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Abdelmalek Aboussoror, Samir Adly, Vincent Jalby. Weak Nonlinear Bilevel Problems: Existence of Solutions via Reverse Convex and Convex Maximization Problems. *Journal of Industrial and Management Optimization*, AIMS, 2011, 7 (3), pp.559-571. 10.3934/jimo.2011.7.559 . hal-00682742

HAL Id: hal-00682742

<https://hal.archives-ouvertes.fr/hal-00682742>

Submitted on 21 Oct 2018

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# WEAK NONLINEAR BILEVEL PROBLEMS : EXISTENCE OF SOLUTIONS VIA REVERSE CONVEX AND CONVEX MAXIMIZATION PROBLEMS

ABDELMALEK ABOUSSOROR

Université Cadi Ayyad  
Faculté Polydisciplinaire de Safi  
B.P. 4162, Sidi Bouzid, Safi, Morocco

SAMIR ADLY AND VINCENT JALBY

Laboratoire XLIM UMR-CNRS 6172  
Université de Limoges  
Département de Mathématiques  
123 Avenue Albert Thomas  
87060 Limoges Cedex, France

ABSTRACT. In this paper, for a class of weak bilevel programming problems we provide sufficient conditions guaranteeing the existence of global solutions. These conditions are based on the use of reverse convex and convex maximization problems.

**1. Introduction.** We consider the following weak nonlinear bilevel optimization problem (weak in the sense of [8])

$$(S) \quad \min_{x \in X} \sup_{y \in \mathcal{M}(x)} F(x, y)$$

called weak Stackelberg problem, where  $\mathcal{M}(x)$  denotes the set of solutions to the lower level problem

$$\mathcal{P}(x) \quad \min_{y \in Y} f(x, y)$$

with  $F, f : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}$ ,  $X$  and  $Y$  be two subsets of  $\mathbb{R}^p$  and  $\mathbb{R}^q$  respectively.

The formulation of the problem that we consider, called the pessimistic formulation, corresponds to a static uncooperative two player game in which one of the players has the leadership and full information about the second player. Player 1 (the leader) with the objective function  $F$  firstly announces a strategy  $x$  in  $X$ , and after, the player 2 (the follower) with the objective function  $f$  reacts optimally by selecting a strategy  $y(x) \in Y$ . Assume that the solution set  $\mathcal{M}(x)$  is not always a singleton and the leader can not influence the choice of the follower. Then, the leader provides himself against the possible worst follower's choice by minimizing

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2000 *Mathematics Subject Classification.* 91A65, 90C26, 52A41.

*Key words and phrases.* Two-level optimization, convex analysis, reverse convex programs, d.c. programs.

This article was finished during the visit of A. Aboussoror to the university of Limoges (July 2010).

the marginal function  $\sup_{y \in \mathcal{M}(x)} F(x, y)$ . The presence in the first level of the implicit constraint set  $\mathcal{M}(x)$ , which is an output of the problem  $\mathcal{P}(x)$ , makes the problem (S) difficult to solve, which in general is not differentiable and not convex, and hence belongs to the class of nondifferentiable global optimization problems. The difficulty encountered in the investigation of weak nonlinear bilevel problems remains in finding suitable conditions which are not strong and depend only on the problem's data. In contrast, the strong Stackelberg problem

$$\min_{x \in X} \inf_{y \in \mathcal{M}(x)} F(x, y)$$

which corresponds to the optimistic formulation, presents less difficulties and hence is more considered in the literature than the weak Stackelberg problem. It corresponds for example to the case where the leader can influence the follower in his choice of the strategies in  $\mathcal{M}(x)$ . Another interesting formulation of the leader's problem corresponds to the case where the leader evaluates the performance of the follower by his optimal value (see for example [6],[21] and [26]). For different applications of bilevel optimization problems, we refer to [9] and [20].

It is well-known that the existence of solutions for weak bilevel programming problems is a difficult task. So, our investigation in this paper, which is a continuation of previous works [1]–[4] dealing with the same subject, is to give sufficient conditions ensuring the existence of solutions to weak nonlinear bilevel optimization problems. For this purpose, we will establish some relationships between problem (S) and some other well-known global optimization problems. More precisely, under certain assumptions, firstly, we show that the existence of solutions to appropriate parameterized reverse convex and convex maximization problems imply the existence of solutions to (S). Similar results using d.c. problems are given in [4]. We note that these relationships between weak bilevel programming problems and such well-known global optimization problems that we provide in this paper are new in the literature. Relating to the same subject, we note that an interesting class of weak nonlinear bilevel optimization problems which admit solutions is given in [17]. On the other hand, reverse convex and convex maximization problems have received a great interest by several authors. Nowadays there exists a great number of interesting theoretical and numerical results for such problems. For papers dealing with reverse convex and convex maximization problems, we refer for example respectively to [5], [16], [23]–[25], [27]–[29] and [10]–[14], [22]. We also refer to the interesting books of global optimization [15] and [30] and references therein.

The paper is organized as follows. In Section 2, we recall some results related to convex analysis and reverse convex problems. In Section 3, under appropriate assumptions, we show the existence of solutions for problem (S) using reverse convex and convex maximization problems.

Throughout the paper, we assume that  $X$  and  $Y$  are compact and compact convex subsets of  $\mathbb{R}^p$  and  $\mathbb{R}^q$ , respectively. For a given optimization problem, we will use the term solution to mean global solution.

**2. Background of convex analysis and reverse convex problems.** In this section, we recall some definitions and fundamental results related to convex analysis and reverse convex problems that we will use in the sequel.

Let  $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function. The set  $\text{dom}(g)$  defined by

$$\text{dom}(g) = \left\{ y \in \mathbb{R}^n / g(y) < +\infty \right\}$$

is called the effective domain of  $g$ . The function  $g$  is said to be proper if  $\text{dom}(g) \neq \emptyset$ . A vector  $y^* \in \mathbb{R}^n$  is said to be a subgradient of  $g$  at  $\bar{y} \in \text{dom}(g)$ , if

$$g(y) \geq g(\bar{y}) + \langle y^*, y - \bar{y} \rangle, \forall y \in \mathbb{R}^n.$$

The set of all subgradients of  $g$  at  $\bar{y}$  is called the subdifferential of  $g$  at  $\bar{y}$ , and is denoted by  $\partial g(\bar{y})$ .

Let  $A$  be a nonempty closed and convex subset of  $\mathbb{R}^n$ .

1. The indicator function of  $A$  denoted by  $\psi_A$ , is the function defined on  $\mathbb{R}^n$  by

$$\psi_A(y) := \begin{cases} 0 & \text{if } y \in A, \\ +\infty & \text{if } y \notin A. \end{cases}$$

2. Let  $a \in A$ . The normal cone to  $A$  at  $a$  is the set denoted by  $\mathcal{N}_A(a)$  and defined by

$$\mathcal{N}_A(a) = \begin{cases} \left\{ x^* \in \mathbb{R}^n / \langle x^*, x - a \rangle \leq 0, \forall x \in A, \right\} & \text{if } a \in A, \\ \emptyset & \text{if } a \notin A. \end{cases}$$

3. Let  $a \in \mathbb{R}^n$ , and set (see [27])

$$\mathcal{N}_A^*(a) = \left\{ x^* \in \mathbb{R}^n / \langle x^*, x - a \rangle \leq 0, \forall x \in A \right\}.$$

If  $a \in A$ , then  $\mathcal{N}_A^*(a) = \mathcal{N}_A(a)$ .

For illustration, we consider the following example.

**Example 1.** Let  $A = [1, 2] \times [1, 2] \subset \mathbb{R}^2$ , and  $a = (3, 3)^T$ . Then, since  $a \notin A$ , we have  $\mathcal{N}_A(a) = \emptyset$ , and it is easy to verify that

$$\begin{aligned} \mathcal{N}_A^*(a) &= \left\{ (u^*, v^*)^T \in \mathbb{R}^2 / \left\langle \begin{pmatrix} u^* \\ v^* \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\rangle \leq 0, \forall x, y \in [1, 2] \right\} \\ &= \left\{ (u^*, v^*)^T \in \mathbb{R}^2 / 2u^* + v^* \geq 0 \right\} \cap \left\{ (u^*, v^*)^T \in \mathbb{R}^2 / u^* + 2v^* \geq 0 \right\} \end{aligned}$$

which is nonempty contrarily to  $\mathcal{N}_A(a)$ .

We recall the following results on subdifferential calculus and optimality conditions (see for example [7] and [19])

**Theorem 2.1.** *Let  $h_1, h_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be two proper convex functions. Assume that there exists a point in  $\text{dom}(h_1)$  at which  $h_2$  is continuous. Then, for any  $y \in \mathbb{R}^n$ , we have*

$$\partial(h_1 + h_2)(y) = \partial h_1(y) + \partial h_2(y).$$

**Theorem 2.2.** *Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function and  $Z$  be a nonempty closed convex subset of  $\mathbb{R}^n$ . Then, we have*

- $\bar{y}$  minimizes  $g$  over  $\mathbb{R}^n$  if and only if  $0 \in \partial g(\bar{y})$ ,
- $\bar{z}$  minimizes  $g$  over  $Z$  if and only if  $0 \in \partial g(\bar{z}) + \mathcal{N}_Z(\bar{z})$ .

Now, let us recall some properties and results concerning reverse convex problems.

Let  $\hat{f}, \hat{g} : \mathbb{R}^n \rightarrow \mathbb{R}$  be two convex functions and  $\hat{D}$  be a nonempty convex subset of  $\mathbb{R}^n$ . Let us consider the following optimization problem

$$(\mathcal{P}) \quad \min_{\substack{x \in \hat{D} \\ \hat{g}(x) \geq 0}} \hat{f}(x).$$

We will make the following assumption.

**(2.1)** There exists  $\hat{x} \in \hat{D}$ , such that

$$\hat{f}(\hat{x}) < \inf_{\substack{x \in \hat{D} \\ \hat{g}(x) \geq 0}} \hat{f}(x).$$

When this property is satisfied, we say that the reverse convex constraint  $\hat{g}(x) \geq 0$  is essential, and  $(\mathcal{P})$  is called a reverse convex problem. Otherwise, the problem  $(\mathcal{P})$  is equivalent to an ordinary convex programming problem.

**Remark 1.** [28] Assumption (2.1) implies that if  $\bar{x}$  is a solution of  $(\mathcal{P})$ , then  $\hat{g}(\bar{x}) = 0$ . Indeed, let  $\bar{x}$  be a solution of  $(\mathcal{P})$ , and suppose that  $\hat{g}(\bar{x}) > 0$ . There exists  $\hat{x} \in \hat{D}$ , such that

$$\hat{f}(\hat{x}) < \inf_{\substack{x \in \hat{D} \\ \hat{g}(x) \geq 0}} \hat{f}(x).$$

Hence,  $\hat{g}(\hat{x}) < 0$ . It follows from the continuity of  $\hat{g}$  that there exists  $t \in ]0, 1[$  and  $x^* = t\bar{x} + (1-t)\hat{x}$ , such that  $\hat{g}(x^*) = 0$ . Then, we have

$$\hat{f}(x^*) \leq t\hat{f}(\bar{x}) + (1-t)\hat{f}(\hat{x}) < t\hat{f}(\bar{x}) + (1-t)\hat{f}(\bar{x}) = \hat{f}(\bar{x})$$

with  $x^* \in \hat{D}$ , which contradicts the optimality of  $\bar{x}$ . Hence the problem  $(\mathcal{P})$  is equivalent to the following

$$\min_{\substack{x \in \hat{D} \\ \hat{g}(x) = 0}} \hat{f}(x).$$

Then, any candidate  $x \in \hat{D}$ , for the optimality to problem  $(S)$  must satisfy the condition  $\hat{g}(x) = 0$ .

For  $\bar{x} \in \mathbb{R}^n$ , consider the level set of  $\hat{f}$  relative to  $\hat{D}$ , and passing by  $\bar{x}$

$$S(\hat{f}, \bar{x}) = \left\{ x \in \hat{D} / \hat{f}(x) \leq \hat{f}(\bar{x}) \right\}.$$

**Theorem 2.3.** [27] *Let  $z$  be a feasible point of  $(\mathcal{P})$  verifying  $\hat{g}(z) = 0$ , and  $0 \notin \partial\hat{g}(z)$ . Assume that assumption (2.1) and the following condition are satisfied*

**(2.2)**  $\forall y \in \hat{D}$  verifying  $\hat{g}(y) = 0$ ,  $\exists y^* \in \partial\hat{g}(y)$ ,  $\exists u \in \hat{D}$ , such that  $\langle y^*, u - y \rangle > 0$ .

*Then,  $z$  is a solution of  $(\mathcal{P})$  if and only if  $z$  satisfies the following condition*

$$\forall y \in \mathbb{R}^n \text{ such that } \hat{g}(y) = 0 \text{ we have } \partial\hat{g}(y) \cap \mathcal{N}_{S(\hat{f}, z)}^*(y) \neq \emptyset. \quad (\mathcal{C}_1)$$

**Remark 2.** We note that in [27], the compactness of  $\hat{D}$  is mentioned among the hypotheses, but not used in the corresponding proof. So, we have not included this property in Theorem 2.3.

**3. Existence of solutions to problem  $(S)$ .** In this section, we will give some sufficient conditions ensuring the existence of solutions to problem  $(S)$  via other well-known global optimization problems in the literature. More precisely, under certain assumptions, we show that the existence of solutions to appropriate parameterized reverse convex and convex maximization problems implies the existence of solutions to problem  $(S)$ . Sufficient conditions for  $(S)$  using d.c. problems are studied in [4].

**3.1. Preliminary results.** First, let us introduce the following notations. For  $x \in \mathbb{R}^p$  and  $(y, t) \in \mathbb{R}^q \times \mathbb{R}$ , set

$$\begin{aligned}\hat{F}_x(\cdot) &= -F_x(\cdot) = -F(x, \cdot) & f_x(\cdot) &= f(x, \cdot) \\ g_x(y, t) &= \hat{F}_x(y) - t & h_x(y, t) &= f_x(y) - t.\end{aligned}$$

We will use the following assumptions.

**(3.1)** For any  $x \in X$ , there exists  $(\hat{y}_x, \hat{t}_x) \in Y \times \mathbb{R}$ , such that

$$g_x(\hat{y}_x, \hat{t}_x) < \inf_{\substack{(y, t) \in Y \times \mathbb{R} \\ h_x(y, t) \geq 0}} g_x(y, t),$$

**(3.2)** For any  $x \in X$ , the function  $f_x$  is convex on  $\mathbb{R}^q$ ,

**(3.3)** For any  $x \in X$ , the function  $F_x$  is concave  $\mathbb{R}^q$ .

Recall that the sets  $X$  and  $Y$  are compact and compact convex subsets of  $\mathbb{R}^p$  and  $\mathbb{R}^q$ , respectively. Let assumptions (3.1)–(3.3) hold. For  $x \in X$ , consider respectively the following parameterized d.c. and reverse convex problems

$$\mathcal{DCP}(x) \quad \min_{y \in Y} (\hat{F}_x - f_x)(y) \quad \mathcal{RCP}(x) \quad \min_{\substack{(y, t) \in Y \times \mathbb{R} \\ h_x(y, t) \geq 0}} g_x(y, t).$$

**Remark 3.** The following remarks are obvious.

1) For any  $(x, y) \in X \times Y$ , we have  $h_x(y, f_x(y)) = 0$ , and hence

$$\left\{ (y, t) \in Y \times \mathbb{R} / h_x(y, t) = 0 \right\} \neq \emptyset.$$

2) Let assumptions (3.1)–(3.3) hold. Then,

- i) From assumption (3.1), the reverse convex constraint  $h_x(y, t) \geq 0$  is essential.
- ii) the problem  $\mathcal{RCP}(x)$  is equivalent to the following (see Remark 1)

$$\min_{\substack{(y, t) \in Y \times \mathbb{R} \\ h_x(y, t) = 0}} g_x(y, t)$$

that is, the search of solutions can be restricted to the constraint set

$$\left\{ (y, t) \in Y \times \mathbb{R} / h_x(y, t) = 0 \right\}.$$

Note that the use of assumption (3.1) in the rest of the paper will be implicit. Otherwise, the problem  $\mathcal{RCP}(x)$  will be equivalent to a convex programming problem.

Let  $x \in X$ . Then, we have the following relationship between the problems  $\mathcal{RCP}(x)$  and  $\mathcal{DCP}(x)$ .

**Proposition 1.** *Let assumptions (3.1)–(3.3) hold. Let  $x \in X$ . Then, we have*

- i) *if  $\bar{y}_x \in Y$  solves  $\mathcal{DCP}(x)$ , then,  $(\bar{y}_x, f_x(\bar{y}_x))$  solves  $\mathcal{RCP}(x)$ ,*
- ii) *if  $(\bar{y}_x, \bar{t}_x) \in Y \times \mathbb{R}$  solves  $\mathcal{RCP}(x)$ , then,  $\bar{y}_x$  solves  $\mathcal{DCP}(x)$ .*

*Proof.* Let  $x \in X$ .

i) Assume that  $\bar{y}_x \in Y$  solves  $\mathcal{DCP}(x)$ . Then,

$$\hat{F}_x(\bar{y}_x) - f_x(\bar{y}_x) \leq \hat{F}_x(y) - f_x(y) \quad \forall y \in Y.$$

Let  $(y, t) \in Y \times \mathbb{R}$ , such that  $h_x(y, t) \geq 0$ , i.e.,  $f_x(y) \geq t$ . Then,

$$g_x(\bar{y}_x, f_x(\bar{y}_x)) = \hat{F}_x(\bar{y}_x) - f_x(\bar{y}_x) \leq \hat{F}_x(y) - f_x(y) \leq \hat{F}_x(y) - t = g_x(y, t).$$

Since  $(\bar{y}_x, f_x(\bar{y}_x))$  is a feasible point of  $\mathcal{RCP}(x)$ , it follows that  $(\bar{y}_x, f_x(\bar{y}_x))$  solves  $\mathcal{RCP}(x)$ .

ii) Assume that  $(\bar{y}_x, \bar{t}_x) \in Y \times \mathbb{R}$ , solves  $\mathcal{RCP}(x)$ . Then,  $f_x(\bar{y}_x) \geq \bar{t}_x$ . Let  $y \in Y$ . Since  $f_x(y)$  is a finite real number, let  $t \in \mathbb{R}$  such that  $f_x(y) \geq t$ . Then, we have

$$\hat{F}_x(\bar{y}_x) - \bar{t}_x \leq \hat{F}_x(y) - t.$$

Hence

$$\hat{F}_x(\bar{y}_x) - f_x(\bar{y}_x) \leq \hat{F}_x(\bar{y}_x) - \bar{t}_x \leq \hat{F}_x(y) - f_x(y) \quad \forall y \in Y.$$

That is  $\bar{y}_x$  solves  $\mathcal{DCP}(x)$ . □

For  $x \in X$ , let the following parameterized convex maximization problem

$$\mathcal{CMP}(x) \quad \max_{\substack{(y,t) \in Y \times \mathbb{R} \\ \hat{F}_x(y) - t \leq 0}} (f_x(y) - t)$$

Then, as above, we have the following relationship between the problems  $\mathcal{CMP}(x)$  and  $\mathcal{DCP}(x)$ .

**Proposition 2.** *Let assumptions (3.2) and (3.3) hold. Let  $x \in X$ . Then, we have*

- i) *if  $\bar{y}_x \in Y$  solves  $\mathcal{DCP}(x)$ , then,  $(\bar{y}_x, \hat{F}_x(\bar{y}_x))$  solves  $\mathcal{CMP}(x)$ ,*
- ii) *if  $(\bar{y}_x, \bar{t}_x) \in Y \times \mathbb{R}$  solves  $\mathcal{CMP}(x)$ , then,  $\bar{y}_x$  solves  $\mathcal{DCP}(x)$ .*

*Proof.* Let  $x \in X$ .

i) Let  $\bar{y}_x$  be a solution of  $\mathcal{DCP}(x)$ . Then,  $\bar{y}_x$  is a solution of the problem

$$\max_{y \in Y} (f_x(y) - \hat{F}_x(y))$$

and hence,

$$f_x(\bar{y}_x) - \hat{F}_x(\bar{y}_x) \geq f_x(y) - \hat{F}_x(y) \quad \forall y \in Y.$$

Let  $(y, t) \in Y \times \mathbb{R}$  such that  $t \geq \hat{F}_x(y)$ . Then,

$$f_x(\bar{y}_x) - \hat{F}_x(\bar{y}_x) \geq f_x(y) - \hat{F}_x(y) \geq f_x(y) - t.$$

It follows from the feasibility of  $(\bar{y}_x, \hat{F}_x(\bar{y}_x))$  to  $\mathcal{CMP}(x)$  that  $(\bar{y}_x, \hat{F}_x(\bar{y}_x))$  is a solution of  $\mathcal{CMP}(x)$ .

ii) Let  $(\bar{y}_x, \bar{t}_x) \in Y \times \mathbb{R}$  be a solution of the problem  $\mathcal{CMP}(x)$ . Let  $y \in Y$ . Then,  $(y, \hat{F}_x(y))$  is a feasible point of  $\mathcal{CMP}(x)$ . Hence,

$$f(\bar{y}_x) - \bar{t}_x \geq f_x(y) - \hat{F}_x(y).$$

On the other hand, since  $(\bar{y}_x, \bar{t}_x)$  solves  $\mathcal{CMP}(x)$ , it follows that  $\hat{F}_x(\bar{y}_x) \leq \bar{t}_x$ . Then, we get

$$f_x(\bar{y}_x) - \hat{F}_x(\bar{y}_x) \geq f_x(\bar{y}_x) - \bar{t}_x \geq f_x(y) - \hat{F}_x(y).$$

So

$$\hat{F}_x(\bar{y}_x) - f_x(\bar{y}_x) \leq f_x(y) - \hat{F}_x(y)$$

which means that  $\bar{y}_x$  solves the problem  $\mathcal{DCP}(x)$ . □

**3.2. Existence of solutions to (S) via reverse convex and convex maximization problems.** In this subsection, we give sufficient conditions ensuring the existence of solutions to (S) via reverse convex and convex maximization problems. First, we recall some results from [2] and [4].

**Theorem 3.1.** [2] *Assume that assumptions (3.2), (3.3) and the following assumptions are satisfied*

(3.4) *For any  $(x, y) \in X \times Y$ , and any sequence  $(x_n)$  converging to  $x$  in  $X$ , there exists a sequence  $(y_n)$  converging to  $y$  in  $Y$ , such that*

$$\limsup_{n \rightarrow +\infty} f(x_n, y_n) \leq f(x, y),$$

(3.5) *The function  $f$  is lower semicontinuous on  $X \times Y$ ,*

(3.6) *The function  $F$  is continuous on  $X \times Y$ .*

*If moreover, the following condition is satisfied*

(C<sub>2</sub>) *For any  $x \in X$ , there exists a common solution to the following problems*

$$\min_{y \in Y} f_x(y) \quad \text{and} \quad \max_{y \in Y} F_x(y)$$

*then, the problem (S) has at least one solution.*

**Remark 4.** Since the function  $F_x(\cdot)$  is concave, the problem  $\max_{y \in Y} F_x(y)$  is equivalent to a convex programming problem.

**Definition 3.2.** A multifunction  $\mathcal{N} : X \rightrightarrows 2^Y$  is said to be lower semicontinuous on  $X$ , if for any  $x \in X$ , and any sequence  $(x_n)$  converging to  $x$  in  $X$ , we have

$$\mathcal{N}(x) \subset \liminf_{n \rightarrow +\infty} \mathcal{N}(x_n)$$

where

$$\liminf_{n \rightarrow +\infty} \mathcal{N}(x_n) = \left\{ y \in Y / \exists y_n \in Y, \lim_{n \rightarrow +\infty} y_n = y, \text{ and } y_n \in \mathcal{N}(x_n), \forall n \in \mathbb{N} \right\}.$$

**Remark 5.** As it is well-known, under some mild assumptions, and if moreover, the multifunction  $\mathcal{M}(\cdot) : X \rightrightarrows 2^Y$  associated to the solution set of the lower level problem is lower semicontinuous on  $X$ , the solution set of (S) is nonempty. However, such a property is strong and rarely satisfied. It is shown in [4], that the property (C<sub>2</sub>) and the lower semicontinuity of  $\mathcal{M}(\cdot)$  are independent from each other. Therefore, the property (C<sub>2</sub>) can be considered as an alternative condition which ensures the existence of solutions to problem (S).

For  $y \in \mathbb{R}^q$ , we denote by  $\partial f_x(y)$  the subdifferential of the function  $f_x(\cdot)$  at  $y$ . Then, we have the following result which gives sufficient conditions for the existence of solutions to problem (S) via d.c. problems.

**Theorem 3.3.** [4] *Assume that assumptions (3.2)–(3.6) and the following condition are satisfied*

(3.7) *For any  $x \in X$ , there exists  $\bar{y}_x \in Y$ , such that*  
*i)  $\bar{y}_x$  is a local solution of the problem  $\text{DCP}(x)$ ,*  
*ii)  $0 \in \partial f_x(\bar{y}_x)$ .*

*Then, the problem (S) has at least one solution.*

For the convenience of the reader we give the proof.



*Proof.* Let  $x \in X$  be arbitrarily chosen. First, note that by ii) of assumption (3.7), the point  $\bar{y}_x$  minimizes the function  $f_x$  over  $\mathbb{R}^q$ , and hence over the set  $Y$  (since  $\bar{y}_x \in Y$ ). Let  $\mathcal{N}_Y(\bar{y}_x)$  denote the normal cone to  $Y$  at  $\bar{y}_x$ . Writing problem  $\mathcal{DCP}(x)$  in its equivalent form

$$\min_{y \in \mathbb{R}^q} (\hat{F}_x + \psi_Y - f_x)(y),$$

and since  $\bar{y}_x$  is a local solution of problem  $\mathcal{DCP}(x)$ , it follows from [18] that  $\partial f_x(\bar{y}_x) \subset \partial(\hat{F}_x + \psi_Y)(\bar{y}_x)$ . On the other hand, since  $Y$  is a nonempty convex set, then the function  $\psi_Y$  is proper and convex (see for example [19]). Moreover, we have  $\text{dom}(\psi_Y) = Y$ , and  $\hat{F}_x$  is continuous on  $\mathbb{R}^q$  and hence in particular on  $Y$ . Then, by using Theorem 2.1, we get

$$\partial(\hat{F}_x + \psi_Y)(\bar{y}_x) = \partial\hat{F}_x(\bar{y}_x) + \mathcal{N}_Y(\bar{y}_x).$$

Hence,  $\partial f_x(\bar{y}_x) \subset \partial\hat{F}_x(\bar{y}_x) + \mathcal{N}_Y(\bar{y}_x)$ , and from ii) of assumption (3.7), it follows that  $0 \in \partial\hat{F}_x(\bar{y}_x) + \mathcal{N}_Y(\bar{y}_x)$ . Therefore, by Theorem 2.2  $\bar{y}_x$  minimizes the function  $\hat{F}_x$  over the set  $Y$ , and hence it is a common solution of the following convex minimization problems

$$\min_{y \in Y} f_x(y) \quad \text{and} \quad \min_{y \in Y} \hat{F}_x(y).$$

So, it is a common solution to the problems (since  $\hat{F}_x = -F_x$ )

$$\min_{y \in Y} f_x(y) \quad \text{and} \quad \max_{y \in Y} F_x(y).$$

Finally, using the result of Theorem 3.1, we deduce that the problem (S) has at least one solution.  $\square$

**Theorem 3.4.** *Assume that assumptions (3.1)–(3.6) and the following assumption are satisfied*

**(3.8)** *For any  $x \in X$ , there exists  $(\bar{y}_x, \bar{t}_x) \in Y \times \mathbb{R}$ , such that*

- i)  $(\bar{y}_x, \bar{t}_x)$  is a solution of the problem  $\mathcal{RCP}(x)$ ,*
- ii)  $0 \in \partial f_x(\bar{y}_x)$ .*

*Then, the problem (S) has at least one solution.*

*Proof.* Let  $x \in X$  be arbitrarily chosen. From assumption (3.8) there exists  $(\bar{y}_x, \bar{t}_x) \in Y \times \mathbb{R}$  which solves  $\mathcal{RCP}(x)$ . It follows from Proposition 1 that  $\bar{y}_x$  solves the problem  $\mathcal{DCP}(x)$ . Then, using the fact that  $0 \in \partial f_x(\bar{y}_x)$ , and Theorem 3.3, we deduce the existence of solutions to (S).  $\square$

**Remark 6.** In the case of problem  $\mathcal{RCP}(x)$ , the assumption (2.2) in Theorem 2.3 becomes

$\forall (y, t) \in Y \times \mathbb{R}$  verifying  $h_x(y, t) = 0$ ,  $\exists (y^*, t^*)^T \in \partial h_x(y, t)$ ,  $\exists (\bar{u}, \bar{t}) \in Y \times \mathbb{R}$ , such that

$$\left\langle \begin{pmatrix} y^* \\ t^* \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{t} \end{pmatrix} - \begin{pmatrix} y \\ t \end{pmatrix} \right\rangle > 0. \quad (\mathcal{C}_3)$$

For  $(\bar{y}, \bar{t}) \in Y \times \mathbb{R}$ , recall the following notation that we will use in the sequel

$$S(g_x, (\bar{y}, \bar{t})) = \left\{ (y, t) \in Y \times \mathbb{R} / g_x(y, t) \leq g_x(\bar{y}, \bar{t}) \right\}.$$

Then, we have the following result.

**Theorem 3.5.** *Assume that assumptions (3.1)–(3.6) and the following condition are satisfied*

(3.9) For any  $x \in X$ ,

i) there exists  $\bar{y}_x \in Y$ , such that for any  $y \in Y$ , we have

$$\left[ \partial f_x(y) \times \{-1\} \right] \cap \mathcal{N}_{S(g_x, (\bar{y}_x, f_x(\bar{y}_x))}^*(y, f_x(y)) \neq \emptyset,$$

ii)  $0 \in \partial f_x(\bar{y}_x)$ .

Then, the problem (S) has at least one solution.

*Proof.* Let  $x \in X$  be arbitrarily chosen. First, let us verify that property (C<sub>3</sub>) is satisfied for problem  $\mathcal{RCP}(x)$  (see Remark 6). Let  $(y, t) \in Y \times \mathbb{R}$ , such that  $h_x(y, t) = 0$ , i.e.,  $f_x(y) = t$ . Let  $y^* \in \partial f_x(y)$ . Then,  $\begin{pmatrix} y^* \\ -1 \end{pmatrix} \in \partial h_x(y, t)$ . Let  $\bar{u} \in Y$ ,  $\epsilon > 0$ , and  $\bar{t} = \langle y^*, \bar{u} - y \rangle + f_x(y) - \epsilon$ . Then

$$\left\langle \begin{pmatrix} y^* \\ -1 \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{t} \end{pmatrix} - \begin{pmatrix} y \\ t \end{pmatrix} \right\rangle = \langle y^*, \bar{u} - y \rangle + t - \bar{t} = \langle y^*, \bar{u} - y \rangle + f_x(y) - \bar{t} = \epsilon > 0.$$

Hence, property (C<sub>3</sub>) is satisfied. Let  $x \in X$ , and  $\bar{t}_x = f_x(\bar{y}_x)$ , where  $\bar{y}_x$  is the point given in i) of assumption (3.9). Let us show that the condition (C<sub>1</sub>) in Theorem 2.3 is satisfied for  $(\bar{y}_x, \bar{t}_x)$ . Let  $(y, t) \in \mathbb{R}^q \times \mathbb{R}$ , such that  $h_x(y, t) = 0$ , i.e.,  $t = f_x(y)$ . We have

$$\partial h_x(y, f_x(y)) = \partial f_x(y) \times \{-1\}.$$

So, from i) of assumption (3.9), we get

$$\partial h_x(y, f_x(y)) \cap \mathcal{N}_{S(g_x, (\bar{y}_x, f_x(\bar{y}_x))}^*(y, f_x(y)) \neq \emptyset.$$

Hence, the condition (C<sub>1</sub>) in Theorem 2.3 is satisfied. Moreover, we have  $h_x(\bar{y}_x, \bar{t}_x) = 0$  and  $0 \notin \partial h_x(\bar{y}_x, \bar{t}_x)$ . Therefore, from Theorem 2.3, we deduce that  $(\bar{y}_x, \bar{t}_x)$  is a solution to problem  $\mathcal{RCP}(x)$ . The rest of the proof is identical to the proof of Theorem 3.4  $\square$

Let us consider the following example where all assumptions of Theorems 3.5 are satisfied.

**Example 2.** Let  $X = [0, 1]$ ,  $Y = [0, 1]$ ,  $f, F : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the functions defined by

$$f(x, y) = x + \frac{1}{2}y^2 \quad \text{and} \quad F(x, y) = x^2 + 2x - y.$$

Let  $x \in X$ , and set  $(\hat{y}_x, \hat{t}_x) = (0, x + 1)$ . We have

$$g_x(\hat{y}_x, \hat{t}_x) = -x^2 - 3x - 1 < \inf_{\substack{(y, t) \in Y \times \mathbb{R} \\ h_x(y, t) \geq 0}} g_x(y, t) = -x^2 - 3x.$$

Hence, assumption (3.1) is satisfied. We easily verify that assumptions (3.2)–(3.6) are also satisfied. Let us check that assumption (3.9) is fulfilled.

Let  $x \in X$ . Take  $\bar{y}_x = 0$ , and let  $y \in Y$ . Then,  $\partial f_x(y) = \{y\}$ , and  $f_x(\bar{y}_x) = x$ . On the other hand, we have

$$S(g_x, (\bar{y}_x, f_x(\bar{y}_x))) = S(g_x, (0, x)) = \left\{ (y, t) \in [0, 1] \times \mathbb{R} / x + y \leq t \right\}.$$

Let  $\begin{pmatrix} y^* \\ t^* \end{pmatrix} \in \partial f_x(y) \times \{-1\} = \{y\} \times \{-1\}$ . Let  $\begin{pmatrix} u \\ v \end{pmatrix} \in S(g_x, (0, x))$ . Then,  $u + x \leq v$ , and

$$\left\langle \begin{pmatrix} y^* \\ t^* \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} y \\ f_x(y) \end{pmatrix} \right\rangle = -\frac{1}{2}y^2 + uy + x - v.$$

Since  $-\frac{1}{2}y^2 + uy + x \leq u + x$ , it follows that

$$\left\langle \begin{pmatrix} y^* \\ t^* \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} y \\ f_x(y) \end{pmatrix} \right\rangle \leq 0.$$

Therefore,  $\begin{pmatrix} y^* \\ t^* \end{pmatrix} \in \mathcal{N}_{S(g_x, (\bar{y}_x, \bar{f}_x(\bar{y}_x)))}^*(y, f_x(y))$ . Consequently,

$$\left[ \partial f_x(y) \times \{-1\} \right] \cap \mathcal{N}_{S(g_x, (\bar{y}_x, f_x(\bar{y}_x)))}^*(y, f_x(y)) \neq \emptyset.$$

Moreover, we have  $0 \in \partial f_x(\bar{y}_x) = \{0\}$ , which shows that assumption (3.9) is satisfied.

In the following theorem, we consider a class of problems where the objective functions of the leader and the follower are linked by the following inequality

**(3.10)** For any  $(x, y) \in X \times Y$ , we have  $F(x, y) + f(x, y) \leq 0$ .

**Theorem 3.6.** *Assume that assumptions (3.1)–(3.6), (3.10) and the following assumption are satisfied*

**(3.11)** *For any  $x \in X$ , there exists  $\bar{y}_x \in Y$ , such that such that*

- i)  $f_x(\bar{y}_x) + F_x(\bar{y}_x) = 0$ ,
- ii)  $0 \in \partial f_x(\bar{y}_x)$ .

*Then, the problem (S) has at least one solution.*

*Proof.* Let  $x \in X$  be arbitrarily chosen and  $y \in \mathbb{R}^q$ . Let us show that

$$\partial f_x(y) \times \{-1\} \subset \mathcal{N}_{S(g_x, (\bar{y}_x, f_x(\bar{y}_x)))}^*(y, f_x(y)).$$

Let  $\begin{pmatrix} y^* \\ t^* \end{pmatrix} \in \partial f_x(y) \times \{-1\}$ . Then,  $y^* \in \partial f_x(y)$  and  $t^* = -1$ . For  $\begin{pmatrix} u \\ v \end{pmatrix} \in S(g_x, (\bar{y}_x, f_x(\bar{y}_x)))$ , we have

$$-F(x, u) - v \leq -F(x, \bar{y}_x) - f(x, \bar{y}_x) = 0,$$

where the last equality follows from i) of assumption (3.11). Thus,

$$-F(x, u) - v \leq 0. \tag{1}$$

Since  $y^* \in \partial f_x(y)$ , it follows that

$$f_x(z) \geq f_x(y) + \langle y^*, z - y \rangle \quad \forall z \in \mathbb{R}^q,$$

and hence

$$f_x(u) \geq f_x(y) + \langle y^*, u - y \rangle. \tag{2}$$

On the other hand, we have

$$\left\langle \begin{pmatrix} y^* \\ t^* \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} y \\ f_x(y) \end{pmatrix} \right\rangle = \langle y^*, u - y \rangle + f_x(y) - v.$$

Using (2), we get

$$\left\langle \begin{pmatrix} y^* \\ t^* \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} y \\ f_x(y) \end{pmatrix} \right\rangle \leq f_x(u) - v \leq -F_x(u) - v \leq 0,$$

where the first and the last inequalities follow from (2) and (1) respectively. That is

$$\partial f_x(y) \times \{-1\} \subset \mathcal{N}_{S(g_x, (\bar{y}_x, f_x(\bar{y}_x)))}^*(y, f_x(y)),$$

and hence assumption (3.9) is satisfied. By the same arguments as in Theorem 3.5, we verify that condition  $(\mathcal{C}_3)$  is satisfied. The nonemptiness of the solution set of problem (S) follows from the same theorem.  $\square$

**Remark 7.** Remark that

- 1) if for any  $x \in X$ , the function  $f_x$  is differentiable on  $\mathbb{R}^q$ , then the point  $\bar{y}_x$  given in assumption (3.11) is a solution of the following system of  $q + 1$  equations

$$\begin{cases} \frac{\partial f_x}{\partial y_i}(\bar{y}_x) = 0, \quad i = 1, \dots, q \\ f_x(\bar{y}_x) + F_x(\bar{y}_x) = 0. \end{cases}$$

- 2) assumption (3.11) implies that the function  $f_x(\cdot)$  is bounded from below by  $-F_x(\bar{y}_x)$ .

In the following example, we check that all assumptions of Theorem 3.6 are satisfied.

**Example 3.** Let  $X = [0, 1]$ ,  $Y = [0, 1]$ ,  $f$  and  $F$  be the functions defined on  $\mathbb{R} \times \mathbb{R}$ , by

$$f(x, y) = -2x^2 + y^2 \quad \text{and} \quad F(x, y) = 2x^2 + (x - 4)y^2.$$

We easily verify that assumptions (3.2)–(3.6) are satisfied. Moreover, we have

$$F(x, y) + f(x, y) = (x - 3)y^2 \leq 0, \quad \forall (x, y) \in X \times Y,$$

that is assumption (3.10) is satisfied. Let us verify assumptions (3.1) and (3.11). Let  $x \in X$  and set  $(\hat{y}_x, \hat{t}_x) = (0, 1)$ . Then,

$$g_x(\hat{y}_x, \hat{t}_x) = -2x^2 - 1 < \inf_{\substack{(y, t) \in Y \times \mathbb{R} \\ h_x(y, t) \geq 0}} g_x(y, t) = 0,$$

that is assumption (3.1) is satisfied. For verifying assumption (3.11), let  $x \in X$  and take  $\bar{y}_x = 0$ . We have

$$f_x(0) = -2x^2 \quad \text{and} \quad F_x(0) = 2x^2.$$

Hence,  $f_x(0) + F_x(0) = 0$ . Moreover,  $0 \in \partial f_x(0)$ . Hence assumption (3.11) is satisfied.

Finally, we have the following theorem which gives sufficient conditions for the existence of solutions to (S) via parameterized convex maximization problems.

**Theorem 3.7.** *Assume that assumptions (3.2) and (3.3) and the following property are satisfied*

**(3.12)** *For any  $x \in X$ , there exists  $(\bar{y}_x, \bar{t}_x) \in Y \times \mathbb{R}$ , such that*

- i)  $(\bar{y}_x, \bar{t}_x)$  is a solution of the problem  $\mathcal{CMP}(x)$ ,*
- ii)  $0 \in \partial f_x(\bar{y}_x)$ .*

*Then, problem (S) has at least one solution.*

*Proof.* Let  $x \in X$  be arbitrarily chosen. The condition i) of assumption (3.12) and Proposition 2 imply that  $\bar{y}_x$  solves the problem  $\mathcal{DCP}(x)$ . By using the fact that  $0 \in \partial f_x(\bar{y}_x)$ , we deduce from Theorem 3.3, that problem (S) has at least one solution.  $\square$

**4. Conclusions.** Due to their complex formulation, weak nonlinear bilevel programming problems are known to be difficult in both the theoretical and numerical aspects. For this reason, such problems are rarely studied in the literature in spite of their importance. In the present paper, and in order to contribute to the reduction of such difficulties, we have derived some relationships between a class of weak nonlinear bilevel optimization problems and some well-known other global optimization problems. Therefore, the established results allow us to get the existence of solutions to  $(S)$  via reverse convex and convex maximization problems. If the theoretical study of weak bilevel programming problems has known some development, the numerical one is still in its infancy. We note that finding numerical algorithms for solving this class of problems is of major importance. This is out of the scope of the present paper and will be the subject of a forthcoming work.

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*E-mail address:* [aboussororabdel@hotmail.com](mailto:aboussororabdel@hotmail.com)

*E-mail address:* [samir.adly@unilim.fr](mailto:samir.adly@unilim.fr)

*E-mail address:* [vincent.jalby@unilim.fr](mailto:vincent.jalby@unilim.fr)