INVARIANCE OF THE BFV–COMPLEX

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Abstract. The BFV-formalism was introduced to handle classical systems, equipped with symmetries. It associates a differential graded Poisson algebra to any coisotropic submanifold S of a Poisson manifold (M, Π).

However the assignment (coisotropic submanifold) $\rightsquigarrow$ (differential graded Poisson algebra) is not canonical, since in the construction several choices have to be made. One has to fix: 1. an embedding of the normal bundle NS of S into M as a tubular neighbourhood, 2. a connection ∇ on NS and 3. a special element Ω.

We show that different choices of a connection and an element Ω – but with the tubular neighbourhood fixed – lead to isomorphic differential graded Poisson algebras. If the tubular neighbourhood is changed too, invariance can be restored at the level of germs.

1. Introduction

The Batalin-Vilkovisky-Fradkin complex (BFV-complex for short) was introduced in order to understand physical systems with complicated symmetries ([BF], [BV]). The connection to homological algebra was made explicit in [St] later on. We focus on the smooth setting, i.e. we want to consider arbitrary coisotropic submanifolds of smooth finite dimensional Poisson manifolds. Bordemann and Herbig found a convenient adaptation of the BFV-construction in this framework ([B], [He]): One obtains a differential graded Poisson algebra associated to any coisotropic submanifold. In [Sch] a slight modification of the construction of Bordemann and Herbig was presented. It made use of the language of higher homotopy structures and provided in particular a conceptual construction of the BFV-bracket.

Note that in the smooth setting the construction of the BFV-complex requires a choice of the following pieces of data: 1. an embedding of the normal bundle of the coisotropic submanifold as a tubular neighbourhood into the ambient Poisson manifold, 2. a connection on the normal bundle, 3. a special function on a smooth graded manifold, called a BFV-charge.

We apply the point of view established in [Sch] to clarify the dependence of the resulting BFV-complex on these data. If one leaves the embedding

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fixed and only changes the connection and the BFV-charge, one simply obtains two isomorphic differential graded Poisson algebras, see Theorem 1 in Section 3. Note that the dependence on the choice of BFV-charge was well understood, see [St] for instance. Dependence on the embedding is more subtle. We introduce the notion of "restriction" of a given BFV-complex to an open neighbourhood of the coisotropic submanifold inside its normal bundle (Definition 2) and show that different choices of embeddings lead to isomorphic restricted BFV-complexes – see Theorem 2 in Section 4. As a Corollary one obtains that a germ-version of the BFV-complex is independent of all the choices up to isomorphism (Corollary 4).

It turns out that the differential graded Poisson algebra associated to a fixed embedding of the normal bundle as a tubular neighbourhood, yields a description of the moduli space of coisotropic sections in terms of the BFV-complex – see [Sch2].

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2. Preliminaries

The purpose of this Section is threefold: to recollect some facts about the theory of higher homotopy structures, to recall some concepts concerning Poisson manifolds and coisotropic submanifolds and to outline the construction of the BFV-complex. More details on these subjects can be found in Sections 2 and 3 of [Sch] and in the references cited therein. We assume the reader to be familiar with the theory of graded algebras and smooth graded manifolds.

2.1. L_∞-algebras: Homotopy Transfer and Homotopies.

Let V be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$ (or any other field of characteristic 0); i.e., V is a collection $(V_i)_{i\in\mathbb{Z}}$ of vector spaces V_i over $\mathbb{R}$. The homogeneous elements of V of degree $i \in \mathbb{Z}$ are the elements of V_i. We denote the degree of a homogeneous element $x \in V$ by $|x|$. A morphism $f : V \to W$ of graded vector spaces is a collection $(f_i : V_i \to W_i)_{i\in\mathbb{Z}}$ of linear maps. The nth suspension functor $[n]$ from the category of graded vector spaces to itself is defined as follows: given a graded vector space V, $V[n]$ denotes the graded vector space corresponding to the collection $V[n]_i := V_{n+i}$. The nth suspension of a morphism $f : V \to W$ of graded vector spaces is given by the collection $(f[n]_i := f_{n+i} : V_{n+i} \to W_{n+i})_{i\in\mathbb{Z}}$. The tensor product of two graded vector spaces V and W over $\mathbb{R}$ is the graded vector whose component in degree k is given by

$$(V \otimes W)_k := \bigoplus_{r+s=k} V_r \otimes W_s.$$

The denote this graded vector space by $V \otimes W$.

The structure of a *flat* $L_\infty[1]$-*algebra* on V is given by a family of multilinear maps $(\mu^k : V^{\otimes k} \to V[1])_{k\geq 1}$ that satisfies:

(1) $\mu^k(\cdots\otimes a\otimes b\otimes\cdots) = (-1)^{|a||b|}\mu^k(\cdots\otimes b\otimes a\otimes\cdots)$ holds for all $k\geq 1$ and all homogeneous elements a, b of V.

(2) The family of *Jacobiators* $(J^k)_{k\geq 1}$ defined by

$$J^k(x_1\cdots x_n) := \\ = \sum_{r+s=k}\sum_{\sigma\in(r,s)-\text{shuffles}} \text{sign}(\sigma)\,\mu^{s+1}(\mu^r(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)})$$

vanishes identically. Here $\text{sign}(\cdot)$ is the Koszul sign, i.e. the representation of Σ_n on $V^{\otimes n}$ induced by mapping the transposition $(2,1)$ to $a\otimes b\mapsto(-1)^{|a||b|}b\otimes a$. Moreover (r,s)-shuffles are permutations σ of $\{1,\ldots,k=r+s\}$ such that $\sigma(1)<\cdots<\sigma(r)$ and $\sigma(r+1)<\cdots<\sigma(k)$.

Since we are only going to consider flat $L_\infty[1]$-algebras we will suppress the adjective "flat" from now on. In this case the vanishing of the first Jacobiator implies that μ^1 is a coboundary operator. We remark that an $L_\infty[1]$-algebra structure on V is equivalent to the more traditional notion of an L_∞-algebra structure on $V[-1]$, see [MSS] for instance.

Given an L_∞-algebra structure $(\mu^k)_{k\geq 1}$ on V, there is a distinguished subset of V_1 that contains elements $v\in V_1$ satisfying the *Maurer-Cartan equation* (MC-equation for short)

$$\sum_{k\geq 1}\frac{1}{k!}\mu^k(v\otimes\cdots\otimes v) = 0.$$

This set is called the set of *Maurer-Cartan elements* (MC-elements for short) of V.

Let V be equipped with an L_∞-algebra structure such that the coboundary operator μ^1 decomposes into $d+\delta$ with $d^2=0=\delta^2$ and $d\circ\delta+\delta\circ d=0$. i.e. (V,d,δ) is a double complex. Then – under mild convergence assumptions – it is possible to construct an L_∞-algebra structure on $H(V,d)$ that is "isomorphic up to homotopy" to the original L_∞-algebra structure on V ([GL]). More concretely, one has to fix an embedding i of $H(V,d)$ into V, a projection pr from V to $H(V,d)$ and a homotopy operator h (of degree -1) which satisfies

$$d\circ h + h\circ d = id_V - i\circ pr.$$

We will also impose the following side-conditions for the sake of simplicity: 1.)$h\circ h=0$, 2.)$pr\circ h=0$ and 3.)$h\circ i=0$. Then explicit formulae for the structure maps for an L_∞-algebras on $H(V,d)$ can be written down. These are given in terms of rooted planar trees, see [Sch] for a review. We will explain the construction in more detail later on for the examples which are relevant for our purpose.

Furthermore one obtains L_∞-morphisms between $H(V,d)$ and V that induce inverse maps on cohomology. Such L_∞-morphisms are called L_∞ *quasi-isomorphisms*.

Consider the differential graded algebra $(\Omega([0,1]), d_{DR}, \wedge)$ of smooth forms on the interval $I := [0,1]$. The inclusions of a point $\{*\}$ as $0 \leq s \leq 1$ induces a chain map $ev_s : (\Omega(I), d_{DR}) \to (\mathbb{R}, 0)$ that is a morphisms of algebras. Given any L_∞-algebra structure on V there is a natural L_∞-algebra structure on $V \otimes \Omega(I)$ defined by

$$\tilde{\mu}^1(v \otimes \alpha) := \mu^1(v) \otimes \alpha + (-1)^{|v|} v \otimes d_{DR}\alpha$$

and

$$\tilde{\mu}^k((v_1 \otimes \alpha_1) \otimes \cdots \otimes (v_k \otimes \alpha_k)) := (-1)^{\#} \mu^k(v_1 \otimes \cdots \otimes v_k) \otimes (\alpha_1 \wedge \cdots \wedge \alpha_k)$$

for $k \geq 2$. Here $\#$ denotes the sign one picks up by assigning $(-1)^{|v_{i+1}||\alpha_i|}$ to passing α_i from the left-hand side of v_{i+1} to the right-hand side (and replacing α_{i+1} by $\alpha_i \wedge \alpha_{i+1}$).

Following [MSS], we call two morphisms f and g from an L_∞-algebra A to B *homotopic* if there exists an L_∞-morphism F from A to $B \otimes \Omega(I)$ such that

- $(id \otimes ev_0) \circ F = f$ and
- $(id \otimes ev_1) \circ F = g$ hold.

This defines an equivalence relation on the set of L_∞-morphisms from A to B.

Let F be an L_∞-morphism from A to $B \otimes \Omega(I)$. Consequently $f_s := ev_s \circ F$ is an L_∞ morphism between A and B for any $s \in I$. Given a MC-element v in A one obtains a one-parameter family of MC-elements

$$w_s := \sum_{k \geq 1} \frac{1}{n!} (f_s)_k (v \otimes \cdots \otimes v)$$

of B. Here $(f_s)_k$ denotes the kth Taylor component of f_s.

In the main body of this paper we are only interested in the following particular case: B is a differential graded Lie algebra (i.e. only the first and second structure maps are non-vanishing). Denote the graded Lie bracket by $[\cdot,\cdot]$. Furthermore we assume that the differential D is given by the adjoint action of a degree $+1$ element Γ that satisfies $[\Gamma, \Gamma] = 0$. The MC-equation for an element w of $(B, D = [\Gamma, \cdot], [\cdot,\cdot])$ reads

$$[\Gamma + w, \Gamma + w] = 0.$$

From the one-parameter family of MC-elements w_s in B one obtains a one-parameter family of differential graded Lie algebras on B by setting

$$D_s(\cdot) := [\Gamma + w_s, \cdot]$$

while leaving the bracket unchanged.

How are the differential graded Lie algebras $(B, D_s, [\cdot,\cdot])$ related for different values of $s \in I$? To answer this question we first apply the L_∞ morphism $F : A \rightsquigarrow B \otimes \Omega(I)$ to v and obtain a MC-element $w(t) + u(t)dt$ in $B \otimes \Omega(I)$.

It is straightforward to check that $w(s) = w_s$ for all $s \in I$. Moreover the MC-equation in $B \otimes \Omega(I)$ splits up into

$$[\Gamma + w(t), \Gamma + w(t)] = 0$$

and

$$\frac{d}{dt} w(t) = [u(t), \Gamma + w(t)].$$

The second equation implies that whenever the adjoint action of $u(t)$ on B can be integrated to a one-parameter family of automorphisms $(U(t))_{t \in I}$, $U(s)$ establishes an automorphism of $(B, [\cdot,\cdot])$ that maps $\Gamma + w(0)$ to $\Gamma + w(s)$ (for any $s \in I$). Consequently:

Lemma 1. *Let A and $(B, [\Gamma, \cdot], [\cdot,\cdot])$ be differential graded Lie algebras, v a MC-element in A and F an L_∞ morphism from A to $B \otimes \Omega(I)$ such that*

$$\sum_{k \geq 1} \frac{1}{k!} F_k(v \otimes \cdots \otimes v)$$

is well-defined in $B \otimes \Omega(I)$. Denote this element by $w(t) + u(t)dt$. Furthermore the flow equation

$$X(0) = b, \quad \frac{d}{dt}|_{t=s} X(t) = [u(s), X(s)], s \in I$$

is assumed to have a unique solution for arbitrary $b \in B$.

Then the one-parameter family $U(t)$ of automorphisms of B that integrates the adjoint action by $u(t)$ maps $\Gamma + w(0)$ to $\Gamma + w(t)$. In particular $U(s)$ is an isomorphims of differential graded Lie algebras

$$(B, [\Gamma + w(0), \cdot], [\cdot,\cdot]) \to (B, [\Gamma + w(s), \cdot], [\cdot,\cdot])$$

for arbitrary $s \in I$.

2.2. **Coisotropic Submanifolds.** We essentially follow [W], where more details can be found. Let M be a smooth, finite dimensional manifold. The bivector field Π on M is *Poisson* if the binary operation $\{\cdot,\cdot\}$ on $\mathcal{C}^\infty(M)$ given by $(f,g) \mapsto < \Pi, df \wedge dg >$ satisfies the *Jacobi identity*, i.e.

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$$

holds for all smooth functions f, g and h. Here $<-,->$ denotes the natural pairing between TM and T^*M. Alternatively one can consider the graded algebra $\mathcal{V}(M)$ of multivector fields on M equipped with the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}$. A bivector field Π is Poisson if and only if $[\Pi, \Pi]_{SN} = 0$.

Associated to any Poisson bivector field Π on M there is a vector bundle morphism $\Pi^\# : T^*M \to TM$ given by contraction. Consider a submanifold S of M. The annihilator N^*S of TS is a subbundle of T^*M. This subbundle fits into a short exact sequence of vector bundles:

$$0 \longrightarrow N^*S \longrightarrow T^*M|_S \longrightarrow T^*S \longrightarrow 0 .$$

Definition 1. *A submanifold S of a smooth, finite dimensional Poisson manifold (M, Π) is called* coisotropic *if the restriction of $\Pi^\#$ to N^*S has image in TS.*

There is an equivalent characterization of coisotropic submanifolds: define the vanishing ideal of S by

$$\mathcal{I}_S := \{f \in \mathcal{C}^\infty(M) : f|_S = 0\}.$$

A submanifold S is coisotropic if and only if $\mathcal{I}_C$ is a Lie subalgebra of $(\mathcal{C}^\infty(M), \{\cdot,\cdot\})$.

2.3. **The BFV-Complex.** The BFV-complex was introduced by Batalin, Fradkin and Vilkovisky with application in physics in mind ([BF], [BV]). Later on Stasheff ([St]) gave an interpretation of the BFV-complex in terms of homological algebra. The construction we present below is explained with more details in [Sch]. It uses a globalization of the BFV-complex for arbitrary coisotropic submanifolds found by Bordemann and Herbig ([B], [He]).

Let S be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold (M, Π). We outline the construction a differential graded Poisson algebra, which we call a *BFV-complex for S in* (M, Π). The construction depends on the choice of three pieces of data: 1. an embedding of the normal bundle of S into M as a tubular neighbourhood, 2. a connection on NS and 3. a special smooth function, called the charge, on a smooth graded manifold.

Denote the normal bundle of S inside M by E. Consider the graded vector bundle $E^*[1] \oplus E[-1] \to S$ over S and let $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ be the pull back of $E^*[1] \oplus E[-1] \to S$ along $E \to S$.

We define $BFV(E)$ to be the space of smooth functions on the graded manifold which is represented by the graded vector bundle $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ over E. In terms of sections one has $BFV(E) = \Gamma(\bigwedge(\mathcal{E}) \otimes \bigwedge(\mathcal{E}^*))$. This algebra carries a bigrading given by

$$BFV^{(p,q)}(E) := \Gamma(\wedge^p \mathcal{E} \otimes \wedge^q \mathcal{E}^*).$$

In physical terminology p / q is referred to as the ghost degree / ghost-momentum degree respectively. One defines

$$BFV^k(E) := \bigoplus_{p-q=k} BFV^{(p,q)}(E)$$

and calls k the total degree (in physical terminology this is the "ghost number").

The smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ comes equipped with a Poisson bivector field G given by the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$, i.e. it is defined to be the natural contraction on $\Gamma(\mathcal{E}) \otimes \Gamma(\mathcal{E}^*)$ and extended to a graded skew-symmetric biderivation of $BFV(E)$.

Choice 1. Embedding.
Fix an embedding $\psi : E \hookrightarrow M$ of the normal bundle of S into M. Hence

the normal bundle E inherits a Poisson bivector field which we also denote by Π. (Keep in mind that Π depends on ψ!)
Choice 2. Connection.
Next choose a connection on the vector bundle $E \to S$. This induces a connection on $\wedge E \otimes \wedge E^* \to S$ and via pull back one obtains a connection ∇ on $\wedge \mathcal{E} \otimes \wedge \mathcal{E}^* \to E$. We denote the corresponding horizontal lift of multivector fields by

$$\iota_\nabla : \mathcal{V}(E) \to \mathcal{V}(\mathcal{E}^*[1] \otimes \mathcal{E}[-1]).$$

It extends to an isomorphism of graded commutative unital associative algebras

$$\varphi : \mathcal{A} := \mathcal{C}^\infty(T^*[1]E \oplus \mathcal{E}^*[1] \oplus \mathcal{E}[-1] \oplus \mathcal{E}[0] \oplus \mathcal{E}^*[2]) \to \mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]).$$

Using φ we lift Π to a bivector field on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$. Since φ fails in general to be a morphism of Gerstenhaber algebras, $\varphi(\Pi)$ is not a Poisson bivector field. Similarly the sum $G + \varphi(\Pi)$ fails to be a Poisson bivector field in general. However the following Proposition provides an appropriate correction term:

Proposition 1. *Let $\mathcal{E}$ be a finite rank vector bundle with connection ∇ over a smooth, finite dimensional manifold E. Consider the smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ and denote the Poisson bivector field on it coming from the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$ by G.*

Then there is an L_∞ quasi-isomorphism $\mathcal{L}_\nabla$ between the graded Lie algebra

$$(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$$

and the differential graded Lie algebra

$$(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN}).$$

A proof of Proposition 1 can be found in [Sch]. It immediately implies

Corollary 1. *Let $\mathcal{E} \to E$ be a finite rank vector bundle with connection ∇ over a smooth, finite dimensional Poisson manifold (E, Π). Consider the smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ and denote the Poisson bivector field on it coming from the natural fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$ by G.*

Then there is a Poisson bivector field $\hat{\Pi}$ on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ such that $\hat{\Pi} = G + \varphi(\Pi) + \triangle$ for $\triangle \in \mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$.

For a proof we refer the reader to [Sch] again.

We remark that $\mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ is the ideal of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ generated by multiderivations which map any tensor product of functions of total bidegree (p,q) to a function of bidegree (P,Q) where $P > p$ and $Q > q$. In general, let $\mathcal{V}^{(r,s)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ be the ideal generated by multiderivations of $\mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ with total ghost degree larger than or equal to r and total ghost-momentum degree larger than or equal to s, respectively.

The bivector field $\hat{\Pi}$ from Corollary 1 equips $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ with the structure of a graded Poisson manifold. Consequently $BFV(E)$ inherits a graded

Poisson bracket which we denote by $[\cdot,\cdot]_{BFV}$. It is called the *BFV-bracket*. Keep in mind that the BFV-bracket depends on the connection on $E \to S$ we have chosen.

Choice 3. Charge.

The last step in the construction of the BFV-complex is to provide a special solution to the MC-equation associated to $(BFV(E), [\cdot,\cdot]_{BFV})$, i.e. one constructs a degree $+1$ element Ω that satisfies

$$[\Omega, \Omega]_{BFV} = 0.$$

Additionally, one requires this element Ω to contain the tautological section of $\mathcal{E} \to E$ as the lowest order term. To be more precise, recall that

$$BFV^1(E) = \bigoplus_{k \geq 0} \Gamma(\wedge^k \mathcal{E} \otimes \wedge^{k-1} \mathcal{E}^*).$$

Hence any element of $BFV^1(E)$ contains a (possibly zero) component in $\Gamma(\mathcal{E})$. One requires that the component of Ω in $\Gamma(\mathcal{E})$ is given by the tautological section of $\mathcal{E} \to E$. A MC-element satisfying this requirement is called a *BFV-charge*.

Proposition 2. *Let (E, Π) be a vector bundle equipped with a Poisson bivector field and denote its zero section by S. Fix a connection on $E \to S$ and equip the ghost/ghost-momentum bundle $\mathcal{E}^*[1] \oplus \mathcal{E}[-1] \to E$ with the corresponding BFV-bracket $[\cdot,\cdot]_{BFV}$.*

(1) *There is a degree $+1$ element Ω of $BFV(E)$ whose component in $\Gamma(\mathcal{E})$ is given by the tautological section Ω_0 and that satisfies*

$$[\Omega, \Omega]_{BFV} = 0$$

if and only if S is a coisotropic submanifold of (E, Π).

(2) *Let Ω and Ω' be two BFV-charges. Then there is an automorphism of the graded Poisson algebra $(BFV(E), [\cdot,\cdot]_{BFV})$ that maps Ω to Ω'.*

See [St] for a proof of this proposition.

Given a BFV-charge Ω one can define a differential $D_{BFV}(\cdot) := [\Omega, \cdot]_{BFV}$, called *BFV-differential*. It is well-known that the cohomology with respect to D is isomorphic to the Lie algebroid cohomology of S (as a coisotropic submanifold of (E, Π)).

By the second part of Proposition 2, different choices of the BFV-charge lead to isomorphic differential graded Poisson algebra structures on $BFV(E)$. In the next Section we will establish that different choices of connection on $E \to S$ lead to differential Poisson algebras that lie in the same isomorphism class. The dependence on the embedding of the normal bundle of S is more subtle and will be clarified in Section 4.

3. Choice of Connection

Consider a vector bundle E equipped with a Poisson bivector field Π such that that zero section S is coisotropic. The aim of this Section is to investigate the dependence of the differential graded Poisson algebra $(BFV(E), D_{BFV}, [\cdot,\cdot]_{BFV})$ constructed in Subection 2.3 on the choice of a connection ∇ on $E \to S$.

Recall that in order to lift the Poisson bivector field Π to a bivector field on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$, a connection ∇ on $E \to S$ was used. Furthermore the L_∞ quasi-isomorphism between $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$ and $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$ in Proposition 1 depends on ∇ too. Consequently so does the graded Poisson bracket $[\cdot,\cdot]_{BFV}$.

Let ∇_0 and ∇_1 be two connections on a smooth finite rank vector bundle $\mathcal{E} \to E$. By Proposition 1 we obtain two L_∞ quasi-isomorphisms $\mathcal{L}_{\nabla_0}$ and $\mathcal{L}_{\nabla_1}$ from $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$ to $(\mathcal{V}(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$. Although these morphisms depend on the connections, this dependence is very well-controlled:

Proposition 3. *Let $\mathcal{E}$ be a smooth finite rank vector bundle over a smooth, finite dimensional manifold E equipped with two connections ∇_0 and ∇_1. Denote the associated L_∞ quasi-isomorphisms between $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$ and $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]), [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$ from Proposition 1 by $\mathcal{L}_0$ and $\mathcal{L}_1$ respectively.*

Then there is an L_∞ quasi-isomorphism

$$\hat{\mathcal{L}} : (\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]) \otimes \Omega(I), [G,\cdot]_{SN} + d_{DR}, [\cdot,\cdot]_{SN})$$

such that $(id \otimes ev_0) \circ \hat{\mathcal{L}} = \mathcal{L}_0$ and $(id \otimes ev_1) \circ \hat{\mathcal{L}} = \mathcal{L}_1$ hold.

Proof. Given two connections ∇_0 and ∇_1, one can define a family of connections $\nabla_s := \nabla_0 + s(\nabla_1 - \nabla_0)$ parametrized by the closed unit interval I. Consequently we obtain a one-parameter family of isomorphisms of graded algebras

$$\varphi_s : \mathcal{A} := \mathcal{C}^\infty(T^*[1]E \oplus \mathcal{E}^*[1] \oplus \mathcal{E}[-1] \oplus \mathcal{E}[0] \oplus \mathcal{E}^*[2]) \xrightarrow{\cong} \mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]),$$

extending the horizontal lifting with respect to the connection $\nabla_s \oplus \nabla_s^*$. Via this identification, $\mathcal{A}$ inherits a one-parameter family of Gerstenhaber brackets which we denote by $[\cdot,\cdot]_s$. and a differential $\tilde{Q}$ which can be checked to be independet from s in local coordinates.

For arbitrary $s \in I$ these structures fit into the following commutative diagram:

$$\begin{tikzcd}
& (\mathcal{A}[1], \tilde{Q}, [\cdot,\cdot]_0) \arrow[dl, "Pr"'] \arrow[dr, "\varphi_0"] & \\
(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) & & (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN}) \\
& (\mathcal{A}[1], \tilde{Q}, [\cdot,\cdot]_s) \arrow[ul, "Pr"] \arrow[ur, "\varphi_s"'] \arrow[uu, "\psi_s"] &
\end{tikzcd}$$

where $\psi_s := \varphi_0^{-1} \circ \varphi_s$ is a morphism of differential graded algebras and of Gerstenhaber algebras. Pr denotes the natural projection.

It is straightforward to show that the cohomology of $(\mathcal{A}, \tilde{Q})$ is $\mathcal{V}(E)$ and that the induced L_∞ algebra coincides with $(\mathcal{V}(E)[1], [\cdot,\cdot])$, see the proof of Proposition 1 in [Sch]. Hence we obtain a one-parameter family of L_∞ quasi-isomorphisms $\mathcal{J}_s : (\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{A}[1], \tilde{Q}, [\cdot,\cdot]_s)$. Composition with ψ_s yields a one-parameter family of L_∞ quasi-isomorphisms

$$\mathcal{K}_s : (\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{A}[1], \tilde{Q}, [\cdot,\cdot]_0).$$

We remark that the composition of $\mathcal{J}_s$ with φ_s yields the L_∞ quasi-isomorphism $\mathcal{L}_s$ between $(\mathcal{V}(E), [\cdot,\cdot]_{SN})$ and $(\mathcal{V}(\mathcal{E}^*[1] \oplus E[-1]), [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$ associated to the connection ∇_s from Proposition 1. Consequently $\mathcal{L}_0$ $(\mathcal{L}_1)$ is the composition of $\mathcal{K}_0$ $(\mathcal{K}_1)$ with φ_0.

Next, consider the differential graded Lie algebra $(\mathcal{A}[1] \otimes \Omega(I), \tilde{Q} + d_{DR}, [\cdot,\cdot]_0)$. To prove Proposition 1, a homotopy $\tilde{H}$ for $\tilde{Q}$ was constructed in [Sch] such that

$$\tilde{Q} \circ \tilde{H} + \tilde{H} \circ \tilde{Q} = id - \iota \circ Pr$$

is satisfied. Here, ι denotes the natural inclusion $\mathcal{V}(E) \hookrightarrow \mathcal{A}$. One defines a one-parameter family of homotopies $\tilde{H}_s := \psi_s \circ \tilde{H} \circ \psi_s^{-1}$ and checks that

$$\tilde{Q} \circ \tilde{H}_s + \tilde{H}_s \circ \tilde{Q} = id - \psi_s \circ \iota \circ Pr$$

holds.

We define $\hat{Pr} : \mathcal{A} \otimes \Omega(I) \to \mathcal{V}(E) \otimes \Omega(I)$ to be $Pr \otimes id$ and $\hat{\iota} : \mathcal{V}(E) \otimes \Omega(I) \to \mathcal{A} \otimes \Omega(I)$ to be $\hat{\iota} := (\psi_s \circ \iota) \otimes id$. Clearly $\hat{Pr} \circ \hat{\iota} = id$ and $\tilde{H}_s$ provides a homotopy between id and $\hat{\iota} \circ \hat{Pr}$. Moreover the side-conditions $\tilde{H}_s \circ \tilde{H}_s = 0$, $\hat{Pr} \circ \tilde{H}_s = 0$ and $\tilde{H}_s \circ \hat{\iota} = 0$ are still satisfied. We summarize the situation in the following diagram:

$$(\mathcal{V}(E) \otimes \Omega(I), 0) \underset{\hat{Pr}}{\overset{\hat{\iota}_s}{\rightleftarrows}} (\mathcal{A} \otimes \Omega(I), \tilde{Q})\ , \tilde{H}_s.$$

Following Subsection 2.1 these data can be used to perform homological transfer. The input consists of the differential graded Lie algebra

$$(\mathcal{A}[1] \otimes \Omega(I), \tilde{Q} + d_{DR}, [\cdot,\cdot]_0).$$

To construct the induced structure maps, one has to consider oriented rooted trees with bivalent and trivalent interior vertices. The leaves (the exterior vertices with the root excluded) are decorated by $\hat{\iota}$, the root by $\hat{Pr}$, the interior bivalent vertices by d_{DR}, the interior trivalent vertices by $[\cdot,\cdot]_0$ and the interior edges (i.e. the edges not connected to any exterior vertices) by $-\tilde{H}_s$. One then composes these maps in the order given by the orientation towards the root. The associated L_∞ quasi-isomorphism is constructed in the same manner, however, the root is not decorated by $\hat{Pr}$ but by $-\tilde{H}_s$ instead.

Recall that $\mathcal{V}^{(r,s)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ is the ideal generated by multiderivations of $\mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ with total ghost degree larger than or equal to r and total ghost-momentum degree larger than or equal to s, respectively. One can check inductively that trees decorated with e copies of $-\tilde{H}_s$ increase the filtration index by (e, e). Moreover trees containing more than one interior bivalent vertex do not contribute since d_{DR} increases the form-degree by 1. These facts imply that 1. the induced structure is given by $(\mathcal{V}(E)[1] \otimes \Omega(I), d_{DR}, [\cdot,\cdot]_{SN})$ and 2. there is an L_∞ quasi-isomorphism

$$(\mathcal{V}(E)[1] \otimes \Omega(I), d_{DR}, [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{A}[1] \otimes \Omega(I), \tilde{Q} + d_{DR}, [-,-]_0).$$

We define

$$\tilde{\mathcal{K}} : (\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{V}(\mathcal{A}[1] \otimes \Omega(I), \tilde{Q} + d_{DR}, [\cdot,\cdot]_{SN})$$

to be the composition of this L_∞ quasi-isomorphism and the obvious L_∞ quasi-isomorphism $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) \hookrightarrow (\mathcal{V}(E)[1] \otimes \Omega(I), d_{DR}, [\cdot,\cdot]_{SN})$.

The composition of $\hat{\mathcal{K}}$ with $id \otimes ev_s : \mathcal{A} \otimes \Omega(I) \to \mathcal{A}$ can be computed as follows: first of all only trees without any bivalent interior edges contribute since all elements of form-degree 1 vanish under $id \otimes ev_s$. Using the identities $\psi_s^{-1}([\psi_s(-), \psi_s(-)]_0) = [-,-]_s$, $\tilde{H}_s = \psi_s \circ \tilde{H} \circ \psi_s^{-1}$ and $\hat{\iota} = \psi_s \circ \iota$ it is a straightforward to show that $(id \otimes ev_s) \circ \hat{\mathcal{K}} = \psi_s \circ \mathcal{K}_s$. Hence

$$\varphi_0 \circ (id \otimes ev_s) \circ \hat{\mathcal{K}} = \varphi_s \circ \mathcal{K}_s = \mathcal{L}_s.$$

Finally, we define the L_∞ quasi-isomorphism $\hat{\mathcal{L}}$ between $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$ and $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1] \otimes \Omega(I), [G,\cdot]_{SN} + d_{DR}, [\cdot,\cdot]_{SN})$ to be $(\varphi_0 \otimes id) \circ \hat{\mathcal{K}}$. By construction $(id \otimes ev_0) \circ \hat{\mathcal{L}} = \mathcal{L}_0$ and $(id \otimes ev_1) \circ \hat{\mathcal{L}} = \mathcal{L}_1$ are satisfied. □

We remark that Propositions 1 and 3 seem to permit "higher analogous", where one incorporates the differential graded algebra of differential forms on the n-simplex $\Omega(\triangle^n)$ instead of just $\Omega(\{*\}) = \mathbb{R}$ (Proposition 1) or $\Omega(I)$ (Proposition 3) – see [Co], where this idea was worked out in the context of the BV-formalism.

Corollary 2. *Let $\mathcal{E}$ be a finite rank vector bundle over a smooth, finite dimensional Poisson manifold (E, Π). Suppose ∇_0 and ∇_1 are two connections on $\mathcal{E} \to E$. Denote the associated L_∞ quasi-isomorphisms between $(\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$ and $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$ from Proposition 1 by $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. Applying these L_∞ quasi-isomorphisms to Π yields two MC-elements $\tilde{\Pi}_0$ and $\tilde{\Pi}_1$ of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$. Hence $\hat{\Pi}_0 := G + \tilde{\Pi}_0$ and $\hat{\Pi}_1 := G + \tilde{\Pi}_1$ are MC-elements of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [\cdot,\cdot]_{SN})$, i.e. Poisson bivector fields on $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$.*

There is a diffeomorphism of the smooth graded manifold $\mathcal{E}^*[1] \oplus \mathcal{E}[1]$ such that the induced automorphism of $\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ maps $\hat{\Pi}_0$ to $\hat{\Pi}_1$. Moreover, this diffeomorphism induces a diffeomorphism of the base E which coincides with the identity.

Proof. Apply the L_∞ quasi-isomorphism $\hat{\mathcal{L}}$ from Proposition 3 to Π and add G to obtain a MC-element $\hat{\Pi} + \hat{Z}dt$ of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1] \otimes \Omega(I), d_{DR}, [\cdot,\cdot]_{SN})$. Let $\mathcal{L}_s$ denote the L_∞ quasi-isomorphism from Proposition 1 constructed with the help of the connection $\nabla_0 + s(\nabla_1 - \nabla_0)$. Recall that $(id \otimes ev_s) \circ \hat{\mathcal{L}} = \mathcal{L}_s$ holds for all $s \in I$.

We set $\hat{\Pi}_s := (id \otimes ev_s)(\hat{\Pi})$ and $\hat{Z}_s := (id \otimes ev_s)(\hat{Z})$. Proposition 3 implies that this definition of $\hat{\Pi}_s$ is compatible with $\hat{\Pi}_0$ and $\hat{\Pi}_1$ defined in the Corollary.

We want to apply Lemma 1 to $A := (\mathcal{V}(E)[1], [\cdot,\cdot]_{SN})$, $B := (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$ and $F := \hat{\mathcal{L}}$. To do so, it remains to show that the flow of $\hat{Z}_s$ is globally well-defined for $s \in [0,1]$. Recall that $\hat{Z}$ is the one-form part of the MC-element constructed from the Poisson bivector field Π on E with help of the L_∞ quasi-isomorphism $\hat{\mathcal{L}} : (\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]) \otimes \Omega(I), [G,\cdot]_{SN} + d_{DR}, [\cdot,\cdot]_{SN})$. Only trees with exactly one bivalent interior vertex give non-zero contributions because the form degree must be one. Consequently there is at least one homotopy in the diagram and by the degree estimate in the proof of Proposition 3 this implies that $\hat{Z}$ is contained in $\mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]) \otimes \Omega(I)$. Hence the derivation $[\hat{Z}, -]_{SN}$ is nilpotent and can be integrated. Furthermore the degree estimate directly implies the last claim of the Corollary. □

The following is an immediate consequence of the previous Corollary:

Corollary 3. *Let (E, Π) be a vector bundle $E \to S$ equipped with a Poisson structure Π such that S is a coisotropic submanifold. Fix two connections ∇_0 and ∇_1 on $E \to S$ and denote the corresponding graded Poisson brackets on $BFV(E)$ by $[\cdot,\cdot]^0_{BFV}$ and $[\cdot,\cdot]^1_{BFV}$ respectively.*

There is an isomorphism of graded Poisson algebra

$$(BFV(E), [\cdot,\cdot]^0_{BFV}) \xrightarrow{\cong} (BFV(E), [\cdot,\cdot]^0_{BFV}).$$

Moreover the induced automorphism of $\mathcal{C}^\infty(E)$ coincides with the identity.

Combining Proposition 2 and Corollary 3 we obtain

Theorem 1. *Let E be a vector bundle equipped with a Poisson bivector Π such that the zero Section S is a coisotropic submanifold. Recall that the pull back of $E \to S$ by $E \to S$ is denoted by $\mathcal{E} \to E$ and*

$$BFV(E) := \mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]) = \Gamma(\wedge \mathcal{E} \otimes \wedge \mathcal{E}^*).$$

Different choices of a connection ∇ on $E \to S$ and of a degree $+1$ element Ω of $(BFV(S), [-,-]_{BFV})$ satisfying

(1) *the lowest order term of Ω is given by the tautological Section Ω_0 of $\mathcal{E} \to E$ and*
(2) $[\Omega, \Omega]^\nabla_{BFV} = 0$,

lead to isomorphic differential graded Poisson algebras

$$(BFV(E), [\Omega, \cdot]^\nabla_{BFV}, [\cdot, \cdot]^\nabla_{BFV}).$$

Proof. Pick two connections ∇_0 and ∇_1 on $E \to S$ and consider the two associated graded Poisson algebras $(BFV(E), [\cdot, \cdot]^0_{BFV})$ and $(BFV(E), [\cdot, \cdot]^1_{BFV})$, respectively. By Corollary 3 there is an isomorphism of graded Poisson algebras

$$\gamma : (BFV(E), [\cdot, \cdot]^0_{BFV}) \xrightarrow{\cong} (BFV(E), [\cdot, \cdot]^1_{BFV}).$$

Moreover the induced automorphism of $\mathcal{C}^\infty(E)$ is the identity.

Assume that Ω and $\tilde{\Omega}$ are two BFV-charges of $(BFV(E), [\cdot, \cdot]^0_{BFV})$ and $(BFV(E), [\cdot, \cdot]^1_{BFV})$, respectively. Applying the automorphism γ to Ω yields another element of $(BFV(E), [\cdot, \cdot]^1_{BFV})$, which can be checked to be a BFV-charge again. By Proposition 2 this implies that there is an inner automorphism β of $(BFV(E), [\cdot, \cdot]^1_{BFV})$ which maps $\gamma(\Omega)$ to $\tilde{\Omega}$.

Hence

$$\beta \circ \gamma : (BFV(E), [\cdot, \cdot]^0_{BFV}) \xrightarrow{\cong} (BFV(E), [\cdot, \cdot]^1_{BFV})$$

is an isomorphism of graded Poisson algebras which maps Ω to $\tilde{\Omega}$. □

4. Choice of Tubular Neighbourhood

Let S be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold (M, Π). Throughout this Section, E denotes the normal bundle of S inside M. As explained in subsection 2.3, the first step in the construction of the BFV-complex for S inside (M, Π) is the choice of an embedding $\psi : E \hookrightarrow M$. Such an embedding equips E with a Poisson bivector field Π_ψ, which is used to construct the BFV-bracket on the ghost/ghost-momentum bundle, see Subsection 2.3.

Let us first consider the case where the embedding is changed by composition with a linear automorphism of the normal bundle E:

Lemma 2. *Let*

$$(BFV(E), [\Omega, \cdot]_{BFV}, [\cdot, \cdot]_{BFV})$$

be a BFV-complex corresponding to some choice of tubular neighbourhood $\psi : E \hookrightarrow M$*, while*

$$(BFV(E), [\Omega^g, \cdot]^g_{BFV}, [\cdot, \cdot]^g_{BFV})$$

is a BFV-complex corresponding to the embedding $\psi \circ g : E \hookrightarrow M$*, where* $g : E \to E$ *is a vector bundle isomorphism covering the identity.*

Then there is an isomorphism of graded Poisson algebras

$$(BFV(E), [\cdot, \cdot]_{BFV}) \to (BFV(E), [\cdot, \cdot]^g_{BFV})$$

which maps Ω *to* Ω^g*.*

Proof. Let Π / Π^g be the Poisson bivector field on E obtained from $\psi : E \hookrightarrow M$ / $\psi \circ g : E \hookrightarrow M$, respectively. Clearly $\Pi^g = (g)_*(\Pi)$.

Choose some connection ∇ of E, which is used to construct the L_∞ quasi-isomorphism

$$\mathcal{L} : (\mathcal{V}(E)[1], [\cdot, \cdot]_{SN}) \rightsquigarrow (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [-,-]_{SN}, [G, -]_{SN}).$$

Plugging in Π results into the BFV-bracket $[\cdot, \cdot]_{BFV}$. On the other hand, we can use $\nabla^g := (g^{-1})^*\nabla$ to construct another L_∞ quasi-isomorphism $\mathcal{L}^g$. Plugging in Π^g results into another BFV-bracket $[\cdot, \cdot]^g_{BFV}$.

We claim that $[\cdot, \cdot]_{BFV}$ and $[\cdot, \cdot]^g_{BFV}$ are isomorphic graded Poisson brackets. First, observe that the isomorphism $g : E \to E$ lifts to an vector bundle isomorphism

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\hat{g}} & \mathcal{E} \\ \downarrow & & \downarrow \\ E & \xrightarrow{g} & E, \end{array}$$

such that the tautological section gets mapped to itself under $(\hat{g})^*$. We denote the induced automorphism of $E^*[1] \oplus E[-1]$ by $\hat{g}$ as well.

By naturality of the pull back of connections, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{V}(E) & \xrightarrow{\iota_\nabla} & \mathcal{V}(E^*[1] \oplus E[-1]) \\ \downarrow{\scriptstyle (g)_*} & & \downarrow{\scriptstyle (\hat{g})_*} \\ \mathcal{V}(E) & \xrightarrow{\iota_{\nabla^g}} & \mathcal{V}(E^*[1] \oplus E[-1]), \end{array}$$

where ι_∇ (ι_{∇^g}) is the horizontal lift induced by ∇ (∇^g). Using this together with the explicit description of the L_∞ quasi-isomorphism $\mathcal{L}$ from Proposition 1 contained in [Sch], or in the proof of Proposition 3, one concludes that

$$(\mathcal{L}^g)_k = (\hat{g})_* \circ (\mathcal{L})_k \circ \left((g)_*^{-1} \otimes \cdots \otimes (g)_*^{-1}\right).$$

Here, $(\mathcal{L})_k$ denotes the kth structure map of the L_∞ quasi-isomorphism $\mathcal{L}$.

This immediately implies that $\hat{g}$ induces an isomorphism between $[\cdot,\cdot]_{BFV}$ and $[\cdot,\cdot]^g_{BFV}$, respectively. Moreover, since $\hat{g}$ maps the tautological section to itself, it maps any BFV-charge to another one.

Finally, Theorem 1 implies the statement of Lemma 2. $\square$

In general, a different choice of embedding can cause drastic changes in the associated BFV-complexes. Consider $S = \{0\}$ inside $M = \mathbb{R}^2$ equipped with the smooth Poisson bivector field

$$\Pi(x,y) := \begin{cases} 0 & x^2+y^2 \leq 4 \\ \exp\left(-\frac{1}{x^2+y^2-4}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} & x^2+y^2 \geq 4 \end{cases}.$$

Let ψ_0 be the embedding of $E \cong \mathbb{R}^2$ into $\mathbb{R}^2$ given by the identity and ψ_1 the embedding given by

$$(x,y) \mapsto \frac{1}{\sqrt{1+x^2+y^2}}(x,y).$$

The image of ψ_1 is contained in the disk of radius 1. Hence Π_{ψ_1} vanishes identically whereas Π_{ψ_0} does not.

The ghost/ghost-momentum bundle $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$ is of the very simple form

$$\mathbb{R}^2 \times \big((\mathbb{R}^2)^*[1] \oplus \mathbb{R}^2[-1]\big) \to \mathbb{R}^2.$$

Denote the Poisson bivector field coming from the natural pairing between $(\mathbb{R}^2)^*[1]$ and $\mathbb{R}^2[-1]$ by G. We choose the standard flat connection on the bundle $\mathbb{R}^2 \to 0$. Then the Poisson bivector fields for the BFV-brackets $[\cdot,\cdot]^0_{BFV}$ and $[\cdot,\cdot]^1_{BFV}$ are simply given by the sums $G + \Pi_{\psi_0}$ and $G + \Pi_{\psi_1}$, respectively.

Any isomorphism of graded Poisson algebras between $(BFV(E), [\cdot,\cdot]^0_{BFV})$ and $(BFV(E), [\cdot,\cdot]^1_{BFV})$ yields an induced isomorphism of Poisson algebras between $(\mathcal{C}^\infty(\mathbb{R}^2), \{\cdot,\cdot\}_{\Pi_{\psi_0}})$ and $(\mathcal{C}^\infty(\mathbb{R}^2), \{\cdot,\cdot\}_{\Pi_{\psi_0}})$. Since Π_{ψ_1} vanishes, the induced automorphism would have to map something non-vanishing to 0, which is a contradiction. Hence there is no isomorphism of graded Poisson algebras between $(BFV(E), [\cdot,\cdot]^0_{BFV})$ and $(BFV(E), [\cdot,\cdot]^1_{BFV})$.

Although different choices of embeddings can lead to differential graded Poisson algebras that are not isomorphic, it is always possible to find appropriate "restrictions" of the BFV-complexes such that the corresponding differential graded Poisson algebras are isomorphic. To this end we define

Definition 2. *Let E be a finite rank vector bundle over a smooth manifold S. Assume E is equipped with a Poisson bivector field Π such that S is a coisotropic submanifold of E. Moreover let $(BFV(E), D_{BFV}, [\cdot,\cdot]_{BFV})$ be a BFV-complex for S in (E,Π) and U an open neighbourhood of S inside E.*

Then the restriction *of the BFV-complex on U is the differential graded Poisson algebra*

$$(BFV^U(E), D^U_{BFV}(\cdot) = [\Omega^U, \cdot]^U_{BFV}, [\cdot,\cdot]^U_{BFV})$$

given by the following data:

(a) *$BFV^U(E)$ is the space of smooth functions on the graded vector bundle $(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_U$ fitting into the following Cartesian square:*

$$\begin{tikzcd} \mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_U \arrow[r] \arrow[d] & \mathcal{E}^*[1] \oplus \mathcal{E}[-1] \arrow[d] \\ U \arrow[r] & E. \end{tikzcd}$$

(b) *$BFV^U(E)$ inherits a graded Poisson bracket $[\cdot,\cdot]^U_{BFV}$ from $BFV(E)$: one restricts the Poisson bivector field corresponding to $[\cdot,\cdot]_{BFV}$ to the graded submanifold $(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_U$ of $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]$.*

(c) *An element Ω^U of $BFV^U(E)$ is called a* restricted BFV-charge *if it is of degree $+1$, $[\Omega^U, \Omega^U]^U_{BFV} = 0$ holds and the component of Ω^U in $\Gamma(\mathcal{E}|_U)$ is equal to the restriction of the tautological section $\Omega_0 \in \Gamma(\mathcal{E})$ to U.*

Proposition 4. *Let S be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold (M, Π). Denote the normal bundle of S by E and fix a connection ∇ on E. Moreover let ψ_0 and ψ_1 be two embeddings of E into M as tubular neighbourhoods of S.*

Using these data one constructs two graded Poisson algebra structures on $BFV(E)$ following subsection 2.3 (in particular one applies Proposition 1). Denote the two corresponding graded Poisson brackets by $[\cdot,\cdot]^0_{BFV}$ and $[\cdot,\cdot]^1_{BFV}$ respectively.

Then there are two open neighbourhoods A_0 and A_1 of S in E such that an isomorphism of graded Poisson algebras

$$(BFV^{A_0}(E), [\cdot,\cdot]^{0,A_0}_{BFV}) \xrightarrow{\cong} (BFV^{A_1}(E), [\cdot,\cdot]^{1,A_1}_{BFV})$$

exists.

Proof. We make use of the fact that any two embeddings of E as a tubular neighbourhood are homotopic up to inner automorphisms of E, i.e. given two embeddings ψ and ϕ of E into M as a tubular neighbourhood, one can find

- a vector bundle isomorphism g of E and
- a smooth map $F : E \times I \to M$

satisfying

- $F|_{E\times\{0\}} = \psi$ and $F|_{E\times\{1\}} = \phi \circ g$,
- $\psi_s := F|_{E\times\{s\}} : E \to M$ is an embedding for all $s \in I$ and
- $\psi_s|_S = id_S$ for all $s \in I$.

The construction of F can be found in [Hi] for instance.

Since vector bundle automorphisms of E yield isomorphic BVF-complexes by Lemma 2, we can assume without loss of generality that the two embeddings $\psi := \psi_0$ and $\phi =: \psi_1$ are homotopic (i.e. $g = id$).

Denote the images of ψ_s by V_s. Since ψ_s is an embedding of a manifold of the same dimension as M, the image V_s is an open subset of M. Moreover

$S \subset V_s$ holds for arbitrary $s \in I$, i.e. V_s is an open neighbourhood of S in M. Because F is continuous, one can find an open neigbourhood V of S in M which is contained in $\bigcap_{s\in I} V_s$.

One defines $\hat{F} : E \times I \to M \times I$, $(e,t) \mapsto (F(e,t),t)$ and checks that $\hat{F}$ is an embedding, hence its image is a submanifold W of $M \times I$ and $\hat{F}$ is a diffeomorphism between $E \times I$ and W. Consider the restriction of $\hat{F}^{-1} : W \xrightarrow{\cong} E \times I$ to $V \times I$ which we denote by G. If one restricts G to "slices" of the form $V \times \{s\}$ one obtains $\psi_s^{-1}|_V$. The images of $\psi_s^{-1}|_V$ are denoted by W_s. By continuity of G there is an open neighbourhood W of S in E which is contained in $\bigcap_{s\in I} W_s$.

We define the following one-parameter family of local diffeomorphisms of E:

$$\phi_s : W_0 \xrightarrow{\psi_0|_{W_0}} V \xrightarrow{(\psi_s|_V)^{-1}} W_s.$$

Moreover E inherits a one-parameter family of Poisson bivector fields defined by $\Pi_s := (\psi_s|_{V_s}^{-1})_*(\Pi|_{V_s})$. The restriction $\Pi_s|_{W_s}$ is equal to $(\psi_s|_V^{-1})_*(\Pi|_V)$. Consequently

$$\Pi_s|_{W_s} = (\phi_s)_*(\Pi_0|_{W_0}) \tag{1}$$

holds for all $s \in I$.

Differentiating ϕ_s yields a smooth one-parameter family of local vector fields $(Y_s)_{s\in I}$ on E. By (1) the smooth one-parameter family

$$\Pi_t|_W - Y_t|_W dt$$

is a MC-element of $(\mathcal{V}(W)[1] \otimes \Omega(I), d_{DR}, [\cdot,\cdot]_{SN})$.

The L_∞ quasi-isomorphism

$$\mathcal{L}_\nabla : (\mathcal{V}(E)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN})$$

from Proposition 1 restricts to an L_∞ quasi-isomorphism

$$\mathcal{L}_\nabla|_W : (\mathcal{V}(W)[1], [\cdot,\cdot]_{SN}) \rightsquigarrow (\mathcal{V}((\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_W)[1], [G,\cdot]_{SN}, [\cdot,\cdot]_{SN}).$$

Hence we obtain an L_∞ quasi-isomorphism

$$\begin{aligned} \mathcal{L}_\nabla|_W \otimes id : \quad & (\mathcal{V}(W)[1] \otimes \Omega(I), d_{DR}, [\cdot,\cdot]_{SN}) \rightsquigarrow \\ & (\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_W)[1] \otimes \Omega(I), d_{DR} + [G,\cdot]_{SN}, [\cdot,\cdot]_{SN}). \end{aligned}$$

Applying $\mathcal{L}_\nabla|_W \otimes id$ to the MC-element $\Pi_t|_W - Y_t|_W dt$ and adding G yields a MC-element $\hat{\Pi}_t - \hat{Y}_t dt$ of $(\mathcal{V}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_W)[1] \otimes \Omega(I), d_{DR}, [\cdot,\cdot]_{SN})$.

It is straightforward to check that $\hat{\Pi}_s$ is the restriction of $\mathcal{L}_\nabla(\sum_{k\geq 1} \frac{1}{k!}\Pi_s^{\otimes k})$ to W and that $\hat{Y}_s$ is the sum of the horizontal lift $\iota_\nabla(Y_s)$ of Y_s with respect to ∇ restricted to W plus a part in $\mathcal{V}^{(1,1)}(\mathcal{E}^*[1] \oplus \mathcal{E}[-1])$ (that acts as a nilpotent derivation).

Using parallel transport with respect to ∇, $(\iota_\nabla(Y_t))_{t\in I}$ can be integrated to a one-parameter family of vector bundle automorphisms

$$\hat{\phi}_s : \mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{W_0} \to \mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{W_s}$$

covering $\phi_s : W_0 \to W_s$ for arbitrary $s \in I$. Similar to the construction of V and W one finds an open neighbourhood A_0 of S in W such that $\phi_t|_{A_0}$: $A_0 \xrightarrow{\cong} A_t$ with $\bigcup_{s\in I} A_s \subset W$. So the restriction of $\hat{\phi}_s$ to $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_0}$ has image $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_s}$ which is a submanifold of $\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_W$ for arbitrary $s \in I$.

Hence the one-parameter family of local vector fields

$$(\iota_\nabla(Y_t)|_{(\mathcal{E}^*[1]\oplus\mathcal{E}[-1])|_{A_t}})_{t\in I}$$

can be uniquely integrated to a one-parameter family of local diffeomorphisms $(\hat{\phi}_t)_{t\in I}$ and consequently the one-parameter family of local vector fields $(\hat{Y}_t|_{A_t})_{t\in I}$ can be uniquely integrated to a one-parameter family of local diffeomorphisms which we denote by

$$\varphi_s : (\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_{A_0} \to (\mathcal{E}^*[1] \oplus \mathcal{E}[-1])|_{A_s}$$

for $s \in I$.

Applying Lemma 1 shows that $\hat{\Pi}_s|_{A_s} = (\varphi_s)_*(\hat{\Pi}_0|_{A_0})$ holds for all $s \in I$. Hence

$$(\varphi_1)_* : \mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_0}) \to \mathcal{C}^\infty(\mathcal{E}^*[1] \oplus \mathcal{E}[-1]|_{A_1})$$

is an isomorphism of Poisson algebras. □

Theorem 2. *Let S be a coisotropic submanifold of a smooth, finite dimensional Poisson manifold (M,Π). Suppose $(BFV(E), D^0_{BFV}, [\cdot,\cdot]^0_{BFV})$ and $(BFV(E), D^1_{BFV}, [\cdot,\cdot]^1_{BFV})$ are two BFV-complexes constructed with help of two arbitrary embeddings of E into M, two arbitrary connections on $E \to S$ and two arbitrary BFV-charges.*

Then there are two open neighbourhoods B_0 and B_1 of S in E such that an isomorphism of differential graded Poisson algebras

$$(BFV^{B_0}(E), D^{0,B_0}_{BFV}, [\cdot,\cdot]^{0,B_0}_{BFV}) \xrightarrow{\cong} (BFV^{B_1}(E), D^{1,B_1}_{BFV}[\cdot,\cdot]^{1,B_1}_{BFV})$$

exists.

Proof. By Theorem 1 we can assume without loss of generality that the two chosen connections coincide. Furthermore it suffices to prove that there is an isomorphism of graded Poisson algebras from some restriction of $(BFV(E), [\cdot,\cdot]^0_{BFV})$ to some restriction of $(BFV(E), [\cdot,\cdot]^0_{BFV})$ which maps a restricted BFV-charge to another restricted BFV-charge. This is a consequence of the fact that Theorem 1 holds also in the restricted setting as long as the open neighbourhood U of S in E, to which we restrict, is contractible to S along the fibres of E.

By Lemma 2, we may assume without loss of generality that the two embeddings under consideration are homotopic. Hence there is a smooth one-parameter family of isomorphisms of graded Poisson algebras

$$(\varphi_s)_* : (BFV^{A_0}(E), [\cdot,\cdot]^{0,A_0}_{BFV}) \to (BFV^{A_s}(E), [\cdot,\cdot]^{s,A_s}_{BFV}),$$

which we constructed in the proof of Proposition 4. The smoothness of this family and the fact that the zero section S is fixed under $(\varphi_s)_{s\in I}$ imply that there is a open neighbourhood A of S in E satisfying $A \subset \bigcap_{s\in I} A_s$.

Fix a restricted BFV-charge Ω of $(BFV^{A_0}(E), [\cdot,\cdot]^{0,A_0}_{BFV})$. The restriction of

$$(\Omega(t) := (\varphi_t)_*(\Omega))_{t\in I}$$

to A yields a smooth one-parameter family of sections of $\bigwedge \mathcal{E} \otimes \bigwedge \mathcal{E}^*|_A$. Although $[\Omega(s)|_A, \Omega(s)|_A]^{s,A}_{BFV} = 0$ holds for all $s \in I$, $\Omega(s)|_A$ is in general not a BFV-charge since its component in $\Gamma(\mathcal{E}|_W)$ is $\Omega_0(s) := (\varphi_s)_*(\Omega_0)$ which does not need to be equal to Ω_0 as required – see Definition 2. In particular $\Omega(1)$ might not be a restricted BFV-charge of $(BFV(E), [\cdot,\cdot]^1_{BFV})$. However we will show that $\Omega(1)$ can be "gauged" to a BFV-charge in the remainder of the proof.

We have to recall some of the ingredients involved in the proof of Proposition 2: The first observation is that $\delta := [\Omega_0, \cdot]_G$ is a differential. Here Ω_0 denotes the tautological section of $\mathcal{E} \to E$, G is the Poisson bivector field associated to the fibre pairing between $\mathcal{E}$ and $\mathcal{E}^*$, and $[\cdot,\cdot]_G$ denotes the graded Poisson bracket on $BFV(E)$ corresponding to G. Second it is possible to construct a homotopy h for δ, i.e. a degree -1 map satisfying

$$\delta \circ h + h \circ \delta = id - i \circ pr \tag{2}$$

where i is an embedding of the cohomology of δ into $BFV(E)$ and pr is a projection from $BFV(E)$ onto cohomology. We remark that h does not restrict to arbitrary open neighbourhoods of S in E. However one can check that it does restrict to open neighbourhoods that can be contracted to S along the fibres of E. Without loss of generality we can assume that A has this property.

We are interested in the smooth one-parameter family

$$h(\Omega_0(s)) \in \Gamma(\mathcal{E} \otimes \mathcal{E}^*|_A) \cong \Gamma(\mathrm{End}(\mathcal{E}|_A))$$

with $s \in I$. Since Ω_0 intersects the zero section of $\mathcal{E} \to E$ transversally at S, so does $\Omega_0(s)$ for arbitrary $s \in I$. This implies 1.) the evaluation of $\Omega_0(s)$ at S is zero and 2.) $h(\Omega_0(s))|_S \in \Gamma(\mathcal{E} \otimes \mathcal{E}^*|_S)$ is fibrewise invertible, i.e. it is an element of $\Gamma(GL(\mathcal{E}|_S))$.

For any $s \in I$ we have $\delta(\Omega_0(s)) = [\Omega_0, \Omega_0(s)]_G = 0$ since both Ω_0 and $\Omega_0(s)$ are sections of $\mathcal{E}|_A$ and G is the Poisson bivector given by contraction between $\mathcal{E}$ and $\mathcal{E}^*$. Moreover $(i \circ pr)(\Omega_0(s)) = 0$ since the projection pr involves evaluation of the section at S, where $\Omega_0(s)$ vanishes. Consequently (2) reduces to $\delta(h(\Omega_0(s))) = \Omega_0(s)$ for all $s \in I$. However this means that if we interpret $h(\Omega_0(s))$ as a fibrewise endomorphism of $\mathcal{E}|_A$ the image of Ω_0 under $-h(\Omega_0(s))$ is $\Omega_0(s)$.

We define $M_s := -h(\Omega_0(s))$ – as already observed, $(M_t)_{t\in I}$ is a smooth one-parameter family of sections of $\mathrm{End}(\mathcal{E}|_A)$ and the restriction to S is a smooth one-parameter family of $GL(\mathcal{E}|_S)$. By smoothness of the one-parameter family it is possible to find an open neighbourhood B of S in E such that the restriction of $(M_t)_{t\in I}$ to B is always fibrewise invertible. Since $M_0 = id|_A$ we know that $(M_t|_B)_{t\in I}$ is a smooth one-parameter family of sections in $GL_+(\mathcal{E}|_B)$, i.e. fibrewise invertible automorphisms of $E|_B$ with positive determinante. In particular $M_1 \in \Gamma(GL_+(\mathcal{E}|_B))$.

Consider the smooth one-parameter family $(m_t)_{t\in I}$ of sections of $\mathrm{End}(\mathcal{E}|_B)$ given by

$$m_t := -M_t^{-1} \circ \left(\frac{d}{dt} M_t\right).$$

It integrates to a smooth one-parameter family of sections of $GL_+(\mathcal{E}|_B)$ that coincides with $(M_t)_{t\in[0,1]}$. The adjoint action of m_t on $(BFV^B(E), [\cdot,\cdot]^{1,B}_{BFV})$ can be integrated to an automorphism of $(BFV^B(E), [\cdot,\cdot]^{1,B}_{BFV})$ and this automorphism maps the restriction of $\Omega_0(1)$ to B to the restriction of Ω_0 to B. Hence $(\exp(m) \circ (\varphi_1)_*)$ maps the restricted BFV-charge Ω to another restricted BFV-charge of $(BFV^B(E), [\cdot,\cdot]^{1,B}_{BFV})$. □

Definition 3. *Let $(BFV(E), D_{BFV}, [\cdot,\cdot]_{BFV})$ be a BFV-complex associated to a coisotropic submanifold S of a smooth Poisson manifold (M,Π). We define a differential graded Poisson algebra $(BFV^{\mathfrak{g}}(E), D^{\mathfrak{g}}_{BFV}, [\cdot,\cdot]^{\mathfrak{g}}_{BFV})$ as follows:*

(a) *$BFV^{\mathfrak{g}}(E)$ is the algebra of equivalence classes of elements of $BFV(E)$ under the equivalence relation: $f \sim g :\Leftrightarrow$ there is a open neighbourhood U of S in E such that $f|_U = g|_U$.*
(b) *$D^{\mathfrak{g}}_{BFV}([\cdot]) := [D_{BFV}(\cdot)]$ where $[\cdot]$ denotes the equivalence class of $\cdot$ under $\sim$.*
(c) *$[[\cdot],[\cdot]]^{\mathfrak{g}}_{BFV} := [[\cdot,\cdot]_{BFV}]$.*

Given a differential graded Poisson algebra with unit $(A, \wedge, d, [\cdot,\cdot])$ we define the corresponding *abstract differential graded Poisson algebra with unit* $[(A, \wedge, d, [\cdot,\cdot])]$ to be the isomorphism class of $(A, \wedge, d, [\cdot,\cdot])$ in the category of differential graded Poisson algebras with unit. In particular $[(A, \wedge, d, [\cdot,\cdot])]$ is a object in the category of differential graded Poisson algebras with unit up to isomorphisms.

Theorem 2 immediately implies

Corollary 4. *Consider a coisotropic submanifold S of a smooth, finite dimensional Poisson manifold (M,Π) and let $(BFV(E), D_{BFV}, [\cdot,\cdot]_{BFV})$ be a BFV-complex associated to S inside (M,Π).*

The abstract differential graded Poisson algebra

$$[(BFV^{\mathfrak{g}}(E), D^{\mathfrak{g}}_{BFV}, [\cdot,\cdot]^{\mathfrak{g}}_{BFV})]$$

is independent of the specific choice of a BFV-complex and hence is an invariant of S as a coisotropic submanifold of (M,Π).

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