

PERMUTATION GROUPS AND TRANSFORMATION SEMIGROUPS: RESULTS AND PROBLEMS

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Abstract

J.M. Howie, the influential St Andrews semigroupist, claimed that we value an area of pure mathematics to the extent that (a) it gives rise to arguments that are deep and elegant, and (b) it has interesting interconnections with other parts of pure mathematics.

This paper surveys some recent results on the transformation semigroup generated by a permutation group G and a single non-permutation a . Our particular concern is the influence that properties of G (related to homogeneity, transitivity and primitivity) have on the structure of the semigroup. In the first part of the paper, we consider properties of $S = \langle G, a \rangle$ such as regularity and generation. The second is a brief report on the synchronization project, which aims to decide in what circumstances S contains an element of rank 1. The paper closes with a list of open problems on permutation groups and linear groups, and some comments about the impact on semigroups are provided.

These two research directions outlined above lead to very interesting and challenging problems on primitive permutation groups whose solutions require combining results from several different areas of mathematics, certainly fulfilling both of Howie's elegance and value tests in a new and fascinating way.

1 Regularity and generation

1.1 Introduction

How can group theory help the study of semigroups?

If a semigroup has a large group of units, we can apply group theory to it. But there may not be any units at all! According to a widespread (but never properly proved) belief, $\lim_{n \rightarrow \infty} \frac{z(n)}{s(n)} = 1$, where $z(n)$ is the number of 3-nilpotent semigroups of order n (semigroups with zero satisfying $xyz = 0$), and $s(n)$ is the number of semigroups of order n . Roughly speaking, this conjecture (which is for sure true!) implies that *almost all finite semigroups have only one idempotent, which is a zero* (see [25] and [18]). This observation, however, should not deter us from

the general goal of investigating how the group of units shapes the structure of the semigroup. There are many families of finite semigroups with group of units. In addition, some of those families of semigroups are interesting enough to keep many mathematicians busy their entire lives; in fact a unique family of finite semigroups, the endomorphism semigroups of vector spaces over finite fields, has been keeping experts in linear algebra busy for more than a century.

Regarding the general question of how the group of units can shape the structure of the semigroup, an especially promising area is the theory of *transformation semigroups*, that is, semigroups of mappings $\Omega \rightarrow \Omega$ (subsemigroups of the *full transformation semigroup* $T(\Omega)$, where $\Omega := \{1, \dots, n\}$). This area is especially promising for two reasons. First, in a transformation semigroup S , the units are the permutations; if there are any, they form a *permutation group* G and we can take advantage of the very deep recent results on them, chiefly the classification of finite simple groups (CFSG). Secondly, even if there are no units, we still have a group to play with, the *normaliser* of S in $\text{Sym}(\Omega)$, the set of all permutations g such that $g^{-1}Sg = S$.

The following result of Levi and McFadden [28] is the prototype for results of this kind. Let S_n and T_n denote the symmetric group and full transformation semigroup on $\Omega := \{1, 2, \dots, n\}$.

Theorem 1.1 *Let $a \in T_n \setminus S_n$, and let S be the semigroup generated by the conjugates $g^{-1}ag$ for $g \in S_n$. Then*

- (a) S is idempotent-generated;
- (b) S is regular;
- (c) $S = \langle a, S_n \rangle \setminus S_n$.

In other words, semigroups of this form, with normaliser S_n , have *very nice* properties!

Inspired by this result, we could formulate a general problem:

- Problem 1.2**
- (a) Given a semigroup property P , for which pairs (a, G) , with $a \in T_n \setminus S_n$ and $G \leq S_n$, does the semigroup $\langle g^{-1}ag : g \in G \rangle$ have property P ?
 - (b) Given a semigroup property P , for which pairs (a, G) as above does the semigroup $\langle a, G \rangle \setminus G$ have property P ?
 - (c) For which pairs (a, G) are the semigroups of the preceding parts equal?

The following portmanteau theorem lists some previously known results on this problem. The first part is due to Levi [26], the other two to Araújo, Mitchell and Schneider [8].

- Theorem 1.3**
- (a) *For any $a \in T_n \setminus S_n$ the semigroups $\langle g^{-1}ag : g \in S_n \rangle$ and $\langle g^{-1}ag : g \in A_n \rangle$ are equal.*
 - (b) *$\langle g^{-1}ag : g \in G \rangle$ is idempotent-generated for all $a \in T_n \setminus S_n$ if and only if $G = S_n$ or $G = A_n$ or G is one of three specific groups of low degrees.*

(c) $\langle g^{-1}ag : g \in G \rangle$ is regular for all $a \in T_n \setminus S_n$ if and only if $G = S_n$ or $G = A_n$ or G is one of eight specific groups of low degrees.

Recently, we have obtained several extensions of these results. The first theorem is proved in [3].

Theorem 1.4 *Given k with $1 \leq k \leq n/2$, the following are equivalent for a subgroup G of S_n :*

- (a) for all rank k transformations a , a is regular in $\langle a, G \rangle$;
- (b) for all rank k transformations a , $\langle a, G \rangle$ is regular;
- (c) for all rank k transformations a , a is regular in $\langle g^{-1}ag : g \in G \rangle$;
- (d) for all rank k transformations a , $\langle g^{-1}ag : g \in G \rangle$ is regular.

Moreover, we have a complete list of the possible groups G with these properties for $k \geq 5$, and partial results for smaller values.

It is worth pointing out that in the previous theorem the equivalence between (a) and (c) is not new (it appears in [27]). Really surprising, and a great result that semigroups owe to the classification of finite simple groups, are the equivalences between (a) and (b), and between (c) and (d).

The four equivalent properties above translate into a transitivity property of G which we call the *k -universal transversal property*, which we will describe in the Subsection 1.3.

In the framework of Problem 1.2, let P be the following property: the pair (a, G) , with $a \in T_n \setminus S_n$ and $G \leq S_n$, satisfies $\langle a, G \rangle \setminus G = \langle a, S_n \rangle \setminus S_n$.

The classification of the pairs (a, G) with this property poses a very interesting group theoretical problem. Recall that the rank of a map $a \in T_n$ is $|\Omega a|$ and the kernel of a is $\ker(a) := \{(x, y) \in \Omega^2 \mid xa = ya\}$; by the usual correspondence between equivalences and partitions, we can identify $\ker(a)$ with a partition $\{A_1, \dots, A_k\}$. Suppose $|\Omega| > 2$ and we have a rank 2 map $a \in T_n$. It is clear that $g^{-1}a \in \langle a, S_n \rangle$, for all $g \in S_n$. In addition, if $\ker(a) = \{A_1, A_2\}$, then $\ker(g^{-1}a) = \{A_1g, A_2g\}$. Therefore, in order to classify the groups with property P above we need to find the groups G such that

$$\{\{A_1, A_2\}g \mid g \in G\} = \{\{A_1, A_2\}g \mid g \in S_n\}. \quad (1)$$

If $|A_1| < |A_2|$, this is just $|A_1|$ -homogeneity; but if these two sets have the same size, the property is a little more subtle.

Extending this analysis, we see that the group-theoretic properties we need to investigate are transitivity on ordered partitions of given shape (this notion was introduced by Martin and Sagan [32] under the name *partition-transitivity*) and the weaker notion of transitivity on unordered partitions of given shape. This is done in Section 1.4, where we indicate the proof of the following theorem from [1].

Theorem 1.5 *We have a complete list (in terms of the rank and kernel type of a) for pairs (a, G) for which $\langle a, G \rangle \setminus G = \langle a, S_n \rangle \setminus S_n$.*

As we saw, the semigroups $\langle a, S_n \rangle \setminus S_n$ have very nice properties. In particular, the questions of calculating their automorphisms and congruences, checking for regularity, idempotent generation, etc., are all settled. Therefore the same happens for the groups G and maps $a \in T_n \setminus S_n$ such that $\langle a, G \rangle \setminus G = \langle a, S_n \rangle \setminus S_n$, and all these pairs (a, G) have been classified.

Another long-standing open question was settled by the following theorem, from [5].

Theorem 1.6 *The semigroups $\langle a, G \rangle \setminus G$ and $\langle g^{-1}ag : g \in G \rangle$ are equal for all $a \in T_n \setminus S_n$ if and only if $G = S_n$, or $G = A_n$, or G is the trivial group, or G is one of five specific groups.*

Problem 1.7 It would be good to have a more refined version of this where the hypothesis refers only to all maps of rank k , or just a single map a .

1.2 Homogeneity and related properties

A permutation group G on Ω is *k-homogeneous* if it acts transitively on the set of k -element subsets of Ω , and is *k-transitive* if it acts transitively on the set of k -tuples of distinct elements of Ω .

It is clear that k -homogeneity is equivalent to $(n-k)$ -homogeneity, where $|\Omega| = n$; so we may assume that $k \leq n/2$. It is also clear that k -transitivity implies k -homogeneity.

We say that G is *set-transitive* if it is k -homogeneous for all k with $0 \leq k \leq n$. The problem of determining the set-transitive groups was posed by von Neumann and Morgenstern [33] in the first edition of their influential book on game theory. In the second edition, they refer to an unpublished solution by Chevalley, but the first published solution was by Beaumont and Peterson [10]. The set-transitive groups are the symmetric and alternating groups, and four small exceptions with degrees 5, 6, 9, 9.

In an elegant paper in 1965, Livingstone and Wagner [30] showed:

Theorem 1.8 *Let G be k -homogeneous, where $2 \leq k \leq n/2$. Then*

- (a) G is $(k - 1)$ -homogeneous;
- (b) G is $(k - 1)$ -transitive;
- (c) if $k \geq 5$, then G is k -transitive.

In particular, part (a) of this theorem is proved by a short argument using character theory of the symmetric group. This can be translated into combinatorics, and generalised to linear and affine groups: see Kantor [23].

The k -homogeneous but not k -transitive groups for $k = 2, 3, 4$ were determined by Kantor [21, 22]. All this was pre-CFSG.

The k -transitive groups for $k > 1$ are known, but the classification uses CFSG. Lists can be found in various references such as [11, 15].

1.3 The k -universal transversal property

Let $G \leq S_n$, and k an integer smaller than n .

The group G has the k -universal transversal property, or k -ut for short, if for every k -element subset S of $\{1, \dots, n\}$ and every k -part partition P of $\{1, \dots, n\}$, there exists $g \in G$ such that Sg is a transversal or section for P : that is, each part of P intersects Sg in a single point.

Theorem 1.9 *For $k \leq n/2$, the following are equivalent for a permutation group $G \leq S_n$:*

- (a) *for all $a \in T_n \setminus S_n$ with rank k , a is regular in $\langle a, G \rangle$;*
- (b) *G has the k -universal transversal property.*

In order to get the surprising equivalence (noted after Theorem 1.4) of “ a is regular in $\langle a, G \rangle$ ” and “ $\langle a, G \rangle$ is regular”, we need to know that, for $k \leq n/2$, a group with the k -ut property also has the $(k - 1)$ -ut property. This fact, the analogue of Theorem 1.8(a), is not at all obvious.

We go by way of a related property: G is $(k - 1, k)$ -homogeneous if, given any two subsets A and B of $\{1, \dots, n\}$ with $|A| = k - 1$ and $|B| = k$, there exists $g \in G$ with $Ag \subseteq B$.

Now the k -ut property implies $(k - 1, k)$ -homogeneity. (Take a partition with k parts, the singletons contained in A and all the rest. If Bg is a transversal for this partition, then $Bg \supseteq A$, so $Ag^{-1} \subseteq B$.)

The bulk of the argument involves these groups. We show that, if $3 \leq k \leq (n - 1)/2$ and G is $(k - 1, k)$ -homogeneous, then either G is $(k - 1)$ -homogeneous, or G is one of four small exceptions (with $k = 3, 4, 5$ and $n = 2k - 1$).

It is not too hard to show that such a group G must be transitive, and then primitive. Now careful consideration of the orbital graphs shows that G must be 2-homogeneous, at which point we invoke the classification of 2-homogeneous groups (a consequence of CFSG).

One simple observation: if G is $(k - 1, k)$ -homogeneous but not $(k - 1)$ -homogeneous of degree n , then colour one G -orbit of $(k - 1)$ -sets red and the others blue; by assumption, there is no monochromatic k -set, so n is bounded by the Ramsey number $R(k - 1, k, 2)$. The values $R(2, 3, 2) = 6$ and $R(3, 4, 2) = 13$ are useful here; $R(4, 5, 2)$ is unknown, and in any case too large for our purposes.

Now we return to considering the k -ut property.

First, we note that the 2-ut property says that every orbit on pairs contains a pair crossing between parts of every 2-partition; that is, every orbital graph is connected. By Higman’s Theorem, this is equivalent to primitivity.

For $2 < k < n/2$, we know that the k -ut property lies between $(k - 1)$ -homogeneity and k -homogeneity, with a few small exceptions. In fact k -ut is equivalent to k -homogeneous for $k \geq 6$; we classify all the exceptions for $k = 5$, but for $k = 3$ and $k = 4$ there are some groups we are unable to resolve (affine, projective and Suzuki groups), which pose interesting problems (see Problems 3.1 and 3.2).

For large k we have:

Theorem 1.10 *For $n/2 < k < n$, the following are equivalent:*

- (a) *G has the k -universal transversal property;*
- (b) *G is $(k - 1, k)$ -homogeneous;*
- (c) *G is k -homogeneous.*

In the spirit of Livingstone and Wagner, we could ask:

Problem 1.11 Without using CFSG, show any or all of the following implications:

- (a) k -ut implies $(k - 1)$ -ut for $k \leq n/2$;
- (b) $(k - 1, k)$ -homogeneous implies $(k - 2, k - 1)$ -homogeneous for $k \leq n/2$;
- (c) k -ut (or $(k - 1, k)$ -homogeneous) implies $(k - 1)$ -homogeneous for $k \leq n/2$.

1.4 Partition transitivity and homogeneity

Let λ be a partition of n (a non-increasing sequence of positive integers with sum n). A partition of $\{1, \dots, n\}$ is said to have *shape* λ if the size of the i th part is the i th part of λ .

The group G is λ -*transitive* if, given any two (ordered) partitions of shape λ , there is an element of G mapping each part of the first to the corresponding part of the second. (This notion is due to Martin and Sagan [32].) Moreover, G is λ -*homogeneous* if there is an element of G mapping the first partition to the second (but not necessarily respecting the order of the parts).

Of course λ -transitivity implies λ -homogeneity, and the converse is true if all parts of λ are distinct. If $\lambda = (n - t, 1, \dots, 1)$, then λ -transitivity and λ -homogeneity are equivalent to t -transitivity and t -homogeneity.

The connection with semigroups is given by the next result, from [1]. Let G be a permutation group, and $a \in T_n \setminus S_n$, where r is the rank of a , and λ the shape of the kernel partition.

Theorem 1.12 *For $G \leq S_n$ and $a \in T_n \setminus S_n$, the following are equivalent:*

- (a) $\langle a, G \rangle \setminus G = \langle a, S_n \rangle \setminus S_n$;
- (b) *G is r -homogeneous and λ -homogeneous.*

So we need to know the λ -homogeneous groups. First, we consider λ -transitive groups.

If the largest part of λ is greater than $n/2$ (say $n - t$, where $t < n/2$), then G is λ -transitive if and only if it is t -homogeneous and the group H induced on a t -set by its setwise stabiliser is λ' -transitive, where λ' is λ with the part $n - t$ removed.

So if G is t -transitive, then it is λ -transitive for all such λ .

If G is t -homogeneous but not t -transitive, then $t \leq 4$, and examination of the groups in Kantor's list gives the possible λ' in each case.

So what remains is to show that, if G is λ -transitive but not S_n or A_n , then λ must have a part greater than $n/2$.

If $\lambda \neq (n), (n - 1, 1)$, then G is primitive.

If $n \geq 8$, then by *Bertrand's Postulate*, there is a prime p with $n/2 < p \leq n - 3$. If there is no part of λ which is at least p , then the number of partitions of shape λ (and hence the order of G) is divisible by p . A theorem of Jordan (see Wielandt [40], Theorem 13.9) now shows that G is symmetric or alternating.

The classification of λ -homogeneous but not λ -transitive groups is a bit harder. We have to use

- (a) a little character theory to show that either G fixes a point and is transitive on the rest, or G is transitive;
- (b) the argument using Bertrand's postulate and Jordan's theorem as before;
- (c) CFSG (to show that G cannot be more than 5-homogeneous if it is not S_n or A_n).

The outcome is a complete list of such groups.

1.5 Normalising groups

We define a permutation group G to be *normalising* if $\langle g^{-1}ag : g \in G \rangle = \langle a, G \rangle \setminus G$ for all $a \in T_n \setminus S_n$.

The classification of normalising groups given by Theorem 1.6 is a little different; although permutation group techniques are essential in the proof, we didn't find a simple combinatorial condition on G which is equivalent to this property. We will not discuss it further here.

2 Synchronization

2.1 Introduction

In this section, we give a brief report on synchronization.

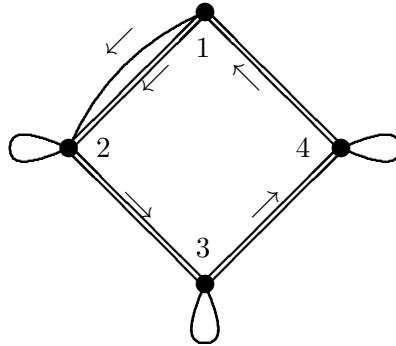
A (finite deterministic) *automaton* consists of a finite set Ω of *states* and a finite set of maps from Ω to Ω called *transitions*, which may be composed freely.

In other words, it is a transformation semigroup with a distinguished set of generators.

An automaton is *synchronizing* if there is a map of rank 1 (image of size 1) in the semigroup. A word in the generators expressing such a map is called a *reset word*.

We will also call a transformation semigroup *synchronizing* if it contains an element of rank 1.

Example 2.1 This example has four (numbered) states, and two transitions A and B , shown as double and single lines respectively.



The reader can check easily that, irrespective of the starting state, following the path $BAAABAAAB$ always ends in state 2, and hence this is a reset word of length 9. In fact, this is the shortest reset word.

The *Černý Conjecture* asserts that if an n -state automaton is synchronizing, then it has a reset word of length at most $(n - 1)^2$. The above example, with the square replaced by an n -gon, shows that this would be best possible. The problem has been open for about 45 years. The best known bound is cubic.

It is known that testing whether an automaton is synchronizing is in P, but finding the length of the shortest reset word is NP-hard.

2.2 Graph homomorphisms and transformation semigroups

All graphs here are undirected simple graphs (no loops or multiple edges).

A *homomorphism* from a graph X to a graph Y is a map f from the vertex set of X to the vertex set of Y which carries edges to edges. (We don't specify what happens to a non-edge; it may map to a non-edge, or to an edge, or collapse to a vertex.) An *endomorphism* of a graph X is a homomorphism from X to itself.

Let K_r be the complete graph with r vertices. The *clique number* $\omega(X)$ of X is the size of the largest complete subgraph, and the *chromatic number* $\chi(X)$ is the least number of colours required for a proper colouring of the vertices (adjacent vertices getting different colours).

- (a) There is a homomorphism from K_r to X if and only if $\omega(X) \geq r$.
- (b) There is a homomorphism from X to K_r if and only if $\chi(X) \leq r$.

There are correspondences in both directions between graphs and transformation semigroups (not quite functorial, or a Galois correspondence, sadly!)

First, any graph X has an *endomorphism semigroup* $\text{End}(X)$.

In the other direction, given a transformation semigroup S on Ω , its *graph* $\text{Gr}(S)$ has Ω as vertex set, two vertices v and w being joined if and only if there is no element of S which maps v and w to the same place.

- (a) $\text{Gr}(S)$ is complete if and only if $S \leq S_n$;
- (b) $\text{Gr}(S)$ is null if and only if S is synchronizing;
- (c) $S \leq \text{End}(\text{Gr}(S))$ for any $S \leq T_n$;

(d) $\omega(\text{Gr}(S)) = \chi(\text{Gr}(S))$; this is equal to the minimum rank of an element of S .

Now the main theorem of this section describes the unique obstruction to synchronization for a transformation semigroup.

Theorem 2.2 *A transformation semigroup S on Ω is non-synchronizing if and only if there is a non-null graph X on the vertex set Ω with $\omega(X) = \chi(X)$ and $S \leq \text{End}(X)$.*

In the reverse direction, the endomorphism semigroup of a non-null graph cannot be synchronizing, since edges can't be collapsed. In the forward direction, take $X = \text{Gr}(S)$; there is some straightforward verification to do. (For details see [4].)

2.3 Maps synchronized by groups

Let $G \leq S_n$ and $a \in T_n \setminus S_n$. We say that G *synchronizes* a if $\langle a, G \rangle$ is synchronizing.

By abuse of language, we say that G is *synchronizing* if it synchronizes every element of $T_n \setminus S_n$.

Our main problem is to determine the synchronizing groups. From the theorem, we see that G is non-synchronizing if and only if there is a G -invariant graph whose clique number and chromatic number are equal.

Rystsov [36] showed the following result, which implies that synchronizing groups are necessarily primitive.

Theorem 2.3 *A permutation group G of degree n is primitive if and only if it synchronizes every map of rank $n - 1$.*

We give a brief sketch of the proof, to illustrate the graph endomorphism technique. The backward implication is trivial; so suppose, for a contradiction, that G is primitive but fails to synchronize the map a of rank $n - 1$. Then there are two points x, y with $xa = ya$, and a is bijective on the remaining points. Choose a graph X with $\langle G, a \rangle \leq \text{End}(X)$. Note that X is a regular graph. Since a is an endomorphism, x and y are non-adjacent; so a maps the neighbours of x bijectively to the neighbours of xa , and similarly the neighbours of y to those of ya . Since $xa = ya$, we see that x and y have the same neighbour set. Now “same neighbour set” is an equivalence relation preserved by G , contradicting primitivity.

So a synchronizing group must be primitive.

We have recently improved this: a primitive group synchronizes every map of rank $n - 2$. The key tool in the proof is graph endomorphisms. Also, a primitive group synchronizes every map of kernel type $(k, 1, \dots, 1)$. For both results, and further information, see [4].

Also, G is synchronizing if and only if there is no G -invariant graph, not complete or null, with clique number equal to chromatic number. For more on this see [9, 13, 34, 35, 36, 38, 39]. Thus, a 2-homogeneous group is synchronizing, and a synchronizing group is primitive. For if G is 2-transitive, the only G -invariant graphs are complete or null; and if G is imprimitive, then it preserves a complete multipartite graph.

Furthermore, a synchronizing group is *basic* in the O’Nan–Scott classification, that is, not contained in a wreath product with the product action. (For non-basic primitive groups preserve Hamming graphs, which have clique number equal to chromatic number.) By the O’Nan–Scott Theorem, such a group is affine, diagonal or almost simple.

None of the above implications reverses. Indeed, there are non-synchronizing basic groups of all three O’Nan–Scott types.

We are a long way from a classification of synchronizing groups. The attempts to classify them lead to some interesting and difficult problems in extremal combinatorics, finite geometry, computation, etc. But that is another survey paper! We content ourselves here with a single result about an important class of primitive groups, namely the classical symplectic, orthogonal and unitary groups, acting on their associated polar spaces. The implicit geometric problem has not been completely solved, despite decades of work by finite geometers. We refer to Thas [37] for a survey.

Theorem 2.4 *A classical group, acting on the points of its associated polar space, is non-synchronizing if and only if the polar space possesses either an ovoid and a spread, or a partition into ovoids.*

2.4 A conjecture

We regard the following as the biggest open problem in the area. A map $a \in T_n$ is *non-uniform* if its kernel classes are not all of the same size.

Conjecture 2.5 A primitive permutation group synchronizes every non-uniform map.

We have some partial results about this but are far from a proof!

3 Problems

In addition to the problems included above, we collect here a number of problems on the general interplay between properties of the group of units and properties of the semigroup containing it.

We start by proposing a problem to experts in number theory. If this problem can be solved, the results on $\text{AGL}(1, p)$, in [3], will be dramatically sharpened.

Problem 3.1 Classify the prime numbers p congruent to 11 (mod 12) such that for some $c \in \text{GF}(p)^*$ we have $|\langle -1, c, c - 1 \rangle| < p - 1$.

The primes less than 500 with this property are 131, 191, 239, 251, 311, 419, 431, and 491.

Problem 3.2 Do the Suzuki groups $\text{Sz}(q)$ have the 3-universal transversal property?

Classify the groups G that have the 4-ut property, when $\text{PSL}(2, q) \leq G \leq \text{PFL}(2, q)$, with either q prime (except $\text{PSL}(2, q)$ for $q \equiv 1 \pmod{4}$, which is not 3-homogeneous), or $q = 2^p$ for p prime.

A group $G \leq S_n$ has the $(n - 1)$ -universal transversal property if and only if it is transitive. And $\langle a, G \rangle$ (for a rank $n - 1$ map a) contains all the rank $n - 1$ maps of T_n if and only if G is 2-homogeneous. In this last case $\langle a, G \rangle$ is regular for all $a \in T_n$, because $\langle a, G \rangle = \{b \in T_n \mid |\Omega b| \leq n - 1\} \cup G$, and this semigroup is well known to be regular.

Problem 3.3 Classify the groups $G \leq S_n$ such that G together with any rank $n - k$ map, where $k \leq 5$, generate a regular semigroup. We already know that such G must be k -homogeneous; so we know which groups to look at (see Theorem 1.10).

The difficulty here (when rank $k > \lfloor \frac{n+1}{2} \rfloor$) is that a k -homogeneous group is not necessarily $(k - 1)$ -homogeneous. Therefore a rank k map $a \in T_n$ might be regular in $\langle a, G \rangle$, but we are not sure that there exists $g \in G$ such that $\text{rank}(bgb) = \text{rank}(b)$, for $b \in \langle a, G \rangle$ such that $\text{rank}(b) < \text{rank}(a)$.

Problem 3.4 The most general problem that has to be handled is the classification of pairs (a, G) , where $a \in T_n$ and $G \leq S_n$, such that $\langle a, G \rangle$ is a regular semigroup.

When investigating $(k - 1)$ -homogeneous groups without the k -universal transversal property (k -ut property), it was common that some of the orbits on the k -sets have transversals for all the partitions. Therefore the following definition is natural.

A group $G \leq S_n$ is said to have the weak k -ut property if there exists a k -set $S \subseteq \Omega$ such that the orbit of S under G contains a section for all k -partitions. Such a set is called a G -universal transversal set. A solution to the following problem would have important consequences in semigroup theory.

Problem 3.5 Classify the groups with the weak k -ut property; in addition, for each of them, classify their G -universal transversal sets.

In McAlister's celebrated paper [31] it is proved that, if $e^2 = e \in T_n$ is a rank $n - 1$ idempotent, then $\langle G, e \rangle$ is regular for all groups $G \leq S_n$. In addition, assuming that $\{\alpha, \beta\}$ is the non-singleton kernel class of e and $\alpha e = \beta$, if α and β are not in the same orbit under G , then $\langle e, G \rangle$ is an orthodox semigroup (that is, the idempotents form a subsemigroup); and $\langle e, G \rangle$ is inverse if and only if α and β are not in the same orbit under G and the stabilizer of α is contained in the stabilizer of β .

Problem 3.6 Classify the groups $G \leq S_n$ that together with any idempotent [rank k idempotent] generate a regular [orthodox, inverse] semigroup.

Classify the pairs (G, a) , with $a \in T_n$ and $G \leq S_n$, such that $\langle e, G \rangle$ is inverse [orthodox].

The theorems and problems in this paper admit linear versions that are interesting for experts in groups and semigroups, but also to experts in linear algebra and matrix theory.

Problem 3.7 Prove (or disprove) that if $G \leq \text{GL}(n, q)$ such that for all singular matrix a there exists $g \in G$ with $\text{rank}(a) = \text{rank}(aga)$, then G contains the special linear group.

For $n = 2$ and for $n = 3$, this condition is equivalent to irreducibility of G . But we conjecture that, for sufficiently large n , it implies that G contains the special linear group.

Problem 3.8 Classify the groups $G \leq \text{GL}(n, q)$ such that for all rank k (for a given k) singular matrix a we have that a is regular in $\langle G, a \rangle$ [the semigroup $\langle G, a \rangle$ is regular].

To handle this problem it is useful to keep in mind the following results. Kantor [23] proved that if a subgroup of $\text{P}\Gamma\text{L}(d, q)$ acts transitively on k -dimensional subspaces, then it acts transitively on l -dimensional subspaces for all $l \leq k$ such that $k + l \leq n$; in [24], he showed that subgroups transitive on 2-dimensional subspaces are 2-transitive on the 1-dimensional subspaces with the single exception of a subgroup of $\text{PGL}(5, 2)$ of order $31 \cdot 5$; and, with the second author [12], he showed that such groups must contain $\text{PSL}(d, q)$ with the single exception of the alternating group A_7 inside $\text{PGL}(4, 2) \cong A_8$. Also Hering [19, 20] and Liebeck [29], using CFSG, classified the subgroups of $\text{PGL}(d, p)$ which are transitive on 1-spaces.

Regarding synchronization, the most important question (in our opinion) is the following conjecture, stated earlier.

Problem 3.9 Is it true that every primitive group of permutations of a finite set Ω synchronizes every non-uniform transformation on Ω ?

Assuming the previous question has an affirmative answer (as we believe), an intermediate step in order to prove it would be to solve the following set of connected problems:

- Problem 3.10**
- (a) Prove that every map of rank $n - 3$, with non-uniform kernel, is synchronized by a primitive group. This is known for idempotent maps (see [4]).
 - (b) Prove that a primitive group synchronizes every non-uniform map of rank 5.
 - (c) Prove that if in $S = \langle f, G \rangle$ there is a map of minimal rank $r > 1$, there can be no map in S with rank $r + 2$.

The next class of groups lies strictly between primitive and synchronizing.

Problem 3.11 Is it possible to classify the primitive groups which synchronize every rank 3 map?

Note that there are primitive groups that do not synchronize a rank 3 map (see [34]). And there are non-synchronizing groups which synchronize every rank 3 map. Take for example $\text{PGL}(2, 7)$ of degree 28; this group is non-synchronizing, but synchronizes every rank 3 map, since 28 is not divisible by 3.

There are very fast polynomial-time algorithms to decide if a given set of permutations generates a primitive group, or a 2-transitive group.

Problem 3.12 Find an efficient algorithm to decide if a given set of permutations generates a synchronizing group.

It would be quite remarkable if such an algorithm exists; as we saw, it would in particular resolve questions about ovoids and spreads in certain polar spaces (among other things).

There are a number of natural problems related to λ -homogeneity.

Problem 3.13 Let $H \leq S_n$ be a 2-transitive group. Classify the pairs (a, G) , where $a \in S_n$ and $G \leq S_n$, such that $\langle a, G \rangle = H$.

Problem 3.14 Let $G \leq S_n$ be a 2-transitive group. (The list of those groups is available in [11, 15].) For every $a \in T_n$ describe the structure of $\langle G, a \rangle \setminus G$. In particular (where G is a 2-transitive group and $a \in T_n$):

- (a) classify all the pairs (a, G) such that $\langle a, G \rangle$ is a regular semigroup (that is, for all $x \in \langle a, G \rangle$ there exists $y \in \langle a, G \rangle$ such that $x = xyx$);
- (b) classify all the pairs (a, G) such that $\langle a, G \rangle \setminus G$ is generated by its idempotents;
- (c) classify all the pairs (a, G) such that $\langle a, G \rangle \setminus G = \langle g^{-1}ag \mid g \in G \rangle$;
- (d) describe the automorphisms, congruences, principal right, left and two-sided ideals of the semigroups $\langle a, G \rangle$ (when G is a 2-transitive group).

Problem 3.15 For each 2-transitive group G classify the G -pairs, that is, the pairs (a, H) such that $H \leq S_n$, $a \in T_n$ and $\langle a, G \rangle \setminus G = \langle a, H \rangle \setminus H$.

Problem 3.16 Let V be a finite dimension vector space. A pair (a, G) , where a is a singular endomorphism of V and $G \leq \text{Aut}(V)$, is said to be an $\text{Aut}(V)$ -pair if

$$\langle a, G \rangle \setminus G = \langle a, \text{Aut}(V) \rangle \setminus \text{Aut}(V).$$

Classify the $\text{Aut}(V)$ -pairs.

Problem 3.17 Formulate and prove analogues of the results in this paper, but for semigroups of linear maps on a vector space.

Problem 3.18 Solve the analogue of Problem 3.17 for independence algebras (for definitions and fundamental results see [6, 7, 14, 16, 17]).

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