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The velocity and other fields of steady two-dimensional surface gravity waves in irrotational motion are investigated numerically. Only symmetric waves with one crest per wavelength are considered, i.e. Stokes waves of finite amplitude, but not the highest waves, nor subharmonic and superharmonic bifurcations of Stokes waves. The numerical results are analysed, and several conjectures are made about the velocity and acceleration fields.

Keywords: surface waves; velocity field; acceleration field; numerical solution

1. Introduction

Although quite a lot seems to be known by physicists regarding the velocity field induced by steady irrotational surface gravity waves, some results remain to be rigorously demonstrated mathematically. The purpose of this paper is to provide some numerical evidence of the structure of the velocity and related fields, in order to help mathematicians to guess which properties are more likely to be true and should thus be demonstrated. We shall pay particular attention to the vertical velocity and acceleration fields, for which rigorous results are lacking.

Of course, numerical 'evidence' is not rigorous mathematical proof. However, accurate computations can give some insights into what may (or may not) be expected, and thus provide hints about what one should try to prove. For instance, waves with different crests [1] and asymmetric waves [2] were first discovered numerically. For extreme waves, such as the two examples just given above, there are several questions that remain to be answered. But, even for non-extreme waves of finite amplitude, there are some results that remain to be rigorously demonstrated. One example is the structure of the vertical velocity field, whereas the horizontal one is already quite well understood [3]. A fairly complete review on exact results and numerical models can be found in recent publications [4–6] and the references therein.

Note that a deep knowledge of steady surface waves is not only of purely academic interest. Indeed, accurate experiments and numerical simulations of large-amplitude irregular unsteady surface waves have shown that their structure can be relatively well described by suitably adapted steady irrotational solutions

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One contribution of 13 to a Theme Issue 'Nonlinear water waves'.

[7,8]. Thus, a better understanding of steady irrotational surface waves should be of interest to all practitioners in the field.

Several efficient numerical methods have been developed to compute steady water waves [9–14]. Among these methods, we decided to use Fenton's program because: (i) it is freely available; and (ii) it gives easy access to various physical quantities. Fenton's algorithm is not the most efficient for computing extreme waves, but it is good enough for symmetric waves of finite amplitude, as considered here.

The paper is organized as follows. In §2, the hypothesis and notation are introduced. Section 3 describes the methodology and the parameters related to the results given in the following section. Section 4 is devoted to discussing the results and making some conjectures.

2. Definitions and notation

We consider steady two-dimensional potential flows due to surface gravity waves in fluid of constant depth. The fluid is homogeneous, the surface tension is neglected and the pressure is zero at the impermeable free surface, whereas the seabed is fixed, horizontal and impermeable. Here, we consider only waves that are symmetric around each crest and trough, and that have only one crest per period; i.e. we do not consider extreme waves that are subharmonic or superharmonic bifurcations of Stokes waves.

Let (x, y) be a Cartesian coordinate system, moving with the wave; x being the horizontal coordinate and y the upward vertical one. The wave is L-periodic and $x=0$ is the abscissa of a wave crest. We denote $k\equiv 2\pi/L$ the wavenumber ($k\to 0$, i.e. $L\to\infty$, for solitary waves). The equations $y=-d$, $y=\eta(x)$ and $y=0$ denote, respectively, the equations of the bottom, of the free surface and of the mean water level. The latter implies that $\langle\eta\rangle=0$ ($\langle\,\cdot\,\rangle$ is the Eulerian average operator over one period), i.e.

$$\langle\eta\rangle \equiv \frac{k}{2\pi}\int_{-\pi/k}^{\pi/k} \eta(x)\,\mathrm{d}x = 0. \tag{2.1}$$

Finally, $H\equiv\eta(0)-\eta(\pi/k)$ denotes the total wave height (trough-to-crest elevation), and the wave steepness ε is classically defined as $\varepsilon\equiv kH/2$. Another dimensionless parameter, L/d, characterizes the relative length of the wave.

Let ϕ, ψ, u and v be the velocity potential, the stream function, the horizontal and vertical velocities, respectively, such that $u=\partial_x\phi=\partial_y\psi$ and $v=\partial_y\phi=-\partial_x\psi$. We denote with over 'breves' the quantities written at the seabed, e.g. $\breve{\phi}(x)=\phi(x,y=-d)$, whereas over 'tildes' denote the quantities written at the surface, e.g. $\tilde{\phi}(x)=\phi(x,y=\eta(x))$.[1] The quantities $\tilde{\psi}$ and $\breve{\psi}$ are constants, because the free surface and the bottom are streamlines.

The dynamic condition can be expressed in term of the Bernoulli equation

$$2p + 2gy + u^2 + v^2 = B, \tag{2.2}$$

[1]Note that, for example, $\tilde{u}=\widetilde{\partial_x\phi}\neq\partial_x\tilde{\phi}=\tilde{u}+\tilde{v}\partial_x\eta$.

where p is the pressure divided by the density, $g > 0$ is the acceleration due to gravity and B is the Bernoulli constant. At the free surface, the pressure is zero, i.e. $\tilde{p} = 0$. The latter condition, together with (2.1), yields a definition for the Bernoulli constant

$$B = \langle \tilde{u}^2 + \tilde{v}^2 \rangle. \tag{2.3}$$

Let $-c_{\mathrm{S}}$ be the mean flow velocity defined as

$$c_{\mathrm{S}} \equiv -\left\langle \frac{1}{d} \int_{-d}^{\eta} u(x, y)\, \mathrm{d}y \right\rangle = \frac{\breve{\psi} - \tilde{\psi}}{d}. \tag{2.4}$$

Thus, c_{S} is the phase velocity of the wave observed in the frame of reference without mean flow. Another important quantity is the phase velocity c_{E} observed in the frame of reference without mean velocity at the seabed, i.e.

$$c_{\mathrm{E}} \equiv -\langle \breve{u} \rangle = -\langle u(x, y = -d) \rangle. \tag{2.5}$$

Many other phase velocities can, of course, be defined, but c_{S} and c_{E} are two velocities of special interest. Note that $B = c_{\mathrm{S}}^2 = c_{\mathrm{E}}^2$ in deep water ($d \to \infty$) and for solitary waves ($k \to 0$). Also note that neither c_{S} nor c_{E} is the linear phase velocity $c_0 \equiv \sqrt{(g/k) \tanh(kd)}$. Note finally that the definition (2.5) of c_{E} implies the existence of a function Φ such that

$$\breve{\phi} = -c_{\mathrm{E}} x + \breve{\Phi}, \quad \langle \breve{\Phi} \rangle = 0. \tag{2.6}$$

The irrotationality yields that the relations in (2.5) and (2.6) written at $y = -d$ also hold at any horizontal line $y = \text{constant}$; thus, the function $\Phi \equiv c_{\mathrm{E}} x + \phi$ is $(2\pi/k)$-periodic in the x-direction. We shall consider the horizontal velocity U observed in the frame of reference without mean velocity at the bottom, i.e.

$$U \equiv u + c_{\mathrm{E}} = \partial_x \Phi, \tag{2.7}$$

as well as the horizontal and vertical components of the acceleration,

$$a_x \equiv \mathrm{D}_t u \quad \text{and} \quad a_y \equiv \mathrm{D}_t v, \tag{2.8}$$

where D_t is the temporal derivative following the motion.

3. Numerical resolution

The equations shown above cannot be solved analytically in closed form. Thus, a numerical resolution is necessary to obtain such solutions. Here, we use a method and program provided by Fenton [11]. This method is efficient enough for all but the highest waves, the latter being of limited practical interest. Fenton's program has been modified to quadruple precision (about 32-digit accuracy) and some numerical parameter adjustments have been made by the present author in order to increase somewhat the program's range of applicability and to guarantee (as much as possible) the accuracy of the results presented here.

We computed large waves of various wavelength-to-depth ratios in order to cover a broad range from deep to shallow water, i.e. we consider the cases $L/d = \{0, 5, 7.5, 10, 15, 20, 30, 40\}$. For each of these ratios, about the largest wave computable with Fenton's program is considered, i.e. we consider the respective height-to-depth ratios $H/d = \{0, 0.4, 0.45, 0.45, 0.5, 0.55, 0.6, 0.65\}$ with the corresponding steepnesses $\varepsilon \approx \{0.360, 0.251, 0.188, 0.141, 0.105, 0.086, 0.063, 0.051\}$. The relative shallowness of these particular solutions can also be gauged from the parameter $\tanh(kd) \approx \{1, 0.85, 0.68, 0.56, 0.4, 0.3, 0.2, 0.16\}$.

Because only symmetric waves are considered, the numerical solutions are displayed for half a period $0 \leqslant kx \leqslant \pi$, where $x = 0$ is the abscissa of a wave crest. Also, without loss of generality, we consider that $c_{\mathrm{E}} > 0$, i.e. the waves propagate towards the increasing x-direction. All the comments below are valid for this (non-restrictive) particular choice.

4. Discussion

Consider first the horizontal velocity field U observed in the frame of reference where the wave travels with speed c_{E}. The horizontal velocity at the free surface (figure 1) and at the bottom (figure 2) decays monotonically from crest to trough. This is known to be rigorously true in deep water [15,16], in finite depth [17] and for solitary waves [18]. The present numerical results (figures 3 and 4) show that it should also be true along any horizontal line, and not only along streamlines.

The computation of vertical velocity v at the free surface (figure 5) and along any horizontal line (figure 6) shows that v has only one maximum and one minimum per wavelength. The maximum is somewhere at the free surface (this is obvious because v is a harmonic function) as shown in figure 7, but it does not seem that its position is simply connected to the abscissa where the slope of the surface is extremum.

Considering the variations between one crest and the following trough, it appears that v is not a convex function in general, but that may be the case in deep water ($kd \gg 1$). This conjecture is supported by the variations of the gradient of the vertical velocity (figures 8–11). From figures 10 and 11, one can infer that this property is shared by the function $\partial_y u = -\partial_x v$. The magnitude of the gradient of v (i.e. $\|\operatorname{grad} v\|$) is maximum at the crest, minimum at the trough and decays monotonically along the free surface (figure 12). These properties do not seem to be true in the bulk of the fluid (figure 13).

The horizontal acceleration field a_x (figures 14 and 15) seems to share the same qualitative variations and convexity as the vertical velocity field, whereas the vertical acceleration field a_y resembles somewhat the field of $\partial_y u = -\partial_x v$ (figures 16 and 17). If true, these conjectures should be more difficult to prove because, unlike v and $\partial_x v$, a_x and a_y are not harmonic functions.

Finally, we consider the pressure field (figure 18), which has been investigated by Escher & Schlurmann [19] under an infinitesimal periodic wave, and under a solitary wave by Constantin [20]. The pressure increases monotonically with depth and the isobars are maximum under the crest. This means that the pressure is dominated by its hydrostatic component. This has been demonstrated for Stokes waves [21] and for solitary waves [22]. If we consider the total pressure minus the hydrostatic pressure, then the resulting dynamic pressure $p + gy = \frac{1}{2}(B - u^2 - v^2)$ is shown in figures 19 and 20. It seems that the dynamic pressure decays monotonically from crest to trough along any horizontal line and probably also along any streamline, as is the case for the free surface and the bottom.

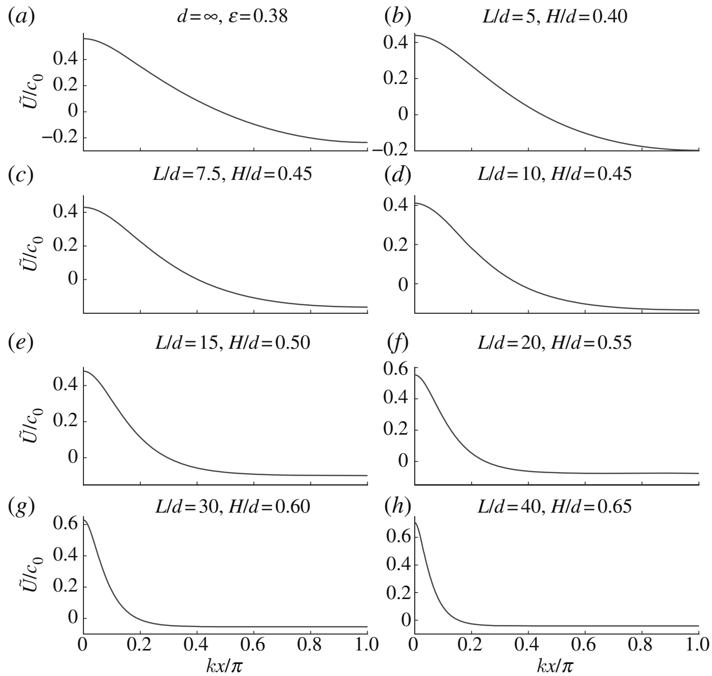

Figure 1. Horizontal velocity at the free surface.

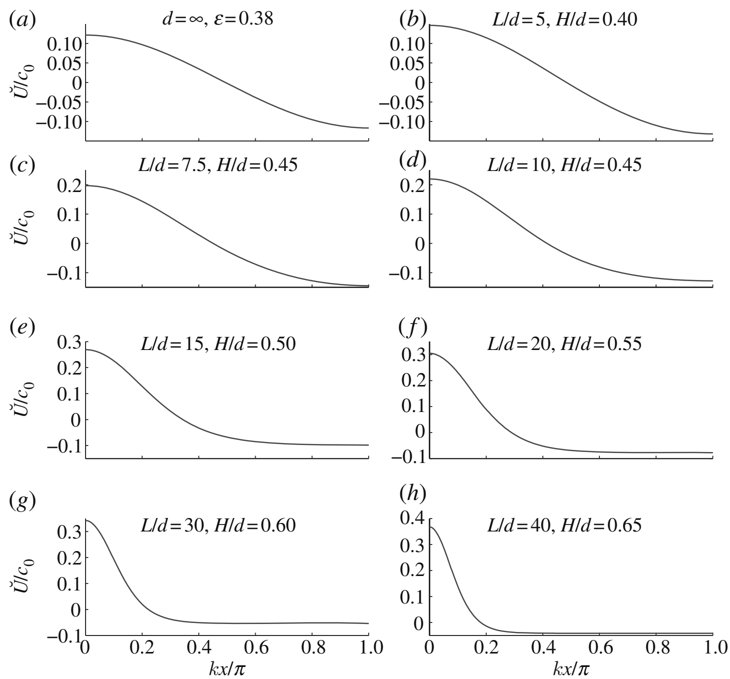

Figure 2. Horizontal velocity at the bottom.

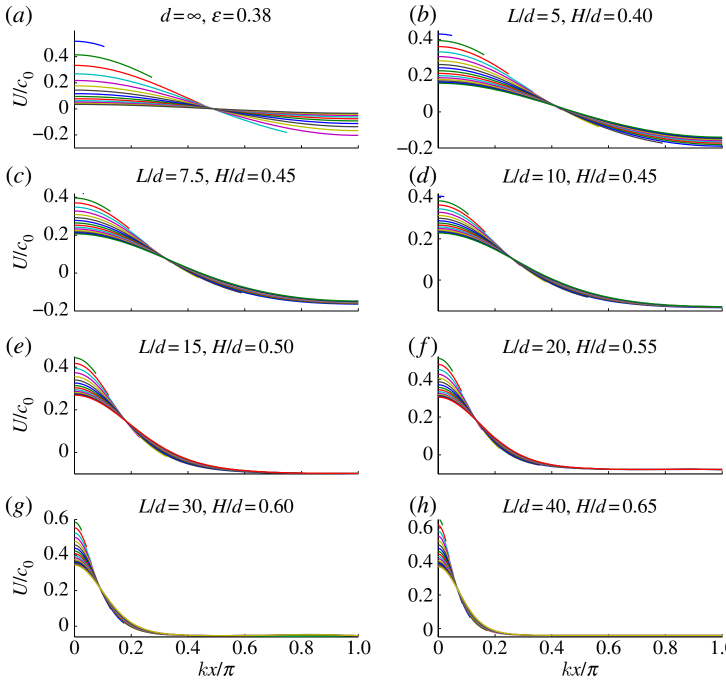

Figure 3. Horizontal velocity along various horizontal lines. Plots are discontinued where the horizontal line crosses the free surface. (Online version in colour.)

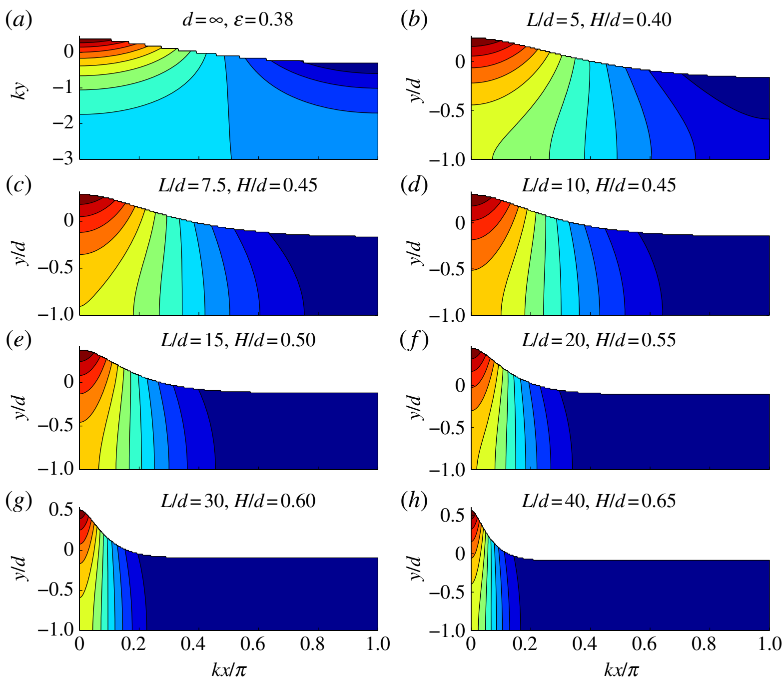

Figure 4. Iso-horizontal velocity field. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

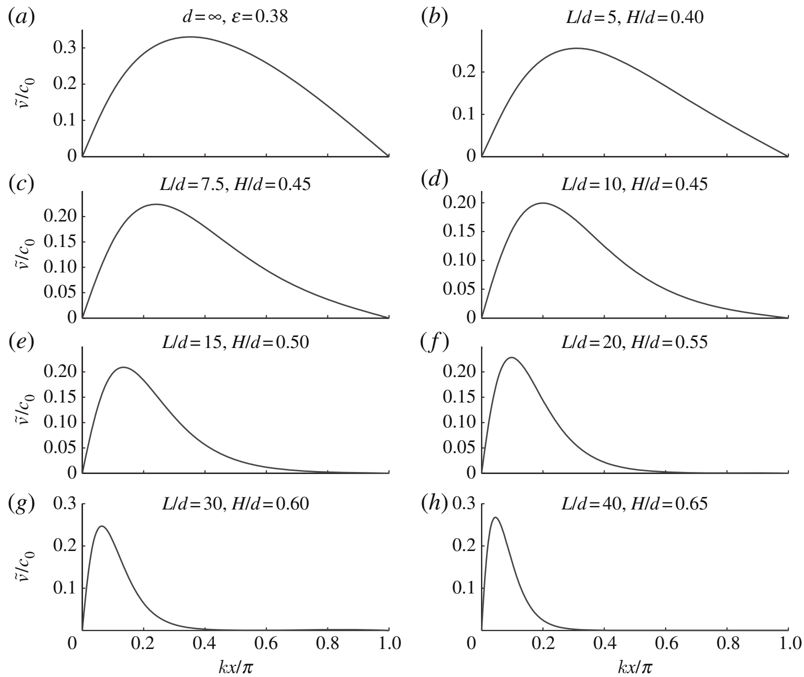

Figure 5. Vertical velocity at the free surface.

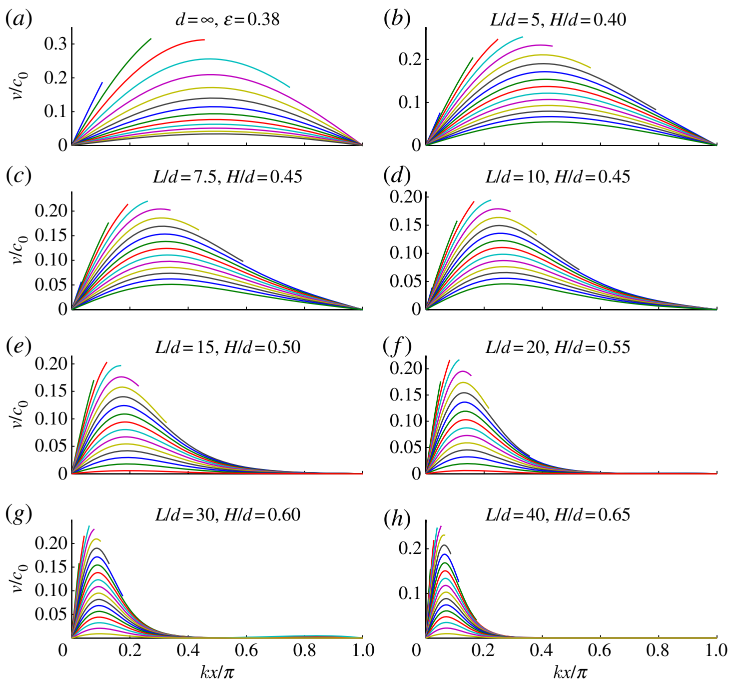

Figure 6. Vertical velocity along various horizontal lines. Plots are discontinued where the horizontal line crosses the free surface. (Online version in colour.)

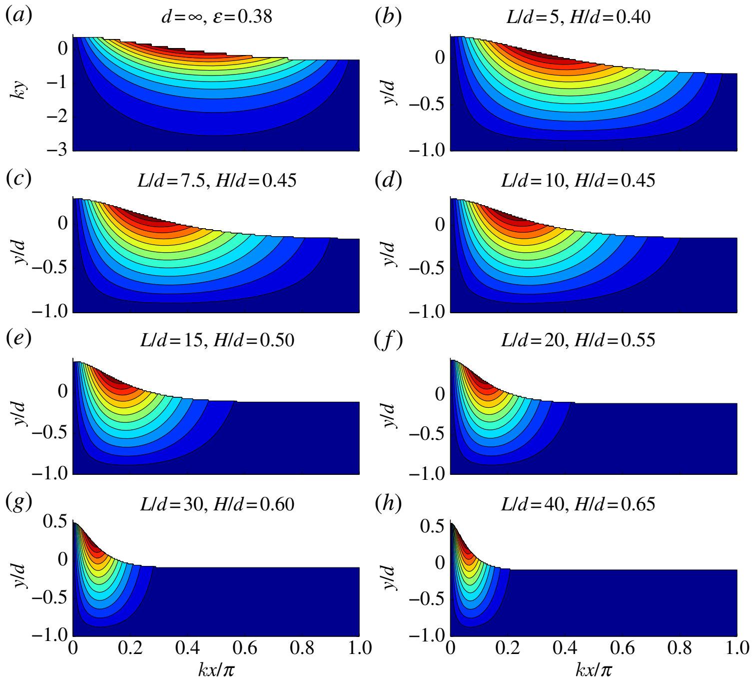

Figure 7. Iso-vertical velocity field. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

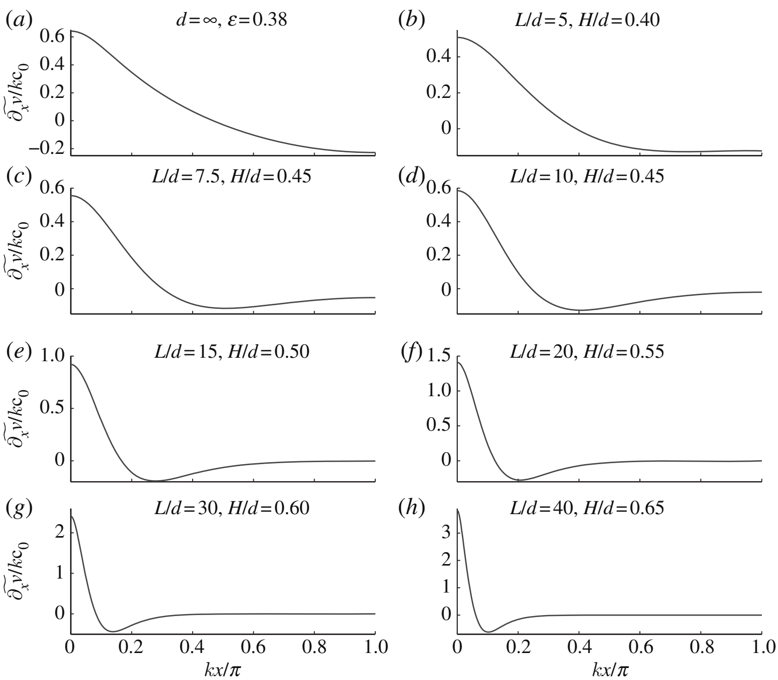

Figure 8. Horizontal derivative of the vertical velocity at the free surface.

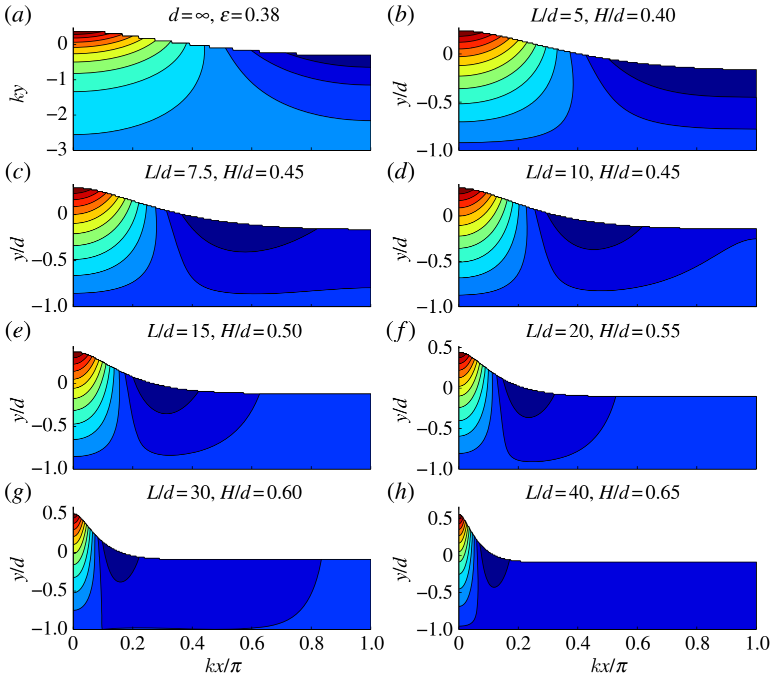

Figure 9. Iso-horizontal derivative of the vertical velocity field. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

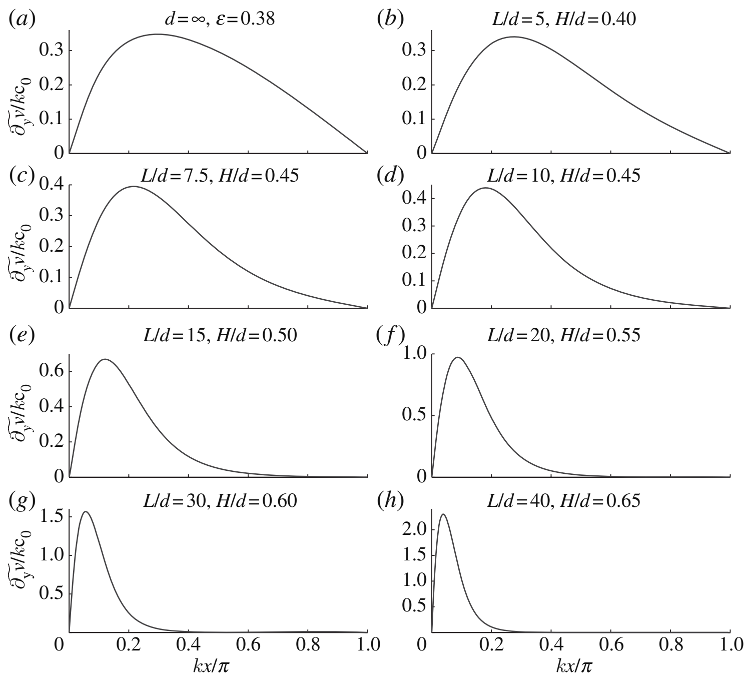

Figure 10. Vertical derivative of the vertical velocity at the free surface.

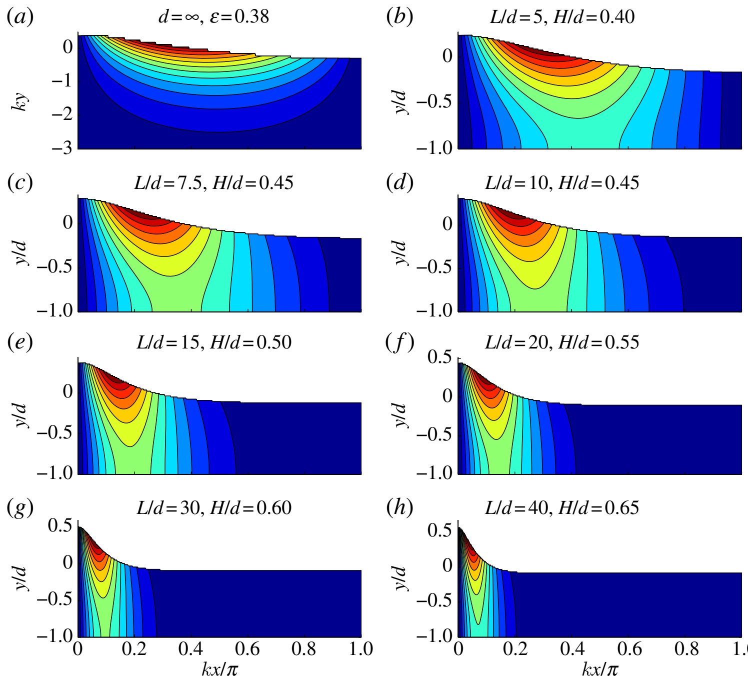

Figure 11. Iso-vertical derivative of the vertical velocity field. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

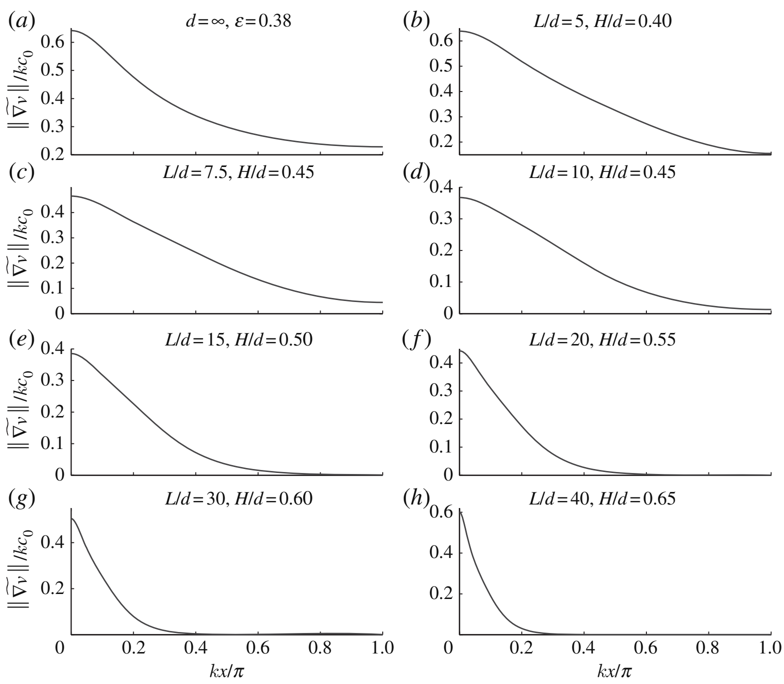

Figure 12. Magnitude of gradient $\|\text{grad } v\|$ at the free surface.

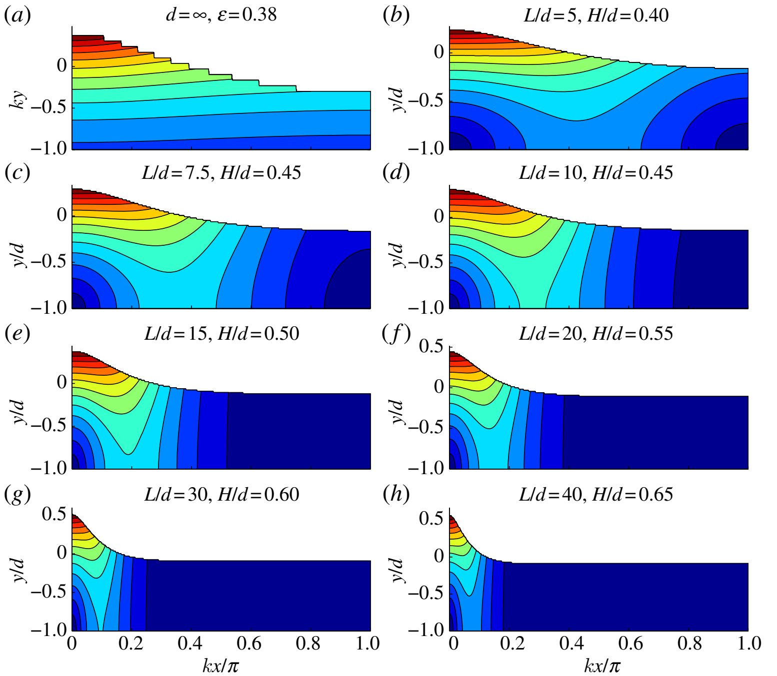

Figure 13. Iso $\|\text{grad } v\|$. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

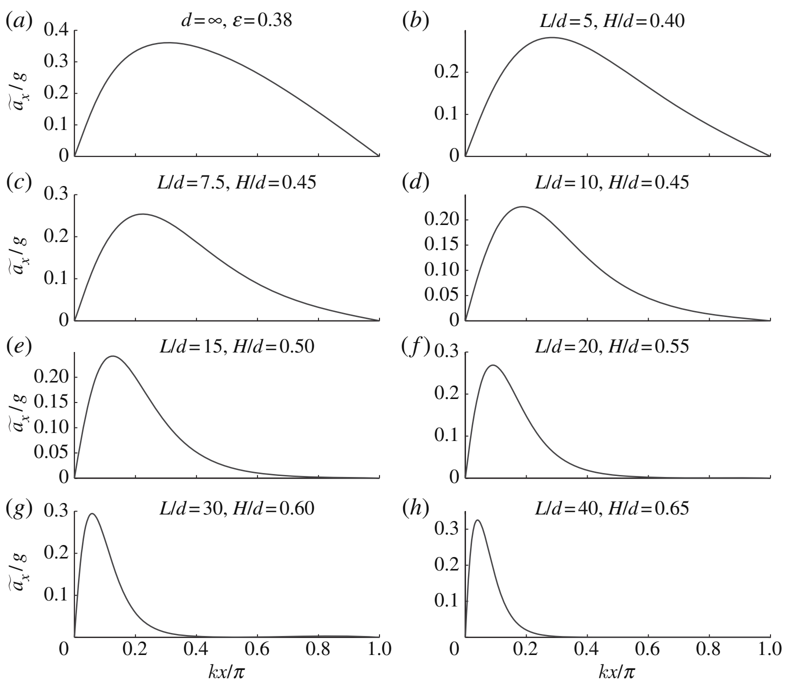

Figure 14. Horizontal acceleration at the free surface.

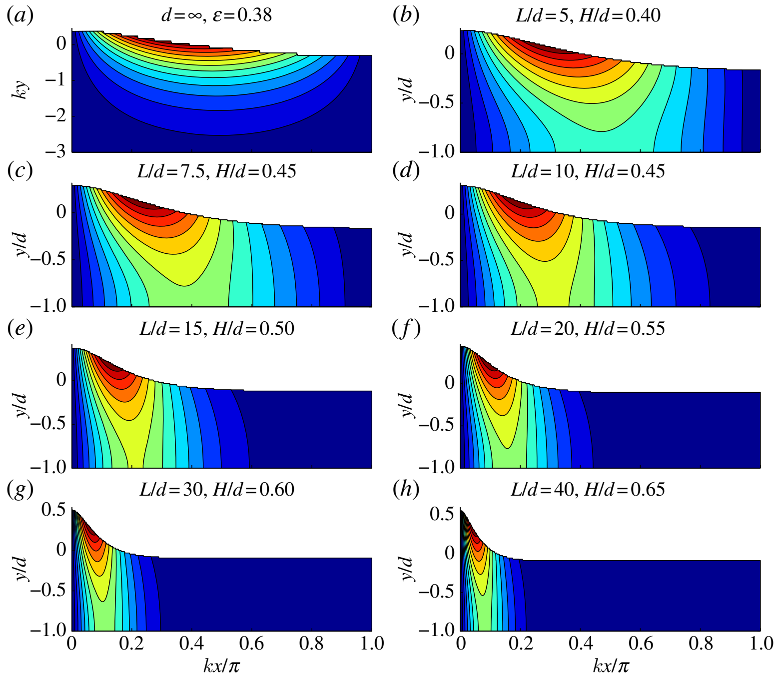

Figure 15. Iso-horizontal accelerations. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

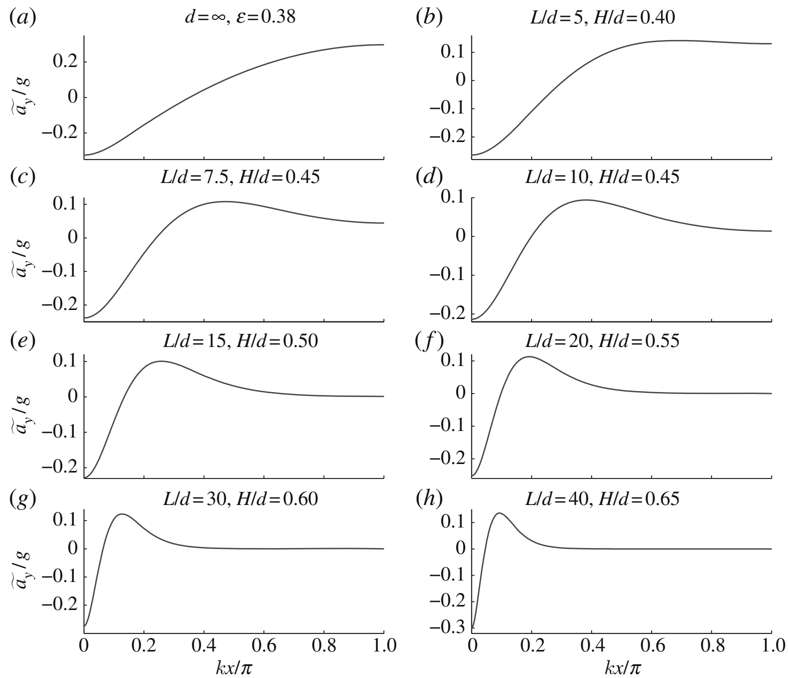

Figure 16. Vertical acceleration at the free surface.

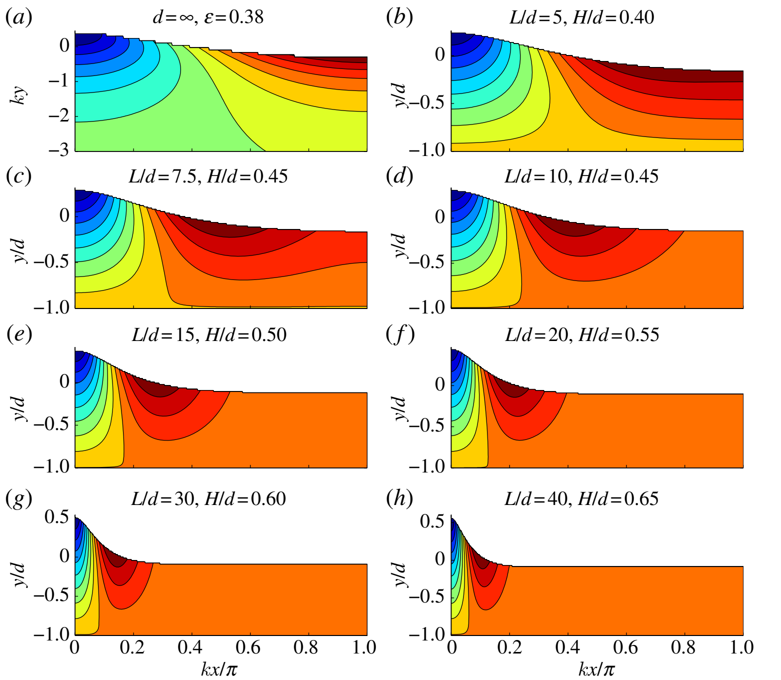

Figure 17. Iso-vertical accelerations. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

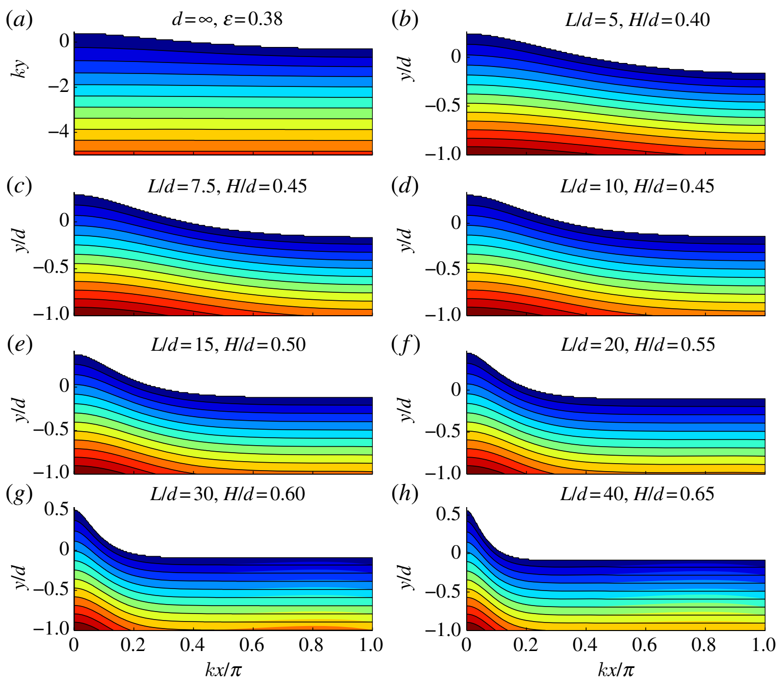

Figure 18. Isobars. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

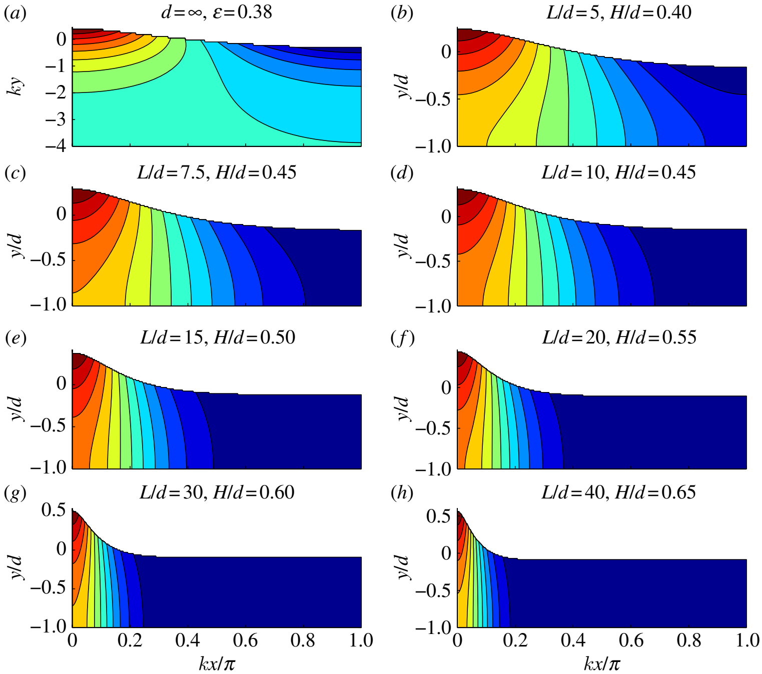

Figure 19. Iso-dynamic pressure $(p + gy)$. Black lines: 12 equally spaced iso-values (different on each panel). (Online version in colour.)

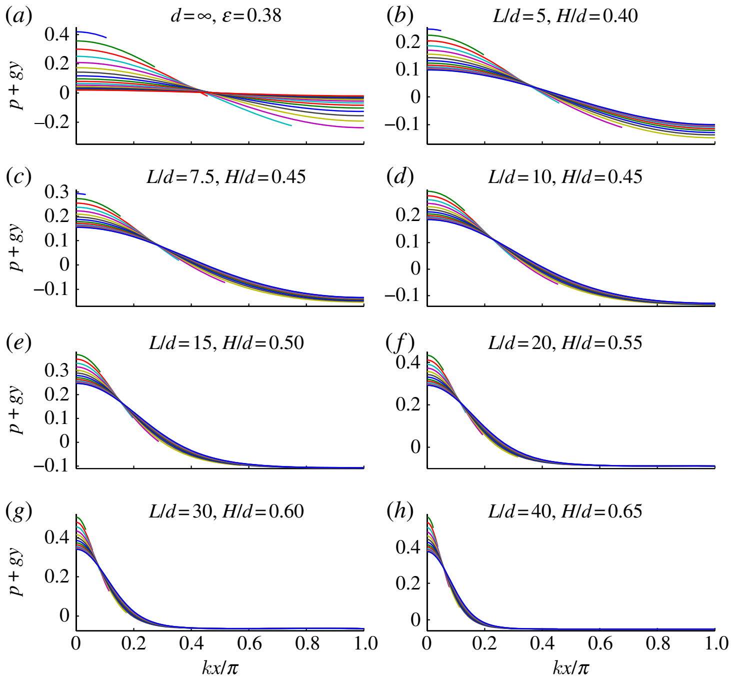

Figure 20. Dynamic pressure along various horizontal lines. Plots are discontinued where the horizontal line crosses the free surface. (Online version in colour.)

This short paper should be concluded by reminding the reader that the conjectures made here are based on observations of numerical 'evidence' and may not be true, in general. However, for symmetric waves with one crest per wavelength, these conjectures should be true at least up to a finite (not small) amplitude, i.e. not only for infinitesimal waves.

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