Reflecting the Semantic Features of S5 at the Syntactic Level

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In this paper we present two different sequent calculi for modal logic **S5**, each of which reflects, at the syntactic level, one of the two ways of describing **S5** semantically. We will analyze both these sequent calculi in detail and we will briefly sketch the proofs of: (i) adequacy of the calculi, (ii) admissibility of the structural rules, cut-rule included. All results are proved in a purely syntactic way.

Contraction-free, Cut-free, Modal logic, Sequent Calculus, Tree-hypersequents.

1 Introduction

Modal logic is one of the non-classical logics that has most flourished in recent years. There are many interesting systems of modal logic, but usually attention is focussed on the ones that are called *normal* modal systems. These systems, that can be easily presented in Hilbert-style, enjoy simple and interesting semantic properties. Moreover they can be set out in a cube, known as the cube of modal logic. In this cube each system extends and is extended by another system, except the weakest one **K** and the strongest one **S5**. This last system represents the main concern of this paper.

S5 is an important modal system, not only for being the most powerful of the cube of modal logic and for having deep philosophical issues, but also for enjoying peculiar Kripke semantics features. It is a well-known fact that there are two different kinds of Kripke frames for **S5**: Kripke frames where the accessibility relation is an equivalence relation, i.e. it enjoys the properties of reflexivity, transitivity and symmetry (or, equivalently, the properties of reflexivity and euclideaness), and Kripke frames where the accessibility relation is the universal relation, i.e. it can simply be omitted.

Unfortunately so much cannot be said of the syntactic level. There are many attempts at finding a Gentzen calculus for this system but each of them is unsatisfactory for one of the following two reasons: either it presents several defects, e.g. it not cut-free ([5], [2]) or it does not not enjoy the subformula property ([9]), or it does not fully reflect[1] the semantic richness

[1] We underline that the word reflect should here be understood in a broad sense, i.e.

of **S5**, i.e it can only treat **S5** as a system whose accessibility relation satisfies several conditions ([3], [4]), or it can only treat **S5** as a system where there is no accessibility relation ([1]).

Our goal in this paper is threefold. (i) We want to exploit the tree-hypersequent method, introduced in [6], to present a new calculus for **S5** that reflects the more complex way of describing this system semantically. (ii) We want to show how the tree-hypersequent method leads us to the construction of a second sequent calculus for **S5** (introduced in [7]) that reflects the second way of describing this system semantically. (iii) We want to emphasize analogies and differences between the two calculi mentioned above. This goal will be realized by exposing the several results obtainable in these calculi in a parallel.

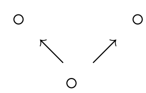
We start our task by informally introducing the tree-hypersequent method. More precisely we explain what a tree-hypersequent is by constructing this object step-by-step. Let us, then, refresh the simple notion of empty hypersequent. An empty hypersequent is a syntactic object of the following form:

$$\overbrace{-/-/-}^{n}$$

which is to say: n slashes that separate $n+1$ dashes. If the order of the dashes is taken into account (as it is not standardly done), we can look to this structure as a tree-frame of Kripke semantics, where the dashes are meant to be the worlds of the tree-frame and the slashes the relations between worlds in the tree-frame. Following this analogy the dash that is at the extreme left of the empty hypersequent denotes a world at distance one in the corresponding tree-frame, the dash after denotes a world a distance two in the corresponding tree-frame, and so on.

In a tree-frame, at every distance, except the first one, we may find n different possible worlds: how do we express this fact in our syntactic object? We separate different dashes that are at the same distance with a semi-colon and get, this way, an empty tree-hypersequent. An example of empty tree-hypersequent is an object of the following form (see figure on the left):

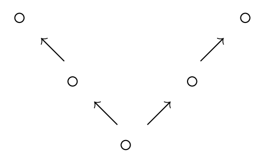
$$- \,/\, - \,;\, - \qquad \rightsquigarrow$$



as we can use it to state that the logical rules of the sequent calculus for classical logic reflect the truth tables of the logical connectives e.g. see [8].

that corresponds to a tree-frame (see figure on the right) with a world at distance one related with two different worlds at distance two. Another example of empty tree-hypersequent is the following (see figure on the left):

$$-/(-/-);(-/-) \qquad \rightsquigarrow$$



that corresponds to a tree-frame (see figure on the right) with a world at distance one related with two different worlds at distance two, each of which is, in its turn, related with another world at distance three.

In order to obtain a tree-hypersequent we fill the dashes with sequents which are objects of the form $M \Rightarrow N$, where M and N are multisets of formulas.

Next section will be used to present the first calculus for the system **S5**. Third section will be dedicated to the explanation of the passage from the first calculus for **S5** to the second one. The last sections will be exploited to briefly present the mains results that are obtainable with these calculi.

2 The first sequent calculus for S5

We define the modal propositional language $\mathcal{L}^{\Box}$ in the following way:

atoms: p^0, p^1, ...

logical constant: $\Box$

connectives: $\neg$, $\wedge$

The other classic connectives can be defined as usual, as well as the constant $\diamond$ and the formulas of the modal language $\mathcal{L}^{\Box}$.

`Syntactic Conventions`

α, β, ...: formulas,

M, N, ...: finite multisets of formulas,

Γ, Δ, ...: sequents(SEQ). The empty sequent ($\Rightarrow$) is included.

G, H, ...: tree-hypersequents (THS),

$\underline{X}$, $\underline{Y}$, ...: finite multisets of tree-hypersequents (MTHS), $\emptyset$ included.

We point out that for the sake of brevity we might use the following notation: given $\Gamma \equiv M \Rightarrow N$ and $\Pi \equiv P \Rightarrow Q$, we will write $\Gamma \centerdot \Pi$ instead of $M, P \Rightarrow N, Q$.

DEFINITION The notion of tree-hypersequent is inductively defined in the following way:

- if $\Gamma \in$ SEQ, then $\Gamma \in$ THS,
- if $\Gamma \in$ SEQ and $\underline{X} \in$ MTHS, then $\Gamma/\underline{X} \in$ THS.

DEFINITION The intended interpretation of a tree-hypersequent is:

- $(M \Rightarrow N)^{\delta} := \bigwedge M \rightarrow \bigvee N$,
- $(\Gamma/G_1; ...; G_n)^{\delta} := \Gamma^{\delta} \vee \Box G_1^{\delta} \vee ... \vee \Box G_n^{\delta}$.

In order to display the rules of the calculi, we will use the notation $G[*]$ to refer to a tree-hypersequent G together with one hole $[*]$, where the hole should be understood, metaphorically, as a zoom that allows us to focus attention on a particular point, $*$, of G. Such an object becomes a real tree-hypersequent whenever the symbol $*$ is *appropriately* replaced by (i) a sequent Γ, and in this case we will write $G[\Gamma]$ to denote the tree-hypersequent G together with a specific occurrence of a sequent Γ in it; (ii) two sequents, Γ/Σ, one after another and separated by a slash, and in this case we will write $G[\Gamma/\Sigma]$ to denote the tree-hypersequent G together with a specific occurrence of a sequent Γ immediately followed by a specific occurrence of a sequent Σ; (iii) a tree-hypersequent H, and in this case we will write $G[H]$ to denote the tree-hypersequent G together with a specific occurrence of a tree-hypersequent H in it.

The calculus *CSS5* is composed of:

Initial Tree-hypersequents

$$G\ [p\ ,\ M \Rightarrow N\ ,\ p]$$

Propositional Rules

$$\frac{G[M \Rightarrow N, \alpha]}{G[\neg\alpha, M \Rightarrow N]}\ \neg A \qquad\qquad \frac{G[\alpha, M \Rightarrow N]}{G[M \Rightarrow N, \neg\alpha]}\ \neg K$$

$$\frac{G[\alpha, \beta, M \Rightarrow N]}{G[\alpha \wedge \beta, M \Rightarrow N]}\wedge A \qquad \frac{G[M \Rightarrow N, \alpha] \qquad G[M \Rightarrow N, \beta]}{G[M \Rightarrow N, \alpha \wedge \beta]}\wedge K$$

Modal Rules

$$\frac{G[\Box\alpha, M \Rightarrow N/\alpha, S \Rightarrow T]}{G[\Box\alpha, M \Rightarrow N/S \Rightarrow T]}\Box A \qquad \frac{G[M \Rightarrow N/ \Rightarrow \alpha; \underline{X}]}{G[M \Rightarrow N, \Box\alpha/\underline{X}]}\Box K$$

Special Logical Rules

$$\frac{G[\Box\alpha, \alpha, M \Rightarrow N]}{G[\Box\alpha, M \Rightarrow N]}\,t \qquad \frac{G[\Box\alpha, M \Rightarrow N/\Box\alpha, S \Rightarrow T]}{G[\Box\alpha, M \Rightarrow N/S \Rightarrow T]}\,4$$

$$\frac{G[\alpha, M \Rightarrow N/\Box\alpha, S \Rightarrow T]}{G[M \Rightarrow N/\Box\alpha, S \Rightarrow T]}\,b \qquad \frac{G[\Box\alpha, M \Rightarrow N/\Box\alpha, S \Rightarrow T]}{G[M \Rightarrow N/\Box\alpha, S \Rightarrow T]}\,5$$

As the reader can easily observe, the calculus $CSS5$ reflects at the syntactic level the first way of describing **S5** semantically: indeed the four special logical rules, t, 4, b and 5, are meant to capture the frame properties of reflexivity, transitivity, symmetry and euclideaness, respectively. It is interesting to note that each of these special logical rules have a (admissible) structural counterpart:

Special Structural Rules

$$\frac{G[\Gamma/(\Sigma/\underline{X}); \underline{X}^{'}]}{G[\Gamma \centerdot \Sigma/\underline{X}; \underline{X}^{'}]}\,\tilde{t} \qquad \frac{G[\Gamma/(\Sigma/\underline{X}); \underline{X}^{'}]}{G[\Gamma/(\Rightarrow /\Sigma/\underline{X}); \underline{X}^{'}]}\,\tilde{4}$$

$$\frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}^{'}); \underline{X}^{''}]}{G[\Gamma \centerdot \Delta/(\Sigma/\underline{X}^{'}); \underline{X}; \underline{X}^{''}]}\,\tilde{b} \qquad \frac{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}^{'}); \underline{X}^{''}]}{G[\Gamma/(\Delta/\underline{X}); (\Sigma/\underline{X}^{'}); \underline{X}^{''}]}\,\tilde{5}$$

LEMMA The rules $\tilde{t}$ and $\tilde{b}$ are height preserving admissible in the calculus $CSS5$. The rule $\tilde{4}$ and $\tilde{5}$ are admissible in the calculus $CSS5$.

Proof. By induction on the height of the derivation of the premiss. In case the last applied rule is the modal rule $\Box A$, we exploit one of the special logical rules to solve the case. $\boxtimes$

As we will see in the last section the existence of these special structural rules is crucial for the proof of cut-admissibility.

3 The second sequent calculus for S5

Let us concentrate on the special structural rule $\tilde{5}$. Roughly speaking this rule allows us to move from the symbol / to the symbol ;. In more intuitive terms: this rule allows us to move from the presence of the accessibility relation of Kripke semantics to its absence.[2] Given this fact, an idea seems to naturally arise: we could construct an alternative sequent calculus for **S5** where we still have n different sequents a time, but there is no longer an order on these sequents, (there is no longer an accessibility relation over the set of worlds), i.e. a sequent calculus where we no longer need to deal with the two symbols / and ;, but with just one of them.

This section will be dedicated to the realization of such an idea by means of the development of another Gentzen system for modal logic **S5**, which, by contrast with $CSS5$, reflects, at the syntactic level, the simplicity of the Kripke frames where the accessibility relation is absent. In this new sequent calculus we will use hypersequents where precisely we only have the meta-linguistic symbol |.[3] Let us emphasize that the return to hypersequents is motivated by work with tree-hypersequents. In other words, hypersequents stand to tree-hypersequents, as Kripke frames with universal relation stand to Kripke frames.

DEFINITION An hypersequent is a syntactic object of the form:

$$M_1 \Rightarrow N_1 | M_2 \Rightarrow N_2 | \ ... \ | M_n \Rightarrow N_n$$

where $M_i \Rightarrow N_i$ $(i = 1, ..., n)$ is a classical sequent.

DEFINITION The intended interpretation of a hypersequent is definable in the following inductive way:

- $(M \Rightarrow N)^{\tau}\!: = \bigwedge M \rightarrow \bigvee N$,
- $(\Gamma_1 | \Gamma_2 | \ ... \ | \Gamma_n)^{\tau}\!: = \Box\Gamma_1^{\tau} \vee \Box\Gamma_2^{\tau} \vee \ ... \ \vee \Box\Gamma_n^{\tau}$

A hypersequent is then just a *multiset* of classical sequents, which is to say the order of the sequents in a hypersequents does not count.

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[2]We underline that it is an easy (but quite tedious) work to show that the rule $\tilde{5}$ is invertible, which is to say, that the following rule:

$$\frac{G[\Gamma/(\Delta/\underline{X}); (\Sigma/\underline{X}'); \underline{X}'']}{G[\Gamma/(\Sigma/(\Delta/\underline{X}); \underline{X}'); \underline{X}'']}$$

is also admissible in the calculus $CSS5$.

[3]We follow the tradition in choosing the symbol |. It would have been the same to choose the slightly different symbol / or the symbol ;.

The postulates of the calculus $CSS5_s$ are:

Initial Hypersequents

$G \mid p, M \Rightarrow N, p$

Propositional Rules

$$\frac{G \mid M \Rightarrow N, \alpha}{G \mid \neg\alpha, M \Rightarrow N}\ {\scriptstyle \neg A} \qquad \frac{G \mid \alpha, M \Rightarrow N}{G \mid M \Rightarrow N, \neg\alpha}\ {\scriptstyle \neg K}$$

$$\frac{G \mid \alpha, \beta, M \Rightarrow N}{G \mid \alpha \wedge \beta, M \Rightarrow N}\ {\scriptstyle \wedge A} \qquad \frac{G \mid M \Rightarrow N, \alpha \qquad G \mid M \Rightarrow N, \beta}{G \mid M \Rightarrow N, \alpha \wedge \beta}\ {\scriptstyle \wedge K}$$

Modal Rules

$$\frac{G \mid \alpha, \Box\alpha, M \Rightarrow N}{G \mid \Box\alpha, M \Rightarrow N}\ {\scriptstyle \Box A_1} \qquad \frac{G \mid M \Rightarrow N \mid\ \Rightarrow \alpha}{G \mid M \Rightarrow N, \Box\alpha}\ {\scriptstyle \Box K}$$

$$\frac{G \mid \Box\alpha, M \Rightarrow N \mid \alpha, S \Rightarrow T}{G \mid \Box\alpha, M \Rightarrow N \mid S \Rightarrow T}\ {\scriptstyle \Box A_2}$$

4 Admissibility of the structural rules

In this section we will show which rules are height-preserving admissible in the calculi $CSS5$ and $CSS5_s$ (the proofs of height-preserving admissibility are developed by straightforward induction on the height of the derivation of the premise); moreover we will prove that the logical and modal rules are height-preserving invertible.

LEMMA The rules of internal weakening:

$$\frac{G[M \Rightarrow N]}{G[M, P \Rightarrow N, Q]}\ {\scriptstyle W} \qquad \frac{G \mid M \Rightarrow N}{G \mid M, P \Rightarrow N, Q}\ {\scriptstyle W_s}$$

are height-preserving admissible in, respectively, $CSS5$ and $CSS5_s$.

LEMMA The rules of external weakening:

$$\frac{G[\Gamma/\underline{X}]}{G[\Gamma/\underline{X};\Sigma]}\ EW \qquad\qquad \frac{G}{G \mid M \Rightarrow N}\ EW_s$$

are height-preserving admissible in, respectively, $CSS5$ and $CSS5_s$.

LEMMA The rules of merge:

$$\frac{G[\Delta/(\Gamma/\underline{X});(\Pi/\underline{X'});\underline{Y}]}{G[\Delta/(\Gamma\centerdot\Pi/\underline{X};\underline{X'});\underline{Y}]}\ merge \qquad\qquad \frac{G \mid M \Rightarrow N \mid P \Rightarrow Q}{G \mid M,P \Rightarrow N,Q}\ merge_s$$

are height-preserving admissible in, respectively, $CSS5$ and $CSS5_s$.

LEMMA The rule of necessitation:

$$\frac{G}{\Rightarrow /G}\ rn$$

is height-preserving admissible in $CSS5$.

LEMMA All the logical and modal rules of $CSS5$ and $CSS5_s$ are height-preserving invertible.

Proof. The proof proceeds by induction on the derivation of the premise of the rule considered. The cases of logical rules of $CSS5$ and $CSS5_s$ are dealt with in the classical way. The only differences - the fact that we are dealing with tree-hypersequents, and hypersequent, respectively, and the cases where last rule applied is one of the modal rules or one of the special logical rules - are dealt with easily.

The rule $\Box A$ and the special logical rules of $CSS5$, as well as the two modal rules $\Box A_1$ and $\Box A_2$ of $CSS5_s$, are all trivially height-preserving invertible since the premise is concluded by weakening from the conclusion, and weakening is height-preserving admissible.

We show in detail the invertibility of the rule $\Box K$ of the calculus $CSS5$ (the proof of the invertibility of the rule $\Box K$ of the calculus $CSS5_s$ is analogous). If $G[M \Rightarrow N, \Box\alpha/\underline{X}]$ is an initial tree-hypersequent, then so is $G[M \Rightarrow N/ \Rightarrow \alpha;\underline{X}]$. If $G[M \Rightarrow N, \Box\alpha/\underline{X}]$ is obtained by a logical rule $\mathcal{R}$, we apply the inductive hypothesis on the premise(s), $G[M^{'} \Rightarrow N^{'}, \Box\alpha/\underline{X}]$ ($G[M^{''} \Rightarrow N^{''}, \Box\alpha/\underline{X}]$) and we obtain derivation(s), of height $n-1$, of $G[M^{'} \Rightarrow N^{'}/ \Rightarrow \alpha;\underline{X}]$ ($G[M^{''} \Rightarrow N''/ \Rightarrow \alpha;\underline{X}]$). By applying the rule $\mathcal{R}$, we obtain a derivation of height n of $G[M \Rightarrow N/ \Rightarrow \alpha;\underline{X}]$. If $G[M \Rightarrow N, \Box\alpha/\underline{X}]$ is of the form $G[\Box\beta, M^{'} \Rightarrow N^{'}, \Box\alpha/S \Rightarrow T]$ and is obtained by the modal rule $\Box A$, we apply the inductive hypothesis to $G[\Box\beta, M^{'} \Rightarrow N^{'}, \Box\alpha/\beta, S \Rightarrow$

$T]$ and we obtain a derivation of height $n-1$ of $G[\Box\beta, M' \Rightarrow N' / \Rightarrow \alpha; \beta, S \Rightarrow T]$. By applying the rule $\Box A$, we obtain a derivation of height n of $G[\Box\beta, M' \Rightarrow N' / \Rightarrow \alpha; S \Rightarrow T]$. If $G[M \Rightarrow N, \Box\alpha/\underline{X}]$ is obtained by one of the special logical rules or by the modal rule $\Box K$ in which $\Box\alpha$ is not the principal formula, then the case can be dealt with analogously to the one of the rule $\Box A$. Finally, if $G[M \Rightarrow N, \Box\alpha/\underline{X}]$ is preceded by the modal rule $\Box K$ and $\Box\alpha$ is the principal formula, the premise of the last step gives the conclusion. ⊠

LEMMA The rules of contraction:

$$\frac{G[\alpha, \alpha, M \Rightarrow N]}{G[\alpha, M \Rightarrow N]}\, CA \qquad \frac{G \mid \alpha, \alpha, M \Rightarrow N}{G \mid \alpha, M \Rightarrow N}\, CA_s$$

and

$$\frac{G[M \Rightarrow N, \alpha, \alpha]}{G[M \Rightarrow N, \alpha]}\, CK \qquad \frac{G \mid M \Rightarrow N, \alpha, \alpha}{G \mid M \Rightarrow N, \alpha}\, CK_s$$

are height-preserving admissible in, respectively, $CSS5$ and $CSS5_s$.

Finally we remind the reader that in the calculus $CSS5$ we have shown the (height-preserving) admissibility of four special structural rules.

5 Adequacy of the calculi

In this section we prove that the sequent calculi $CSS5$ and $CSS5_s$ prove exactly the same formulas as the corresponding Hilbert system **S5**.

THEOREM

(i) If $\vdash \alpha$ in **S5**, then $\vdash\Rightarrow \alpha$ in $CSS5$.

(ii) If $\vdash \alpha$ in **S5**, then $\vdash\Rightarrow \alpha$ in $CSS5_s$.

Proof. By induction on the height of proofs in $CSS5$ and $CSS5_s$, respectively. The classical axioms and the modus ponens are proved as usual; the axioms T, the axiom 4, the axiom B are proved by exploiting the corresponding special logical rules in $CSS5$, and the modal rules in $CSS5_s$. We present the proof of the axiom 5.[4]

$CSS5 \vdash\Rightarrow \neg\Box\neg\alpha \rightarrow \Box\neg\Box\neg\alpha$

[4]Notice that we use the derived rules for the connective $\rightarrow$. Moreover in the case where repeated running applications of a same rule $\mathcal{R}$ take place, we write the rule $\mathcal{R}$ with the symbol * as index.

$$
\begin{array}{cl}
\Box\neg\alpha \Rightarrow /\alpha \Rightarrow \alpha; \Box\neg\alpha \Rightarrow & \\ \hline
\Box\neg\alpha \Rightarrow /\neg\alpha \Rightarrow \neg\alpha; \Box\neg\alpha \Rightarrow & {\scriptstyle \neg^*} \\ \hline
\Box\neg\alpha \Rightarrow / \Rightarrow \neg\alpha; \Box\neg\alpha \Rightarrow & {\scriptstyle \Box A} \\ \hline
\Rightarrow / \Rightarrow \neg\alpha; \Box\neg\alpha \Rightarrow & {\scriptstyle 5} \\ \hline
\Rightarrow / \Rightarrow \neg\alpha; \Rightarrow \neg\Box\neg\alpha & {\scriptstyle \neg K} \\ \hline
\Rightarrow \Box\neg\alpha / \Rightarrow \neg\Box\neg\alpha & {\scriptstyle \Box K} \\ \hline
\Rightarrow \Box\neg\alpha, \Box\neg\Box\neg\alpha & {\scriptstyle \Box K} \\ \hline
\neg\Box\neg\alpha \Rightarrow \Box\neg\Box\neg\alpha & {\scriptstyle \neg A} \\ \hline
\Rightarrow \neg\Box\neg\alpha \to \Box\neg\Box\neg\alpha & {\scriptstyle \to K}
\end{array}
$$

$CSS5_s \vdash \Rightarrow \neg\Box\neg\alpha \to \Box\neg\Box\neg\alpha$

$$
\begin{array}{cl}
\Rightarrow |\Box\neg\alpha \Rightarrow |\alpha \Rightarrow \alpha & \\ \hline
\Rightarrow |\Box\neg\alpha \Rightarrow | \Rightarrow \neg\alpha, \alpha & {\scriptstyle \neg K} \\ \hline
\Rightarrow |\Box\neg\alpha \Rightarrow |\neg\alpha \Rightarrow \neg\alpha & {\scriptstyle \neg A} \\ \hline
\Rightarrow |\Box\neg\alpha \Rightarrow | \Rightarrow \neg\alpha & {\scriptstyle \Box A_2} \\ \hline
\Rightarrow | \Rightarrow \neg\Box\neg\alpha| \Rightarrow \neg\alpha & {\scriptstyle \neg K} \\ \hline
\Rightarrow \Box\neg\alpha| \Rightarrow \neg\Box\neg\alpha & {\scriptstyle \Box K} \\ \hline
\Rightarrow \Box\neg\alpha, \Box\neg\Box\neg\alpha & {\scriptstyle \Box K} \\ \hline
\neg\Box\neg\alpha \Rightarrow \Box\neg\Box\neg\alpha & {\scriptstyle \neg A} \\ \hline
\Rightarrow \neg\Box\neg\alpha \to \Box\neg\Box\neg\alpha & {\scriptstyle \to K}
\end{array}
$$

⊠

THEOREM

(i) If $\vdash G$ in $CSS5$, then $\vdash (G)^{\tau}$in **S5**.

(ii) If $\vdash G$ in $CSS5_s$, then $\vdash (G)^{\tau}$in **S5**.

Proof. By induction on the height of proofs in **S5**. (i) The technique to develop this proof consists of the following two steps: first of all, the sequent(s) affected by the rule should be isolated and the corresponding implication proved, then the implication should be transported up all along the tree so that, by modus ponens, the desired result is immediately achieved. (ii) The case of the axioms is trivial, while for the inductive steps with the propositional rules all we need is classical logic and the fact that if **S5**$\vdash \alpha_1 \to (\alpha_2 \to ... \to (\alpha_n \to \beta)...)$, then **S5** $\vdash \Box\alpha_1 \to (\Box\alpha_2 \to ... \to (\Box\alpha_n \to \Box\beta)...)$. As for the inductive steps for modal rules, we again exploit the fact that, if $\mathbf{S5} \vdash \alpha_1 \rightarrow (\alpha_2 \rightarrow ... \rightarrow (\alpha_n \rightarrow \beta)...)$, then $\mathbf{S5} \vdash \Box\alpha_1 \rightarrow (\Box\alpha_2 \rightarrow ... \rightarrow (\Box\alpha_n \rightarrow \Box\beta)...)$ and the axioms T and 4. ⊠

6 Cut-admissibility

We dedicate this last section to the proof of the admissibility of the cut-rule in $CSS5$ and $CSS5_s$. We then have to present the two cut-rules. In order to introduce the cut-rule for the calculus $CSS5$ we firstly need the following two definitions.

DEFINITION Given two tree-hypersequents, $G[\Gamma]$ and $G^{'}[\Gamma^{'}]$, the relation of *equivalent position* between two of their sequents, in this case Γ and $\Gamma^{'}$, $G[\Gamma] \sim G^{'}[\Gamma^{'}]$, is defined inductively in the following way:

- $\Gamma \sim \Gamma^{'}$
- $\Gamma/\underline{X} \sim \Gamma^{'}/\underline{X}^{'}$
- If $H[\Gamma] \sim H^{'}[\Gamma^{'}]$, then $\Delta/H[\Gamma]; \underline{X} \sim \Delta^{'}/H^{'}[\Gamma^{'}]; \underline{X}^{'}$

DEFINITION Given two tree-hypersequents $G[\Gamma]$ and $G^{'}[\Gamma^{'}]$ such that $G[\Gamma] \sim G^{'}[\Gamma^{'}]$, the operation of *product*, $G[\Gamma] \otimes G^{'}[\Gamma^{'}]$, is defined inductively in the following way:

- $\Gamma \otimes \Gamma^{'} := \Gamma \centerdot \Gamma^{'}$
- $(\Gamma/\underline{X}) \otimes (\Gamma^{'}/\underline{X}^{'}) := \Gamma \centerdot \Gamma^{'}/\underline{X}; \underline{X}^{'}$
- $(\Delta/H[\Gamma]; \underline{X}) \otimes (\Delta^{'}/H^{'}[\Gamma^{'}]; \underline{X}^{'}) :=$

  $\Delta \centerdot \Delta^{'}/(H[\Gamma] \otimes H^{'}[\Gamma^{'}]); \underline{X}; \underline{X}^{'}$

Cut-rule of the calculus $CSS5$. Given two tree-hypersequents $G[M \Rightarrow N, \alpha]$ and $G^{'}[\alpha, P \Rightarrow Q]$ such that $G[M \Rightarrow N, \alpha] \sim G^{'}[\alpha, P \Rightarrow Q]$, the cut-rule is:

$$\frac{G[M \Rightarrow N, \alpha] \quad G^{'}[\alpha, P \Rightarrow Q]}{G \otimes G^{'}[M, P \Rightarrow N, Q]} \, cut_\alpha$$

Cut-rule of the calculus $CSS5_s$. The cut-rule of the calculus $CSS5_s$ is simpler than the previous one and it is the following:

$$\frac{G \mid M \Rightarrow N, \alpha \quad G^{'} \mid \alpha, P \Rightarrow Q}{G \mid G' \mid M, P \Rightarrow N, Q} \; cut^{s}_{\alpha}$$

Contrary to the other structural rules (that we were given a chance to observe in section 4), the cut-rules of the calculi $CSS5$ and $CSS5_s$ are not so similar between each other. The reason for this is quite simple. In the calculus $CSS5$ we deal with tree-hypersequents and then, when we have to fuse two tree-hypersequents by means of an application of a cut-rule, we should ensure that the tree-shape is kept. In the calculus $CSS5_s$, instead, we deal with hypersequents, which are just multisets of sequents, therefore when we fuse two hypersequents by means of a cut-rule, we can do it arbitrarily since there is no particular structure to keep.

Each cut-rule is admissible in the corresponding calculus, as the following theorems state.

THEOREM The rule cut_α is admissible in the calculus $CSS5$.

Proof. The proof is developed by induction on the complexity of the cut-formula, which is the number (≥ 0) of the occurrences of logical symbols in the cut-formula α, with subinduction on the sum of the heights of the derivations of the premises of the cut-rule. The proof has the same structure as the proof of admissibility of cut for the sequent calculus of first-order logic, (see for example [10]). However, for the sake of clarity, we consider in details two cases: (i) the case of a cut with cut-formula principal in modal rules in both premisses of cut; (ii) the case of a cut with cut-formula principal in the modal rule $\Box K$ and in the rule t in the left and right premisses of cut, respectively. With this second case we want to underline the indispensability of the special structural rules introduced in Section 3.

(i):[5]

$$\frac{\dfrac{G[M \Rightarrow N / \Rightarrow \beta]}{G[M \Rightarrow N, \Box\beta]} \; \Box K \qquad \dfrac{G^{'}[\Box\beta, \Pi/\beta, \Psi]}{G'[\Box\beta, \Pi/\Psi]} \; \Box A}{G \otimes G'[M \Rightarrow N \centerdot \Pi/\Psi]} \; cut_{\Box\beta}$$

We reduce to:

[5]In the cases (i) and (ii), we assume to write, for the sake of clarity, the rule $\Box K$ without the aid of the multiset of tree-hypersequents $\underline{X}$. We rely on the reader for understanding the rule correctly, anyway.

$$\dfrac{\dfrac{G[M \Rightarrow N/ \Rightarrow \beta] \qquad \dfrac{G[M \Rightarrow N, \Box\beta] \quad G^{'}[\Box\beta, P \Rightarrow Q/\beta, Z \Rightarrow W]}{G \otimes G^{'}[M, P \Rightarrow N, Q/\beta, Z \Rightarrow W]}\, cut_{\Box\beta}}{G \otimes G \otimes G^{'}[M, M, P \Rightarrow N, N, Q/Z \Rightarrow W]}\, cut_{\beta}}{G \otimes G^{'}[M, P \Rightarrow N, Q/Z \Rightarrow W]}\, C^{*}{+}merge^{*}$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of cut and the second cut is eliminable by induction on the complexity of the cut formula.

(ii):

$$\dfrac{\dfrac{G[M \Rightarrow N/ \Rightarrow \beta]}{G[M \Rightarrow N, \Box\beta]}\, \Box K \qquad \dfrac{G^{'}[\Box\beta, \beta, P \Rightarrow Q]}{G'[\Box\beta, P \Rightarrow Q]}\, t}{G \otimes G'[M, P \Rightarrow N, Q]}\, cut_{\Box\beta}$$

We reduce to:

$$\dfrac{\dfrac{\dfrac{G[M \Rightarrow N/ \Rightarrow \beta]}{G[M \Rightarrow N, \beta]}\, \tilde{t} \qquad \dfrac{G[M \Rightarrow N, \Box\beta] \quad G^{'}[\Box\beta, \beta, P \Rightarrow Q]}{G \otimes G^{'}[\beta, M, P \Rightarrow N, Q]}\, cut_{\Box\beta}}{G \otimes G \otimes G^{'}[M, M, P \Rightarrow N, N, Q]}\, cut_{\beta}}{G \otimes G^{'}[M, P \Rightarrow N, Q]}\, C^{*}{+}merge^{*}$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of cut and the second cut is eliminable by induction on the complexity of the cut formula.

In those cases where the last applied rule on the left premise is the rule $\Box K$ and the last applied rule on the right premise is the the rule 4 or the rule 5, and the cut-formula is principal in both the left and the right premises, the situation is a little bit more complicated but can be dealt with by adopting the technique showed in [6]. $\boxtimes$

THEOREM The rule cut^{s}_{α} is admissible in the calculus $CSS5_s$.

Proof. The proof is developed by induction on the complexity of the cut-formula, with subinduction on the sum of the heights of the derivations of the premises of the cut-rule. The proof has been fully developed in [7]. However, for the sake of clarity, we show in detail two cases, the ones of a cut with cut-formula principal in modal rules in both premisses of cut.

(i):

$$\dfrac{\dfrac{G \mid M \Rightarrow N \mid \ \Rightarrow \beta}{G \mid M \Rightarrow N, \Box\beta}\,\Box K \qquad \dfrac{G^{'} \mid \Box\beta, P \Rightarrow Q \mid \beta, Z \Rightarrow W}{G' \mid \Box\beta, P \Rightarrow Q \mid Z \Rightarrow W}\,\Box A_2}{G \mid G' \mid M, P \Rightarrow N, Q \mid Z \Rightarrow W}\,cut^s_{\Box\beta}$$

We reduce to:

$$\dfrac{\dfrac{G \mid M \Rightarrow N \mid \ \Rightarrow \beta \qquad \dfrac{G \mid M \Rightarrow N, \Box\beta \quad G^{'} \mid \Box\beta, P \Rightarrow Q \mid \beta, Z \Rightarrow W}{G \mid G^{'} \mid M, P \Rightarrow N, Q \mid \beta, Z \Rightarrow W}\,cut^s_{\Box\beta}}{G \mid G \mid G^{'} \mid M \Rightarrow N \mid M, P \Rightarrow N, Q \mid Z \Rightarrow W}\,cut^s_{\beta}}{G \mid G^{'} \mid M, P \Rightarrow N, Q \mid Z \Rightarrow W}\,merge^*{+}C^*$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of cut and the second cut is eliminable by induction on the complexity of the cut formula.

(ii):

$$\dfrac{\dfrac{G \mid M \Rightarrow N \mid \ \Rightarrow \beta}{G \mid M \Rightarrow N, \Box\beta}\,\Box K \qquad \dfrac{G^{'} \mid \Box\beta, \beta, P \Rightarrow Q}{G' \mid \Box\beta, P \Rightarrow Q}\,\Box A_1}{G \mid G' \mid M, P \Rightarrow N, Q}\,cut^s_{\Box\beta}$$

We reduce to:

$$\dfrac{\dfrac{G \mid M \Rightarrow N \mid \ \Rightarrow \beta \qquad \dfrac{G \mid M \Rightarrow N, \Box\beta \quad G^{'} \mid \Box\beta, \beta, P \Rightarrow Q}{G \mid G^{'} \mid \beta, M, P \Rightarrow N, Q}\,cut^s_{\Box\beta}}{G \mid G \mid G^{'} \mid M \Rightarrow N \mid M, P \Rightarrow N, Q}\,cut^s_{\beta}}{G \mid G^{'} \mid M, P \Rightarrow N, Q}\,merge^*{+}C^*$$

where the first cut is eliminable by induction on the sum of the heights of the derivations of the premises of cut and the second cut is eliminable by induction on the complexity of the cut formula. ⊠

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