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Harmonic Inferentialism and the Logic of Identity

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Abstract. Inferentialism claims that the rules for the use of an expression express its meaning without any need to invoke meanings or denotations for them. Logical inferentialism endorses inferentialism specifically for the logical constants. Harmonic inferentialism, as the term is introduced here, usually but not necessarily a subbranch of logical inferentialism, follows Gentzen in proposing that it is the introduction-rules whch give expressions their meaning and the elimination-rules should accord harmoniously with the meaning so given. It is proposed here that the logical expressions are those which can be given schematic rules that lie in a specific sort of harmony, general-elimination (ge) harmony, resulting from applying a certain operation, the ge-procedure, to produce ge-rules in accord with the meaning defined by the I-rules. Griffiths (2014) claims that identity cannot be given such rules, concluding that logical inferentialists are committed to ruling identity a non-logical expression. It is shown that the schematic rules for identity given in Read (2004), slightly amended, are indeed ge-harmonious, so confirming that identity is a logical notion.
Keywords: inferentialism, identity, inversion principle, harmony, proof-theoretic semantics; Gentzen, Lorenzen, Prawitz, Dummett, Brandom, Griffiths.

§1. Inferentialism Inferentialism claims that the meaning of any expression is given not by identifying some object as its meaning, but by stating the rules for its use in inference, in particular, by stating the grounds on which statements containing an expression can be asserted—that is, under what conditions such statements can be inferred from others—and the consequences which such assertions entail—that is, what statements can be inferred from such an assertion. Logical inferentialists make the restricted claim that such an inferentialist account is appropriate at least for the logical constants, whatever the appropriate story is for non-logical expressions. Logical inferentialism champions a proof-theoretic semantics in terms of rules of inference, as opposed to a model-theoretic semantics, whereby the meaning of the logical constants is to be given by the recursive clauses in the definition of a model.

Broad-brush inferentialism is espoused in Brandom (1994, 2000). There Brandom contrasts material inferences with formal inference. The latter can be defined in terms of the former, but not vice versa, he says (Brandom, 2000, p. 55). A formally valid inference is a valid inference that "cannot be turned into a materially bad one by substituting non-privileged for non-privileged vocabulary" (Brandom, 1994, p. 104). But what determines validity itself? The inferentialist idea espoused here is this: when we add a word to our vocabulary, we (implicitly or explicitly) associate it with grounds for asserting statements containing it. The collection of statements warranted in this way is the introduction- (or I-)fragment. Certain statements

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involving the new term will be derivable in the I-fragment, but as yet, no inferences from such assertions will be possible. Nonetheless, some such inferences will be admissible: that is, if a certain statement involving the new term is derivable, then certain other statements, containing the term or not, will also be derivable using the rules for assertion. For example, suppose the rule for 'true' is that, if a statement s is assertable, so is 's is true'. Call this **Tr**-I, or **Tr**-Introduction. Then it is clear that if 's is true' is assertable, so too will be s, since that is the only ground on which 's is true' can be asserted. That thought justifies adding the inference of s from 's is true' as an elimination- (or E-)rule.

Prawitz (1973, p. 234) summarizes the idea as follows:

> "The main idea is this: while the introduction inferences represent the form of proofs of compound formulas by the very meaning of the logical constants ... and hence preserve validity, other inferences have to be justified by the evidence of operations of a certain kind."

Gentzen (1969, p. 81) proposed that "by making these ideas more precise, it should be possible to display the E-inferences as unique functions of their corresponding I-inferences on the basis of certain assumptions,"[1] exhibiting classic examples. The first attempt at generalising and making Gentzen's ideas precise was due to Paul Lorenzen with his idea of the inversion principle:

> "A general formulation of an 'inversion principle' would be for instance: given a system of rules such that for the derivation of an expression p_0 only the rules
>
> $$p_1 \to p_0; \ldots; p_n \to p_0$$
>
> (possibly containing bound variables) are needed, then for every expression p, in which certain variables do not occur free, the meta-rule
>
> $$[p_1 \to p; \ldots; p_n \to p] \to (p_0 \to p)$$
>
> is valid."[2]

Satisfaction of the inversion principle ensures that the E-rules generated are not only individually no stronger than the I-rules warrant but collectively no weaker than are warranted too. Lorenzen's proposal has been developed in recent years into the conception of general-elimination rules and of a general-elimination procedure for generating those ge-rules.[3] Suppose a logical expression '$*$', forming a wff $*\vec{\alpha}$

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[1] "Durch Präzisierung dieser Gedanken dürfte es möglich sein, die B-Schlüsse auf Grund gewisser Anforderungen als eindeutige Funktionen der zugehörigen E-Schlüsse nachzuweisen." (Gentzen, 1935, p. 189)

[2] "Eine allgemeine Formulierung eines 'Inversionsprinzips' wäre etwa: Ist ein System von Regeln vorgegeben, so daß zur Ableitung einer Aussage p_0 nur die Regeln $p_1 \to p_0; \ldots; p_n \to p_0$ (evtl. mit gebundenen Variablen) benutzt werden können, so gilt für jede Aussage p, in der gewisse Variable nicht frei vorkommen, die Metaregel $p_1 \to p; \ldots; p_n \to p \dot{\to} p_0 \to p$." (Lorenzen, 1950, p. 176) In Lorenzen (1955, §1.4), he attempted to spell out the necessary conditions on the bound variables, later corrected in Hermes (1959).

[3] See, e.g., von Plato (2001) and Read (2010). The general-elimination procedure proposed in Francez & Dyckhoff (2012) is significantly different from that given here.

from a collection $\vec{\alpha}$ of 0 or more arguments, has m I-rules ($m \geq 0$), each of the form:

$$\frac{\pi_{i1} \quad \cdots \quad \pi_{in_i}}{*\vec{\alpha}} \; *\mathrm{I}_i$$

Here each $\pi_{ij}, 0 \leq j \leq n_i$, the j-th premise of the rule, is a derivation of some wff, γ_{ij}, from assumptions $\{\pi_k : k \in K_{ij}\}$ which are discharged by the rule. (In the simplest case, $K_{ij} = \varnothing$.) These I-rules generate a collection of E-rules each of the form:

$$\frac{*\vec{\alpha} \quad \begin{matrix}[\pi_{1j(1)}]\\ \vdots \\ \zeta\end{matrix} \quad \cdots \quad \begin{matrix}[\pi_{mj(m)}]\\ \vdots \\ \zeta\end{matrix}}{\zeta} \; *\mathrm{E}_j$$

($j \in \bigtimes_{i=1}^{m} n_i$) each minor premise of which is a derivation of a common wff ζ on the assumption of one of the premises of each I-rule, where that assumption is discharged by the E-rule.[4]

That the E-rules are no stronger than is warranted is shown by a local reduction, demonstrating that anything following from $*\vec{\alpha}$ by $*\mathrm{E}_j$ already follows from the grounds for assertion of $*\vec{\alpha}$:[5]

$$\frac{\dfrac{\begin{matrix}\vdots\\ \pi_{i1}\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\\ \pi_{in_i}\end{matrix}}{*\vec{\alpha}} \; *\mathrm{I}_i \quad \begin{matrix}[\pi_{1j(1)}]\\ \vdots \\ \zeta\end{matrix} \quad \cdots \quad \begin{matrix}[\pi_{mj(m)}]\\ \vdots \\ \zeta\end{matrix}}{\zeta} \; *\mathrm{E}_j$$

simplifies to

$$\begin{matrix}\vdots\\ \pi_{ij(i)}\\ \vdots\\ \zeta\end{matrix}$$

Thus $\{*\mathrm{E}_j : j \in \bigtimes_{i=1}^{m} n_i\}$ warrant assertion of no more than is already warranted by the grounds for asserting $*\vec{\alpha}$ given in $\{*\mathrm{I}_i : i \leq m\}$.

But more: recall that we are adding the logical constant '$*$' by the rules $\{*\mathrm{I}_i : i \leq m\}$ to an atomic basis which lacks the expression '$*$'—and if there are other logical constants, we are considering so far only the I-fragment which their I-rules define. Hence the rules in $\{*\mathrm{I}_i : i \leq m\}$ give the *only* grounds for asserting $*\vec{\alpha}$. Accordingly, if some application of $*\mathrm{E}_j$ warrants assertion of ζ, then $*\vec{\alpha}$ must already be warranted, and so once again, what warrants ζ is the warrant for $*\vec{\alpha}$ (plus warrants for any parametric assumptions for the premises). Thus $\{*\mathrm{E}_j : j \in \bigtimes_{i=1}^{m} n_i\}$ warrant assertion of anything which grounds the assertion of $*\vec{\alpha}$. The rules for '$*$' are complete. Moriconi & Tesconi (2008, p. 105) put the point as follows:

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[4] Where for some i, $*\mathrm{I}_i$ allows discharge of an assumption, $*\mathrm{E}_j$ will be a higher-level rule in the sense of Schroeder-Heister (1984).

[5] See Prawitz (1965, ch. II §2). In general, in the presence of other connectives, we may need to perform so-called permutative reductions to permute the application of $*\mathrm{E}_j$ with other E-rules to bring $*\mathrm{I}_i$ and $*\mathrm{E}_j$ into contact. See, e.g., Dummett (1977, p. 112) and (1991, p. 250).

> "*[A]ll* $*$-propositions are obtained ... [that is,] completeness is attained if whenever a $*$-proposition [$*\vec{\alpha}$] satisfies a certain condition [namely, that expressed in the set of E-rules] and the [introduction-]rule for '$*$' states that [$*\vec{\alpha}$ can be inferred from Y] then the underlying implicational structure guarantees that [Y satisfies the condition too]."

§2. Harmonic Inferentialism It is tempting to think that the inferential system for a term introduced in this way will be a conservative extension of the system to which the term was added. Prawitz, in his review of Dummett (1991), pointed out that this temptation is mistaken, noting in particular the case of adding terms like 'true':

> "... because we know from Gödel's incompleteness theorem ... that the addition to arithmetic of higher order concepts may lead to an enriched system that is not a conservative extension of the original in spite of the fact that some of these concepts are governed by rules that must be said to satisfy the requirement of harmony."[6]

'Harmony' was the epithet coined by Dummett (1973, p. 396) to capture the idea that the two sorts of rule (I-rules and E-rules) should exhibit "a certain consonance between the two aspects of the use of a given form of expression" (Dummett, 1973, p. 397), that is, between the grounds for assertion and the consequences of such assertions. In fact, for Dummett harmony is a relatively weak notion:

> "Harmony is an excessively modest demand ... It does not show that ... we are accustomed to draw all those consequences we should be entitled to draw." (Dummett, 1991, p. 287)

When the E-rules allow all and only the warranted consequences to be drawn, Dummett called the set of I- and E-rules 'stable'. We have seen that the inversion principle ensures that the ge-rules generated by a set of I-rules form with them a stable set of rules. Stability ensures that not only do the I-rules justify the E-rules, but in addition, the E-rules justify the (very same) I-rules. It ensures the consonance behind the original conception of harmony.[7]

Steinberger (2011, p. 619) claims that the basis of harmony is a principle of innocence:

[6] Prawitz (1994, p. 374). See also Brandom (1994, p. 127), who expresses similar reservations. Sundholm (1998, p. 202) spelled the point out: "The truth of the [Gödel] proposition has not been demonstrated solely according to the rules and axioms of the original formalism ... [W]hat is called for ... is the use of the concept of truth for sentences in the arithmetical language."

[7] See also Zucker & Tragesser (1978, p. 506), who propose that the E-rules "*stabilize or delimit* the meaning of the logical constant concerned, by saying, in effect, of the given I-rules: 'These are the only ways in which this constant can be introduced'."

> "It should not be possible, solely by engaging in deductive logical reasoning, to discover hitherto unknown (atomic) truths that we would have been incapable of discovering independently of logic."

That the harmony of a term's rules does not mean that its addition always constitutes a conservative extension of the atomic basis shows that Steinberger's principle is mistaken. The point of harmony is that the consequences drawn from an assertion (by the E-rules) should be no more than is justified by the grounds for their assertion (encapsulated in the I-rules), but those I-rules themselves can be creative, as is the case with the truth predicate of arithmetic. Addition of the truth predicate allows one to prove the Gödel sentence of the original theory, which is a purely number-theoretic truth. Often the rules are not creative in that way (as with the standard connectives of propositional logic), but nothing in Dummett's conception of harmony as consonance between the two aspects of an assertion rules this out.

Steinberger (2011, p. 629) goes on to speak of a core notion of ideal harmony:

> "Let us, in this hopeful vein, refer to the conjunction of levelling procedures [i.e., harmony] and a stability requirement designed to ward off E-weak disharmony [i.e., that the E-rules are weaker than the I-rules warrant] as *ideal* harmony."

He notes (2011, p. 634) that Dummett conjectured that ideal harmony entails what Dummett calls total harmony (*viz* conservative extension), but concedes (2011, p. 630) that this conjecture is mistaken: "Conceptual progress in the sciences ... leads to non-conservative extensions" (2011, p. 619). Consequently, Steinberger is wrong to argue that innocence is essential to the role logic plays. Two mistaken but related ideas dominated thought about logic in the twentieth century. One was that all logical consequence is formal; the other that logic is empty. The connection between the two ideas is the thought that all content is contained in the non-logical terms, and logical consequence is simply truth-preservation through all substitutions for the non-logical terms. This is the real basis of Steinberger's principle of innocence—more a principle of impotence, of the impotence and emptiness of logic as proclaimed by Wittgenstein and trumpeted by the logical positivists.[8] Logic is seen as a purely formal matter and not one of sense or content. But this is a mistake. Inferentialism recognises that the logical constants also have content, that content being implicitly defined by the rules of usage. Adding harmonious rules for truth allows us to prove the Gödel sentence and relative consistency.

What has to be accepted is that, although their ultimate aim is the same, namely, an account of the meaning of logical constants in purely proof-theoretical terms, different authors have different conceptions of harmony—intrinsic harmony, total harmony, ideal harmony and so on. The expression "general-elimination harmony" was coined by Francez & Dyckhoff (2012) to characterise the notion of harmony captured by the ge-procedure. However articulated, harmony restricts inferentialism to exclude connectives like Prior's notorious tonk.[9] I will use "harmonic inferentialism" to refer to the subclass of (logical) inferentialism by which the meaning of

[8] See, e.g., Wittgenstein (1961, §5.43): "All the propositions of logic say the same thing, to wit nothing." See also §6.11.

[9] See Prior (1960) and Read (2010, p. 561).

each logical constant is wholly given by its I-rules, and the E-rules are no stronger and no weaker than that meaning warrants, and so are in harmony with them.

Brandom (1994, p. 130) endorses Dummett's requirement that the two aspects of use of an expression should lie in such an intuitive harmony:

> "The expressive task of making material inferential commitments explicit plays an essential role in the reflectively rational Socratic practice of harmonizing our commitments."

Within the class of materially valid inferences, as we noted, Brandom distinguishes a class of formal validities, namely, those in which truth is preserved through all substitutions for the non-privileged vocabulary. Where the privileged vocabulary consists of logical constants, we have a logically valid inference.

§3. Logical Constants Having clarified harmonic inferentialism as a special form of logical inferentialism, we now need to consider what marks out the logical constants. The issue is particularly pressing for those who advocate what Etchemendy (1990, ch. 4) called interpretational semantics. Logical consequence holds, Tarski said, when truth is preserved through all possible interpretations of the non-logical terms. The view was articulated by Tarski (2002, p. 186) and before him by Bolzano.[10] But as Etchemendy (1990, pp. 109-10) observed, the plausibility of this definition depends on where the line is drawn between the logical terms and the others. Tarski tried to address this concern in his posthumous paper (Tarski, 1986), suggesting that the logical notions are those preserved under all permutations of the domain. The upshot, he realised (Tarski, 1986, p. 151) is that logicality becomes identified with questions of cardinality—which should give one pause, for that seems to identify logic with mathematics (or at least, set theory).

Many inferentialists propose that the logical constants are precisely those that can be characterised by harmonious inference rules.[11] Dummett (1991, p. 247) writes:

> "The demand that the introduction rules and the elimination rules be in harmony is ... compelling when it is being maintained that the meaning of the logical constant in question can be completely determined by laying down the fundamental laws governing it."[12]

This is too strong, given Dummett's account of harmony as conservative extension. Again, even though classical negation is given inharmonious rules in Gentzen's NK and Prawitz's ND, it was argued in Read (2000, §3.3) that it can be given harmonious rules in a multiple-conclusion natural deduction system. Nor does Dummett's alternative suggestion of identifying harmony (under the name, 'intrinsic harmony') with normalization give a necessary or sufficient condition for logicality. The Curry-Fitch-Prawitz (CFP) rules for '$\Box$' and '$\Diamond$' normalize,[13] but as argued in Read (2008) they are not really harmonious, for the I-rules confer different meanings from what

[10] See Tarski (2002, p. 193 n. I).
[11] Prawitz, Dummett and Zucker and Tragesser are listed among those advocating this view in MacFarlane (2015, n. 20).
[12] However, he does qualify his demand: he says (Dummett, 1991, p. 215) that they ought to satisfy the condition, but that often they don't, yet are still meaningful.
[13] Prawitz (1965, ch. VI §1). See also Curry (1950, ch. V) and Fitch (1952, ch. 3).

would justify the corresponding E-rule. The meaning of $\Diamond\alpha$ is not given by the CFP $\Diamond$I-rule, nor does the meaning of '$\Box$' really justify their $\Box$E-rule except in those logics as strong as T.[14] Thus on the general-elimination account of harmony, inferentially equivalent rules need not be equally harmonious. Whether a set of I- and E-rules is ge-harmonious is not simply a matter of what can be inferred; it is also required that the full meaning is expressed by the I-rules, and that the E-rules are generated in accordance with that meaning by the ge-procedure.

Recall the contrast drawn in §1. above between formal validity and material validity. This too depends on distinguishing between logical and non-logical vocabulary. What is formally valid is what is valid by the meaning of the logical constants alone, regardless of the non-logical vocabulary. The distinction goes back to the middle ages. Buridan (2014, I 4, p. 68) wrote:

> "An inference (*consequentia*) is called formal if it is valid in all terms retaining a similar form,"

distinguishing the (non-logical) terms from the (logical) form. The success of this definition depends on two things: first, a prior notion of validity; secondly, a clear characterization of 'form'. The former was given in the observation that the rules are complete in §1. The latter requires clarification of the notion of logical form.

Consider Aristotle's account of syllogistic validity. He first distinguishes the four forms of categorical (that is, subject-predicate) proposition, A, E, I and O, where the predicate is said of all, none, some and not all of the subject, and then characterizes the three figures (*schemata*) of pairs of such propositions by whether the middle (or shared) term is predicate of one and subject of the other, predicate of both, or subject of both. Syllogistic validity is formal, depending on whether truth is preserved whatever (real, material) terms are substituted for the schematic letters in those three *schemata*. We can see this most clearly in Aristotle's method of counterinstances, showing which pairs of subject-predicate forms are not productive of a conclusion of the appropriate shape. What Aristotle was able to do was to give inference rules involving conversion, *reductio* and *ecthesis*, for deriving all the productive pairs, the categorical syllogisms.

Aristotle later added chapters attempting the same task for the modal syllogism. Reflection on a similar challenge for modern modal logic is salutary. What are the grounds on which $\Diamond\alpha$ can be asserted? The CFP I-rule is too weak, inferring $\Diamond\alpha$ from α itself. If that were the sole ground for asserting $\Diamond\alpha$, then modalities would collapse and '$\Diamond$' would just mean 'true', not 'possibly true'. One needs to discern more in the logical form of $\Diamond\alpha$, and introduce more structure into the proof theory. There are many ways to do this: labelled deductive systems, hybrid logic, tree-hypersequents and so on.[15] What this means for our present reflections is that the concept of form, and with it, formal validity, is underspecified. Propositions do not have a single logical form, or even a single most specific form. It is clear that the logic of some concepts can be captured purely schematically, 'and' for example.

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[14] Indeed, for that very reason, Prawitz (1979, pp. 34-36) rejects the modal operators as not properly logical.

[15] See, e.g., Negri (2005), Braüner (2014), Poggiolesi (2011, ch. 10).

(Even this needs qualifying: the schematic I-rule for '∧':

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta} \wedge\mathrm{I}$$

captures a certain refined, abstracted, concept of conjunction, ignoring niceties such as the implications of word order, for example.) Perhaps the logic of others, e.g., colour terms, cannot.[16] In between we have notions such as 'possible' whose logical behaviour is amenable to formal treatment given sufficient apparatus.

Ge-harmony requires that the introduction-rules give the full meaning of the expression in question, and the elimination-rule be constructed by the ge-procedure to allow one to infer no more and no less than is warranted by the meaning so conferred. On such a view, the logical constants are those which can be given schematic rules satisfying general-elimination harmony. In consequence, the meanings encapsulated in the I- and E-rules will be the same, and that meaning can easily be read off from the I-rules, the rules for asserting statements with the expression as main connective. Their meaning consists in the various grounds for their assertion.

§4. Identity In a recent article, Griffiths (2014) challenges this proposal, with specific reference to rules proposed in Read (2004) for identity. Griffiths (2014, §7) claims that if a connective is defined by inharmonious rules, then harmonious rules cannot be given for it. This is a bold claim, and one that needs to be examined with care. Much was wrong, indeed, with the treatment of identity in Read (2004),[17] or rather with its articulation, for, as I will argue, the basic idea was correct. There are also major problems with Griffiths' arguments. It is, therefore, worth turning to discuss the logic of identity in some detail.

The problem identified for the logic of identity in Read (2004) was that the meaning conferred on '=' by the standard introduction-rule for identity:

$$\frac{}{a = a} \text{Refl}$$

does not seem to justify the standard elimination-rule:[18]

$$\frac{a = b \qquad \alpha^a_x}{\alpha^b_x} \text{Congr}$$

The E-rule was described as an expression of the indiscernibility of identicals (Read, 2004, p. 115). This is misleading. Congr is formulated in a language with limited expressive power, essentially extensional. Although the proposal was not specific

[16] Brandom generalizes the inferentialist account to include so-called language-entry and -exit transitions, but as such they are not schematic. See, e.g., Brandom (2007, p. 658): "Thus the visible presence of red things warrants the applicability of the concept red ... the point is that the connection between those circumstances of application and whatever consequences of application the concept may have can be understood to be inferential in a broad sense, even when the items connected are not themselves sentential."

[17] See, e.g., Kremer (2007).

[18] Here, α^a_x denotes the result of replacing every free occurrence of 'x' by 'a' if 'x' is free for 'a' in α, and if not, the result of replacing every free occurrence of 'x' by 'a' in a well-defined bound alphabetic variant of α in which 'x' is free for 'a'.

about the logics to which the rules might be added, it was intended for some form of first-order non-modal language, suitable for the expression of classical, intuitionist or relevant logic.[19] So Congr expresses the indiscernibility of identicals by first-order non-modal expressions. Its converse is therefore the identity of objects indiscernible by first-order non-modal predicates. The suggestion (Read, 2004, p. 116) was that a suitable I-rule that would generate Congr as E-rule by the ge-procedure would be:

$$\frac{\begin{array}{c}[Fa]\\ \vdots\\ Fb\end{array}}{a=b}\ {=}\mathrm{I}$$

where the predicate variable 'F' does not occur in any parametric assumptions (and so is arbitrary).[20] This was a simplification of an apparently weaker rule with two premises:

$$\frac{\begin{array}{c}[Fa]\\ \vdots\\ Fb\end{array}\qquad \begin{array}{c}[Fb]\\ \vdots\\ Fa\end{array}}{a=b}\ {=}\mathrm{I}'$$

to which it was argued to be equivalent. They have a different effect, however, on the ge-procedure, which generates two E-rules from $=\mathrm{I}'$:

$$\frac{a=b \qquad \begin{array}{c}\left[\begin{array}{c}Fa\\ \vdots\\ Fb\end{array}\right]\\ \vdots\\ \zeta\end{array}}{\zeta} \qquad \text{and} \qquad \frac{a=b \qquad \begin{array}{c}\left[\begin{array}{c}Fb\\ \vdots\\ Fa\end{array}\right]\\ \vdots\\ \zeta\end{array}}{\zeta}$$

which simplify by reasoning elaborated in Read (2015, §3) to the "flattened rules":

$$\frac{a=b \qquad Fa}{Fb}\ {=}\mathrm{E}_1 \qquad \text{and} \qquad \frac{a=b \qquad Fb}{Fa}\ {=}\mathrm{E}_2$$

The difference in the ge-procedure turns out to be important. In Read (2004), appeal was made to a second-order comprehension principle to show that the E-rule, stated there and above for atomic wffs in minor premise and conclusion, generalised to Congr, that is, holds for arbitrary wffs. But the pair of rules, $=\mathrm{E}_1$ and $=\mathrm{E}_2$, allow one to derive Congr by a simple induction over the degree of α, as one of a pair of valid rules:

$$\frac{a=b \qquad \alpha_x^a}{\alpha_x^b}\ \mathrm{Congr} \qquad \text{and} \qquad \frac{a=b \qquad \alpha_x^b}{\alpha_x^a}\ \mathrm{Congr}'$$

[19] For the theory of identity in a modal language, see, e.g., da Costa & Mortensen (1980). Identity can be defined in a second-order language: see, e.g., Shapiro (1991, pp. 67-8).

[20] Griffiths (2014, p. 502) writes: "F is a schematic variable ranging over predicates of any adicity and degree of complexity." Not so: 'F' ranges simply over monadic predicate letters, such that 'Fa' and 'Fb' are atomic wffs.

$=\text{E}_1$ and $=\text{E}_2$ give the atomic basis. Now suppose Congr and Congr′ hold for arbitrary α. To show that they hold for $\neg\alpha$, we proceed as follows:

$$\dfrac{\dfrac{a = b \quad \dfrac{}{\alpha^b_x}\,(1)}{\alpha^a_x}\,\text{Ind.Hyp.} \qquad \neg\alpha^a_x}{\dfrac{\bot}{\neg\alpha^b_x}\,\neg\text{I}(1)}\,\neg\text{E}$$

and similarly for Congr′. Moreover, if Congr and Congr′ hold for α and β, then they hold for $\alpha \wedge \beta$:

$$\dfrac{\dfrac{a = b \quad \dfrac{(\alpha\wedge\beta)^a_x}{\alpha^a_x}\,\wedge\text{E}_1}{\alpha^b_x}\,\text{Ind.Hyp.} \qquad \dfrac{a = b \quad \dfrac{(\alpha\wedge\beta)^a_x}{\beta^a_x}\,\wedge\text{E}_2}{\beta^b_x}\,\text{Ind.Hyp.}}{(\alpha\wedge\beta)^b_x}\,\wedge\text{I}$$

and similarly for Congr′. The same method extends to the other connectives.

The rules =I′, Congr and Congr′ are ge-harmonious. But as we noted, Griffiths claims that harmonious rules cannot be given for a connective already defined by inharmonious rules. How is this disagreement to be explained? Part of the explanation lies in the fact that Griffiths and I spell out the notion of harmony in different ways. But that is not the full story. Griffiths observes that =I as given in Read (2004) is in fact equivalent to Refl: the only wffs provable with =I have the form '$a = a$'; and the =E-rule is the same, *viz* Congr. We noted earlier that inferentially equivalent rules can differ in that one set is harmonious, the other not. But here it is different: the I-rules on their own are inferentially equivalent, and so give the same meaning to '='. Thus if one I-rule is harmonious with Congr then so is the other. The meaning of '=' is given by the rules for its assertion; and Refl and =I′ (and =I) are equivalent in that they each only permit assertions of self-identity, $a = a$. So we must agree with Griffiths that the pairs of rules Refl and Congr, on the one hand, and =I′ and Congr, on the other, cannot differ in one pair being harmonious, the other not, for the meaning defined by each I-rule is the same. Consequently, he writes (Griffiths, 2014, pp. 499, 501, 504):

> "Because the old rules [i.e., Refl and Congr] are not harmonious (as Read argues), nor are his [i.e., =I and Congr] . . .
> [Refl] and [Congr] are clearly not harmonious . . .
> [I]t is uncontroversial that the original rules are unharmonious by this intuitive notion [*viz* "two rules' being exactly commensurate in strength"]: all the inferentialists I can find who comment on identity, including Read, admit this."

So we do; but none of us proves it, and no argument was given for this claim in Read (2004). Griffiths has shown we were wrong. Refl is not in the right form of an I-rule, *viz* specifying the grounds for assertion of wffs of the form '$a = b$', to which to apply the ge-procedure to generate the appropriate E-rule(s), that is, a set of E-rules "commensurate in strength". Once the I-rule is put in the right form, *viz* =I′, the procedure shows that Congr is (one part of) the correct E-rule. Then =I′, Congr and Congr′ lie in ge-harmony, so Refl and Congr must also be harmonious, for $=I'$ and Refl, being inferentially equivalent, give the same meaning to '=', and Congr (and Congr′) are justified by the meaning so given.

Griffiths (2014, p. 507) infers from his claim that harmonious rules cannot be given that inferentialists must deny that identity is a logical notion. He attributes to me (2014, pp. 505-6) the view that "formulability in terms of harmonious inference rules is neither necessary nor sufficient for logical constanthood ... [but] desirable." But here he conflates rules for an expression's being actually harmonious with whether harmonious rules can be formulated. E.g., I argued (2010, p. 561) that the CFP rules for '$\Box$' and '$\Diamond$' are not harmonious, but in Read (2008) I showed that the labelled rules for modality given in, e.g., Basin et al. (1997) were harmonious. The modal case is not the same as that with identity, where the rules Refl and =I are equivalent. The failure of harmony in the case of the CFP rules for possibility is one of I-weak disharmony, as dubbed by Steinberger (2011, p. 621): that is, the CFP $\Diamond$I-rule is too weak to justify the restrictions that need to be placed on the $\Diamond$E-rule to ensure that between them they characterise possibility. But it does not follow that '$\Diamond$' cannot be given harmonious rules. It can, but only by ensuring that the I-rule properly captures the meaning of '$\Diamond$'. The real basis of harmony is to ensure that the meaning justifying the E-rules should be the same as that given by the I-rules, so that the full meaning of the term in question is contained in the grounds for assertion. This wasn't true in the case of the CFP rules for '$\Diamond$'. We now see, thanks to Griffiths' observations, that it is in the case of Refl, but obscurely so, made clear by reformulating the I-rule for '=' as $=I'$ (or =I).

Harmonious formulability (specifically, ge-harmony) was proposed above as a necessary condition for logicality to ensure that the ge-procedure can be applied. What is not necessary, as I argued (2010, §7), is normalization and the eliminability of local peaks (Dummett's "intrinsic harmony"), contrary to what Griffiths (2014, p. 504) suggests. He writes:

> "The inference rules for \$ are in GE harmony if there is a reduction procedure by which all local peaks with respect to \$ can be eliminated."

Not so: I wrote (Read, 2010, p. 525): "harmony is not normalization," giving counterexamples where peaks can be levelled, but not eliminated. Sometimes when one peak is levelled, another arises of equal height.

Moreover, ge-harmony is not in itself sufficient for logicality: broad-brush inferentialism seeks harmonious rules for every meaningful expression. What distinguishes the logical expressions is their formal nature, that is, their schematic formulability. This is clearly an open-ended criterion: for example, the language may need to be extended, as in the case of the labelled systems for modal logic. The concept of form distinguishing the logical from the non-logical is to that extent underspecified.

The existence of ge-harmonious and schematic rules is both necessary and sufficient for an expression to prove itself as a meaningful logical constant; moreover, the above rules, $=I'$, Congr and Congr′, are ge-harmonious and schematic and so show that identity is indeed a logical notion.

§5. Conclusion Inferentialism rejects the idea that semantics should consist in a mapping of each expression onto a range of objects, and proposes instead that the rules for an expression's use, to wit, the assertion-conditions of statements containing it and the consequences of those assertions, express its meaning. In particular, logical inferentialism claims that the introduction- and elimination-rules for a logical constant give its semantics in proof-theoretic terms. They may do so even if the rules are inharmonious, as happens with the CFP-rules for modality. The rules are harmonious when no more can be inferred from a proposition than is warrented by the grounds for its assertion. Harmonic inferentialism is the restriction of inferentialism to the case where the I-rules fully specify the meaning of the term in question and the E-rules are fully justified (so are no stronger and no weaker than is warranted) by the meaning so conferred. It was proposed here that the logical expressions are those which can be given schematic rules that accordingly satisfy Lorenzen's inversion principle, that what follows from an assertion is no more and no less than what follows severally from its grounds.

Owen Griffiths (2014) claimed that not only are the standard rules for identity not harmonious, but harmonious rules cannot be given for it. We have seen that this is not so. The notion of general-elimination harmony captures the idea expressed in the inversion principle, and gives specific content to the suggestion by Gentzen that the E-rules for a logical constant are uniquely determined by the meaning conferred on it by the I-rules. The rule $=\mathrm{I}'$ given in §4. states the assertion-conditions for statements of the form '$a = b$', and generates the rules Congr and Congr′ by the ge-procedure. But $=\mathrm{I}'$ is in fact equivalent to Refl, as Griffiths observed. So although Refl is not in the right form for application of the ge-procedure, Congr and Congr′ allow inference of no more and no less than the meaning encapsulated in $=\mathrm{I}'$, or equivalently in Refl. Refl and Congr are indeed harmonious, but one can only see that fact when the I-rule is put in a form that makes clear what the meaning of '$a = b$' is, that is, a form that makes explicit the grounds for asserting wffs of the form '$a = b$', as is done by (the equivalent rule) $=\mathrm{I}'$. Thus '=' has been shown, *contra* Griffiths, to be expressible by harmonious rules and so to be a logical notion, as claimed in Read (2004).

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