New limit cycles of dry friction oscillators under harmonic load

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Abstract We consider a system composed of two masses connected by linear springs. One of the masses is in contact with a driving belt moving at a constant velocity. Friction force, with Coulomb's characteristics, acts between the mass and the belt. Moreover, the mass is also subjected to a harmonic external force. Several periodic orbits including stick phases and slip phases are obtained. In particular, the existence of periodic orbits including a part where the mass in contact with the belt moves in the same direction at a higher speed than the belt itself is proved. Non-sticking orbits are also found for a non-moving belt. We prove that this kind of solution is symmetric in space and in time.

Keywords Dry friction · Periodic orbits · Coupled oscillator · Stick-slip motion

1 Introduction

This paper is a continuation of several investigations [5–7], related to vibrating systems excited by dry friction. These systems are frequently encountered in many industrial applications. One of the most popular models of stick-slip oscillators consists of several masses connected by linear springs, one (or more) of the masses is in contact with a driving belt moving at a constant velocity. In the past, several authors investigated the behavior of this system, with different friction laws and with or without external actions and damping [1, 4], mainly via numerical approach.

However, assuming Coulomb's laws of dry friction, the corresponding dynamical model is a piecewise linear system, and even for multi-degree-of-freedom cases, some analytical results about the existence and the stability of periodic orbits including stick-slip phases have been obtained [5–7].

One interesting phenomenon is the existence, inside periodic orbits with stick and slip parts, of an "overshooting" slip phase. During this part of the orbit, the mass in contact with the belt moves in the same direction at a higher speed than the belt itself.

Up to now [9], this phenomenon has been observed only for more complex friction characteristics than Coulomb's ones. In [7], a self-excited stick-slip oscillator including two degrees of freedom is considered. Assuming Coulomb's laws of dry friction, a set of periodic orbits including an overshooting part is obtained using analytical methods.

In this work, we consider the same model of dry friction oscillator subjected to a harmonic external force. Several periodic orbits containing stick phases and slip phases are obtained. In particular, the existence of periodic orbits including an overshooting part is proved.

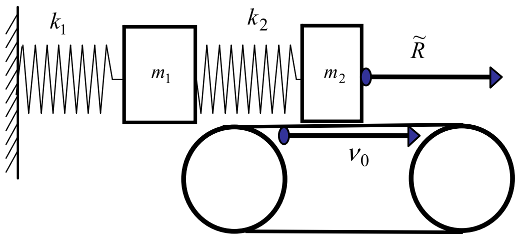

Fig. 1 Dry friction oscillator

2 Problem formulation

The system (Fig. 1) is composed of two masses m_1, m_2 connected by two linear springs of stiffness k_1, k_2. The second mass is in contact with a belt moving at a constant velocity v_0. A friction force $\tilde{F}$ acts between the mass and the belt. Moreover, the second mass is also subjected to an external force $\tilde{R}$ given by

$$\tilde{R} = \tilde{p}\cos(\tilde{\omega}t' + \varphi)$$
$$(\tilde{p}, \tilde{\omega}, \varphi \text{ are constant parameters}) \tag{1}$$

The equations of motion related to this system are written as

$$\begin{aligned} &x_1'' + x_1 - \chi x_2 = 0, \\ &x_2'' + \chi\eta(x_2 - x_1) = \eta u + p\cos(\omega t + \varphi) \end{aligned} \tag{2}$$

x_1, x_2 are the displacements of the masses,

$$\begin{aligned} &\eta = \frac{m_1}{m_2}, \quad \chi = \frac{k_2}{k_1 + k_2}, \quad u = \frac{\tilde{F}}{k_1 + k_2}, \\ &p = \eta\frac{\tilde{p}}{k_1 + k_2}, \quad t = \Omega t', \quad \Omega = \sqrt{\frac{k_1 + k_2}{m_1}}, \\ &(O)' = \frac{d(O)}{dt}, \quad \omega = \frac{\tilde{\omega}}{\Omega} \end{aligned} \tag{3}$$

The dry friction force u is obtained from Coulomb's laws:

$$u = \begin{cases} u_s \operatorname{sign}(V - x_2'), & \text{if } V - x_2' \neq 0, \\ \chi(x_2 - x_1) - \frac{p}{\eta}\cos(\omega t + \varphi), & \text{if } V - x_2' = 0, \ \left|\chi(x_2 - x_1) - \frac{p}{\eta}\cos(\omega t + \varphi)\right| < u_r, \\ u_s, & \text{if } V - x_2' = 0, \quad \chi(x_2 - x_1) - \frac{p}{\eta}\cos(\omega t + \varphi) > u_r, \\ -u_s, & \text{if } V - x_2' = 0, \ \chi(x_2 - x_1) - \frac{p}{\eta}\cos(\omega t + \varphi) < -u_r, \ 0 < u_s < u_r, \ V = \frac{v_0}{\Omega}, \end{cases} \tag{4}$$

u_r is the friction force at rest (sticking), u_s is the slipping friction force.

3 Prediction of the oscillations exhibited by the system

The dynamical behavior of this oscillator includes several phases of slip and stick motion of m_2. For each kind of motion, a close form solution is available.

3.1 Slip motion of m_2 with $x_2' < V$

The solution is obtained from a modal analysis of (2) where $u = u_s$

$$\begin{aligned} &Z(t) = H(t)(Z_0 - F_0) + F(t), \\ &F(t) = \begin{pmatrix} R(t) \\ R'(t) \end{pmatrix}, \\ &R(t) = Q\cos(\omega t + \varphi), \quad F_0 = F(0), \end{aligned} \tag{5}$$

$$\begin{aligned} &Z = \begin{pmatrix} z \\ z' \end{pmatrix}, \quad Z_0 = Z(0), \\ &z = X - d_0, \quad X = (x_1, x_2)^t, \\ &H(t) = \begin{pmatrix} H_1(t) & H_2(t) \\ H_3(t) & H_1(t) \end{pmatrix}, \quad d_0 = (d_{01}, d_{02})^t, \\ &d_{01} = \frac{u_s}{1 - \chi}, \quad d_{02} = \frac{d_{01}}{\chi}, \end{aligned} \tag{6}$$

$$\begin{aligned} &Q = (q_1, q_2)^t, \quad q_1 = \frac{p\chi}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}, \\ &q_2 = q_1\frac{(1 - \omega^2)}{\chi} \end{aligned} \tag{7}$$

The two by two matrices $H_i(t)$ $(i = 1, 2, 3)$ and the natural frequencies (ω_1, ω_2) are obtained in analytical form (see Appendix 1).

3.2 Slip motion of m_2 with $x_2' > V$ (overshooting)

The solution is obtained from (2) where $u = -u_s$

$$\begin{aligned} &Z(t) = H(t)(Z_0 - F_0) + F(t) + 2L(t)d_0, \\ &L(t) = \begin{pmatrix} H_1(t) - I \\ H_3(t) \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{8}$$

3.3 Stick motion of m_2 ($x_2' = V$)

This motion is related to the following dynamical system:

$$x_1'' + x_1 - \chi x_2 = 0, \qquad x_2'' = 0 \tag{9}$$

The solution [5] is given by

$$Z(t) = \Gamma(t) Z_0, \quad \Gamma(t) = \begin{pmatrix} \Gamma_1(t) & \Gamma_2(t) \\ \Gamma_3(t) & \Gamma_1(t) \end{pmatrix} \tag{10}$$

The two by two matrices $\Gamma_i(t)$ $(i = 1, 2, 3)$ are given in Appendix 1. Moreover, during all this kind of motion, the following constraint holds:

$$\left|\chi\eta(x_2 - x_1) - p\cos(\omega t + \varphi)\right| < \eta u_r \tag{11}$$

4 Symmetrical periodic solutions

A first set of periodic orbits of period $\Theta = 2\pi/\omega$ is obtained. These motions involve for each period first a slip motion of m_2 with $x_2' < V$ followed by a phase of stick motion of the mass ($x_2' = V$). At the beginning ($t = 0$) of the slip motion, the initial conditions are given by

$$\begin{aligned} &z_2'(0) = V, \\ &\chi\eta\big(z_2(0) - z_1(0)\big) = p\cos\varphi + \eta(u_r - u_s) \end{aligned} \tag{12}$$

For $0 < t < \tau$, the system undergoes a slip motion defined by (5).

At the end ($t = \tau$) of the slip motion, the following condition is assumed:

$$Z(\tau) = EZ_0, \quad E = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{13}$$

The time lag φ of the external force is given by $\varphi = (\pi - \omega\tau)/2$. From (12) and (13), we deduce:

$$\begin{aligned} &-\eta(u_r + u_s) < \chi\eta(z_{2c} - z_{1c}) - p\cos(\omega\tau + \varphi) \\ &\quad = -\eta(u_r - u_s) < \eta(u_r - u_s) \end{aligned} \tag{14}$$

For $\tau < t < \tau + T$, the system motion is a sticking motion given by

$$Z(t) = \Gamma(t - \tau) E Z_0$$

A periodic solution of period Θ is obtained if the following relation is fulfilled:

$$Z_0 = \Gamma(T) E Z_0, \quad (T = \Theta - \tau) \tag{15}$$

Taking into account the properties (Appendix 1) of the matrices $H(t), \Gamma(t)$, the conditions for the existence of this periodic solution are given by the following system of four scalar equations:

$$\begin{aligned} &(H_1 + I)(z_0 - Qs_0) + H_2(z_0' + Q\omega c_0) = 0, \\ &\quad H_i = H_i(\tau) \\ &(\Gamma_1 + I)z_0 - \Gamma_2 z_0' = 0, \quad \Gamma_i = \Gamma_i(T), \\ &\quad (i = 1, 2) \\ &s_0 = \sin(\omega\tau/2), \quad c_0 = \cos(\omega\tau/2) \end{aligned} \tag{16}$$

τ and hence the time duration $T = \Theta - \tau$ of the stick motion, together with the initial conditions $z(0), z'(0)$ are computed. As in the case of the self-excited dry friction oscillator considered in [5], an interesting property of symmetry is proved for these orbits (see Appendix 2):

$$\begin{aligned} &Z(t) = EZ(\tau - t), \quad 0 < t < \tau/2, \quad \text{(slip motion)} \\ &Z(t_1) = EZ(T - t_1), \quad 0 < t_1 < T/2, \\ &\quad \text{(stick motion)}, \quad t_1 = t - \tau \end{aligned} \tag{17}$$

A numerical validation is made for the following set of data:

$$\begin{aligned} &\chi = 0.5, \quad \eta = 3.8, \quad u_s = 0.02, \quad u_r = 0.1607, \\ &V = 1, \quad \omega = 0.6, \quad p = 0.05 \end{aligned}$$

The other parameters related to this orbit are computed:

$$\begin{aligned} &\tau = 7.749, \quad T = 2.723, \quad \Theta = 10.472, \\ &z_1(0) = 1.0609, \quad z_2(0) = 1.3615, \quad z_1'(0) = 0.581 \end{aligned}$$

The corresponding phase portraits $(z_i, z_i'), i = (1, 2)$ of the two masses are shown on Fig. 2: these curves are symmetrical with respect to the vertical line $z = 0$. (The thick parts of the curves are related to the slip motion while the thin parts show the stick motion.)

The constraints derived from (11) during the stick motion:

$$\begin{aligned} &(t < t_1 < T, \ t_1 = t - \tau) \\ &f_1 \equiv \chi\eta(z_2 - z_1) + p\sin\big(\omega(t_1 + \tau/2)\big) \\ &\qquad - \eta(u_r - u_s) \le 0 \\ &f_2 \equiv \chi\eta(z_2 - z_1) + p\sin\big(\omega(t_1 + \tau/2)\big) \\ &\qquad + \eta(u_r + u_s) > 0 \end{aligned}$$

are fulfilled.

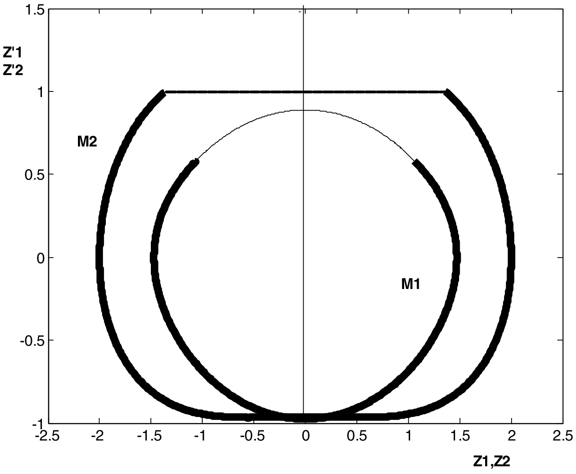

Fig. 2 Phase portraits of the symmetrical solution

5 Periodic orbits including an overshooting part

A second set of periodic orbits of period $\Theta = 2\pi/\omega$ is obtained. For each period, the motion is composed of three parts. The first one is a slip motion of m_2 with $x_2' < V$ for $0 < t < \tau$; the next part ($0 < t - \tau < \tau_1$) is an overshooting slip motion of the mass ($x_2' > V$); the last part $0 < t - \tau - \tau_1 < T$

($T = \Theta - \tau - \tau_1$) is a stick motion of m_2.

At the beginning of the motion for $t = 0$, the conditions (12) are fulfilled. If at $t = \tau$, instead of (13), the following conditions:

$$\begin{aligned} &z_2'(\tau) = V, \\ &\chi\eta\big(z_2(\tau) - z_1(\tau)\big) - p\cos(\omega\tau + \varphi) \\ &\quad + \eta(u_r + u_s) < 0 \end{aligned} \tag{18}$$

are fulfilled, we get an overshooting motion for $t > \tau$.

This motion ends at $t = \tau + \tau_1$ if, at this time:

$$\begin{aligned} &z_{2c}' \equiv z_2'(\tau + \tau_1) = V, \\ &-\eta(u_r + u_s) < \chi(z_{2c} - z_{1c}) \\ &\qquad - p\cos\big(\omega(\tau + \tau_2) + \varphi\big) \\ &\qquad < \eta(u_r - u_s) \\ &z_{ic} \equiv z_{ic}(\tau + \tau_1), \quad (i = 1, 2) \end{aligned} \tag{19}$$

For $\tau + \tau_1 < t < \tau + \tau_1 + T$, the system undergoes a sticking motion. A periodic solution of period $\tau + \tau_1 + T = \Theta$ is obtained if:

$$Z(\Theta) = Z_0 \tag{20}$$

Taking into account the constraints deduced from (18), (19) and (20), the solution is defined by five linear equations with respect to $(z_{10}, z_{20}, z_{10}')$:

$$a_i z_{10} + b_i z_{20} + c_i z_{10}' + d_i = 0, \quad (i = 1, \ldots, 5) \tag{21}$$

(a_i, b_i, c_i, d_i) are given in Appendix 3.

Assuming that $\Delta = \det(a_j, b_j, c_j) \neq 0$, $(j = 1, 2, 3)$, the values of $(z_{10}, z_{20}, z_{10}')$ are obtained:

$$\begin{aligned} &z_{10} = \frac{\Delta_1}{\Delta}, \quad z_{20} = \frac{\Delta_2}{\Delta}, \quad z_{10}' = \frac{\Delta_3}{\Delta} \\ &\Delta_1 = \det - \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \\ &\Delta_2 = \det \begin{vmatrix} d_1 & a_1 & c_1 \\ d_2 & a_2 & c_2 \\ d_3 & a_3 & c_3 \end{vmatrix}, \\ &\Delta_3 = \det \begin{vmatrix} d_1 & b_1 & a_1 \\ d_2 & b_2 & a_2 \\ d_3 & b_3 & a_3 \end{vmatrix} \end{aligned} \tag{22}$$

The parameters (τ, τ_1) are the roots of the compatibility conditions:

$$\begin{aligned} F_k(\tau, \tau_1) &\equiv a_{k+3}\Delta_1 + b_{k+3}\Delta_2 + c_{k+3}\Delta_3 + d_{k+3}\Delta \\ &= 0, \quad (k = 1, 2) \end{aligned} \tag{23}$$

Assuming that (χ, η, V, u_s) are given data, u_r is deduced from the relation:

$$u_r = u_s + f, \quad f = \chi(z_{20} - z_{10}) - \frac{p}{\eta}\cos\varphi \tag{24}$$

An example of overshooting orbit is obtained for the values

$$\begin{aligned} &\chi = 0.2, \quad \eta = 4, \quad u_s = 0.05, \quad V = 1, \\ &\omega = 0.6289, \quad p = 0.05, \quad \varphi = 0 \end{aligned}$$

The other parameters related to this solution are computed:

$$\begin{aligned} &\tau = 4.909, \quad \tau_1 = 2.8275, \quad T = 2.2543, \\ &\Theta = 9.9908, \quad u_r = 0.4585, \quad z_1(0) = 1.2809, \\ &z_2(0) = 3.3861, \quad z_1'(0) = -0.4117 \end{aligned}$$

The phase portraits (z_i, z_i'), $i = (1, 2)$ of the two masses are shown on Fig. 3. (The heavy thick part of

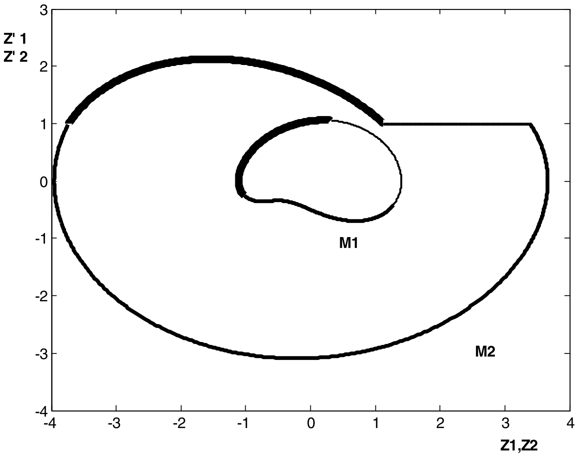

Fig. 3 Phase portraits of the overshooting periodic solution

the curves shows the overshooting motion while the thin one is related to the stick motion.)

6 Non-sticking periodic solutions

In industrial applications, avoiding sticking phases of motion is sometimes necessary. In the past, several authors [2, 3, 8] investigated the existence of periodic non-sticking solutions of a one-degree-of-freedom oscillator subjected to simple harmonic loading. The mass is in contact with a fixed surface and a dry friction force acts between the mass and the surface. The aim of these works is to obtain some estimates about the minimum external force amplitude needed to prevent this sticking motion. The non-sticking orbit involves for each period a slipping motion with a negative mass velocity, and a slipping motion with a positive mass velocity (overshooting motion). Moreover, the authors assumed that the motion is symmetric in space [2, 3, 8] and time [3, 8].

In the following, this problem is revisited for the two-degree-of-freedom oscillator considered in this work.

Let us consider the system described in Fig. 1, and assume the following initial conditions:

$$z'_{20} = V, \quad \chi\eta(z_{20} - z_{10}) > p\cos\varphi + \eta(u_r - u_s) \tag{25}$$

For $0 < t < \tau$, the system undergoes a slipping motion with $z'_2 < V$, given by (5).

This motion ends at $t = \tau$ if $z'_{2B} \equiv z'_2(\tau) = V$.

Let us assume that $\tau = \pi/\omega = \Theta/2$ and that

$$\begin{aligned} &\chi\eta(z_{2B} - z_{1B}) < p\cos(\omega\tau + \varphi) - \eta(u_r + u_s) \\ &\qquad \equiv -p\cos\varphi - \eta(u_r + u_s), \\ &z_{iB} = z_i(\tau), \quad (i = 1, 2) \end{aligned} \tag{26}$$

For $t > \tau$ the system undergoes an overshooting slipping motion ($z'_2 > V$). This motion is given by

$$\begin{aligned} Z(t) &= H(t-\tau)(Z_B - F_B) \\ &\quad + 2L(t-\tau)d_0 + F(t) \end{aligned} \tag{27}$$

A periodic motion of period $\Theta = 2\pi/\omega$ is obtained if

$$\begin{aligned} &Z_0 = Z(\Theta) \equiv H(\tau)(Z_B - F_B) + 2L(\tau)d_0 + F_0, \\ &Z_B = Z(\tau) = H(\tau)(Z_0 - F_0) + F_B, \\ &F_B = F(\tau) = -F_0 \end{aligned} \tag{28}$$

From (28) we deduce:

$$\begin{aligned} &\big[H(\tau) - H(-\tau)\big][Z_B - F_B] + 2L(\tau)d_0 = 0, \\ &H(\tau) - H(-\tau) = 2\begin{vmatrix} 0 & H_2 \\ H_3 & 0 \end{vmatrix}, \\ &H_i = H_i(\tau), \quad (i = 1, 2, 3) \end{aligned} \tag{29}$$

From (29), we obtain

$$\begin{aligned} &H_2(z'_B - Q\omega\sin\varphi) + (H_1 - I)d_0 = 0 \\ &H_3(z_B + Q\cos\varphi + d_0) = 0 \end{aligned} \tag{30}$$

From (30), if $\det(H_3) \neq 0$, the following relations hold:

$$\begin{aligned} &z_B = -Q\cos\varphi - d_0 \quad \text{i.e. } x_B = -Q\cos\varphi, \\ &z'_B = Q\omega\sin\varphi + H_2^{-1}(I - H_1)d_0 \end{aligned} \tag{31}$$

From (28), (31) and the relation:

$$H_1^2 - I = H_2H_3, \quad \text{hence } H_3 = H_2^{-1}H_1^2 - H_2^{-1} \tag{32}$$

the following results are obtained

$$\begin{aligned} &z_0 = Q\cos\varphi - d_0, \quad \text{i.e. } x_0 = -x_B = Q\cos\varphi \\ &z'_0 = -Q\omega\sin\varphi + H_2^{-1}(H_1 - I)d_0 = -z'_B \end{aligned} \tag{33}$$

This last condition and the relation $z'_{20} \equiv z'_{2B} = V$ lead to $V = 0$.

In conclusion, non-sticking periodic orbits are obtained only for $V = 0$, and the motion is symmetric in space and time (see Appendix 4).

The initial conditions and the time lag φ of the external force are deduced from (33):

$$\begin{aligned} &z_{10} = q_1\cos\varphi - d_{01}, \quad z_{20} = q_2\cos\varphi - d_{02}, \\ &z'_{10} = -q_1\omega\sin\varphi - a_1d_{01} - a_2d_{02} \\ &a_1 = d(\varphi_1\lambda_2 - \varphi_2\lambda_1), \quad a_2 = d(\varphi_2 - \varphi_1) \end{aligned} \tag{34}$$

$$\sin\varphi = -(b_1 d_{01} + b_2 d_{02})/q_2\omega,$$
$$\left|(b_1 d_{01} + b_2 d_{02})/q_2\omega\right| < 1,$$
$$b_1 = d(\varphi_1 - \varphi_2)\lambda_1\lambda_2, \quad b_2 = d(\lambda_2\varphi_2 - \lambda_1\varphi_1) \tag{35}$$
$$\varphi_i = \omega_i \tan g(\omega_i \pi/2\omega),$$
$$\lambda_i = \frac{1-\omega_i^2}{\chi} \quad (i=1,2), \quad d = \chi/\left(\omega_1^2 - \omega_2^2\right)$$

The constraints deduced from (25) and (26) lead to the same condition:

$$\chi\eta(q_2 - q_1)\cos\varphi - p\cos\varphi - \eta u_r > 0 \tag{36}$$

and from (36), a condition about the minimum value of the external force amplitude needed to avoid a sticking motion is obtained:

$$p > \left| D\frac{b_1 d_{01} + b_2 d_{02}}{(1-\omega^2)\omega} \right|, \quad D = \left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right) \tag{37}$$

A numerical validation is performed for the following values of the parameters:

$$\chi = 0.3, \quad \eta = 4, \quad \omega = 0.6, \quad p = 1,$$
$$u_s = 0.1, \quad u_r = 0.2996, \quad \Theta = 10.472$$

The corresponding values of the initial conditions and of the time lag φ are obtained:

$$x_{10} = 1.5608, \quad x_{20} = 3.3295,$$
$$x'_{10} = 0.1523, \quad \varphi = 0.3925$$

The phase portraits (x_i, x'_i), $i = (1,2)$ of the two masses are shown on Fig. 4 (the thick parts of the curves are related to the overshooting motion). These curves are symmetrical with respect to the origin 0.

Under the assumption that there are only two transitions of motion during one period, we prove that non-sticking orbits are symmetric in space and time for almost all values of ω (see Appendix 5).

7 Conclusion

In this work, the steady state response of a two-degree-of-freedom oscillator subjected to dry friction and harmonic load is considered. Assuming Coulomb's laws of dry friction, the existence of several interesting periodic orbits, including stick and slip phases, is proved. In particular, periodic solutions with a phase during which the mass in contact with the belt moves faster than the belt are obtained. Moreover, in the case of a non-moving belt, a set of non-sticking periodic solutions is obtained, and we prove that these orbits are symmetrical in space and in time.

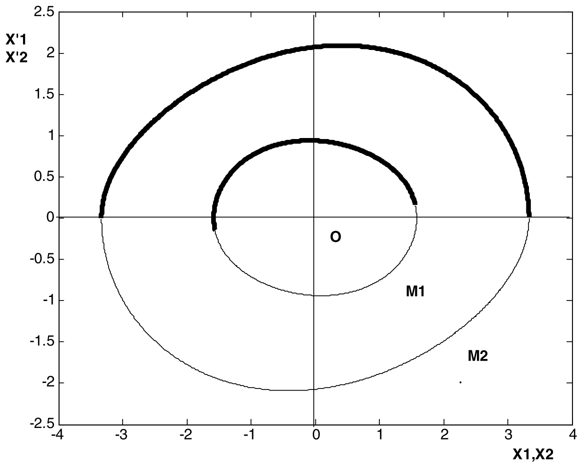

Fig. 4 Phase portrait of the non-sticking solution

Appendix 1

$$H_i(t) = \Lambda B_i(t)\Lambda^{-1}, \quad (i=1,2,3),$$
$$B_2(t) = \begin{pmatrix} s_1/\omega_1 & 0 \\ 0 & s_2/\omega_2 \end{pmatrix}, \tag{38}$$
$$s_j = \sin(\omega_j t) \ (j=1,2),$$
$$B_1(t) = B'_2(t), \quad B_3(t) = B''_2(t)$$

The natural frequencies (ω_1, ω_2) are the roots of the characteristic equation:

$$D\left(s^2\right) \equiv \det\left(K - Is^2\right) = 0,$$
$$K = \begin{pmatrix} 1 & -\chi \\ -\chi\eta & \chi\eta \end{pmatrix} \tag{39}$$

The eigenvectors $\psi_j = \binom{1}{\lambda_j}$, $(j=1,2)$ are defined by $(K - I\omega_j^2)\psi_j = 0$.

These matrices fulfil the following property:

$$H_1^2(t) - H_2(t)H_3(t) = 0, \tag{40}$$
$$\Gamma_i(t) = \Sigma\gamma_i(t)\Sigma^{-1}, \quad (i=1,2,3),$$
$$\gamma_2(t) = \begin{pmatrix} \sin t & 0 \\ 0 & t \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}, \tag{41}$$
$$\gamma_1(t) = \gamma'_2(t), \quad \gamma_3(t) = \gamma''_2(t)$$

The matrices $\Gamma_i(t)$ fulfil also the property:

$$\Gamma_1^2(t) - \Gamma_2(t)\Gamma_3(t) = 0 \tag{42}$$

Appendix 2

For $\tau/2 < t < \tau$, the periodic solution is defined by

$$Z(\tau - t) = H(\tau - t)(Z_0 - F_0) + F(\tau - t) \quad (43)$$

From the identities:

$$\begin{aligned} &F(\tau - t) = EF(t), \quad \big(\text{i.e. } F(\tau) = EF_0\big) \\ &H(\tau - t) = H(\tau)H(-t), \quad H(-t)E = EH(t) \end{aligned} \quad (44)$$

the first relation (17) is deduced.

For $T/2 < t_1 < T$, the solution is defined by

$$Z(t) = \Gamma(T - t_1)EZ_0 \quad (45)$$

From the identities:

$$\begin{aligned} &\Gamma(T - t_1) = \Gamma(T)\Gamma(-t_1), \\ &E\Gamma(-t_1) = \Gamma(t_1)E, \\ &Z(\Theta) = \Gamma(T)EZ_0 = Z_0 \end{aligned} \quad (46)$$

the last relation (17) follows.

Appendix 3

$$\begin{aligned} a_1 &= \tilde{H}_{11} - \Gamma_{11}, \qquad b_1 = \tilde{H}_{12} - \Gamma_{12}, \\ c_1 &= \tilde{H}_{13} - \Gamma_{13} \\ d_1 &= -\big(\tilde{H}_{11}q_1 + \tilde{H}_{12}q_2\big)\cos\varphi \\ &\quad + \omega\big(\tilde{H}_{13}q_1 + \tilde{H}_{14}q_2\big)\sin\varphi + q_1\cos\varphi_C \\ &\quad + 2(h_{11} - 1)d_{01} + 2h_{12}d_{02} + \big(\tilde{H}_{14} + \Gamma_{14}\big)V, \end{aligned} \quad (47)$$

$$\begin{aligned} a_2 &= \tilde{H}_{21}, \qquad b_2 = \tilde{H}_{22} - 1, \qquad c_2 = \tilde{H}_{23} \\ d_2 &= -\big(\tilde{H}_{21}q_1 + \tilde{H}_{22}q_2\big)\cos\varphi \\ &\quad + \omega\big(\tilde{H}_{23}q_1 + \tilde{H}_{24}q_2\big)\sin\varphi + q_2\cos\varphi_C \\ &\quad + 2h_{21}d_{01} + 2\big(h_{22} - 1\big)d_{02} + (\tilde{H}_{24} + T)V, \end{aligned} \quad (48)$$

$$\begin{aligned} a_3 &= \tilde{H}_{31} + \Gamma_{31}, \qquad b_3 = \tilde{H}_{32} + \Gamma_{32}, \\ c_3 &= \tilde{H}_{11} - \Gamma_{11} \\ d_3 &= -\big(\tilde{H}_{31}q_1 + \tilde{H}_{32}q_2\big)\cos\varphi \\ &\quad + \omega\big(\tilde{H}_{11}q_1 + \tilde{H}_{12}q_2\big)\sin\varphi - q_1\sin\varphi_C \\ &\quad + 2h_{31}d_{01} + 2h_{32}d_{02} + \big(\tilde{H}_{12} - \Gamma_{12}\big)V, \end{aligned} \quad (49)$$

$$\begin{aligned} a_4 &= H_{41}, \qquad b_4 = H_{42}, \qquad c_4 = H_{21}, \\ d_4 &= (H_{22} - 1)V - q_2\omega\sin(\varphi_B) \\ &\quad - (H_{41}q_1 + H_{42}q_2)\cos\varphi \\ &\quad + (H_{21}q_1 + H_{22}q_2)\omega\sin\varphi, \end{aligned} \quad (50)$$

$$\begin{aligned} a_5 &= \tilde{H}_{41}, \qquad b_5 = \tilde{H}_{42}, \qquad c_5 = \tilde{H}_{21}, \\ d_5 &= \big(\tilde{H}_{22} - 1\big)V - q_2\omega\sin(\varphi_C) \\ &\quad - \big(\tilde{H}_{41}q_1 + \tilde{H}_{42}q_2\big)\cos\varphi \\ &\quad + \big(\tilde{H}_{21}q_1 + \tilde{H}_{22}q_2\big)\omega\sin\varphi \\ &\quad + 2(h_{41}d_{01} + h_{42}d_{02}) \end{aligned} \quad (51)$$

$$H_{ij} = H_{ij}(\tau), \quad h_{ij} = H_{ij}(\tau_1), \quad \tilde{H}_{ij} = H_{ij}(\tau + \tau_1)$$

$$\varphi_B = \omega\tau + \varphi, \quad \varphi_C = \varphi_B + \omega\tau_1$$

Appendix 4

From (5), (19) and (30), the non-sticking periodic orbit obtained with the assumption that $\tau = \pi/\omega = \Theta/2$ is described by

$$\begin{aligned} &X(t) = H(t)(X_0 - F_0 - \Delta_0) + F(t) + \Delta_0, \\ &X(t + \tau) = H(t)(X_B - F_B + \Delta_0) \\ &\qquad\qquad + F(t + \tau) - \Delta_0, \\ &0 \le t \le \tau, \quad \Delta_0 = \begin{pmatrix} d_0 \\ 0 \end{pmatrix} \end{aligned} \quad (52)$$

Taking into account the properties

$$F(t + \tau) = -F(t), \quad 0 \le t \le \tau, \quad X_B = -X_0 \quad (53)$$

the symmetry of the motion is deduced:

$$X(t) = -X(t + \tau), \quad 0 \le t \le \tau \quad (54)$$

Appendix 5

Let us consider the same kind of non-sticking periodic orbits as in Sect. 6, with $V = 0$ but without the assumption $\tau = \Theta/2$.

For $0 < t < \tau$, the motion is given by (5), while for $\tau < t < \Theta$, the motion is described by

$$\begin{aligned} &\bar{Z}(t) = H(t - \tau)(\bar{Z}_B - F_B) + F(t), \\ &\bar{Z}(t) = \begin{pmatrix} \bar{z}(t) \\ z'(t) \end{pmatrix}, \quad \bar{z}(t) = z(t) + 2d_0, \\ &\bar{Z}_B = \bar{Z}(\tau) \end{aligned} \quad (55)$$

A periodic motion of period $\Theta = 2\pi/\omega$ is obtained if $Z(\Theta) = Z_0$ or:

$$\begin{aligned} &\bar{Z}_0 = H(\tau_1)\big(\bar{Z}_B - F_B\big) + F_0, \\ &Z_B = Z(\tau) = H(\tau)(Z_0 - F_0) + F_B, \\ &F_B = F(\tau), \quad \tau_1 = \Theta - \tau, \quad \bar{Z}_0 = \bar{Z}(0) \end{aligned} \quad (56)$$

From (56), we deduce:

$$\begin{aligned}\xi_1 &\equiv Z_B - F_B + E(Z_0 - F_0)\\ &= \big(H(\tau) + E\big)(Z_0 - F_0),\\ \xi_2 &\equiv \bar{Z}_B - F_B + E(\bar{Z}_0 - F_0)\\ &= \big(H(-\tau_1) + E\big)(\bar{Z}_0 - F_0)\end{aligned} \tag{57}$$

From the relations:

$$H(t)E = E\big(H(t)\big)^{-1} = EH(-t) \tag{58}$$

it follows:

$$\begin{aligned}&\big(H(-t) - E\big)\big(H(t) + E\big) = 0,\\ &\big(H(t) - E\big)\big(H(-t) + E\big) = 0\end{aligned} \tag{59}$$

From (57) and (59), we deduce

$$\big(H(-\tau) - E\big)\xi_1 = 0, \qquad \big(H(\tau_1) - E\big)\xi_2 = 0 \tag{60}$$

On the other hand, it is not difficult to show that

$$\xi_1 - \xi_2 = Z_B - \bar{Z}_B + E(Z_0 - \bar{Z}_0) = 0 \tag{61}$$

Let us introduce the following notations:

$$\begin{aligned}\xi_i &= \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \quad (i = 1, 2)\\ X_1 &= (H_1 - I)(z_0 - Q\cos\varphi)\\ &\quad + H_2\big(z_0' + Q\omega\sin\varphi\big),\\ Y_1 &= H_3(z_0 - Q\cos\varphi)\\ &\quad + (H_1 + I)\big(z_0' + Q\omega\sin\varphi\big),\\ X_2 &= (h_1 - I)(\bar{z}_0 - Q\cos\varphi)\\ &\quad - h_2\big(z_0' + Q\omega\sin\varphi\big),\\ Y_2 &= -h_3(\bar{z}_0 - Q\cos\varphi)\\ &\quad + (h_1 + I)\big(z_0' + Q\omega\sin\varphi\big),\\ H_j &= H_j(\tau), h_j = h_j(\tau_1), \quad (j = 1, 2, 3)\end{aligned} \tag{62}$$

From (60), it results:

$$\begin{aligned}Y_1 &= P_1X_1, \quad P_1 = H_2^{-1}(H_1 + I),\\ Y_2 &= P_2X_2, \quad P_2 = -h_2^{-1}(h_1 + I)\end{aligned} \tag{63}$$

The relations $X_1 = X_2, Y_1 = Y_2$, deduced from (61) are equivalent to:

$$X_1 = X_2, \quad PX_1 = 0, \quad P = P_1 - P_2 \tag{64}$$

From (64), two cases are obtained:

1. $\mathrm{Det}(P) \neq 0$ leads to $X_1 = 0$, hence

$$X_2 = 0, \qquad Y_1 = 0, \qquad Y_2 = 0 \tag{65}$$

From (56) and (64), we deduce

$$\begin{aligned}&z_B - Q\cos\varphi_B = z_0 - Q\cos\varphi,\\ &z_B' + Q\omega\sin\varphi_B = -\big(z_0' + Q\omega\sin\varphi\big),\\ &\bar{z}_B - Q\cos\varphi_B = \bar{z}_0 - Q\cos\varphi\end{aligned} \tag{66}$$

From the relation:

$$\begin{aligned}&z_B' = -Q\omega\sin\varphi_B - \big(z_0' + Q\omega\sin\varphi\big),\\ &z_{2B}' = z_{20}' = 0\end{aligned} \tag{67}$$

we obtain $\sin\varphi_B = -\sin\varphi$, hence

$$\begin{aligned}&\varphi_B \equiv \omega\tau + \varphi = -\varphi, \quad \varphi = -\omega\tau/2 \quad \text{or}\\ &\varphi_B \equiv \omega\tau + \varphi = \pi + \varphi, \quad \text{hence } \tau = \pi/\omega = \tau_1\end{aligned}$$

In the first case, ($\varphi_B \equiv -\varphi, \varphi = -\omega\tau/2$), from (67) $z_B' = -z_0'$ and from (66):

$$z_B = z_0 - Q\cos\varphi + Q\cos\varphi_B = z_0$$

which is impossible because for $0 < t < \tau$, $z_2' < 0$, hence: $z_{2B} < z_{20}$.

In the second case, ($\tau = \pi/\omega = \tau_1$), hence $H_i = h_i,\ i = 1, 2, 3$

$$\begin{aligned}&z_B + Q\cos\varphi = z_0 - Q\cos\varphi, \quad \text{i.e.,}\\ &x_B + Q\cos\varphi = x_0 - Q\cos\varphi.\end{aligned}$$

From (56), we get

$$\begin{aligned}&z_B + \bar{z}_B + 2Q\cos\varphi\\ &\quad = H_1(z_0 + \bar{z}_0 - 2Q\cos\varphi) \quad \text{i.e.}\\ &x_B + Q\cos\varphi\\ &\quad = H_1(x_0 - Q\cos\varphi) = x_0 - Q\cos\varphi\end{aligned} \tag{68}$$

Hence, if $\det(H_1 - I) \neq 0$, we obtain

$$x_0 = Q\cos\varphi, \qquad x_B = -Q\cos\varphi = -x_0 \tag{69}$$

2. $\mathrm{Det}(P) = 0$

From $P = H_2^{-1}(H_1 + I) + h_2^{-1}(h_1 + I)$: we deduce

$$\begin{aligned}&P = \Lambda\tilde{P}\Lambda^{-1}, \quad \tilde{P} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix},\\ &\Lambda = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\\ &\rho_i = \omega_i \frac{\sin(\alpha_i + \beta_i)}{(\sin\alpha_i)(\sin\beta_i)}, \quad (i = 1, 2)\\ &\alpha_i = \omega_i\tau/2, \quad \beta_i = \omega_i\tau_1/2\end{aligned} \tag{70}$$

It results $\mathrm{Det}(P) = 0$ if $\sin(\alpha_i + \beta_i) \equiv \sin(\omega_i\pi/\omega) = 0$, i.e., $\omega_i = k\omega$, $(k = 1, 2, \ldots)$ (resonance).

Except this particular case of resonance with the natural frequencies of the system, only symmetrical periodic solutions with a phase of slipping motion and a phase of overshooting motion exist.

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