$ from $\mathcal{F}(G)$ into itself is continuous for a fixed compact set K. Let K_j be an increasing sequence of compact sets in G such that $G = \bigcup_j K_j$.

We define $f(t)$ by applying the previous construction to the map $t \to F_t \cap K_1$, on the set $\{t : F_t \cap K_1 \neq \emptyset\}$, then to the map $t \to F_t \cap K_2$, on the set $\{t : F_t \cap K_2 \neq \emptyset\} \cap \{t : F_t \cap K_1 = \emptyset\}$, and so on ... □

7.2 A lemma on conditional expectation

Lemma 7.2.1 *Let $\mathbb{P}$ be a probability measure on a measurable space $(E,\mathcal{F})$ and h a measurable positive function such that $\int h\,d\mathbb{P}=1$. Then, for every sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$ and every measurable nonnegative (or $h\,\mathbb{P}$-integrable) function f, we have:*

$$\mathbb{E}_{h\mathbb{P}}[f|\mathcal{B}] = \mathbb{E}_{\mathbb{P}}[fh|\mathcal{B}]/\mathbb{E}_{\mathbb{P}}[h|\mathcal{B}],\ \mathbb{P}-\text{a.e.}. \tag{47}$$

If θ is a bijective bi-measurable map from E onto itself such that $\theta\mathbb{P}\sim\mathbb{P}$, then, for any $\mathbb{P}$-integrable function f, we have:

$$\mathbb{E}_{\mathbb{P}}[f|\mathcal{B}] = \mathbb{E}_{\mathbb{P}}\Big[\Big(\frac{d\theta\mathbb{P}}{d\mathbb{P}}\Big)^{-1}\circ\theta|\mathcal{B}\Big]\ \mathbb{E}_{\mathbb{P}}\Big[f\circ\theta^{-1}\ \frac{d\theta\mathbb{P}}{d\mathbb{P}}|\theta\mathfrak{B}\Big]\circ\theta = \frac{\mathbb{E}_{\mathbb{P}}\Big[f\circ\theta^{-1}\ \frac{d\theta\mathbb{P}}{d\mathbb{P}}|\theta\mathfrak{B}\Big]\circ\theta}{\mathbb{E}_{\mathbb{P}}\Big[\frac{d\theta\mathbb{P}}{d\mathbb{P}}|\theta\mathfrak{B}\Big]\circ\theta}. \tag{48}$$

If $\mathbb{P}=h\lambda$, where λ is a σ-finite θ-invariant measure and $\mathcal{B}=\mathfrak{J}$ the σ-algebra of θ-invariant sets in $\mathfrak{E}$, we have:

$$\mathbb{E}_{h\lambda}(f\circ\theta|\mathfrak{J}) = \mathbb{E}_{h\lambda}(f\frac{h\circ\theta^{-1}}{h}|\mathfrak{J}). \tag{49}$$

<u>Proof</u> We prove only the second assertion. For every bounded $\mathcal{B}$-measurable ψ we have

$$\begin{aligned}
\int \mathbb{E}_{\mathbb{P}}[f\circ\theta|\mathcal{B}]\ \psi\ d\mathbb{P} &= \int_E f\circ\theta\ \psi\ d\mathbb{P} = \int_E f\ \psi\circ\theta^{-1}\ \frac{d\theta\mathbb{P}}{d\mathbb{P}}\ d\mathbb{P}\\
&= \int_E \mathbb{E}_{\mathbb{P}}\Big[f\ \frac{d\theta\mathbb{P}}{d\mathbb{P}}|\theta\mathcal{B}\Big]\ \ \psi\circ\theta^{-1}\ d\mathbb{P} = \int_E \frac{d\theta^{-1}\mathbb{P}}{d\mathbb{P}}\ \mathbb{E}_{\mathbb{P}}\Big[f\ \frac{d\theta\mathbb{P}}{d\mathbb{P}}|\theta\mathfrak{B}\Big]\circ\theta\ \ \psi\ d\mathbb{P}\\
&= \int_E \mathbb{E}_{\mathbb{P}}\Big[\frac{d\theta^{-1}\mathbb{P}}{d\mathbb{P}}|\mathcal{B}\Big]\ \mathbb{E}_{\mathbb{P}}\Big[f\ \frac{d\theta\mathbb{P}}{d\mathbb{P}}|\theta\mathfrak{B}\Big]\circ\theta\ \ \psi\ d\mathbb{P},
\end{aligned}$$

which implies (48):

$$\mathbb{E}_{\mathbb{P}}(f\circ\theta|\mathcal{B}) = \mathbb{E}_{\mathbb{P}}\Big[\frac{d\theta^{-1}\mathbb{P}}{d\mathbb{P}}|\mathcal{B}\Big]\ \mathbb{E}_{\mathbb{P}}\Big[\frac{d\theta\mathbb{P}}{d\mathbb{P}}f|\theta\mathfrak{B}\Big]\circ\theta.$$

□

7.3 Regular conditional probability

Definition 7.3.1 Let $(E,\mathcal{F},\mathbb{P})$ be a probability space and $\mathfrak{B}$ a sub σ-algebra of $\mathcal{F}$. A *regular conditional probability* relatively to $\mathcal{B}$ and $\mathbb{P}$ is a map P from $E\times\mathcal{F}$ to $[0,1]$ such that :

$i)$ For every $x\in E$, $P(x,\cdot)$ is a probability measure on $\mathcal{F}$.

$ii)$ For every $A\in\mathcal{F}$, the map $x\in E\longrightarrow P(x,A)$ is a version of the conditional expectation of 1_A with respect to the σ-algebra $\mathcal{B}$. This map is thus $\mathcal{B}$-measurable and satisfies for every $\mathfrak{B}$-measurable function φ:

$$\int_E 1_A(x)\,\varphi(x)\,\mathbb{P}(dx) = \int_E P(x,A)\,\varphi(x)\,\mathbb{P}(dx).$$

For every $\mathcal{F}$-measurable nonnegative or bounded function f, Pf defined by $Pf(x) := \int_E f(y)\, P(x, dy)$ is then a version of the conditional expectation of f with respect to $\mathcal{B}$.

For the existence of a regular conditional probability, we can refer to the general setting used in Neveu's book (Corollaire, Proposition V-4-4, [Ne64]):

In the following, we will assume that there exists an approximating compact class in $(E, \mathcal{F}, \mathbb{P})$ (see [Ne64] for this notion) and that $\mathcal{F}$ is generated by a countable family.

Theorem 7.3.2 *[Ne64] For every σ-algebra $\mathcal{B}$ in $\mathcal{F}$, there exists a regular conditional probability with respect to $\mathcal{B}$.*

This result applied to the product space $(X \times G, \mathfrak{X} \times \mathcal{B}_G)$, the probability $h\lambda$ on $X \times G$, where $h > 0$ on $X \times G$ is such that $\int h(x, g)\, \mu_X(dx)\, \chi(g)\, m_G(dg) = 1$, and the sub-$\sigma$-algebra $\mathfrak{J}$ of τ_φ-invariant sets (see Notations 2.1.1 and 2.1.2) gives the regular conditional probability P^h used in section 2.

Now we have to show that the probability measures $P^h((x, g), \cdot)$ are τ_φ-ergodic. For the action of a single transformation, this can be done by applying the ergodic theorem (cf. [Aa97]). For the sake of completeness we give a proof in the last subsection below.

Remark 7.3.3 When the action of a single transformation τ on X is replaced by the Borel action of a countable group, the proof of the ergodicity of $P^h((x, g), \cdot)$ is more difficult. A reference is [GrSc00].

7.4 Ergodic theorem and ergodic decomposition

Notations 7.4.1 *Let θ be a bijective bi-measurable transformation on a measurable space $(E, \mathcal{F})$, μ a positive σ-finite θ-quasi-invariant measure, $\mathfrak{J} := \{B \in \mathcal{F} : \theta^{-1}B = B\}$.*

Let h be a measurable function on E such that $h(x) > 0$ and $\mu(h) = 1$. Let P^h be a regular conditional probability with respect to the probability measure $h\,\mu$ and the σ-algebra $\mathfrak{J}$ of θ-invariant measurable subsets.

Let T_h be the contraction of $\mathbb{L}^1(E, \mathcal{F}, h\,\mu)$, in duality with the operator of composition by θ acting on $\mathbb{L}^\infty(E, \mathcal{F}, \mu)$, defined by:

$$T_h f(x) = f \circ \theta^{-1}(x)\, \frac{d(\theta(h\mu))}{d(h\mu)}(x) = T(hf)(x).$$

Replacing θ by θ^{-1} we get the inverse operator T_h^{-1}.

Proposition 7.4.2 *For every $f \in \mathbb{L}^1(E, \mathcal{F}, h\mu)$ and μ-a.e. $x \in E$,*

$$\sum_{k=-n}^{n} T_h^k f(x) / \sum_{k=-n}^{n} T_h^k 1(x) \underset{n \to +\infty}{\longrightarrow} \mathbb{E}_{h\,\mu}[f|\mathfrak{J}](x).$$

Proof Applying Hurewicz's ergodic theorem to the contraction T_h, we get that the sequence

$$(\sum_{k=0}^{n} T_h^k f(x) / \sum_{k=0}^{n} T_h^k 1(x))_{n\geq 1}$$

converges μ-a.e. and the same result holds for the contraction T_h^{-1}.

On the conservative part C the limit in both directions is equal to $\mathbb{E}_{h\,\mu}[f|\mathfrak{I}](x)$, so that on C,

$$\lim_{n\to+\infty} (\sum_{k=-n}^{n} T_h^k f / \sum_{k=-n}^{n} T_h^k 1) \;=\; \mathbb{E}_{h\,\mu}[f|\mathfrak{I}],\; \mu\text{—a.e.}.$$

On the dissipative part D, the limit is the quotient of the series. For j, k in $\mathbb{Z}$, we have

$$T_h^j f \circ \theta^k = T_h^{j-k} f \; \frac{d(h\,\mu)}{d(\theta^{-k}(h\,\mu))} = \frac{T_h^{j-k} f}{T_h^{-k} 1}.$$

This implies that on D the quotient of the series is a θ-invariant function and that, for every measurable θ-invariant function φ which is null on C, we have :

$$\begin{aligned}\int_E \frac{\sum_{k\in\mathbb{Z}} T_h^k f}{\sum_{j\in\mathbb{Z}} T_h^j 1}\, \varphi \; d(h\mu) \;&=\; \sum_{k\in\mathbb{Z}} \int_E \frac{T_h^k f}{\sum_{j\in\mathbb{Z}} T_h^j 1}\, \varphi \; d(h\mu) = \sum_{k\in\mathbb{Z}} \int_E \frac{f\;\varphi}{\sum_{j\in\mathbb{Z}} T_h^j 1 \circ \theta^k} \; d(h\mu) \\ &=\; \sum_{k\in\mathbb{Z}} \int_E \frac{f\;\varphi}{\sum_{j\in\mathbb{Z}} T_h^j 1}\; T_h^{-k} 1 \; d(h\mu) = \int_E f \;\varphi \; d(h\,\mu).\end{aligned}$$

On D the quotient of the series is therefore equal to $\mathbb{E}_{h\,\mu}[f|\mathfrak{I}]$. □

Lemma 7.4.3 *For μ-a.e. $x \in E$, the measure $\theta P^h(x,\cdot)$ is absolutely continuous with respect to $P^h(x,\cdot)$ and*

$$\frac{d(\theta P^h(x,\cdot))}{dP^h(x,\cdot)} = \frac{d(\theta(h\,\mu))}{d(h\,\mu)} = \frac{h\circ\theta^{-1}}{h}\;\frac{d(\theta\mu)}{d\mu}. \tag{50}$$

Proof For a positive $\mathcal{F}$-measurable f and a $\mathfrak{I}$-measurable positive function φ, we have :

$$\int_E f\circ\theta \;\varphi \, d(h\,\mu) = \int_E f\circ\theta \;\varphi\circ\theta \; d(h\,\mu) = \int_E f\,\varphi\, \frac{d(\theta(h\,\mu))}{d(h\,\mu)}\, d(h\,\mu).$$

This shows, μ-a.e.,

$$\theta(P^h)(f) = P^h(f\circ\theta) = \mathbb{E}_{h\,\mu}[f\circ\theta|\mathfrak{I}] = \mathbb{E}_{h\,\mu}[f\,\frac{d(\theta(h\,\mu))}{d(h\,\mu)}|\mathfrak{I}] = P^h(f\,\frac{d(\theta(h\,\mu))}{d(h\,\mu)}).$$

□

For the elements $x \in E$ for which (50) holds, T_h is a positive contraction of $\mathbb{L}^1(E, \mathcal{F}, P^h(x,\cdot))$. We have then:

Corollary 7.4.4 *For the elements $x \in E$ for which (50) holds, T_h is a positive contraction of $\mathbb{L}^1(E, \mathcal{F}, P^h(x, \cdot))$. For every $f \in \mathbb{L}^1(E, \mathcal{F}, P^h(x, \cdot))$ and for $P^h(x, \cdot)$-a.e. $y \in E$,*

$$\sum_{k=-n}^{n} T_h^k f(y) / \sum_{k=-n}^{n} T_h^k 1(y) \underset{n \to +\infty}{\longrightarrow} \mathbb{E}_{P^h(x,\cdot)}[f|\mathfrak{J}](y).$$

Theorem 7.4.5 *A decomposition of the measure μ in ergodic components is given by*

$$\mu(dy) = \int_E (h(.))^{-1} \; P^h(x, \cdot)\, h(x)\, \mu(dx). \tag{51}$$

<u>Proof</u> The equality is clear. It remains to prove the ergodicity of the probabilities $P^h(x, \cdot)$ for μ-a.e. $x \in E$.

From (51) and Proposition 7.4.2, we have, for every $f \in \mathbb{L}^1(E, \mathcal{F}, h\mu)$ and for μ-a.e. $x \in E$,

$$\sum_{k=-n}^{n} T_h^k f(y) / \sum_{k=-n}^{n} T_h^k 1(y) \underset{n \to +\infty}{\longrightarrow} \mathbb{E}_{h\mu}[f|\mathfrak{J}](y) = P^h f(y), \text{ for } P(x, .)-\text{a.e. } y \in E.$$

The functions $g = P^h f$ and $g^2 = (P^h f)^2$ are $\mathfrak{J}$-measurable and therefore P^h-invariant μ-a.e.: $P^h g(x) = g(x)$ and $P^h g^2(x) = g^2(x)$, for μ-a.e. $x \in E$. By the Cauchy-Schwarz inequality, this implies $g(y) = g(x)$, for $P(x, .)$-a.e. $y \in E$.

Let $\mathcal{F}_0$ be a countable Boole algebra which generates $\mathcal{F}$. For $x \in E$, let Q_x^h be a regular conditional probability with respect to the probability $P^h(x, .)$ and the σ-algebra $\mathfrak{J}$. From the previous property and Corollary 7.4.4, we obtain, for μ-a.e. $x \in E$ and $P^h(x, .)$-a.e. $y \in E$,

$$\forall A \in \mathcal{F}_0, \; Q_x^h(y, A) = P^h(x, A)$$

and consequently, for μ-a.e. $x \in E$ and $P^h(x, .)$-a.e. $y \in E$, we have the same property for every $A \in \mathcal{F}$.

For every $I \in \mathfrak{J}$, we know that

$$Q_x^h(y, I) = \mathbb{E}_{P^h(x,.)}[1_I|\mathfrak{J}](y) = 1_I(y), \text{ for } P(x, .)-\text{a.e. } y \in E$$

and therefore as above

$$Q_x^h(y, I) = 1_I(x), \text{ for } P(x, .)-\text{a.e. } y \in E.$$

It follows that, for μ-a.e. $x \in E$,

$$\forall I \in \mathfrak{J}, P^h(x, I) = Q_x^h(y, I) = 1_I(y), \text{ for } P(x, .)-\text{a.e. } y \in E.$$

This implies the ergodicity of the measures $P^h(x, .)$, for μ-a.e. x. □

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