# Regularity of minimizing maps with values in $S^2$ and some numerical simulations

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# REGULARITY OF MINIMIZING MAPS WITH VALUES IN $S^2$ AND SOME NUMERICAL SIMULATIONS

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## 0. Introduction

Let $\Omega$ be the unit ball $B^3(0,1)$ of $\mathbb{R}^3$. Let $H^1(\Omega, S^2)$ be the set of all $u \in H^1(\Omega, \mathbb{R}^3)$ with $u(x) \in S^2$ a.e. where $S^2$ is the unit sphere of $\mathbb{R}^3$. For $\lambda \geq 0$ and $f \in L^2(\Omega, S^2)$, let

$$F_\lambda(u,f) = \int_\Omega |\nabla u|^2 + \lambda \int_\Omega |u-f|^2 \tag{0.1}$$

The critical points of $F_\lambda( \, . \, , f)$ satisfy the following Euler-Lagrange equation

$$-\Delta u = u \, |\nabla u|^2 + \lambda \, [f - < u, f > u] \tag{$E_{\lambda,f}$}$$

Notice that $F_\lambda$ is lower semi-continuous on $H^1(\Omega, S^2)$, so that

$$\inf_{u \in H^1(\Omega, S^2)} F_\lambda(u,f) \tag{0.3}$$

is achieved by some $u_\lambda$ which satisfies $(E_{\lambda,f})$.

In this paper, we are interested in studying the regularity of $u_\lambda$. Recall that in [BB], Bethuel and Brezis have shown that there exists some regular function $f$ with values in $\mathbb{R}^3$ such that $u_\lambda(f) = u_\lambda$ is not smooth on $\Omega$. In [HZ], Hadiji and Zhou considered the same problem and they obtained that there is $\lambda_1 > 0$ such that for every $\lambda \geq \lambda_1$, for any function $f$ in $H^1(\Omega, S^2)$ which is not a strong limit of smooth maps, every $u_\lambda$ is not regular in $\Omega$, and they obtained that there is $\lambda_0 > 0$ such that for every $0 < \lambda \leq \lambda_0$, $u_\lambda$ is regular in $\Omega$ provided that $f$ satisfied some conditions. In the first part, we prove that these conditions are not necessary and we give a numerical value of $\lambda_0$.

Note that it is well known (see [SU1], [SU2]) that $u_\lambda$ is smooth accept at a finite number of points.

The regularity of minimizing maps and some related phenomena are studied by many authors (see [BB], [B87], [B89], [BBC], [DH], [HKL] and [HL]).

In the second part, we propose a numerical study of the problem (0.3). In order to minimize the energy, we have followed a strategy due to F. Alouges [A], developed to solve the problem $\min_{u \in H^1_{n_0}(\Omega, S^2)} \int_\Omega |\nabla u(x)|^2 \, dx$.

The principal difficulty is the lack of convexity of the constraint. The iterative algorithm used allows us to decrease the energy at each step, on the contrary of algorithms developed by others authors [CLL],[CHKLL],[DGL].

The proof of the convergence will be also given and numerical results will be presented with $\Omega = (0,1)^3$ and different functions $f$.

## 1. Regularity of minimizing applications with values in $S^2$

Our main result is the following:

**Theorem 1.1** Let $f$ be any measurable function with values into $S^2$ then we have, for every $0 \leq \lambda \leq \frac{3}{5}$, every minimizer solution of (0.3) is regular in $\Omega$.

**Proof of the theorem 1.1:** We start by quoting two results. The first lemma concerns the behavior of $u_\lambda$ near each singularities and requires some modifications of the result of [BCL].

**Lemma 1.1** Suppose that $y \in \Omega$ is a singularity of $u_\lambda$ then, we have $u_\lambda(x) \simeq \pm R\left(\frac{x-y}{|x-y|}\right)$ as $x$ goes to $y$ where $R$ is a rotation of $\mathbb{R}^3$. In particular, the degree of $u_\lambda$ around each singularity is $\pm 1$.

The second is a variant of the well known monotonicity formula for standard minimizing harmonic map.

**Lemma 1.2** For any $a \in \Omega$ and $r < dist(a, \partial\Omega)$ we have

$$\frac{d}{dr}\Big(\frac{1}{r}\int_{B(a,r)} |\nabla u_\lambda|^2 + \frac{\lambda}{r}\int_{B(a,r)} |u_\lambda - f|^2 + \frac{8\pi}{3}\lambda r^2\Big) \geq 0$$

In particular $\frac{1}{r}\int_{B(a,r)} |\nabla u_\lambda|^2 + \frac{32\pi}{3}\lambda r^2$ is nondecreasing in $r$.

The proof of these lemmas are contained in [HZ]. Hence, the two proofs are omitted.

Setting $v_\lambda(x) = u_\lambda\big((1-r)x\big)$ then we have

$$\int_\Omega |\nabla v_\lambda|^2 = \frac{1}{1-r}\int_{B(0,1-r)} |\nabla u_\lambda|^2, \tag{1.1}$$

and,

$$\begin{aligned}\int_\Omega |v_\lambda - f|^2 &= \int_\Omega |v_\lambda - u_\lambda + u_\lambda - f|^2 \\ &= \int_\Omega |v_\lambda - u_\lambda|^2 + 2\int_\Omega (v_\lambda - u_\lambda)\cdot(u_\lambda - f) + \int_\Omega |u_\lambda - f|^2 \qquad (1.2)\\ &= 2\int_\Omega (u_\lambda - v_\lambda)\cdot f + \int_\Omega |u_\lambda - f|^2.\end{aligned}$$

Since $u_\lambda$ is minimizer, it follows that

$$\frac{1}{1-r}\int_{B(0,1-r)} |\nabla u_\lambda|^2 + 2\lambda\int_\Omega (u_\lambda - v_\lambda)\cdot f \geq \int_\Omega |\nabla u_\lambda|^2,$$

thus

$$\frac{r}{1-r}\int_{B(0,1-r)} |\nabla u_\lambda|^2 + 2\lambda\int_\Omega (u_\lambda - v_\lambda)\cdot f \geq \int_{\Omega\setminus B(0,1-r)} |\nabla u_\lambda|^2.$$

writing

$$\frac{r}{1-r}\cdot\left(\int_\Omega |\nabla u_\lambda|^2 - \int_{\Omega\setminus B(0,1-r)} |\nabla u_\lambda|^2\right) + 2\lambda\int_\Omega (u_\lambda - v_\lambda)\cdot f \geq \int_{\Omega\setminus B(0,1-r)} |\nabla u_\lambda|^2,$$

we obtain

$$\int_\Omega |\nabla u_\lambda|^2 + 2\lambda.\frac{(1-r)}{r}\cdot\int_\Omega (u_\lambda - v_\lambda)\cdot f \geq \frac{1}{r}\int_{\Omega\setminus B(0,1-r)} |\nabla u_\lambda|^2.$$

Using Poincaré inequality

$$\frac{1}{r}\int_\Omega (u_\lambda - v_\lambda)\cdot f \leq \frac{1}{r}\left(\int_\Omega \Big|u_\lambda(x) - u_\lambda\big((1-r)x\big)\Big|^2\right)^{\frac{1}{2}}\cdot\left(\int_\Omega |f|^2\right)^{\frac{1}{2}}.$$

We can write for almost everywhere $x$ in $\Omega$

$$u_\lambda(x) - u_\lambda\big((1-r)x\big) = \int_{1-r}^1 x\cdot\nabla u_\lambda(tx)\ dt$$

and then, for a.e. $x \in \Omega$

$$\Big|u_\lambda(x) - u_\lambda\big((1-r)x\big)\Big|^2 \leq \left(\int_{1-r}^1 |x|\ |\nabla u_\lambda|(tx)\ dt\right)^2$$

using again Poincaré inequality

$$\Big|u_\lambda(x) - u_\lambda\big((1-r)x\big)\Big|^2 \leq \left(\int_{1-r}^1 \Big|\nabla u_\lambda(tx)\Big|^2 dt\right)\cdot\left(\int_{1-r}^1 |x|^2 dt\right)$$

then

$$\int_\Omega \Big| u_\lambda(x) - u_\lambda\big((1-r)x\big)\Big|^2 \, dx \le \int_\Omega r \cdot |x|^2 \Big( \int_{1-r}^1 |\nabla u_\lambda|^2 (tx) \, dt \Big) \, dx.$$

Using the change of variable $y = tx$, and Fubini-Tonelli theorem, we obtain

$$\begin{aligned}
\int_\Omega \Big| u_\lambda(x) - u_\lambda\big((1-r)x\big)\Big|^2 \, dx &\le r \cdot \int_{1-r}^1 \Big( \int_{B(0,t)} |y|^2 \; |\nabla u_\lambda|(y) \; \frac{dy}{t^5} \Big) \; dt \\
&\le \int_{1-r}^1 \frac{dt}{t^3} \left( \int_\Omega |\nabla u_\lambda|^2 \right) \\
&\le \frac{r}{2} \left( \frac{1}{(1-r)^2} - 1 \right) \cdot \left( \int_\Omega |\nabla u_\lambda|^2 \right)
\end{aligned}$$

This leads to

$$2\frac{1-r}{r} \int_\Omega (u_\lambda - v_\lambda) \cdot f \le \sqrt{4 - 2r} \cdot \left( \int_\Omega |\nabla u_\lambda|^2 \right)^{\frac{1}{2}} \cdot \left( \int_\Omega |f|^2 \right)^{\frac{1}{2}}. \tag{1.3}$$

On the other hand, we have for all $P \in S^2$

$$F_\lambda(u_\lambda) \le \lambda \int_\Omega |P - f|^2,$$

hence

$$\int_\Omega |\nabla u_\lambda|^2 \le C\lambda \|u_\lambda - P\|_{L^2}.$$

In particular, $u_\lambda$ tends to a constant $P_0 \in S^2$ strongly in $H^1(\Omega, S^2)$.Since $u_\lambda$ is a minimizer we have $\int_\Omega u_\lambda \cdot f \ge 0$. Thus we deduce that

$$\int_\Omega |\nabla u_\lambda|^2 \le \lambda \int_\Omega |P_0 - f|^2 \le \lambda \left( \|P_0\|_2^2 + \|f\|_2^2 \right) \tag{1.4}$$

then using the fact $|f| = 1$, we have

$$\int_\Omega |\nabla u_\lambda|^2 \le \lambda \frac{8\pi}{3}.$$

Combining this inequality and (1.3), we obtain

$$2\frac{1-r}{r} \int_\Omega (u_\lambda - v_\lambda) \cdot f \le 2\sqrt{\frac{8\pi\lambda}{3}} \sqrt{4 - 2r} \cdot \|f\|_2.$$

Finally, using $|f| = 1$, we are led to

$$\begin{aligned}
\frac{1}{r} \int_{\Omega \setminus B(0,1-r)} |\nabla u_\lambda|^2 &\le \frac{\lambda 8\pi}{3} \left( 1 + \sqrt{\lambda} \, \sqrt{2 - r} \right) \\
&\le \lambda \; 4\pi \; g(r),
\end{aligned} \tag{1.5}$$

where $g(r) = \frac{2}{3}\Big(1 + \sqrt{2-r}\Big)$, if we assume $\lambda \leq 1$.

Let $x \in \Omega$ such that $|x| > \frac{1}{2}$. Define $r_x = 1 - |x|$, since

$$B(x, r_x) \subset \Omega \setminus B(0, 1 - 2r_x)$$

we have

$$\frac{1}{2r_x} \int_{B(x,r_x)} |\nabla u_\lambda|^2 \leq \frac{1}{2r_x} \int_{\Omega \setminus B(0,1-2r_x)} |\nabla u_\lambda|^2,$$

by (1.5)

$$\frac{1}{r_x} \int_{B(x,r_x)} |\nabla u_\lambda|^2 \leq \lambda \; 8\pi \; g(2r_x).$$

Using the monotonicity formula (see Lemma 1.2), we obtain for all $r < r_x$

$$\frac{1}{r} \int_{B(x,r)} |\nabla u_\lambda|^2 \leq \lambda \; 8\pi \; g(2r_x) + \frac{32}{3} \; \lambda\pi(r_x^2 - r^2). \tag{1.6}$$

Then, for all $x$ such that $|x| \geq \alpha_0$,

$$\lim_{r\to 0} \frac{1}{r} \int_{B(x,r)} |\nabla u_\lambda|^2 \leq \lambda \; 8\pi \; \left(g(2r_x) + \frac{4}{3} r_x^2\right). \tag{1.7}$$

Let $x \in B(0, \frac{1}{2})$, we have $B(x, \frac{1}{2}) \subset B^3(0,1)$. Using again the monotonicity formula and (1.4), we see for all $r < \frac{1}{2}$

$$\begin{aligned}\frac{1}{r} \int_{B(x,r)} |\nabla u_\lambda|^2 &\leq 2 \int_{B(x,\frac{1}{2})} |\nabla u_\lambda|^2 + \frac{32}{3} \; \lambda\pi(\frac{1}{4} - r^2)\\ &\leq 8\pi \; \lambda,\end{aligned}$$

then

$$\lim_{r\to 0} \frac{1}{r} \int_{B(x,r)} |\nabla u_\lambda|^2 \leq 8\pi\lambda \qquad \forall x \in B(0, \frac{1}{2}). \tag{1.8}$$

Direct computations show that for all $\lambda \leq \frac{3}{5}$ the right hand sides in (1.7) and (1.8), are strictly less than $8\pi$, so, for all $x$ in $\Omega$

$$\lim_{r\to 0} \frac{1}{r} \int_{B(x,r)} |\nabla u_\lambda|^2 < 8\pi.$$

(We note that the right hand side in (1.8) is strictly less than $8\pi$ for all $\lambda < 1$.) Now, applying Lemma 1.1, we obtain the desired conclusion.

## 2. Remarks and generalizations

### 2.1 Remark on the domain $\Omega$

We have a similar result as Theorem 1.1 if we only assume that $f$ is a function in $L^2(\Omega, \mathbb{R}^3)$ and if we replace the domain $\Omega$ by any unit ball associated to another norm in $\mathbb{R}^3$ :

**Theorem 2.1** Let $f$ be any function in $L^2(B_N, \mathbb{R}^3)$ not necessary with values into $S^2$ defined on $B_N = \{x \in \mathbb{R}^3, N(x) \leq 1\}$ where $N$ is a norm in $\mathbb{R}^3$, then there exists a constant $\lambda_0 > 0$, depending only on $\|f\|_2$ and $N$, such that every minimizer $u_\lambda \in H^1(B_N, S^2)$ of the functional $F_\lambda(., f)$ for $\lambda \leq \lambda_0)$ is regular in $B_N$.

The proof is the same as for $B^3$. Using $v_\lambda(x) = u_\lambda((1-r)x)$ we obtain inequalities (1.3) and (1.4) thus we prove that there exists a function $G(r) = \left(\|P_0\|_2^2 + \|f\|_2^2\right) + \sqrt{(\|P_0\|_2^2 + \|f\|_2^2)} \cdot \|f\|_2 \cdot \sqrt{2-r}$, such that

$$\frac{1}{r}\int_{\Omega\setminus(1-r)\Omega} |\nabla u_\lambda|^2 \leq \lambda G(r).$$

Thus for $x \in \Omega$ such that $N(x) > \dfrac{1}{2}$, if we set $r_x = 1 - N(x)$ then we see that there exists a constant $k$ which depends only on the norm $N$ such that

$$B^3(x, kr_x) \subset B_N \setminus B_N(0, 1-2r_x), \tag{2.1}$$

and we can conclude as in Theorem 1.1 (if $N$ is the uniform norm, then we can choose $k = 1$).

### 2.2 Remark on the equation

We have seen that any solution of problem (0.3) satisfies weakly, the equation:

$$-\Delta u = u|\nabla u|^2 + \lambda[f - < u, f > u]. \tag{$E_{\lambda,f}$}$$

Then, if we take the exterior product of $(E_{\lambda,f})$ by $u$ we obtain :

$$(\Delta u + \lambda f) \times u = 0. \tag{$E^*_{\lambda,f}$}$$

Conversely, any solution of $(E^*_{\lambda,f})$ shall be collinear to $u$ thus such a map satisfies :

$$\Delta u + \lambda f = \mu u,$$

so taking the scalar product of this equation we find that $\mu$ had to be equal to $-|\nabla u|^2 + < u, f >$, thus the equations $(E^*_{\lambda,f})$ and $(E_{\lambda,f})$ are equivalent.

As a consequence, if $(u_n)$ is a solution of $(E_{\lambda,f_n})$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(B^3, S^2),$$

$$f_n \rightharpoonup f \quad \text{weakly in } L^2(B^3, S^2),$$

then $u$ is a solution of $(E_{\lambda,f})$. With the formulation $(E^*_{\lambda,f})$, we have just to note that by compact injection of $H^1$ in $L^2$ that $u_n$ tends strongly to $u$ in $L^2(B^3, S^2)$ then we can pass to the limit in $f_n \times u_n$, it is well known that $\Delta u_n \times u_n$ tends also to $-\Delta u \times u$ (see [BBC], [C]).

### 2.3 Remark on the solutions

The regularity obtained in Theorem 1.1 (and Theorem 2.1) is really a consequence of the minimization problem and not of the equation.

Indeed, if we consider the following minimization problem on $H^1(B^3, S^2)$:

$$(P_\lambda) \qquad \inf_{u=\varphi|_{\partial B^3}} \int_{B^3} |\nabla u|^2 + \lambda \int_{B^3} |u - f|^2,$$

where $\varphi$ is a given smooth boundary condition. Any solution of problem $(P_\lambda)$ satisfies the equation $(E_{\lambda,f})$.

Then, for $\varphi$ equal to the identity on the boundary and $f$ constant for example, the solutions $u_\lambda$ of $(P_\lambda)$ converge to the solution of $(P_0)$ when $\lambda$ tends to zero, thus there exists singular solutions of $(E_{\lambda,f})$ for $\lambda$ small.

## 3. Numerical minimization of the energy $F_\lambda$

In this part, we propose a numerical study of the problem (0.3). The strategy used here is based on the works of F. Alouges [A]. The principal difficulties of finding numerically the minimizer are:

• Non convexity of the constraint $|u(x)| = 1$ a.e. which avoid us to use standard algorithms directly.

• The minimizer $u_\lambda$ may be non regular (non continuous) for some $\lambda$, $f$.

• Non uniqueness. For some $\lambda$, $f$ (if $f$ have symmetries for example), $u_\lambda$ need not to be unique.

Most of the methods to solve this kind of problems can be split into two steps:

1. Let $u_0$ be an initial guess.
2. For $n = 0$ ... until convergence:
   - 3.1 Find $v_n$ such that $F_\lambda(v_n) \leq F_\lambda(u_n)$ where $v_n$ may not belong to $H^1(\Omega, S^2)$;
   - 3.2 Set $u_{n+1}(x) = \frac{v_n(x)}{|v_n(x)|}$.

The minimization problem we will solve at the step 3.1 allows us to decrease the energy at the step 3.2. In other words, for all iterations $n$, we have

$F_\lambda(u_{n+1}) \leq F_\lambda(v_n) \leq F_\lambda(u_n)$. Other methods ([CLL], [CHKLL], [DGL] for example) do not have this property. In particular, $F_\lambda\left(\frac{v_n}{|v_n|}\right) \leq F_\lambda(v_n)$ is not assured for all iterations $n$.

The step 3.1 will be solved by a conjugate gradient method because there is no parameter to optimize (on the contrary of a saddle-point or relaxation technique for example). Moreover, the numerical tests of F. Alouges seems to prove that it is the better method.

### 3.1 An energy decreasing algorithm

Here, we want to solve the problem: "Find $v_n$ such that $F_\lambda(v_n) \leq F_\lambda(u_n)$" in order to assume that :

$$F_\lambda(u_{n+1}) = F_\lambda\Big(\frac{v_n}{|v_n|}\Big) \leq F_\lambda(v_n) \leq F_\lambda(u_n) \tag{3.1}$$

at each step $n$. This can be done using the following proposition given by F. Alouges :

**Proposition 3.1** If $v \in H^1(\Omega, \mathbb{R}^3)$ verifies $|v(x)| \geq 1$ a.e., then $\frac{v}{|v|}$ belongs to $H^1(\Omega, S^2)$. Moreover, we have

$$\left|\nabla\left(\frac{v(x)}{|v(x)|}\right)\right|^2 \leq |\nabla v(x)|^2 \qquad \text{a.e.}$$

and for all function $f \in H^1(\Omega, S^2)$

$$|v(x) - f(x)|^2 \leq \left|\frac{v(x)}{|v(x)|} - f(x)\right|^2 \qquad \text{a.e.}$$

So, if $v$ verifies $|v(x)| \geq 1$ a.e., one easily have the condition (3.1).

**Proof of Proposition 3.1:** This result can be shown by direct computations (See [A]).

Now, the following result allows us to minimize $F_\lambda$ with a function $v_n$ verifying $|v(x)| \geq 1$ a.e.:

**Proposition 3.2** Let $K_u$ be the set:

$$K_u = \{w \in H^1(\Omega, \mathbb{R}^3) \text{ such that } w(x) \cdot u(x) = 0 \text{ a.e.}\}$$

Let $v = u - w$ where $w$ belongs to $K_u$, then

$$|v(x)|^2 = |u(x) - w(x)|^2 = 1 + |w(x)|^2 \geq 1 \quad \text{a.e.} \tag{3.2}$$

and the (convex) problem:

$$\text{Minimize } F_\lambda(u-w) \text{ for } w \in K_u \tag{3.3}$$

possesses an unique solution, called $w(u)$.

**Proof of Proposition 3.2:** The proof of (3.2) is obvious because $w(x) \cdot u(x) = 0$ a.e. For the point (3.3), we have

$$I(w) = F_\lambda(u-w) = \int_\Omega |\nabla(u-w)|^2 + \lambda|u-w-f|^2 \, dx.$$

Expanding this expression gives

$$I(w) = \int_\Omega |\nabla w|^2 + \lambda|w|^2 \, dx - 2\int_\Omega \nabla u \cdot \nabla w + \lambda(u-f)\cdot w \, dx + \int_\Omega |\nabla u|^2 + \lambda|u-f|^2 \, dx.$$

So, minimize $I(w)$ is equivalent to minimize $J(w)$ defined by

$$J(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 + \lambda|w|^2 \, dx - \int_\Omega \nabla u \cdot \nabla w + \lambda(u-f)\cdot w \, dx.$$

Let $a(w,\psi) = \int_\Omega \nabla w \cdot \nabla\psi + \lambda w \cdot \psi \, dx$ and $L(\psi) = \int_\Omega \nabla u \cdot \nabla \psi + \lambda(u-f)\cdot \psi \, dx$ for all $\psi \in K_u$. Then $a$ is clearly continuous, coercive on $K_u$ (because $\lambda > 0$), and $L$ is continuous on $K_u$. Moreover $K_u$ is a linear subspace of $H^1(\Omega, \mathbb{R}^3)$, so we can use the Lax-Milgram theorem to prove the uniqueness of $w$. Furthermore, $w$ is also the solution of the variational problem:

$$a(w,\psi) = L(\psi) \qquad \text{for all } \psi \in K_u. \tag{3.4}$$

### 3.2 Convergence of the algorithm

Now, the algorithm can be wrote as follow:

$$\left[\begin{array}{l} \text{1. Let } u_0 \text{ be an initial guess.} \\ \text{2. For } n = 0 \text{ ... until convergence:} \\ \quad \text{3.1 Find } w_n\text{, solution of the problem (3.3) with } u = u_n; \\ \quad \text{3.2 Set } u_{n+1}(x) = \dfrac{u_n(x) - w_n(x)}{|u_n(x) - w_n(x)|}. \end{array}\right. \tag{3.5}$$

and we will prove the convergence of the algorithm by the following result:

**Theorem 3.1** The algorithm (3.5) converges in the sense that $(u_n)$ (up to a subsequence) weakly converges in $H^1(\Omega, \mathbb{R}^3)$ to a map $u_\infty \in H^1(\Omega, S^2)$ verifying the equation $(E_{\lambda,f})$. Moreover, the full sequence $(w_n)_{n\geq 0}$ strongly converges to 0 in $H^1(\Omega, \mathbb{R}^3)$.

**Proof:** The proof is similar to these of F. Alouges (see [A]). We first need the lemma:

**Lemma 3.1** We have, for all $n \geq 0$

$$F_\lambda(u_n) = F_\lambda(u_n - w_n) + \int_\Omega |\nabla w_n|^2 + \lambda |w_n|^2 \; dx$$

**Proof:** Expanding $F_\lambda(u_n - w_n)$ gives

$$F_\lambda(u_n - w_n) = F_\lambda(u_n) + \int_\Omega |\nabla w_n|^2 + \lambda |w_n|^2 \; dx - 2\int_\Omega \nabla u_n \cdot \nabla w_n + \lambda w_n \cdot (u_n - f) \; dx,$$

and using the variational formulation (3.4), we have

$$\int_\Omega \nabla u_n \cdot \nabla w_n + \lambda w_n \cdot (u_n - f) \; dx = \int_\Omega |\nabla w_n|^2 + \lambda |w_n|^2 \; dx.$$

So, because $F_\lambda(u_{n+1}) \leq F_\lambda(u_n - w_n)$,

$$\int_\Omega |\nabla w_n|^2 + \lambda |w_n|^2 \; dx \leq F_\lambda(u_n) - F_\lambda(u_{n+1}).$$

Summing this relation, we obtain

$$\sum_{n=0}^{n=N} \int_\Omega |\nabla w_n|^2 + \lambda |w_n|^2 \; dx \leq F_\lambda(u_0)$$

and this serie converges.

Since $\lambda > 0$, $\int_\Omega |\nabla w_n|^2 + \lambda |w_n|^2 \; dx$ is equivalent to the usual norm on $H^1(\Omega, \mathbb{R}^3)$ and $w_n \longrightarrow 0$ strongly in $H^1(\Omega, \mathbb{R}^3)$.

The proof of the weakly convergence of $(u_n)$ in $H^1(\Omega, S^2)$ is given in [A]. It is based on the fact that $(u_n)$ is bounded in $H^1(\Omega, S^2)$.

Finally we have to prove that the limit $u_\infty$ is a critical point of $F_\lambda$. Using the variational formulation (3.4), and taking $\psi = \phi \times u_n$ where $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^3)$, we have:

$$\int_\Omega \nabla w_n \cdot \nabla(\phi \times u_n) + \lambda w_n \cdot \phi \times u_n \; dx = \int_\Omega \nabla u_n \cdot \nabla(\phi \times u_n) + \lambda (u_n - f) \cdot \phi \times u_n \; dx.$$

Expanding this expression, we obtain

$$\int_\Omega \nabla\phi\cdot(u_n\times\nabla(w_n-u_n))+\phi\cdot(\nabla u_n\times\nabla w_n+\lambda u_n\times(w_n-f))\;dx=0,$$

so $u_n$, $v_n$ satisfy the following Euler-Lagrange equation

$$div(u_n\times\nabla(u_n-w_n))=\nabla w_n\times\nabla u_n+\lambda(w_n-f)\times u_n$$

in the sense of distributions. Using the facts that

$$\begin{aligned}&u_n\rightharpoonup u_\infty \text{ weakly in } H^1,\\ &u_n\to u_\infty \text{ strongly in } L^2,\\ &w_n\to 0 \text{ strongly in } H^1,\end{aligned}$$

$u_\infty$ satisfies the Euler-Lagrange equation:

$$div(u_\infty\times\nabla u_\infty)+\lambda f\times u_\infty=0$$

in the sense of distributions, which is equivalent to the fact that $u_\infty$ verifies the relation $(E_{\lambda,f})$ (see section 2).

### 3.3 Discretization

We use finite elements method because we absolutely need to have a symmetric matrix. Indeed, because of the lack of Dirichlet conditions on $\partial\Omega$, finite differences method produces a non-symmetric matrix preventing us using a conjugate gradient technique.

The finite elements used are linear on cubes (8 nodes) with a constant space-step in each direction.

If we call $\{\varphi_i\}_{i=1...N}$ the set of interpolation functions, and $V_h=span\{\varphi_i;i=1...N\}$, a function $u$ is approximated by :

$$u(x)\simeq u_h(x)=\sum_{i=1}^N u_i^h\varphi_i(x).$$

We can also define the set:

$$K_{u^h}^h=\{w^h\in V_h\;:\;w^h(x)\cdot u^h(x)\text{ for all }x\in\Omega\}.$$

With all these notations, the discretized algorithm can be wrote as follow:

$$(3.6)\qquad\left[\begin{array}{l}1.\text{ Let } u_0^h \text{ be an initial guess.}\\ 2.\text{ For } n=0\;...\text{ until convergence:}\\ \quad 2.1\text{ Find } w_n^h \text{ such that } F_\lambda(w_n^h)=\min\limits_{w^h\in K^h_{u_n^h}}F_\lambda(u_n^h-w^h);\\ \quad 2.2\text{ Set } u_{n+1}^h(x)=\dfrac{u_n^h(x)-w_n^h(x)}{|u_n^h(x)-w_n^h(x)|}.\end{array}\right.$$

The discrete version of the Lax-Milgram theorem allows us to say that the solution $w_n^h$ of (3.1) is unique and satisfies the problem

$$a(w_n^h, \phi^h) = L(\psi^h) \text{ for all } \psi^h \in K_{u_n^h}^h.$$

**Remark:** When a $u_0^h$ is given, at each step $n$, the solution $w_n^h$ is unique, so the limit $u_\infty^h$ is also unique. But, with an another initial guess $u_0^h$, we can obtain a different limit $u_\infty^h$ since the solution of the problem (0.3) may be not unique. Furthermore, the present algorithm may converges to a critical point (but not necessary a global minimizer) of $F_\lambda$. Examples of this phenomena will be given in the numerical results.

### 3.4 Resolution of the convex problem : a conjugate gradient technique

Here, we follow the algorithm given by F. Alouges [A], based on the remark bellow:

Suppose we want to minimize $F(X) = \frac{1}{2}(AX, X) - (b, X)$ where $(.,.)$ is the inner product on $\mathbb{R}^N$, $A$ is a positive definite $N \times N$ matrix, $b$ is a vector in $\mathbb{R}^N$, and $X \in \mathbb{R}^N$, subject to the constraint $BX = 0$.

The solution $X$ may be obtained by applying a conjugate gradient method procedure to the functional

$$\tilde{F}(X) = \frac{1}{2}(\pi A \pi X, X) - (\pi b, X),$$

where $\pi$ stands for the orthogonal projector onto the linear space

$$K = \{X \in \mathbb{R}^N \text{ such that } BX = 0\}$$

provided the algorithm is started with $X_0 \in K$.

In our case, since $u^h(x) = \sum_{i=1}^N u_i^h \varphi_i(x)$, $w^h(x) = \sum_{i=1}^N w_i^h \varphi_i(x)$, the projector $\pi^h$ onto the linear space $K_{u^h}^h$ can be wrote

$$\pi^h w^h(x) = w^h(x) - \big(w^h(x)\cdot u^h(x)\big) u^h(x) = w^h(x) - \sum_{i,j=1}^N u_i^h w_j^h \big(\varphi_i(x)\cdot\varphi_j(x)\big) u^h(x).$$

## 4. Numerical results

In this section, numerical results are given with $\Omega = (0,1)^3$ and different functions $f$. That allows us to have an idea of the behavior of the energy $F_\lambda$ and the solution $u_\lambda$ as a function of $\lambda$ and $f$. Three cases will be studied :

1. $f(x) = \frac{x-x_0}{|x-x_0|}$ where $x_0 \in \Omega$. $f$ has a singularity of degree 1 inside $\Omega$.

2. $f$ is the stereographic projection onto $(0,0,1)$ shifted to the point (0.5, 0.5, 1). Here, $f$ have a singularity of degree 1 on $\partial\Omega$.

3. $f$ is the dipole. $f$ has two singularities into $\Omega$, one of degree $+1$, one of degree -1 such that $f$ has a global degree 0 (see [B87]).

In that three cases, $u_\lambda$ is regular for a $\lambda$ small enough and singular for a large enough one. The singularities are always inside $\Omega$ (never on $\partial\Omega$), of degree $\pm 1$, and locally equivalent to $\pm R(\frac{x-y}{|x-y|})$ where $R$ is a rotation and $y$ the point where the singularity appears.

**Remark:** We have also used a simulated annealing method to solve the problem (0.3). It gives exactly the same results but with a CPU times much more costly. That is why we have chosen to not present this algorithm here.

### 4.1 The function $f$ has one singularity inside $\Omega$

In this first example, we have $f(x) = \frac{x-x_0}{|x-x_0|}$ where $x_0 = (0.6,\ 0.8\ , 0.5) \in \Omega$.

**Remark:** In order to have an idea of the properties of the solution $u_\lambda$, we have decided to draw only a section of two components of the solution. Here, the section is $\begin{cases} u^1_\lambda(x_1, x_2, 0.5) \\ u^2_\lambda(x_1, x_2, 0.5) \end{cases}$, $(x_1, x_2) \in (0,1)^2$, because the point $x_0$ where the singularity may appear is inside this section.

Figure 2 shows that the solution $u_\lambda$ tends to a limit $P_0(x) = P_0 = \frac{\int_\Omega f(x)\ dx}{|\int_\Omega f(x)\ dx|}$ as $\lambda$ goes to 0.

When $\lambda$ increases, the solution $u_\lambda$ becomes more variable but still regular (figure 3) while $\lambda$ is smaller than a certain $\lambda_\ell$. Then, when $\lambda > \lambda_\ell$, the solution $u_\lambda$ has a singularity at $x_0 = (0.6,\ 0.8\ , 0.5)$, and $u_\lambda(x) \simeq \frac{x-x_0}{|x-x_0|}$ as $x$ goes to $x_0$. Notice that the singularities of $u_\lambda$ and $f$ are at the same point $x_0$.

Figure 1 shows the energy of the minimizing function $u_\lambda$ for different values of $\lambda$, obtained with two different initializations. The first initialization (init 1) is $u_{n=0}(x) = (0, -1, 0)$ ($u_{n=0}$ is regular), and the second (init 2) is $u_{n=0} = f$ ($u_{n=0}$ is singular).

When $\lambda$ is far from $\lambda_\ell$, the two different sequences $(u_n)$ converge to the same limit $u_\lambda$, and we obtain the same energy.

On the contrary, when $\lambda$ is near from $\lambda_\ell$, with the first initialization the sequence $(u_n)$ converges to a regular function $u^1_\lambda$ and with the second initialization, $(u_n)$ converges to a singular function $u^2_\lambda$.

If $\lambda < \lambda_\ell$, we have $E_\lambda(u^1_\lambda) < E_\lambda(u^2_\lambda)$ so the global minimizer is $u^1_\lambda$ whereas if $\lambda > \lambda_\ell$, we have $E_\lambda(u^1_\lambda) > E_\lambda(u^2_\lambda)$ so the global minimizer is $u^2_\lambda$.

### 4.2 The function $f$ has one singularity on $\partial\Omega$

This example, where $f$ has a singularity of degree +1 on the boundary $\partial\Omega$ at the point $x_0 = (0.5, 0.5, 1)$ (see figure 5) allows us to numerically verify that for $\lambda$ large enough, $u_\lambda$ have a singularity localized inside $\Omega$ and not on the boundary $\partial\Omega$.

Figure 6 shows that, as above, around a certain $\lambda_\ell$, we obtain two different minima according to the initialization, and only one of the two is the global minimum (except of course for $\lambda = \lambda_\ell$ where the two are global minima. In this case, the problem $(P)$ have not an unique solution).

In order to represent $u_\lambda$, we have chosen to plot a section of two components of $u_\lambda$: $\begin{cases} u_\lambda^1(x_1, 0.5, x_3) \\ u_\lambda^3(x_1, 0.5, x_3) \end{cases}$, $(x_1, x_3) \in (0,1)^2$.

In figure 7, we can see that $u_\lambda$ tends to $\frac{\int_\Omega f(x)\ dx}{|\int_\Omega f(x)\ dx|}$ as $\lambda$ goes to 0 and in figure 8 that when $\lambda$ is small enough, $u_\lambda$ is regular. For $\lambda$ large enough, $u_\lambda$ becomes singular (figure 9). The singularity of $u_\lambda$ is at a $x_0$ inside $\Omega$, and of degree +1. When $\lambda$ increases, the singularity $x_0^\lambda$ draw near to $\partial\Omega$ but it never reach $\partial\Omega$.

### 4.3 The function $f$ has two singularities inside $\Omega$

Here, the function $f$ is the dipole. Figure 10 represents the section of $f$ defined by $\begin{cases} f^1(x_1, 0.5, x_3) \\ f^3(x_1, 0.5, x_3) \end{cases}$, $(x_1, x_3) \in (0,1)^2$. $f$ has two singularities, one of degree +1, one the degree -1. Figure 12 (the section of $u_\lambda$ represented is the same as in figure 10) shows that $u_\lambda$ is regular if $\lambda < \lambda_\ell$. In this example, $\lambda$ is much larger than in the previous ones, because the energy increases slowly.

Figure 13 shows that the two singularities appears at the same times, such that the solution $u_\lambda$ has always a global degree equal to 0.

Around the $\lambda_\ell$ where the behavior of the solution $u_\lambda$ changes, we obtain different limit according to the initialization. The first initialization is $u_0 = f$ ($u_0$ is singular), the second is $u_0 = (0, 0, -1)$ and the third is $u_0 = (0, 0, 1)$.

### Conclusion

The algorithm developed here is very efficient and well suited to solve the minimization problem (0.3). The non-linear initial problem has been solved using a sequence of linear problem, much easier to treat (but each iteration requires the resolution of a linear system), and the rate of convergence is very good. The numerical results confirm the result of part 1, and the conjecture that the close set $I(f)$ of $\lambda$ where $u_\lambda$ is regular is an interval of $\mathbb{R}^+$.

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