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Jean Lerbet, Marwa Aldowaji, Noël Challamel, F. Nicot, Florent Prunier, Félix Darve

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J. Lerbet[1,*], M. Aldowaji[1], N. Challamel[2], F. Nicot[3], F. Prunier[4], and F. Darve[5]

[1] IBISC , UFRST-UEVE, Evry, France
[2] Laboratoire d'Ingénierie des MATériaux de Bretagne, UBS, Lorient, France
[3] Cemagrf, ETNA, Geomechanics Group, Grenoble, France
[4] Laboratoire de Génie Civil et Ingénierie Environnementale, INSA, Lyon, France
[5] Laboratoire Sols Solides Structures, UJF-INPG-CNRS, Grenoble, France

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The bifurcation problem of constrained non-conservative systems with non symmetric stiffness matrices is investigated. It leads to study the subset $D_{p,n}$ of $\mathcal{M}_n(\mathrm{R})$ of the so called p-positive definite matrices ($1 \leq p \leq n$). The main result ($D_{1,n} \subset D_{p,n}$) is proved, the reciprocal result is investigated and the consequences on the stability of elastic nonconservative systems are highlighted.

1 Introduction

This paper investigates the linear static stability of constrained nonconservative mechanical systems. More precisely, the systems studied are elastic systems subjected to nonconservative positional forces (or circulatory forces) and we investigate the loss of stability by divergence but the similar mathematical problem is met in geomechanics for non associated plastic materials. It is well known that for such systems, there is no general inclusion between the domain of instability by divergence and the domain of instability by flutter (see for example [7, 11] for precisions). It is also well known that such systems may present paradoxical behaviors if some damping is introduced in the models: damping may increase the instability domain (see [1, 13] for example and most recently see also [8]). It is, however, less reported that other paradoxical effects may be met for additional constraints. This problem has recently been investigated (see [3, 4, 10] for example) and remarkable new results obtained for one additional kinematic constraint: the additional constraint may destabilize the system and preventing the instability by divergence of the constrained system (ie for any kinematic constraint) leads to the second order work criterion: the symmetric part of the stiffness matrix must be positive definite. In the present paper, the general mathematical framework of the problem is highlighted, which leads to the new linear algebra concept of p-positive definite matrices ($0 < p \leq n$) of size n and the previous result is generalized for any family of p independent kinematic constraints. The paper is organized as follows. In the first part, as introduction to later developments, we start by illustrating on the example of a three degree of freedom Ziegler's column how the known approach is performing and how it does not lead to appropriate results. In the second part, the general mechanical background is given, which leads to the central mathematical problem. In the third part, the new concept is defined with its elementary properties and the main result of the paper is proved. Various other interesting mathematical problems related to the new concept are also suggested. In the last part, we return to mechanics and apply the results to the problem of stability. The initial example of a 3 d.o.f. Ziegler pendulum is again used as an illustration of general results.

2 A motivating example

Let us consider the following 3 d.o.f. Ziegler column as in Fig. 1. The mechanical system Σ consists of three bars OA, AB, BC with $OA = AB = AC = \ell$ linked by three elastic springs of the same stiffness k. The nonconservative external action (the circulatory force) is the follower force $\vec{P}$. The kinetic energy T of Σ reads (suppose an uniform distribution of mass):

* Corresponding author E-mail: jean.lerbet@ibisc.univ-evry.fr

$$
\begin{aligned}
T &= \frac{m\ell^2}{2}\left(\frac{7}{3}\dot\theta_1^2+\frac{4}{3}\dot\theta_2^2+\frac{1}{3}\dot\theta_3^2+3\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)+\dot\theta_1\dot\theta_3\cos(\theta_1-\theta_3)+\dot\theta_2\dot\theta_3\cos(\theta_2-\theta_3)\right)\\
&= \frac{1}{2}\dot\theta^T M(\theta)\dot\theta
\end{aligned}
$$

with

$$
\theta=\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix},\quad M(\theta)=m\ell^2\begin{pmatrix}\frac{7}{3} & \frac{3}{2}(\cos\theta_2-\theta_1) & \frac{1}{2}(\cos\theta_3-\theta_1)\\ \frac{3}{2}(\cos\theta_2-\theta_1) & \frac{4}{3} & \frac{1}{2}(\cos\theta_2-\theta_3)\\ \frac{1}{2}(\cos\theta_3-\theta_1) & \frac{1}{2}(\cos\theta_3-\theta_1) & \frac{1}{3}\end{pmatrix}
$$

the elastic energy of the springs is

$$
U=\frac{k}{2}((\theta_1^2+(\theta_1-\theta_2)^2+(\theta_2-\theta_3)^2)=\frac{k}{2}(2\theta_1^2+2\theta_1^2+\theta_3^2-2\theta_1\theta_2-2\theta_2\theta_3)
$$

and the virtual power of $\vec{P}$ in any configuration $\theta=(\theta_1,\theta_2,\theta_3)$ reads ($P>0$ in compression):

$$
\mathcal{P}_P^*=P\ell(\sin(\theta_3-\theta_1)\theta_1^*+\sin(\theta_3-\theta_2)\theta_2^*).
$$

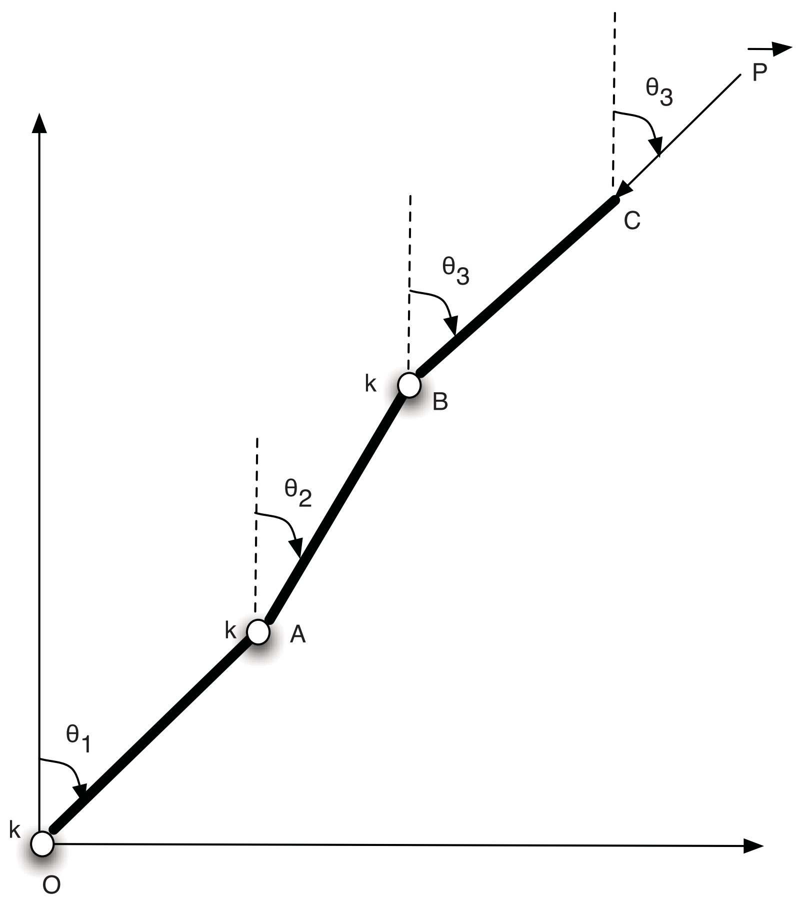

Fig. 1 3 dof Ziegler system.

Dimensions of the parameters m,ℓ,k are independent. Hence one can introduce units of measurement in such a way that all these parameters would be equal to one. Let us do it. Then it remains the only dimensionless parameter $\beta=\frac{P\ell}{k}$ which is thet loading parameter. The chosen equilibrium position is the vertical one $\theta_{eq}=0=(0,0,0)$. The stiffness matrix reads:

$$
K(\beta)=\begin{pmatrix}2-\beta & -1 & \beta\\ -1 & 2-\beta & -1+\beta\\ 0 & -1 & 1\end{pmatrix}.
$$

As it is well known but nevertheless paradoxical, $\det(K(\beta))=1$ is independent of β: the system cannot be unstable by divergence for any value of the loading parameter β (or with the notation of the paper $\beta_s=\infty$): to investigate the stability of the system, it is necessary to take into account the inertia of the system and to make a spectral analysis of the classical dynamical system (see (2) below). As consequence, flutter is here the unique mode of linear instability. With the chosen mass distribution, flutter occurs for the lowest value β_f such that the spectral equation $P(x) = \det(M(0)x + K(\beta)) = 0$ with $x = \mu^2$ has a multiple root. For the chosen mass distribution (uniform)

$$P(x) = (13/108)x^3 + (131/36 - (19/18)\beta)x^2 + (12 - (41/6)\beta^2)x + 1.$$

This equation possesses a multiple root for $\beta_f \approx 1.483$ such that the discriminant

$$\Delta = (1/46656)(45548392 - 89906052\beta + 70924849\beta^2 - 29092268\beta^3 + 6603748\beta^4 - 790848\beta^5 + 39168\beta^6)$$

vanishes.

However, as it is also known but still paradoxical, adding a kinematical constraint may destabilize the system. For example, consider the constraint $\theta_3 = 0$. For this new two dof system Σ_1, the reduced stiffness matrix reads:

$$K_1(\beta) = \begin{pmatrix} 2-\beta & -1 \\ -1 & 2-\beta \end{pmatrix}$$

and $\det(K_1(\beta)) = (1-\beta)(3-\beta)$ which leads to the value $\beta = 1$ for the instability by divergence of the constrained system.

This simple example leads to the following six questions:

- What about the (in)stability by divergence for another single kinematic constraint?
- What about the (in)stability by divergence for any other single kinematic constraint?
- Is it possible to preview the result with only the matrix K of the system without constraint ie without calculating the reduced stiffness matrices and solving the corresponding problems of divergence instability?
- What about the (in)stability by divergence for two kinematic constraints?
- What about the (in)stability by divergence for any system of two kinematic constraints?
- Is it possible to preview the result with only the matrix K of the system without constraint ie without calculating the reduced stiffness matrices and solving the corresponding problem of divergence instability?

In [3, 4, 10], the answer of the three first questions leads to the beautiful following result. Let Σ be any n dof mechanical system the stiffness matrix of which is $K(\beta)$ not necessary symmetric. Suppose Σ divergence stable for the range of variations of β. As long as the symmetric part $K_s(\beta)$ of $K(\beta)$ is positive definite, no single kinematic constraint may destabilize Σ. For λ_m the lowest positive singular value of $K_s(\beta)$ (root of $\det(K_s(\beta)) = 0$), there exists a kinematic constraint destabilizing the system and we may find this constraint on the cone of the quadratic form $x \mapsto^t xK_s(\beta_m)x$. The main goal of this paper is to generalize this result to any family of p kinematic constraints ($0 < p < n$). In the last part, we will come back to this example and we will apply to it these results for solving the last three previous questions.

3 Structural mechanics

3.1 General framework

Let us come back now to more general considerations. Stability of mechanical systems (body systems, continuum, discretized continuum) is considered to be a significant issue and one of the fundamental ones in mechanics and science in general. Therefore, in mechanics today, two main approaches may be identified. The first one called the “static approach” in the present paper concerns evolutions in which the inertia is not taken into account, either because it is negligible or because it does not contribute to the stability of the system. The second one can be linked to the stability in the sense of Lyapunov and mainly concerns a system’s dynamic evolution. Mechanically, this implies that inertial effects are considered, this called here the “dynamic” or spectral approach.

However, it is clear that the relation between the two approaches is essential for a proper understanding of the phenomena concerned. More specifically, we assume that the studied system Σ is evolving under the action of a parameter β (loading parameter) and we study the stability of an equilibrium state of Σ, a state defined by the position of each particle and the external and internal forces allowing this equilibrium to be reached. Each approach (“static” or “dynamic”) in general leads to two critical different values, β_s and β_d of β. These two values do not highlight exactly the same physical phenomena. The “static” approach makes it possible to understand the bifurcations of equilibrium of Σ and the “dynamic” approach the instability of the equilibrium considered (evolution of the system after a small kinematical (positions and/or velocities) disturbance of the equilibrium), the remarkable property consisting in the identity of these two phenomena for conservative systems.

Let us now consider a mechanical system Σ and an equilibrium reached under the loading β. Let M and $K(\beta)$ be, respectively, the (symmetric definite positive) matrix of mass and the matrix of stiffness of Σ at equilibrium. The framework is linear mechanics and the equation of bifurcation or of "static" instability (or also of the so-called divergence) is:

$$\det(K(\beta)) = 0. \tag{1}$$

The (linear) "dynamic" stability analysis is traditionally done through the spectral analysis of the dynamical system:

$$M\ddot{X} + K(\beta)X = 0. \tag{2}$$

Equation (1) leads to a value β_s of static critical load by divergence (and sometimes it may lead to any critical load; see [1] for example or the previous motivating example) and (2) leads to a value β_d of the dynamic critical load. As previously noted, if the system is conservative, then $\beta_s = \beta_d$, but for elastic nonconservative systems considering β_f as the flutter critical load so that $\beta_d = \text{Min}\{\beta_s, \beta_f\}$, both cases $\beta_s < \beta_f$ or $\beta_s > \beta_f$ may occur. Such nonconservative systems – as first reported by Ziegler [13] or later by Bolotin (see [1] for example) – possess "counterintuitive" behaviors. For example, the introduction of damping in such systems may be unfavorable for the systems stability (see [2] for the first explanation).

Another remarkable point, but less reported, concerns additional kinematic constraints to the system. While, for Rayleigh, it is conventionally accepted that the addition of such constraints reduced the extent of the spectrum and therefore supports to the stabilization (when the equilibrium is the minimum of the potential), some authors such as Tarnai ([12], for example) have called this principle into question even in the conservative case. However, in the given examples, the addition of kinematical constraints also changes the equilibrium position itself and are not within the Rayleigh theory field of applications. On the other hand, as it is showed in the motivating example of Sect. 2, the constrained system may be less stable than the unconstrained system, this counterintuitive result illustrating the nonconservativity of Σ as well.

The goal of this paper is to highlight a number of original mathematical developments of this problem of constrained nonconservative systems.

3.2 Statement of the problem

By [3], we consider the study of stability of a configuration of an elastic structure under any load, that is to say that the linearized equations around the equilibrium configuration studied are (2). The linearized equations of statics become:

$$K(\beta)X = 0 \tag{3}$$

a solution $X \neq 0$ indicating a "static" instability by loss of uniqueness.

It is then a solution of the equation $\det(K(\beta)) = 0$ and this loss of stability is called divergence. If the system is also subjected to nonconservative actions, $K(\beta)$ is any (nonsymmetric) matrix and it will be assumed in the following that, during the evolution of the system (variations of β), this matrix remains invertible: instability by divergence of the unconstrained system is assumed to be impossible.

Now suppose the same system is subjected to additional kinematical constraints such that the equilibrium configuration that we may consider as 0 is not disturbed. If there are p independent constraints $f(q) = f(q_1, \ldots, q_n) = (f_1(q), \ldots, f_p(q)) = 0_{\mathrm{R}^p}$ and if $A \in \mathcal{M}_{np}(\mathbf{R})$ means the Jacobian matrix of f ($A_{i,j} = \frac{\partial f_j(0)}{\partial q_i}$), then the column vector X of infinitesimal displacement must satisfy the following system:

$$\left\{ \begin{array}{rcl} K(\beta)X + A\Lambda & = & 0, \\ {}^tAX & = & 0, \end{array} \right. \tag{4}$$

where $\Lambda = {}^t(\lambda_1, \ldots, \lambda_p)$ is the Lagrange multipliers associated with the constraints in the equilibrium's configuration. Following [3], we can rewrite the previous system in a condensed manner. Let

$$K_A(\beta) = \begin{pmatrix} K(\beta) & A \\ {}^tA & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} X \\ \Lambda \end{pmatrix}.$$

So (4) is equivalent to:

$$K_A(\beta)Y = 0. \tag{5}$$

Note that:

$$\begin{pmatrix} K(\beta) & A \\ {}^tA & 0 \end{pmatrix} \begin{pmatrix} K^{-1}(\beta) & -K^{-1}(\beta)A \\ 0 & I_{pp} \end{pmatrix} = \begin{pmatrix} I_{nn} & 0 \\ {}^tAK^{-1}(\beta) & -{}^tAK^{-1}(\beta)A \end{pmatrix}.$$

Thus:

$$\det(K_A(\beta))\det(K^{-1}(\beta)) = (-1)^p \det({}^tAK^{-1}(\beta)A),$$

so:

$$\det(K_A(\beta)) = (-1)^p \det(K(\beta))\det({}^tAK^{-1}(\beta)A). \tag{6}$$

Let us write $\tilde{A} = K^{-1}(\beta)A$; it follows that:

$$\det(K_A(\beta)) = (-1)^p \det(K(\beta))\det({}^t\tilde{A}K(\beta)\tilde{A}) \tag{7}$$

because:

$$\begin{aligned}\det({}^tAK^{-1}(\beta)A) &= \det({}^tA{}^tK^{-1}(\beta){}^tK(\beta)K^{-1}(\beta)A) \\ &= \det({}^t\tilde{A}{}^tK(\beta)\tilde{A}) \\ &= \det({}^t({}^t\tilde{A}K(\beta)\tilde{A})) \\ &= \det({}^t\tilde{A}K(\beta)\tilde{A}).\end{aligned}$$

A part of these transformations are actually the Schur Complement formula and for more precisions see [14]. The analysis of singularities of the matrix $K_A(\beta)$ highlighting the instability by divergence of the constrained system is brought back to the singularities of the matrix ${}^t\tilde{A}K(\beta)\tilde{A}$, because $A \mapsto \tilde{A} = K^{-1}(\beta)A$ is a bijection in the set of matrices $\mathcal{M}_{np}(\mathbf{R})$ of maximal rank p. We denote $\mathcal{G}_{np}$ this set.
In the next section, we introduce the new concept of p-positive definite matrices.

4 p-positive definite matrices

4.1 Definition and immediate properties: symmetric case

Although the concept of a definite positive matrix is often used for symmetric matrices, it may be defined for any square matrix:

Definition 1 *Let $K \in \mathcal{M}_{nn}$ be a square matrix, K is said to be positive definite if and only if*

$${}^tXKX > 0 \quad \forall X \in \mathcal{M}_{n1} \quad X \neq 0$$

and that criterion concerns only the symmetric part $K_s = \frac{K+{}^tK}{2}$ of K because ${}^tXKX = {}^tXK_sX$.

Any positive definite matrix K is invertible (with a strictly positive real determinant), and K^{-1} is positive definite also.
Moreover, let us recall that if K is symmetric positive definite, it is diagonalizable in an orthonormal basis $(Y_k)_{k=1,\ldots,n}$ of $\mathbf{R}^n$ for the canonical scalar product $\mathbf{R}^n$: ${}^tY_kY_l = \delta_{k,l}$, and K defines a scalar product of $\mathbf{R}^n$.
We now define a sort of generalization of the useful concept of a definite positive matrix.

Definition 2 *Let $K \in \mathcal{M}_n(\mathbf{R})$ be a square matrix, $1 \leq p \leq n$ an integer. K is said to be p-positive definite if, for all matrices $A \in \mathcal{M}_{np}$ of rank p, then $\det({}^tAKA) > 0$. The subset of $\mathcal{M}_n(\mathbf{R})$ constituted by p-positive definite matrices is noted $D_{p,n}$.*
For $p = 1$ it is obvious that 1-positive definite is equivalent to positive definite.

Remark 1 *Let $\phi_K(A) = \det({}^tAKA)\ \forall A \in \mathcal{M}_{np}, 1 \leq p \leq n$. Thus we have:*

$$\phi_{-K}(A) = \det({}^tA(-K)A) = (-1)^p\phi_K(A). \tag{8}$$

Remark 2 *This definition may be extended to $p = 0$ by K is 0-definite positive if $\det(K) > 0$ and the inclusion $D_{0,n} \subset D_{1,n}$ is another way to formulate the well-known Bendixon theorem (sometimes also called the Bromwich theorem), which states that the smallest eigenvalue of the symmetric part K_s of any square matrix K is lower than any real part of the eigenvalues of K ([6] for example).*

4.2 Central result: 1-positive definite implies p-positive definite

We begin now by demonstrating a partial result:

Proposition 1: *if K is (1-)positive definite then K is 2-positive definite. In other words, $D_{1,n} \subset D_{2,n}$.*

Proof. Let $A \in \mathcal{M}_{n2}$ of rank 2, $A = (x_1 \; x_2)$ with $x_i \in \mathcal{M}_{n1}$ for $i = 1, 2$. We will sometimes identify $\mathcal{M}_{n1}$ and $\mathbf{R}^n$.

$$\det\left({}^tAKA\right) = \det\begin{pmatrix} {}^tx_1Kx_1 & {}^tx_1Kx_2 \\ {}^tx_2Kx_1 & {}^tx_2Kx_2 \end{pmatrix} = \det\begin{pmatrix} {}^tx_1K_sx_1 & {}^tx_1Kx_2 \\ {}^tx_2Kx_1 & {}^tx_2K_sx_2 \end{pmatrix},$$

where $K = K_s + K_a$ with K_s (resp. K_a) the symmetric part (resp. skew-symmetric part) of K.

$$\begin{aligned}\det\left({}^tAKA\right) &= \left({}^tx_1K_sx_1\right)\left({}^tx_2K_sx_2\right) - \left({}^tx_1Kx_2\right)\left({}^tx_2Kx_1\right) \\ &= \left({}^tx_1K_sx_1\right)\left({}^tx_2K_sx_2\right) - \left({}^tx_1K_sx_2\right)\left({}^tx_2K_sx_1\right) - \left({}^tx_1K_sx_2\right)\left({}^tx_2K_ax_1\right) \\ &\quad - \left({}^tx_1K_ax_2\right)\left({}^tx_2K_sx_1\right) - \left({}^tx_1K_ax_2\right)\left({}^tx_2K_ax_1\right)\end{aligned}$$

we have ${}^tK_s = K_s$, ${}^tK_a = -K_a$, so by a straightforward calculation, we obtain:

$$\det\left({}^tAKA\right) = \det\left({}^tAK_sA\right) + \left({}^tx_1K_ax_2\right)^2. \tag{9}$$

If we now consider the quadratic form associated to K_s, it defines a scalar product for which the Cauchy-Schwarz inequality gives:

$$\det\left({}^tAK_sA\right) = \left({}^tx_1K_sx_1\right)\left({}^tx_2K_sx_2\right) - \left({}^tx_1K_sx_2\right)\left({}^tx_2K_sx_1\right) > 0,$$

where the inequality is strict, because A being of rank 2, the vectors x_1, x_2 are not collinear.

We deduce that $\det({}^tAK_sA) = \left({}^tx_1K_sx_1\right)\left({}^tx_2K_sx_2\right) - \left({}^tx_1K_sx_2\right)\left({}^tx_2K_sx_1\right) > 0$, and therefore according to (9), $\phi_K(A) > 0$ and K is 2-positive definite. □

This proposition is actually general and constitutes the main result of the paper: every 1-definite positive matrix is p-definite positive for any $1 \leq p \leq n$. The demonstration of this general result is straightforward if K is symmetric:

Proposition 2: *if K is symmetric and 1-positive definite, then for any $1 \leq p < n$, K is p-positive definite.*

Proof. Let us suppose that K is symmetric (1-)definite positive and let $A \in \mathcal{M}_{np}$ be a matrix of rank p. Let us consider S the square root of K. S is also symmetric (1-)definite positive and ${}^tSS = S^2 = K$. Thus:

$$\phi_K(A) = \det({}^tAKA) = \det({}^tA{}^tSSA) = \det({}^tRR),$$

where $R = {}^t(AS)AS \in \mathcal{M}_{pp}$ is a symmetric positive definite matrix, because if $u \in \mathcal{M}_{p1}$ we have ${}^tuRu = {}^tu{}^tSSu = {}^t(Su)(Su) = \|Su\|^2 \geq 0$ and ${}^tuRu = 0$, implies $Su = 0$. It follows that $u = 0$ because rank(S)= p. We then deduce that K is p-positive definite. □

Let us recall that for nonconservative mechanical systems, we are concerned by nonsymmetric matrices. The general theorem holds for any matrix K:

Theorem 1 *If K is 1-positive definite then for any $1 \leq p < n$, K is p-positive definite. In other words, $D_{1,n} \subset D_{p,n}$.*

Proof. Let K be positive definite. Then K_s is symmetric positive definite, so there is a symmetric matrix (the square root of K_s) denoted $M \in \mathcal{M}_{nn}$ such that $K_s = M^2$.

Let $A \in \mathcal{M}_{np}$ of rank p, then the following equalities hold:

$$\begin{aligned}\det\left({}^tAKA\right) &= \det\left({}^tA\left(K_s + K_a\right)A\right) \\ &= \det\left({}^tA\left(M^2 + K_a\right)A\right) \\ &= \det\left({}^tAM\left(I_n + M^{-1}K_aM^{-1}\right)MA\right) \\ &= \det\left({}^tA{}^tM\left(I_n + M^{-1}K_aM^{-1}\right)MA\right)\end{aligned}$$

$$= \det \left({}^t AM \left(I_n + M^{-1} K_a M^{-1} \right) MA \right)$$
$$= \det \left({}^t (AM) \left(I_n + M^{-1} K_a M^{-1} \right) (MA) \right)$$

Let $U_{np} = \left\{ B \in \mathcal{M}_{np} | \text{ rank } (B) = p \right\}$, then $\psi : A \longmapsto B = \psi(A) = MA$ is a bijection from U_{np} into itself, we have $A = M^{-1} B$ with $B \in U_{np}$. Then:

$$\begin{aligned} \det \left({}^t AKA \right) &= \det \left({}^t B \left(I_n + M^{-1} K_a M^{-1} \right) B \right) \\ &= \det \left({}^t B \left(I_n + \tilde{K}_a \right) B \right) \\ &= \chi_{B, \tilde{K}_a}(-1), \end{aligned}$$

where $\tilde{K}_a = M^{-1} K_a M^{-1}$ is also skew-symmetric as K_a because M and M^{-1} are symmetric, and where

$$\chi_{B, \tilde{K}_a}(x) = \det \left({}^t B \left(\tilde{K}_a - x I_n \right) B \right)$$

could be called a generalized characteristic polynomial of $\tilde{K}_a$ relative to B. λ is a root (possibly complex) of $\chi_{B, \tilde{K}_a}$ if and only if the matrix ${}^t B \left(\tilde{K}_a - \lambda I_n \right) B$ is not invertible, this means if there is $U \in \mathcal{M}_{np}(\mathbf{C})$ nonzero such that:

$${}^t B \left(\tilde{K}_a - \lambda I_n \right) BU = 0,$$

then:

$${}^t B \tilde{K}_a BU = \lambda {}^t BBU.$$

Since the matrices are real and the matrix $\tilde{K}_a$ skew-symmetric, by taking the conjugate transpose (or adjoint), this gives:

$$-{}^t \bar{U} {}^t B \tilde{K}_a B = \bar{\lambda} {}^t \bar{U} {}^t BB.$$

By multiplying on the right by U:

$$-{}^t \bar{U} {}^t B \tilde{K}_a BU = \bar{\lambda} {}^t \bar{U} {}^t BBU$$

so:

$$-\lambda {}^t \bar{U} {}^t BBU = \bar{\lambda} {}^t \bar{U} {}^t BBU$$

then:

$$-\lambda \|BU\|^2 = \bar{\lambda} \|BU\|^2.$$

Consequently, $\lambda = -\bar{\lambda}$, meaning that the no null roots are purely imaginary and the only real root possible of $\chi_{B, \bar{K}}$ is then 0.

The roots of $\chi_{B, \bar{K}}$ are complex conjugates, whereas the polynomial is real. So let us group the conjugate roots together, let us separate the null root assumed to be of multiplicity r and the other conjugate roots assumed to be of number s and written as $\lambda_k = i\alpha_k$ with $\alpha_k > 0$. Since the dominant coefficient of $\chi_{B, \bar{K}}$ is $(-1)^p \det({}^t BB)$:

$$\chi_{B, \tilde{K}_a}(x) = (-1)^p \det({}^t BB) x^r \prod_{k=1}^{s} \left(x^2 + \alpha_k^2 \right) \tag{10}$$

and then:

$$\chi_{B, \tilde{K}_a}(-1) = (-1)^{r+p} \det({}^t BB) \prod_{k=1}^{s} \left(1 + \alpha_k^2 \right). \tag{11}$$

We can deduce that the sign of $\chi_{B, \tilde{K}_a}(-1)$ is that of $(-1)^{r+p} \det({}^t BB)$.

But $r = p - 2s$, so $(-1)^{r+p} = (-1)^{2p-2s} = 1 > 0$, and $\det({}^t BB) > 0$ because ${}^t BB$ is a symmetric real matrix, so it is diagonalizable and has real eigenvalues μ_i for $i = 1, \dots, p$.

Let $X_i, i = 1, \dots, p$ be the eigenvectors corresponding to μ_i, then ${}^t BB\, X_i = \mu_i\, X_i$, thus ${}^t X_i^t BB\, X_i = \mu_i\, {}^t X_i X_i \Rightarrow \|BX_i\|^2 = \mu_i \|X_i\|^2 \Rightarrow \mu_i > 0, \forall i = 1, \dots, p$.

We deduce that $\chi_{B, \tilde{K}_a}(-1) > 0$, i.e., $\phi_K(A) = \det \left({}^t AKA \right) > 0$. □

4.3 The reciprocal result

First of all, let us note that the converse of Theorem 1 is false. More precisely, the converse of Proposition 1 is itself false, which implies that the converse of Theorem 1 is false:

Proposition 3: *For any n-square matrix K, K 2- definite positive does not imply K (1)-definite positive. In other words, $D_{2,n} \not\subset D_{1,n}$.*

P r o o f. Let $K = -I_n$. $-K = I_n$ is 1- definite positive. So by Proposition 2, $-K$ is 2-definite positive. But according to relation (8) of remark 1, $\phi_{-K}(A) = \phi_K(A)$ for any $A \in \mathcal{M}_{n2}$. Thus K is 2-definite positive as well and it is not (1)-definite positive: K is actually definite negative! □

It is interesting to highlight the reasons why any (invertible) matrix is not a (p)-definite positive matrix. Indeed, returning to the problem of structural stability, a value of loading parameter λ corresponding for an (invertible) stiffness matrix $K(\lambda)$ to pass from (1)-positive definite to not (1)-positive definite corresponds exactly to a loss of stability by divergence of an adequate constrained system. This very important result has already been proved in [4] and the object of the following section is to investigate the general case for any $1 \leq p < n$. We now investigate the reciprocal result of Theorem 1 by supposing that K is not definite positive but with K_s invertible and by investigating the resulting properties about ϕ_K. The following result holds:

Theorem 2 *If K is any invertible (neither positive definite nor negative definite) matrix, then $\phi_K(A) = \det({}^tAKA)$ is not strictly positive for all $A \in \mathcal{M}_{np}$ for $1 \leq p < n$ and in particular for $p = 2$.*

P r o o f. To rank $K_S = r = n$, the signature of the quadratic form q is $(k, n-k)$ and none of these numbers is zero. Let

$$\mathcal{B} = (e_1, \dots, e_k, e_{k+1}, \dots, e_n)$$

be an orthogonal basis for q. We can also assume that:

$$q(e_i) = 1 \quad \text{if} \quad i = 1, \dots, k$$
$$q(e_i) = -1 \quad \text{if} \quad i = k+1, \dots, n.$$

Let $F^+ = Vec(e_1, \dots, e_k)$ and $F^- = Vec(e_{k+1}, \dots, e_n)$. So the sum $\mathbf{R}^n = F^+ \oplus F^-$ is orthogonal for q, i.e, ${}^te_iK_se_j = 0$ if $i \neq j$.

Let us now consider P is the matrix passage from the canonical basis to the basis $\mathcal{B}$.

$${}^tPK_sP = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}.$$

Moreover, let us consider $x \in \mathbf{R}^n$, and let us write $x = x^+ + x^-$ the only decomposition in $\mathbf{R}^n$, so ${}^t(x^+)K_sx^- = 0$. Let us assume now that $p = 2$ and let $A = {}^t(x_1\ x_2) \in \mathcal{M}_{n2}$ be a matrix of rank 2. Let us also choose ${}^tx_1$ and ${}^tx_2$ in $\mathbf{R}^n$ (identification $\mathbf{R}^n$ and $\mathcal{M}_{1n}$) orthogonal for q or K_s. Then:

- ${}^tx_1K_sx_2 = 0$,
- ${}^tx_1K_ax_1 = 0$, ${}^tx_2K_ax_2 = 0$.

Therefore:

$$\phi_K(A) = \det \begin{pmatrix} q(x_1) & {}^tx_1K_ax_2 \\ {}^tx_2K_ax_1 & q(x_2) \end{pmatrix} = q(x_1)q(x_2) + ({}^tx_1K_ax_2)^2.$$

Thus, $\phi_K(A) \leq 0 \iff ({}^tx_1K_ax_2)^2 \leq -q(x_1)q(x_2)$.

By homogeneity ($x_i \rightarrow \frac{x_i}{q(x_i)}$), we can choose $q(x_1) = 1$, $q(x_2) = -1$. It follows that $\phi_K(A) = -1 + ({}^tx_1K_ax_2)^2$.

1. If K_a is singular (recall that the only real eigenvalue possible of K_a is 0), there is $x \neq 0$ such that $K_ax = 0$, and let us write $x = x^+ + x^-$ the only decomposition in $\mathbf{R}^n$, so we have $K_ax^+ = -K_ax^-$.
    - If $x^+ = 0$, we consider $x_2 = x^-$, it follows that $K_ax^- = 0$, and then ${}^tx_1K_ax_2 = 0$, so $\phi_K(A) = -1$ for all $x_1 \in F^+$.

- If $x^- = 0$, we consider $x_1 = x^+$, it follows that $K_a x^+ = 0$, and then ${}^t x_1 K_a x_2 = 0$, so $\phi_K(A) = -1$ for all $x_2 \in F^-$.
- If neither x^+ nor x^- are not zero, then we take $x_1 = x^+$, $x_2 = x^-$. Then (x_1, x_2) are independent (they are not zero and in two supplementary vector subspaces). The calculation then gives

  $${}^t x_1 K_a x_2 =^t (x^+) K_a x^- = -^t(x^+) K_a x^+ = 0$$

  because $K_a x^+ = -K_a x^-$ and K_a is skew-symmetric. It follows that $\phi_K(A) = -1$.

2. If K_a is nonsingular, we will assume for example that $k > n - k$, i.e., $\dim(F^+) > \dim(F^-)$, so $\dim(K_a(F^-)) = \dim(F^-) = n - k$, because K_a is a bijective. Let us consider H the orthogonal of $K_a(F^-)$ for the canonical scalar product $(x|y) =^t xy$, then we have $\dim(H) = n - (n - k) = k$. Therefore $\dim(F^+ + H) = \dim(F^+) + \dim(H) - \dim(F^+ \cap H) \leq n$ and thus $\dim(F^+ \cap H) \geq \dim(F^+) + \dim(H) - n = k + k - n = 2k - n > 0$. Consider then $0 \neq x_1 \in F^+ \cap H$ and $0 \neq x_2 \in F^-$, so $K_a x_2 \in K_a(F^-)$ and $x_1 \in H$, and thus $(x_1|K_a x_2) =^t x_1 K_a x_2 = 0$.
   The demonstration in the opposite case $k < n - k$ is strictly similar.
3. There remains the very particular case where K_a is singular, $n = 2k$ is even, and the signature of K_s is exactly (k, k). Moreover, because $2 \leq p < n$, we have at least $k \geq 2$, so n is even and ≥ 4, the cases of $n = 2$ and $n = 3$ have already been completely resolved.
   Let $0 \neq x_2 \in F^-$, $z = K_a x_2$, and $z^\perp$ its orthogonal for the canonical scalar product in $\mathbf{R}^n$. This is a vector space of dimension $n-1 = 2k-1$. It is known that $\dim F^+ = k$, so $\dim(F^+ \cap z^\perp) \geq 2k-1+k-2k = k-1$, but $k \geq 2$, thus $\dim(F^+ \cap z^\perp \geq 1)$. So there is a vector $0 \neq x_1 \in F^+ \cap z^\perp$; then it is clear that $x_1 \in F^+$ and ${}^t x_1 z =^t x_1 K_a x_2 = 0$.

In all cases, the result obtained shows that, if the signature of K_s is $(k, n-k)$ with $0 < k < n$, then there is $A \in \mathcal{M}_{n2}$ with rank $(A) = 2$ such that $\phi_K(A) = \det({}^t AKA) = -1 < 0$. We note that for any n and any signature of K_s (of rank n), taking $p = 2$ gives a matrix A such that $\phi_K(A) = \det({}^t AKA) = -1 < 0$. One might ask whether this is true for all $1 \leq p < n$. □

4.4 Open problems

Two interesting open problems can be identified on the new concept of p-definite matrices.

- For $1 \leq p < p' \leq n$, does p-definite positive imply p'-definite positive or in other words, $D_{p,n} \subset D_{p',n}$? The main result of the paper is the proof for $p = 1$.
- For $1 \leq p < p' \leq n$, describe the sets $D_{p',n} \setminus D_{p,n}$, $D_{p,n} \setminus D_{1,n}$.

4.5 Another possible generalization

Although the proposed generalization of definite positive matrices ensues logically from mechanical considerations, another mathematical point of view should have been able to be chosen by considering that a matrix K should be p-definite positive in $\mathcal{M}_n(\mathbf{R})$ if for all $A \in \mathcal{M}_{np}$ with rank$(A) = p$, ${}^t AKA$ would be a (1)-definite positive matrix in $\mathcal{M}_p(\mathbf{R})$. We will then call K a p^*-definite positive matrix.

Proposition 4: *If K is (1)-definite positive then K is p^*-definite positive.*

Proof. Suppose that K is a (1)-definite positive matrix. Let $A \in \mathcal{M}_{np}$ with rank$(A) = p$ be a matrix and $\lambda \in \mathbf{R}^p$. Thus ${}^t\lambda {}^t AKA\lambda = {}^t(A\lambda)K(A\lambda) =^t uKu > 0$ with $u \in \mathbf{R}^n$ because K is a (1)-definite positive matrix. □

Proposition 5: *Every p^*-definite positive matrix is p-positive definite but the converse is false. However, if K is symmetric, then p-definite positive is equivalent to p^*-definite positive.*

Proof. Suppose that K is a p^*-definite positive matrix. Let $A \in \mathcal{M}_{np}$ with rank$(A) = p$ be a matrix. Then, ${}^t AKA$ is definite positive. By triangularizing ${}^t AKA$ in $\mathbf{C}$, we obtain:

$$\det({}^t AKA) = \prod_{k=1}^{p} \mid \lambda_k \mid^2 \prod_{k=1}^{r} \mu_k$$

with $\lambda_k, \bar{\lambda}_k, k = 1\ldots, p$ the complex eigenvalues of ${}^t AKA$ and $\mu_k, k = 1\ldots, r$ its real eigenvalues. Let u_k be a real eigenvector attached to μ_k. Then ${}^t AKAu_k = \mu_k u_k$ and $0 <^t u_k^t AKAu_k = \mu_k \mid\mid u_k \mid\mid^2$. We deduce that $\mu_k > 0$ for all $k = 1\ldots, r$ and finally $\det({}^t AKA) > 0$: K is p-definite positive.

The converse is obviously false: for example every matrix B with $\det(B) > 0$ is not necessarly definite positive and the equivalence for symmetric matrices is easy to prove:

for symmetric matrices, definite positive $\Leftrightarrow$ p-definite positive for all $1 \leq p \leq n \Leftrightarrow p^*$-definite positive for all $1 \leq p \leq n$. □

5 Consequences on the system's mechanical stability

5.1 General considerations

From a mechanical viewpoint, we recalled in the first section that additional constraints in nonconservative elastic systems do not necessarily improve the structure's stability.

Moreover, following several recent papers [3, 4, 9, 10], a new criterion of "static" stability, generalizing the criterion of divergence for nonconservative systems may be used, which may be viewed as a transposition in structural mechanics of the second-order work criterion, well-known in geomechanics. This criterion means, mathematically, that K is 1-definite positive. The above cited studies have proved the efficiency of this criterion in stability of structures for many aspects of the stability problem. This new criterion obviously involves only the symmetric part K_s of K and in [9], a partial relation between this criterion and the Lyapunov stability criterion was also established. As already noted, in [3], this criterion appears as an optimum for stability by divergence of one-constrained nonconservative elastic systems.

To describe the consequences on stability, we then have to distinguish both criteria of "static" stability, which leads to three stability criteria (in the linear stability domain):

a) in classical static instability or divergence (in the case of relation (1)), we will speak of s-instability and β_s is the corresponding value.
b) in linear or weak dynamic instability or flutter, we will speak of f-instability and β_f is the corresponding value.
c) in new criterion of static instability or mixed instability, corresponding to $K(\beta)$ not (1-) definite positive, we will speak of m-instability and β_m is the corresponding value.

$\beta_m \leq \beta_s$ always holds (see [9] for a proof), but any general relation is known concerning β_f.

In the present paper, the introduction of the new concept of p-definite matrices and the associated central result has two distinct consequences concerning structural stability.

- Adopting the new criterion of "static" stability, it proves that the introduction of constraints in the system does not change the "static" stability in the sense of divergence. Indeed, suppose that $K(\beta)$ is 1-definite positive or equivalently that $\beta < \beta_m$. With p additional constraints, Theorem 1 implies $K(\beta)$ is p-definite positive and thus the system is still s-stable. It is then interesting to investigate if the new criterion is itself automatically satisfied for the constrained system, that is to say is Σ m-stable.
  Let us clarify the problem. Let Σ be a mechanical system, $q_e = 0$ an equilibrium configuration such that the stiffness matrix of Σ at 0 is K. Suppose that K is (1-) definite positive (new criterion of "static" stability of this equilibrium) and consider now p additional constraints. Let Σ_p be this new system, the equilibrium configuration assumed to be the same: we assume that the additional constraints do not disturb the equilibrium. The question is then the following: is Σ_p still m-stable? In other words, considering K_p the stiffness matrix of Σ_p, is K_p (1-) still definite positive?
  Eliminating p variables by the relations of constraints, Σ_p is a $n-p$ degrees of freedom system and K_p should be a $n-p \times n-p$ matrix. However, this approach requires eliminating p variables of the system and recalculating K_p. It may be easily proved that K_p is (1-) definite positive if and only if K is (1-) definite positive on the $n-p$ subspace of $\mathbf{R}^n$ defined by the relations ${}^tAX = 0$ of (4), which is obviously satisfied because K is (1-) definite positive on the whole space $\mathbf{R}^n$.
- Without adopting the new criterion, suppose only that Σ is s-stable with only a single additional constraint. Then Theorem 1 shows that it is automatically s-stable with p additional constraints: adding several other kinematical constraints does not change the domain of s-instability by divergence.

Moreover, another important argument of continuity highlights the usefulness of the new concept of p-definite positive matrices. In practice, the loading parameter β increases from 0 to the critical value β^*. At $\beta = 0$, the system may be assumed to be conservative and stable and thus $\det(K(0)) > 0$ and then $K(0)$ is symmetric definite positive and thus the corresponding system is stable when submitted any to p additional kinematic constraints (Rayleigh theory). It follows that $K(0)$ is also p-definite positive: as noted in the last proposition, for symmetric matrices all these concepts are equivalent. S-stability of a constrained system during the loading path (β increasing) is ensured by the condition of p-definite positivity: as long as $K(\beta)$ is p-definite positive, Σ remains s-stable.

The last open problem (describe $D_{p,n} \setminus D_{1,n}$) has the following mechanical counterpart. Suppose that the loading parameter β takes a value $\beta > \beta_m$ such that the symmetric part $K_s(\lambda)$ has one negative eigenvalue. In terms of structural stability, that means that there exists a convenient kinematical constraint ($p = 1, A = A_1 \in \mathcal{M}_{n,1}(\mathbf{R})$) such that ${}^t\tilde{A}K(\beta)\tilde{A} \leq 0$ and the constrained system is no longer stable according to divergence stability. In terms of geomechanics, loading parameters exist (boundary conditions) to go into the isotropic cone. Do other additional kinematic constraints exist ($p \geq 2, A = (A_1, \ldots, A_p) \in \mathcal{M}_{n,p}(\mathbf{R})$) stabilizing the system? If $K(\beta) \notin D_{p,n}$ such additional kinematic constraints do not exist, but if $K(\beta) \in D_{p,n} \setminus D_{1,n}$ such additional constraints do exist.

5.2 Optimization problem

In the previous section, we showed that if a mechanical system Σ is such that its stiffness matrix K (or its symmetric part) is positive definite, then the instability by divergence may not occur for any system Σ' deduced from Σ by applying kinematical constraints on Σ that one could call a subsystem Σ' of Σ. The converse is an interesting problem, the answer being positive for one constraint as was shown in [4]. We claim here that this result is universal for any system of (linear) kinematical constraints: an n dof mechanical system Σ ($K = K(\beta)$ with β an increasing loading parameter) may be kinematically constrained by p constraints ($1 \leq p < n$) such that the constrained system becomes unstable by divergence if and only if β satisfies $\det(K_s(\beta)) = 0$ and we may also clarify the system of constraints that will to destabilize Σ. We recall that β_m is defined as the smallest positive root of $\det(K_s(\beta)) = 0$ so that $\beta \in [0, \beta_m]$ implies that $K = K(\beta)$ is (1)-definite positive.

As in the previous section, we begin with the case $p = 2$.

Proposition 6: *Let $K = K(\beta)$ be a matrix 2-definite positive. Then for all $\beta < \beta_m$, any subsystem Σ' obtained from Σ by applying two kinematical constraints on Σ is stable according to the divergence criterion, and for $\beta = \beta_m$, there exists a system of two independent kinematical constraints such that the corresponding subsystem is divergence unstable. Moreover, the system of constraints allowing the destabilization may be explicitly given.*

Proof. From Proposition 1, we know that for all $\beta < \beta_m$, any subsystem Σ' obtained from Σ by applying two kinematical constraints on Σ is stable and that if the $A \in \mathcal{G}_{n2}$ subset of $\mathcal{M}_{n2}$ of matrices of rank 2, then $\phi_K(A) > 0$. Suppose now that $\beta = \beta_m$ and again use the notations of the demonstration of Proposition 1. Because the spectrum of $K_s(\lambda_m)$ is now included in $\mathbf{R}_+$ and no longer in $\mathbf{R}_+^*$, the Cauchy-Schwarz inequality becomes a large inequality. Let $A \in \mathcal{G}_{n2}$, $A = (x_1\ x_2)$ with $x_i \in \mathcal{M}_{n1}$ for $i = 1, 2$. Thus:

$$\det\left({}^tAK_s(\beta_m)A\right) = \left({}^tx_1K_s(\beta_m)x_1\right)\left({}^tx_2K_s(\beta_m)x_2\right) - \left({}^tx_1K_s(\beta_m)x_2\right)\left({}^tx_2K_s(\beta_m)x_1\right) \geq 0$$

and (9) becomes:

$$\begin{aligned}\det\left({}^tAK(\beta_m)A\right) = &\left({}^tx_1K_s(\beta_m)x_1\right)\left({}^tx_2K_s(\beta_m)x_2\right) - \left({}^tx_1K_s(\beta_m)x_2\right)\left({}^tx_2K_s(\beta_m)x_1\right) \\ &+ \left({}^tx_1K_a(\beta_m)x_2\right)^2 \geq 0.\end{aligned}$$

Let us now choose $x_1 \neq 0 \in \text{Ker}(K_s(\beta_m))$ and $x_2 \in K_a(\beta_m)x_1^\perp$. Note that $K_a(\beta_m)x_1 \neq 0$ because if $K_a(\beta_m)x_1 = 0$, then $K(\beta_m)x_1 = K_s(\beta_m)x_1 + K_a(\beta_m)x_1 = 0$ and $K(\beta_m)$ is then singular, which is a contradiction with the main assumption that the whole system without constraints is divergence stable: $K(\beta_m)$ must be regular. Thus $\dim Vec(K_a(\beta_m)x_1) = 1$ and $\dim Vec(K_a(\beta_m)x_1)^\perp = n - 1$. There are $n - 1$ independent possible choices for x_2. With such choices:

$$\det\left({}^tAK(\beta_m)A\right) = 0,$$

which signs the instability by divergence of the constrained system. □

Let us now examine the general case with a family of $p < n$ constraints. The following result holds:

Theorem 3: *Let $K = K(\beta)$ be a p-definite positive matrix. Then for all $\beta < \beta_m$, any subsystem Σ' obtained from Σ by applying p kinematical constraints on Σ is stable according to the divergence criterion and for $\beta = \beta_m$, a system of p independent kinematical constraints exists such that the corresponding subsystem is divergence unstable. Moreover, the system of constraints allowing destabilization may be explicitly given.*

Proof. From Theorem 1, we know that for all $\beta < \beta_m$, any subsystem Σ' obtained from Σ by applying p kinematical constraints on Σ is stable and that if $A \in \mathcal{G}_{np}$ subset of $\mathcal{M}_{np}$ of matrices of rank p, then $\phi_K(A) > 0$.

Suppose now $\beta = \beta_m$ and suppose, for example, that $\dim \ker K_s(\beta_m) = 1$. Let us choose $x_1 \neq 0 \in \ker K_s(\beta_m)$. Then by a similar reasoning, as for the case $p = 2$, $K_a x_1 \neq 0$. So $\dim Vec(K_a(\lambda_m)x_1) = 1$ and $\dim Vec(K_a(\lambda_m)x_1)^\perp = n - 1$. Let us choose $p - 1$ independent vectors $x_2, \ldots, x_p \in Vec(K_a(\beta_m)x_1)^\perp$. Because $p < n$, this always possible, although x_1 itself belongs to $Vec(K_a(\beta_m)x_1)^\perp$. Thus, the first column of ${}^tAK(\beta_m)A$ is:

- on the first row:
  ${}^tx_1K(\beta_m)x_1 = {}^t x_1K_s(\beta_m)x_1 = 0$ because $K_s(\beta_m)x_1 = 0$;
- on the other rows ($j = 2, \ldots, p$):
  ${}^tx_1K(\beta_m)x_j = {}^t x_1K_s(\beta_m)x_j + {}^t x_1K_a(\beta_m)x_j = {}^t x_jK_s(\beta_m)x_1 + {}^t x_1K_a(\beta_m)x_j = 0 + 0 = 0$ because $K_s(\beta_m)x_1 = 0$ and $x_j \in Vec(K_a(\beta_m)x_1)^\perp$.

The first column of ${}^tAK(\beta_m)A$ is then nil and thus:

$$\det\left({}^tAK(\beta_m)A\right) = 0,$$

which signs the instability by divergence of the constrained system. □

Note that fortunately the result cannot be extended to $p = n$: with n constraints, the system cannot move and is stable. Remark also that this optimization problem could appear as a geometric problem of the optimization of an orthogonal invariant function ϕ on a Grassmann manifold like in [5]. Indeed, the function ϕ_K is (right)$\mathcal{O}(p)$-invariant because if $Q \in \mathcal{O}(p)$ we have:

$$\begin{aligned}\phi_K(AQ) &= \det({}^tQ\,{}^tAKAQ) = \det({}^tQ)\det({}^tAKA)\det(Q) = \det({}^tQ)\det(Q)\phi_K(A)\\ &= \det({}^tQQ)\phi_K(A) = \det(I_p)\phi_K(A)\\ &= \phi_K(A)\end{aligned}$$

but here we do not need to look for a numerical algorithm because we have got an exact algebraic solution due to the special form of the objective map ϕ_K.

To summarize, the second-order work criterion appears as the optimum criterion so that a divergence stable system Σ remains divergence stable when it is submitted to any system of kinematical constraints.

5.3 A 3 d.o.f. system

To illustrate the previous results, we come back to the first motivating example, we apply the previous results and we take advantage of it to answer the last questions of Sect. 2 on hold.

5.3.1 Instability of the constrained system and second-order criterion

The symmetric part $K_s(\beta)$ of $K(\beta)$ is:

$$K_s(\beta) = \begin{pmatrix} 2-\beta & -1 & \frac{\beta}{2} \\ -1 & 2-\beta & -1+\frac{\beta}{2} \\ \frac{\beta}{2} & -1+\frac{\beta}{2} & 1 \end{pmatrix}.$$

The roots of $\det(K_s(\beta)) = 0$ are $1, 1-\sqrt{3}, 1+\sqrt{3}$ and thus, with previous notations, $\beta_m = 1$, which leads to

$$K_s(1) = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

denoting as previously by $Vec(u_1, \ldots, u_r)$ the vector space generated by the family of vectors $u_1, \ldots, u_r$, calculations give: Ker$(K_s(1)) = Vec(x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix})$. The skew symmetric part of $K(1)$ is $K_a(1) = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$ and thus the subspace $Vec(K_a(1)(x_1))$ generated by the vector $K_a(1)(x_1)$ is the one-dimensional subspace $Vec(\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix})$ and its orthogonal is the plane $Vec(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix})$. Following the algorithm given by the proof of Theorem 3, let us choose x_2 in this plane such that (x_1, x_2) is a free family. For example, $x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Now let $A = (x_1 x_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$. Calculations give ${}^tAK(1)A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and obviously $\det({}^tAK(1)A) = 0$. Be careful that the matrix A is actually the matrix $\tilde{A}$ of Sect. 3.2. To come back to the matrix of kinematical constraints we have to calculate $K(1)A$ which leads to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}$ meaning that the 3 dof Ziegler system constrained by $\theta_1 - \theta_2 = 0$ and $-\theta_3 = 0$ is divergence unstable for $\beta = 1$ which may be immediately checked.

5.3.2 Optimization problem

Let us put $A = (x_1 x_2) = \begin{pmatrix} \alpha_1 & \delta_1 \\ \alpha_2 & \delta_2 \\ \alpha_3 & \delta_3 \end{pmatrix}$. $A \mapsto \phi_K(A) = \phi_{K(\beta)}(A)$ may so be considered as a function of six variables $\alpha_1, \delta_1, \alpha_2, \delta_2, \alpha_3, \delta_3$ plus the loading parameter β. The optimization problem is to calculate:

$$\inf_{(\alpha_i,\delta_i)\in U} \phi_{K(\beta)}(\alpha_i, \delta_i)$$

and to prove that $\beta = 1$ is the lower positive value of the loading parameter. U is the open set of $\mathbb{R}^6$ such that $\text{rank}(A) = 2$.

MAPLE may solve the complete problem, i.e., the nonlinear system

$$\frac{\partial \phi_{K(\beta)}(\alpha_i, \delta_i)}{\partial \alpha_i} = 0, \frac{\partial \phi_{K(\beta)}(\alpha_i, \delta_i)}{\partial \delta_i} = 0, \quad i = 1, 2, 3.$$

The nine families of solutions are:

$$S_1 = \{\beta = \beta, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \alpha_3 = \alpha_3, \delta_1 = 0, \delta_2 = 0, \delta_3 = 0\},$$
$$S_2 = \{\beta = \beta, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = \alpha_3, \delta_1 = 0, \delta_2 = 0, \delta_3 = \delta_3\}$$
$$S_3 = \left\{\beta = \beta, \alpha_1 = \alpha_1, \alpha_2 = 0, \alpha_3 = \frac{\alpha_1\delta_3}{\delta_1}, \delta_1 = \delta_1, \delta_2 = 0, \delta_3 = \delta_3\right\},$$
$$S_4 = \left\{\beta = 1 \pm \sqrt{3}, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \alpha_3 = \alpha_2 + \alpha_1, \delta_1 = \delta_1, \delta_2 = 0, \delta_3 = \delta_1\right\},$$
$$S_5 = \left\{\beta = \beta, \alpha_1 = \frac{\alpha_2\delta_1}{\delta_2}, \alpha_2 = \alpha_2, \alpha_3 = \frac{\delta_3\alpha_2}{\delta_2}, \delta_1 = \delta_1, \delta_2 = \delta_2, \delta_3 = \delta_3\right\},$$
$$S_6 = \left\{\beta = 1 \pm \sqrt{3}, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \alpha_3 = \alpha_2 + \alpha_1, \delta_1 = -\delta_2 + \delta_3, \delta_2 = \delta_2, \delta_3 = \delta_3\right\},$$
$$S_7 = \{\beta = 1, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \alpha_3 = 0, \delta_1 = \delta_1, \delta_2 = \delta_2, \delta_3 = 0\},$$
$$S_8 = \left\{\beta = \frac{-2\,\delta_1\delta_2 + 2\,\delta_1^2 + \delta_2^2}{\delta_1\,(\delta_1 - \delta_2)}, \alpha_1 = \alpha_1, \alpha_2 = \frac{\alpha_1\delta_2}{\delta_1}, \alpha_3 = \frac{\alpha_1\delta_3}{\delta_1}, \delta_1 = \delta_1, \delta_2 = \delta_2, \delta_3 = \delta_3\right\},$$
$$S_9 = \left\{\beta = 1 + \sqrt{3}, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \alpha_3 = \alpha_2 + \alpha_1\,,\right.$$
$$\left.\delta_1 = 1/2 \pm 1/2\,\sqrt{3 + 2\,\sqrt{3}}\,\delta_2, \delta_2 = \delta_2, \delta_3 = \delta_2 + 1/2 \pm 1/2\,\sqrt{3 + 2\,\sqrt{3}}\,\delta_2\right\}.$$

For S_1, S_2, S_3, S_5, S_8, $\text{rank}(A) = 1$ the vectors x_1 and x_2 are colinear and these solutions must be rejected. for S_4 and S_9, $\beta = 1 \pm \sqrt{3}$ corresponds to the other two values of the load parameter, which are roots of $\det(K_s)(\beta) = 0$: they are < 0 or > 1. Thus the solution S_7 with $\beta = 1$ is the optimal solution and the value of A ($A = (x_1 x_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$) suggested above corresponds to a possible value of solutions of S_7: $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0, \delta_1 = 1, \delta_2 = 0, \delta_3 = 0$. Figure 2 plots on the z-axis the values of the partial function $(\delta_2, \beta) \mapsto \phi_{K(\beta)}(1, 1, 0, 1, \delta_2, 0)$, which shows the optimal value $\beta = 1$ for all δ_2 as a particular case of the solution S_7.

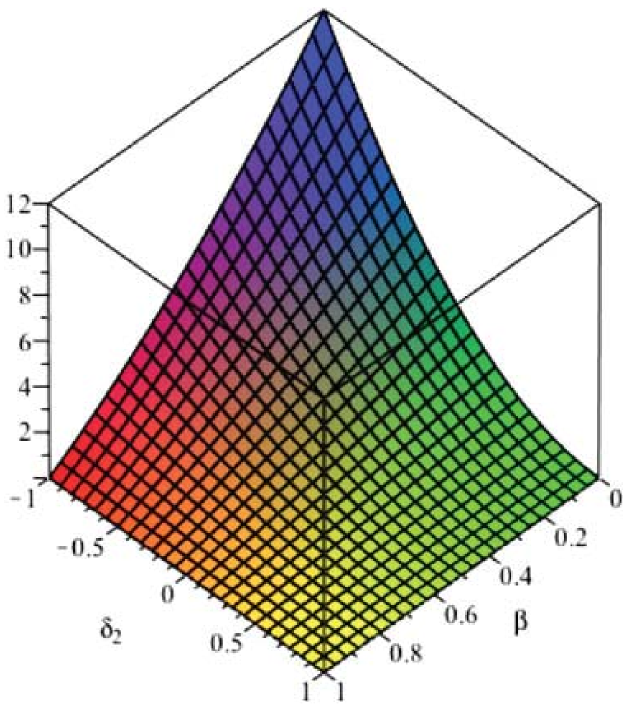

Fig. 2 (online colour at: www.zamm-journal.org) $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 0$, $\delta_1 = 1$, $\delta_3 = 0$, δ_2, β variable.

6 Conclusion

In this paper, we study the properties of p-definite positive matrices. This new concept of linear algebra, generalizing the concept of definite positive matrices, is defined for investigating bifurcations or "static" instabilities of nonconservative mechanical systems submitted to additional kinematical constraints. The main result of the paper is the proof of the fact that any positive definite matrix is p-definite positive for all $1 \leq p \leq n$. In the last part, the consequences for stability of constrained mechanical systems are highlighted and illustrated on a 3 dof system. The second-order work criterion then appears as the optimum criterion for a divergence stable system for remaining divergence stable if subjected to any family of kinematical constraints.

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