POSITION SPACE FEYNMAN QUADRICS AND THEIR MOTIVES

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To Yuri Ivanovitch Manin, on the occasion of his birthday, with admiration

ABSTRACT. In this note, we introduce and study position space Feynman quadrics that are the loci of divergences of the position space Feynman integrals for Euclidean massless scalar quantum field theories. We prove that the Feynman quadrics define objects in the category of mixed Tate motives for complete graphs with a bound on the number of vertices. This result shows a strong contrast with the graph hypersurfaces approach which produces also non-mixed Tate examples.

1. INTRODUCTION

1.1. **The problem: From Feynman integrals to motives and periods.** A *Feynman graph* is a finite 1-dimensional connected CW-complex. For a given Feynman graph Γ, we denote the sets of vertices and edges respectively by $Ver(\Gamma)$ and $Edg(\Gamma)$, and the boundary map by $\partial_\Gamma : Edg(\Gamma) \to Ver(\Gamma)$.

In quantum field theory (QFT), the protocol called *Feynman rules* associates an integral to each Feynman graph Γ of the QFT. In position space setting Feynman rules work as follows: Let X be the spacetime manifold and $d\mathbf{x}$ be a volume form on X.

- First, take the space $X^{Ver(\Gamma)} := \{f : Ver(\Gamma) \to X \text{ where } v \mapsto \mathbf{x}_v\}$ of possibly degenerate configurations of vertices in X;
- Next, attach the *propagator* $G_{\mathbb{R}}(\mathbf{x}^i, \mathbf{x}^j)$ of the QFT (which is simply the Green's function for the Laplacian) to each edge e with $\partial_\Gamma(e) = (ij)$.

The *(unregularized) Feynman integral* associated to Γ is defined as the integral

$$ {}^0W_\Gamma := \int_{X^{Ver(\Gamma)}} \omega_\Gamma \qquad (1.1) $$

of the differential form

$$ \omega_\Gamma := \prod_{\substack{e\in Edg(\Gamma)\\ \partial_\Gamma(e)=(ij)}} G_{\mathbb{R}}(\mathbf{x}^i, \mathbf{x}^j) \bigwedge_{v\in Ver(\Gamma)} d\mathbf{x}_v. \qquad (1.2) $$

These integrals are generally divergent (see for instance [20]). In order to extract meaningful quantities, one takes care of the divergencies of the integrand (1.2) by using a *regularization* procedure which carefully introduces *counterterms* to produce the *regularized values* W_Γ of ${}^0W_\Gamma$.

Kontsevich was the first who suggested that one should consider appropriate regularizations of these integrals, as well as their residues, as periods in the sense of [23]. In this perspective, one expects to apprehend the nature of the numbers arising from Feynman integrals by examining the locus $Z_\Gamma := \{\omega_\Gamma = \infty\}$ of the divergence of the integrand (1.2) as an algebraic variety in an appropriate complexification of the configuration space $X^{Ver(\Gamma)}$.

The first guess was that the motives of Z_Γ are mixed Tate motives for Euclidean massless scalar QFTs where the spacetime X is $\mathbb{R}^{2d}$ $(d > 1)$ and the propagator is

$$ G_{\mathbb{R}}(\mathbf{x}, \mathbf{y}) := G_{\mathbb{R}}(\mathbf{x} - \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||^{2-2d}. \qquad (1.3) $$

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Main Problem: *Confirm whether Z_Γ define objects in the category of mixed Tate motives.*

Attacking the main problem is a plausible strategy in explaining the presence of multiple ζ-values in Feynman integral computations (see [7]), as it would follow from the result of F. Brown [8] combined with confirmation of Z_Γ are in the category of mixed Tate motives.

1.2. **Main results: Feynman quadrics and their motives.** The main focus of this work to examine the geometry of the *Feynman quadric*

(1.4)
$$Z_\Gamma := \left\{ (\mathbf{x}^i = (z_1^i, \ldots, z_d^i, w_1^i, \ldots, w_d^i) : i \in Ver(\Gamma)) \in (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\Gamma)} \mid \prod_{\substack{e \in Edg(\Gamma) \\ \partial e = (ij)}} G(\mathbf{x}^i - \mathbf{x}^j) = \infty \right\}$$

where

$$G(\mathbf{x}) = \frac{1}{(z_1 \cdot w_1 + \cdots + z_d \cdot w_d)^{d-1}}. \tag{1.5}$$

This is a suitable setup to attack the main problem since the fixed point set of the real structure

$$c: \begin{array}{ccc} (\mathbb{A}^{2d}(\mathbb{C}))^{Ver(\Gamma)} & \to & (\mathbb{A}^{2d}(\mathbb{C}))^{Ver(\Gamma)} \\ (z_1^i, \ldots, z_d^i, w_1^i, \ldots, w_d^i) & \mapsto & (\bar{w}_1^i, \ldots, \bar{w}_d^i, \bar{z}_1^i, \ldots, \bar{z}_d^i) \end{array} \quad \text{for all } i \in Ver(\Gamma),$$

can be identified with the configuration space $(\mathbb{A}^{2d}(\mathbb{R}))^{Ver(\Gamma)}$ in the spacetime $\mathbb{R}^{2d}$ and the restriction of (1.5) to the fixed point set of c is the Euclidean massless scalar propagator $G_\mathbb{R}$ in (1.3).

Main Theorem: *Let κ_n denote the complete graph with $|Ver(\kappa_n)| = n$, and let $n \leq d+1$. Then, the Feynman quadric Z_{κ_n} defines an object in the category of mixed Tate motives.*

Considering only the complete graphs with limited number of vertices may seem very restrictive at the first glance, but we can still consider the Feynman integrals of arbitrarily complicated graphs in higher dimensional spacetimes: For a given Γ with $|Ver(\Gamma)| = n$, since the graph Γ is a subgraph of κ_n, the integrand (1.2) can be thought as an algebraic form

$$\prod_{\substack{e \in Edg(\Gamma) \\ \partial_\Gamma(e) = (ij)}} G(\mathbf{x}^i - \mathbf{x}^j) \bigwedge_{v \in Ver(\Gamma)} d\mathbf{x}_v \in \Omega^*((\mathbb{A}^{2d})^n \backslash Z_{\kappa_n}) \text{ with } d\mathbf{x}_v := dz_1^v \wedge \cdots \wedge dz_d^v \wedge dw_1^v \wedge \cdots \wedge dw_d^v \tag{1.6}$$

whenever $d \geq n-1$. In other words, after an appropriate regularization[1], the residues and the regularized values of these Feynman integrals can be given as periods of mixed Tate motives.

This technical result actually implies a bit more: The valid theories describing the same phenomenon must produce the same outcomes. This thesis implies that both momentum and position space formulations of Feynman integrals should produce the same results for any valid regularization techniques. The resulting nature of the motives should be independent from its formulation. However, our main results above indicates otherwise.

1.3. **Graph hypersurfaces vs Feynman quadrics.** In its original form in [22], the main problem refers to the motives of the *graph hypersurfaces*

$$X_\Gamma := \left\{ \alpha := (\alpha_e : e \in Edg(\Gamma)) \mid \Psi_\Gamma(\alpha) := \sum_{\substack{\text{spanning trees} \\ T \subset \Gamma}} \prod_{e \notin Edg(T)} \alpha_e = 0 \right\} \subset \mathbb{A}^{Edg(\Gamma)}$$

that are the loci of the divergences for the *parametric Feynman integrals* in momentum space. The attempts to prove that the graph hypersurfaces are mixed Tate motives failed already at early

[1]One needs here a regularization protocol that preserves the divergence loci, such as the algebraic regularization [14, 25] or analytic regularization, see for instance [27].

stages [3], and the recently found counter examples in [9, 10, 16] quashed all remaining hope for the mid-dimensional (co)homology, the part responsible of Feynman integrals, being mixed Tate. (See [24] and the references therein).

Our main theorem here, and the recent progresses [9, 10, 16] on graph hypersurfaces, indicate that the motivic nature of Feynman integrals depends on the momentum/position space formulations and the regularization methods. The periods of non-mixed Tate motives are expected to be complicated in general (see, for instance §1.2 in [11]). However, the periods of the Feynman quadrics should be given in term of multiple polylogarithms according to Conjecture 1.9 in [18].

The key observation here is that the definition of X_Γ is independent of the dimension of the spacetime. This fact allows us choose the dimension of the Feynman quadrics high enough so that they would provide mixed Tate motives as stated in our main theorem.

1.4. **Organization of the paper.** In Section 2, we examine the simplest possible case, that is the singular quadric $\{z_1 \cdot w_1 + \cdots + z_d \cdot w_d = 0\} \subset \mathbb{A}^{2d}$ associated to the propagator (1.5). In this test example, we illustrate the main techniques; we compute the class of this quadric in Grothendieck ring of varieties as well as the associated mixed motive in the Voevodsky's category. Our computations show that this quadric defines a mixed Tate motive. After resolving the singularity of the quadric, we show that its motive is a mixed Tate motive.

In Section 3, we introduce the Feynman quadric Z_Γ. We then examine the Feynman quadrics Z_{κ_n} associated to the complete graphs κ_n in details. We give a stratification of Z_{κ_n} by using appropriate projections which we illustrate on a simple case in §2. While most of the strata are given by trivial fibrations, one of the strata turn out be highly non-trivial, a *space of hyperplane arrangements in almost general positions*.

In Section 4, we prove that the each of these strata of Z_{κ_n} define mixed Tate motives, and concluded that, the Feynman quadric itself defines also an element in the subcategory of mixed Tate motives when $n \leq d+1$.

In Section 5, we discuss the similarities and the variations between the Feynman quadrics and the alternative geometries arising from the Feynman integrals. We briefly discuss the possible reasons behind the contrasts between the graph hypersurfaces and the Feynman quadrics. We also discuss the possible generalizations to the cases of more general graphs and the metrics with different signature. Finally, we note other alternatives geometries arising from position space Feynman integrals.

Contents

1.5. **Notation and Conventions.** We denote the category of schemes of finite type over $\Bbbk$ by $\mathfrak{Sch}_{\Bbbk}$, and the Grothendieck ring of varieties by $K_0(Var)$. We use the same notation with Voevodsky [29] to denote the motivic categories over $\Bbbk$, such as $DM^{eff}_{gm}(\Bbbk)$, $DM^{-}_{gm}(\Bbbk)$, $DM^{eff}_{gm}(\Bbbk) \otimes \mathbb{Q}$ etc., and denote the motivic functor (resp. with compact support) by $\mathfrak{m} : \mathfrak{Sch}_{\Bbbk} \to DM^{eff}_{gm}(\Bbbk)$ (resp. $\mathfrak{m}^c : \mathfrak{Sch}_{\Bbbk} \to DM^{eff}_{gm}(\Bbbk)$).

1.5.1. *The ambient space.* We fix the dimension $d \geq 1$ and consider the product $\mathbb{P}^d \times \mathbb{P}^d$ of projective spaces with homogeneous coordinates $([\mathbf{z}]; [\mathbf{w}]) := ([z_0, \ldots, z_d]; [w_0, \ldots, w_d])$. We set $\mathbb{A}^d \times \mathbb{A}^d \hookrightarrow \mathbb{P}^d \times \mathbb{P}^d : (\mathbf{z}; \mathbf{w}) := (z_1, \ldots, z_d; w_1, \ldots, w_d) \mapsto ([1 : z_1 : \cdots : z_d]; [1 : w_1 : \cdots : w_d])$ be the embedding of the affine part $\mathbb{A}^d \times \mathbb{A}^d$.

2. Simplest example: A quadric associated to the massless propagator

In this section, we associate a *simple Feynman quadric* to the propagator of the Euclidean massless scalar QFTs. While the simple Feynman quadric is defined as the locus of divergence of the propagator, its geometric and motivic properties are slightly different than its relatives in the literature [5, 12, 13, 14, 24, 25].

We study the geometry of this quadric in this section. We compute its class in the Grothendieck ring of varieties and its motive in the Voevodsky's category of mixed motives. Our computations conclude that the simple Feynman quadric defines a mixed Tate motive in Voevodsky's category. We then resolve the singularity of this quadric and show that its motive is also mixed Tate.

2.1. **Euclidean scalar massless propagator and the simple Feynman quadric.** Let $d > 1$. Our centre of interest will be the quadric $\mathcal{Q} \subset \mathbb{A}^d \times \mathbb{A}^d$ defined by

$$q(\mathbf{z}, \mathbf{w}) := \mathbf{z} \cdot \mathbf{w} = z_1 w_1 + \cdots + z_d w_d = 0. \tag{2.1}$$

We call it *simple Feynman quadric.* We denote the Zariski closure of $\mathcal{Q}$ in $\mathbb{P}^d \times \mathbb{P}^d$ by $\widehat{\mathcal{Q}}$.

Remark 2.1. *The simple Feynman quadric parameterizes the pairs of orthogonal vectors in the vector space $\mathbb{K}^d$, i.e., $\mathbf{z}, \mathbf{w} \in \mathbb{K}^d$ and $\mathbf{z} \perp \mathbf{w}$. The complement of the locus $\{\mathbf{z} = 0\} \cup \{\mathbf{w} = \mathbf{0}\}$ is closely related to the Stiefel manifolds and can be viewed as a homogeneous space. Such a setup would provide an elegant way to examine motive of $\mathcal{Q}$ in terms of the motives of the classical groups. A related approach can be found in* [25]*. However, we follow a more pedestrian approach in this paper.*

2.1.1. *From Feynman quadric to the propagator.* Let $c : \mathbb{A}^{2d}(\mathbb{C}) \to \mathbb{A}^{2d}(\mathbb{C})$ be the real structure

$$(\mathbf{z};\mathbf{w}) \mapsto (\bar{\mathbf{w}};\bar{\mathbf{z}}) := (\bar{w}_1,\ldots,\bar{w}_d;\bar{z}_1,\ldots,\bar{z}_d). \tag{2.2}$$

Lemma 2.2. *The restriction of the rational function q^{1-d} to the real locus $\mathbf{Fix}(c)$ of (2.2) gives the propagator (1.3) of the Euclidean massless scalar QFT in $\mathbb{A}^{2d}(\mathbb{R})$.*

Proof. The fixed point locus $\mathbf{Fix}(c)$ of (2.2) is $\mathbb{A}^d(\mathbb{C}) = \{(\mathbf{z},\mathbf{w}) \mid w_i = \bar{z}_i \ \forall\ i = 1,\ldots,d\}$, that is $\mathbb{A}^{2d}(\mathbb{R})$ as a smooth manifold. Therefore, the restriction of the function q^{1-d} to real locus becomes

$$\frac{1}{(q(\mathbf{z},\bar{\mathbf{z}}))^{d-1}} = \frac{1}{(|z_1|^2+\cdots+|z_d|^2)^{d-1}} = \frac{1}{\|\mathbf{z}\|^{2d-2}}$$

that is exactly the propagator for the scalar massless QFT in $\mathbb{A}^{2d}(\mathbb{R})$ (see, for instance §7 in [17]). □

2.2. **The simple Feynman quadric and its motive.** Consider the projection $\pi : \mathbb{A}^d \times \mathbb{A}^d \to \mathbb{A}^d : (\mathbf{z},\mathbf{w}) \mapsto \mathbf{w}$, and its restriction

$$\pi : \mathcal{Q} \to \mathbb{A}^d \tag{2.3}$$

to our quadric $\mathcal{Q}$.

Lemma 2.3. *The fibres of the morphism $\mathcal{Q} \to \mathbb{A}^d$ are*

$$\pi^{-1}(\mathbf{w}) \cong \begin{cases} \mathbb{A}^d = \{(z_1,\ldots,z_d)\} & \textit{if} \quad \mathbf{w} = (0,\ldots,0), \\ \mathbb{A}^{d-1} = \{(z_1,\ldots,z_d) \mid z_1w_1+\cdots+z_dw_d = 0\} & \textit{if} \quad \mathbf{w} \neq (0,\ldots,0). \end{cases}$$

Proof. The statement directly follows from the defining equation (2.1) of the quadric $\mathcal{Q}$. □

The types of the fibres in Lemma 2.3 of the projection π induce the following decomposition of the base $\mathbb{A}^d$: Let

$$\begin{aligned} S_0 &= \{(0,\ldots,0)\} \subset \mathbb{A}^d = \{\mathbf{w}\} \\ S_1 &= \mathbb{A}^d \setminus S_0 = \mathbb{A}^d \setminus \{0\}, \end{aligned}$$

and, consider the fibrations

$$\begin{array}{ccccc} F_i & \hookrightarrow & U_i & = & \pi^{-1}(S_i) \\ \downarrow \pi & & \downarrow \pi & & \\ \mathrm{pt} & \hookrightarrow & S_i & & \end{array} \tag{2.4}$$

with fibres F_i given in Lemma 2.3, i.e., $F_0 = \mathbb{A}^d$ and $F_1 = \mathbb{A}^{d-1}$.

Lemma 2.4. *The bundles U_i over S_i are trivial for $i = 0, 1$.*

Proof. For $i = 0$ case, the bundle U_0 is trivial as any bundle over a point is trivial.

For case $i = 1$ case, the bundle U_1 is a rank-$(d-1)$ subbundle of a trivial bundle $S_1 \times \mathbb{A}^d \to S_1$ whose fibre over $\mathbf{w}$ consists of the quotient space $\mathbb{A}^d/\langle\mathbf{w}\rangle$. On the other hand, the subbundle N_1 which is "normal" to U_1, that is having the affine line $\mathbb{A}^1 = \langle\mathbf{w}\rangle$ spanned by $\mathbf{w}$ as the fibre over $\mathbf{w}$ is trivial: The map

$$S_1 \times \mathbb{K} \to N_1 : \quad (\mathbf{w}, c) \mapsto (\mathbf{w}, c\cdot\mathbf{w})$$

is an isomorphism and provides the trivialization that is needed. This implies that U_1 is also a trivial bundle over S_1. □

2.2.1. *The class of the simple Feynman quadric in the Grothendieck ring.*

Lemma 2.5. *The class* $[\mathcal{Q}]$ *of the quadric in the Grothendieck ring* $K_0(Var)$ *is given by*

$$[\mathcal{Q}] = [U_0] + [U_1] = \mathbb{L}^{2d-1} + \mathbb{L}^d - \mathbb{L}^{d-1} \in \mathbb{Z}[\mathbb{A}^1] \subset K_0(Var)$$

where $\mathbb{L}$ *denotes the Leftschetz motive* $[\mathbb{A}^1]$.

Proof. The scissor congruence allows us write the class $[\mathcal{Q}]$ as the sum $[U_0] + [U_1]$, see (2.4). The rest follows from the fact that, for a locally trivial fibration $E \to B$ with fibre F, the classes in the Grothendieck ring of varieties $K_0(Var)$ satisfies $[E] = [B] \cdot [F]$:

$$\begin{aligned} [U_0] + [U_1] &= [\mathbb{A}^0] \cdot [\mathbb{A}^d] + ([\mathbb{A}^d] - 1) \cdot [\mathbb{A}^{d-1}] \\ &= \mathbb{L}^{2d-1} + \mathbb{L}^d - \mathbb{L}^{d-1}. \end{aligned}$$

□

Remark 2.6. *The same line of arguments shows that the class* $[\widehat{\mathcal{Q}}]$ *of the quadric in the Grothendieck ring* $K_0(Var)$ *is* $\mathbb{L}^d + [\mathbb{P}^d] \cdot [\mathbb{P}^{d-1}]$*: Consider the restriction* $\widehat{\pi} : \widehat{\mathcal{Q}} \subset \mathbb{P}^d \times \mathbb{P}^d \to \mathbb{P}^d$ *of the projection. Then, the fibres are*

$$\widehat{\pi}^{-1}([\mathbf{w}]) \cong \begin{cases} \mathbb{P}^d = \{([z_0 : \cdots : z_d]\} & \text{if } [\mathbf{w}] = [1 : 0 : \cdots : 0], \\ \mathbb{P}^{d-1} = \{[z_0 : \cdots : z_d] \mid z_1 w_1 + \cdots + z_d w_d = 0\} & \text{if } [\mathbf{w}] \neq [1 : 0 : \cdots : 0]. \end{cases} \tag{2.5}$$

Therefore, we have a decomposition according to the types of the fibres in (2.5)*:*

$$\widehat{S}_{-1} = \emptyset, \quad \widehat{S}_0 = \{[1 : 0 : \cdots : 0]\} \subset \mathbb{P}^d, \quad \widehat{S}_1 = \mathbb{P}^d \setminus \widehat{S}_0,$$

and $\widehat{U}_i = \widehat{\pi}^{-1}(\widehat{S}_i), i = 0, 1$. *The scissor congruence and the classes of fibrations in* $K_0(Var)$ *provides that*

$$\begin{aligned} [\widehat{U}_0] + [\widehat{U}_1] &= [\mathbb{A}^0] \cdot [\mathbb{P}^d] + ([\mathbb{P}^d] - 1) \cdot [\mathbb{P}^{d-1}] \\ &= (\mathbb{L}^d + [\mathbb{P}^{d-1}]) + (\mathbb{L} + \cdots + \mathbb{L}^d) \cdot [\mathbb{P}^{d-1}] \\ &= \mathbb{L}^d + (1 + \mathbb{L} + \cdots + \mathbb{L}^d) \cdot [\mathbb{P}^{d-1}] = \mathbb{L}^d + [\mathbb{P}^d] \cdot [\mathbb{P}^{d-1}]. \end{aligned}$$

2.2.2. *The motive of the simple Feynman quadric in Voevodsky's category.*

Lemma 2.7. *The motives* $\mathfrak{m}^c(U_i), i = 0, 1$ *associated to the fibrations* (2.4) *are given by are*

$$\mathfrak{m}^c(U_i) = \begin{cases} \mathfrak{m}^c(S_0 \times \mathbb{A}^d) = \mathfrak{m}^c(\mathbb{A}^0)(d)[2d] & \& \quad i = 0 \\ \mathfrak{m}^c(S_1 \times \mathbb{A}^{d-1}) = \mathfrak{m}(\mathbb{A}^d \setminus \{0\})(p-1)[2p-2] & \& \quad i = 1. \end{cases}$$

Proof. From Corollary 4.1.8 of [29] we know that taking the product with an affine space $\mathbb{A}^k$ is an isomorphism at the level of the corresponding motives: the motive $\mathfrak{m}^c(X \times \mathbb{A}^k) = \mathfrak{m}^c(X)(k)[2k]$ is obtained from $\mathfrak{m}^c(X)$ by Tate twists and shifts. The result then follows by applying this identity to the fibrations (2.4). □

Corollary 2.8. *The motives* $\mathfrak{m}^c(U_i), i = 0, 1, 2$ *are mixed Tate motives.*

Proof. This is an immediate consequence of Lemma 2.7: The motives $\mathfrak{m}^c(U_i)$ of the fibrations depend only on the motives $\mathfrak{m}^c(S_i)$ of the base and the motives $\mathfrak{m}^c(F_i)$ of the the fibres through products, Tate twists, sums, and shifts. All these operations preserve the subcategory of mixed Tate motives. The result follows from the fact that the motives of S_i and F_i are mixed Tate. □

Proposition 2.9. *The Voevodsky motive* $\mathfrak{m}^c(\mathcal{Q})$ *associated to the quadric* $\mathcal{Q}$ *is a mixed Tate motive.*

Proof. The bundle U_0 is a closed subscheme in $\mathcal{Q}$. We can use the canonical distinguished triangle (Prop 4.1.5 in [29]) that is

$$\mathfrak{m}^c(U_0) \longrightarrow \mathfrak{m}^c(\mathcal{Q}) \longrightarrow \mathfrak{m}^c(\mathcal{Q} \setminus U_0) \longrightarrow \mathfrak{m}^c(U_0)[1].$$

The motives $\mathfrak{m}^c(U_0)$ and $\mathfrak{m}^c(\mathcal{Q} \setminus U_0) = \mathfrak{m}^c(U_1)$ are mixed Tate due to Corollary 2.8 which simply implies that the motive $\mathfrak{m}^c(\mathcal{Q})$ of the simple Feynman quadric is also mixed Tate. □

Remark 2.10. *The same line of arguments shows that the motive $\mathfrak{m}(\widehat{\mathcal{Q}})$ of the Zariski closure $\widehat{\mathcal{Q}}$ of quadric inside $\mathbb{P}^d \times \mathbb{P}^d$ also defines a mixed Tate motive. The main difference here is that the fibres in* (2.5) *are no longer affine spaces and the product formula is usable.*

However, the results in [19] *provides usable setup such cases: The motives of the proper smooth locally trivial fibrations $E \to B$ with fibres F that admit cellular decomposition and satisfy Poincaré duality can be given as*

$$\mathfrak{m}(E) = \bigoplus_{p \geq 0} CH_p(F) \oplus \mathfrak{m}(B)(p)[2p]$$

due to Thm 2.10 in [19].

2.3. **Singularity of the simple Feynman quadric, its resolution and its motive.** The vector

$$(\frac{\partial q}{\partial \mathbf{z}}, \frac{\partial q}{\partial \mathbf{w}}) := (\frac{\partial q}{\partial z_1}, \dots, \frac{\partial q}{\partial z_d}; \frac{\partial q}{\partial w_1}, \dots, \frac{\partial q}{\partial w_d}) = (w_1, \dots, w_d; z_1, \dots, w_d) \in T_{(\mathbf{z};\mathbf{w})}(\mathbb{A}^d \times \mathbb{A}^d)$$

that is normal to $\mathcal{Q}$ at $(\mathbf{z};\mathbf{w}) \in \mathcal{Q}$ does not vanish unless $(\mathbf{z};\mathbf{w}) = (0;0)$, i.e., $\mathcal{Q}$ has no singular points other than $(0;0)$. It is easy check that the same is true for $\widehat{\mathcal{Q}} \setminus (0;0)$. We denote the smooth part $\mathcal{Q} \setminus (0;0)$ (resp. $\widehat{\mathcal{Q}} \setminus (0;0)$) by $\mathcal{Q}_{sm}$ (resp. by $\widehat{\mathcal{Q}}_{sm}$).

The singularity of $\mathcal{Q}$ is a quite simple type: Our affine quadric $\mathcal{Q}$ admits the following multiplicative group $\mathbb{G}_m$ action;

$$\forall \lambda \in \mathbb{G}_m, \quad (z_1, \dots, z_d; w_1, \dots, w_d) \mapsto (\lambda z_1, \dots, \lambda z_d; \lambda w_1, \dots, \lambda w_d,)$$

that provides us the fibration

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & \hookrightarrow & \mathcal{Q}_{sm} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \hookrightarrow & \mathcal{Q}_{sm}/\mathbb{G}_m. \end{array} \tag{2.6}$$

In other words, $\mathcal{Q}$ can be thought as the cone of the smooth quadric

$$\mathcal{Q}_{sm}/\mathbb{G}_m = \{[z_1, \dots, z_d; w_1, \dots, w_d] \mid z_1 w_1 + \cdots + z_d w_d = 0\} \subset \mathbb{P}^{2d-1}.$$

Moreover, it also hints us that the motive of $\mathcal{Q}_{sm}/\mathbb{G}_m$ is mixed Tate.

Lemma 2.11. $[\mathcal{Q}_{sm}/\mathbb{G}_m] = (\mathbb{L}^{2d-1} + \mathbb{L}^d - \mathbb{L}^{d-1} - 1) \cdot (\sum_{r=0}^{\infty} -\mathbb{L}^r)$.

Proof. As we know that the class of the fibres $\mathbb{L} - 1$ and the class $[\mathcal{Q}_{sm}] = [\mathcal{Q}] - 1$ of the total space (see, Remark 2.6), the class $[\mathcal{Q}_{sm}/\mathbb{G}_m]$ of the quotient can be given by $[\mathcal{Q}_{sm}] \cdot (\mathbb{L} - 1)^{-1}$ as stated above. □

2.3.1. *Resolution of the singularity.* The morphism

$$\begin{array}{rcl} \sigma : \mathbb{A}^{2d} & \to & \mathbb{A}^{2d} \times \mathbb{P}^{2d-1} \\ (z_1, \dots, z_D, w_1, \dots, w_D) & \mapsto & ((z_1, \dots, z_D, w_1, \dots, w_D), [z_1 : \cdots : z_D : w_1 : \cdots : w_D]) \end{array}$$

is defined outside the origin of $\mathbb{A}^{2d}$. The Zariski-closure $\mathrm{Bl}_{(0;0)}\mathcal{Q}$ of $\sigma(\mathcal{Q}_{sm})$ inside $\mathbb{A}^{2d} \times \mathbb{P}^{2d-1}$ is a nonsingular subvariety. It is actually blow-up of $\mathcal{Q}$ resolving the singularity at $(0;0)$. We denote the exceptional divisor $(\mathrm{Bl}_{(0;0)}\mathcal{Q}) \setminus \mathcal{Q}_{sm})$ by $\mathcal{D}$.

Proposition 2.12. *The motive $Bl_{(0;0)}\mathcal{Q}$ of the blown-up affine quadric $\mathcal{Q}$ is a mixed Tate motive.*

Proof. The strict transform is an isomorphims away from the singular point of $\mathcal{Q}$. We show that $\mathfrak{m}^c(\mathcal{Q}_{sm})$ is mixed Tate by simply using the canonical distinguished triangle for $(0;0) \hookrightarrow \mathcal{Q}$.

We only need to calculate the motive of the exceptional divisor $\mathcal{D}$ and patch them together using the distingushed triangle for $\mathcal{D} \hookrightarrow \mathrm{Bl}_{(0;0)}\mathcal{Q}$. Consider the blow-up $\mathrm{Bl}_{(0;0)}\mathbb{A}^{2d}$ at $(0;0)$ as a subspace of $\mathbb{A}^{2d} \times \mathbb{P}^{2d-1}$ with the projection $\mathbb{A}^{2d} \times \mathbb{P}^{2d-1} \to \mathbb{A}^{2d}$. If we pick a homogeneous chart $[u_1 : \cdots : u_d : v_1 : \cdots : v_d]$ in the central fibre $\{0\} \times \mathbb{P}^{2d-1}$ of the projection, then, we observe that

$$\mathcal{D} = \mathrm{Bl}_{(0;0)}\mathcal{Q} \cap (\{0\} \times \mathbb{P}^{2d-1}) = \{[u_1 : \cdots : u_d : v_1 : \cdots : v_d] \mid u_1v_1 + \cdots + u_dv_d = 0\}.$$

We can give a motivic decomposition of $\mathcal{D}$ as in §2.2. The same line of arguments in the proof of the Proposition 2.9 applies to this stratification and implies that the exceptional divisor is of type mixed Tate. □

Corollary 2.13. *The motive $\mathfrak{m}^c(\widetilde{\mathcal{Q}})$ of the blow-up of the quadric $\widehat{\mathcal{Q}}$ at its singular point is also a mixed Tate motive.*

Proof. This statement follows from the facts that the motives $\mathfrak{m}^c(\mathrm{Bl}_{(0;0)}\mathcal{Q})$ and $\mathfrak{m}^c(\widehat{\mathcal{Q}} \setminus \mathcal{Q})$ of the strata o $\widetilde{\mathcal{Q}}$ are mixed Tate. □

Remark 2.14. *Proposition 2.12 can be proved alternatively as follows. The fibration* (2.6) *is modified by adding the points at infinity to each fibre $\mathbb{A}^1 \setminus \{0\}$, and that provides an locally trivial $\mathbb{A}^1$-fibration:*

$$\begin{array}{ccccc} \mathbb{A}^1 \cong \mathbb{P}^1 \setminus \{0\} & \hookrightarrow & \widehat{\mathcal{Q}}_{sm} & = & \widehat{\mathcal{Q}} \setminus \{0\} \\ \downarrow & & \downarrow & & \\ pt & \hookrightarrow & \mathcal{Q}_{sm}/\mathbb{G}_m. & & \end{array} \tag{2.7}$$

We can conclude that the motive $\mathfrak{m}(\mathcal{Q}_{sm}/\mathbb{G}_m)$ of the exceptional divisor is mixed Tate as the motives of the locally trivial $\mathbb{A}^1$-fibrations are the same as their bases due to $\mathbb{A}^1$-homotopy invariance.

3. The position space Feynman quadrics

This section is the technical heart of our paper. We associate a Feynman quadric to each Feynman graphs and study them, in particular, in the case of the complete graphs. The reduction to the complete graphs can justified by the fact the periods of Feynman quadrics can be formulated as the periods of the Feynman quadric of complete graph with same number of vertices (see [14] for a similar treatment for configuration space setup). We simply imitate the projections in §2.2 and introduce the complement of the Feynman quadric as a configuration space of certain hyperplane arrangements. Then, we give a stratification of these configuration spaces in terms of the degeneration types of these hyperplane arrangements.

3.1. **Feynman quadric associated to Feynman graphs.** The *Feynman quadric* Z_Γ associated to a given Feynman graph Γ is the quadric

$$Z_\Gamma := \bigcup_{e \in Edg(\Gamma)} \mathcal{H}^e \subset (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\Gamma)}$$

whose irreducible components are

$$\mathcal{H}^e := \{q^{ij} = q(\mathbf{z}^i - \mathbf{z}^j, \mathbf{w}^i - \mathbf{w}^j) = 0 \mid (ij) = \partial_\Gamma(e)\}.$$

Lemma 3.1. *Let $e \in Edg(\Gamma)$ and $\partial_\Gamma(e) = (ij)$. The quadric $\mathcal{H}^e \subset (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\Gamma)}$ is isomorphic to $\mathcal{Q} \times (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\Gamma)\setminus\{j\}}$.*

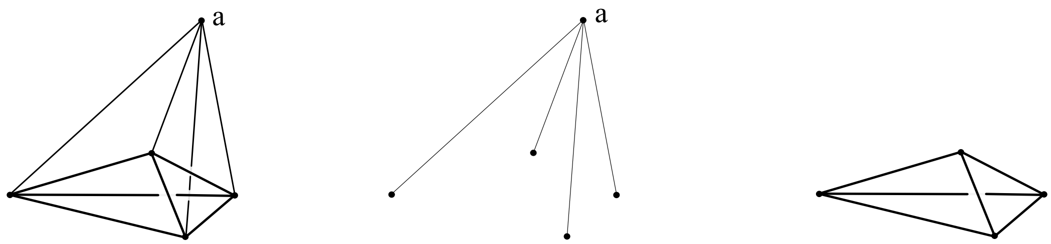

FIGURE 1. (I) κ^+ complete graph with 5 vertices; (II) Star of a; (III) κ, obtained from κ^+ be removing $St(a)$.

Proof. Consider the composition of the morphisms:

$$(\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\Gamma)} \xrightarrow{p_{ij}} (\mathbb{A}^d \times \mathbb{A}^d)^{\{i,j\}} \xrightarrow{tr_i} \mathbb{A}^d \times \mathbb{A}^d$$

$$(\mathbf{z}^m; \mathbf{w}^m \mid m \in Ver(\Gamma)) \longmapsto ((\mathbf{z}^i; \mathbf{w}^i), (\mathbf{z}^j; \mathbf{w}^j)) \longmapsto (\mathbf{z}^i - \mathbf{z}^j; \mathbf{w}^i - \mathbf{w}^j).$$

The morphism p_{ij} simply forgets all factors but $(\mathbf{z}^i; \mathbf{w}^i)$ and $(\mathbf{z}^j; \mathbf{w}^j)$, hence a trivial fibration with fibres $(\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\Gamma)\setminus\{i,j\}}$. On the other hand the morphism tr_iuses the translations to fix $(\mathbf{z}^i; \mathbf{w}^i)$ at $(0; 0)$, therefore it is also a trivial fibration with fibres $\mathbb{A}^d \times \mathbb{A}^d$ parameterizing $(\mathbf{z}^i; \mathbf{w}^i)$. The quadric $\mathcal{H}^e$ is simply the preimage $(tr_i \circ p_{ij})^{-1}(\mathcal{Q})$ of the simple Feynman quadric $\mathcal{Q}$. ☐

Corollary 3.2. *The singular locus of the quadric $\mathcal{H}^e$ is the diagonal*

$$\Delta^e = \{(\mathbf{z}^m, \mathbf{w}^m \mid m \in Ver(\Gamma)) \in (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\Gamma)} \mid \mathbf{z}^i = \mathbf{z}^j \ \&\ \mathbf{w}^i = \mathbf{w}^j\}.$$

Proof. The singularities of $\mathcal{H}^e$ are determined as stated by using the singularities of $\mathcal{Q}$ in §2.3 and Lemma 3.1. ☐

3.2. **A projection for the Feynman quadrics.** In this paragraph, we establish an explicit connection between the (complements of the) Feynman quadrics and a configuration spaces of hyperplane arrangements in "almost general position".

3.2.1. *Projection and its fibres.* This paragraph is a straightforward generalization of §2.2 which essentially examines the case of one-edge graph.

For any Γ with $|Ver(\Gamma)| = n$, the Feynman quadric Z_Γ is contained (set theoretically) in Z_{κ_n} where κ_n is the complete graph with n-vertices. From now on, we will consider only the complete graphs. However, the main strategy below is valid for any Feynman graphs Γ. We will remark the ramifications in §5 after completing case of complete graphs.

Let κ^+, κ be a pair of complete graphs such that κ is obtained from κ^+ by removing one of its vertices $a \in Ver(\kappa^+)$, and the all edges adjacent to the vertex a, i.e., $Ver(\kappa^+) = Ver(\kappa) \cup \{a\}$, $Edg(\kappa^+) = Edg(\kappa) \cup St(a)$ (which is called *the star of the vertex* a) where $St(a)$ denotes the set $\partial^{-1}_{\kappa^+}(a)$ of edges adjacent to the vertex a, and the boundary map ∂_κ is the restriction of ∂_{κ^+} to $Edg(\kappa)$. For an example, see Figure 1 which illustrates the simple case where κ^+ is complete graph with 5-vertices, the star $St(a)$ a and κ.

Consider the projection

$$\begin{array}{rcl} \pi_{\kappa^+} : (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa^+)} & \to & (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa)} \times \mathbb{A}^d \\ (\mathbf{z}^m, \mathbf{w}^{\mathbf{m}} \mid m \in Ver(\kappa^+)) & \mapsto & \mathbf{b} := (\mathbf{z}^m, \mathbf{w}^{\mathbf{m}} \mid m \in Ver(\kappa)) \times \mathbf{w}^a \end{array} \tag{3.1}$$

whose fibers are $\mathbb{P}^d$.

In order to examine the Feynman quadric $Z_{\kappa^+} = \bigcup_{e\in Edg(\kappa^+)} \mathcal{H}^e \subset (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa^+)}$ for a bigger graph κ^+, we study the fibres of the restriction of the forgetful morphism (3.1):

$$\pi_{\kappa^+} : Z_{\kappa^+} \ \to \ (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa)} \times \mathbb{A}^d. \tag{3.2}$$

Lemma 3.3. *The fibre* $\pi^{-1}_{\kappa^+}(\mathbf{b})$ *of* (3.2) *over a point* $\mathbf{b}$ *is*

1. *the affine space* $\mathbb{A}^d$*, if* $\mathbf{b} \in (Z_\kappa \times \mathbb{A}^d)$*,*
2. *the affine space* $\mathbb{A}^d$*, if* $\mathbf{b} \in \bigcup_{e\in St(a)} \{\mathbf{w}^a = \mathbf{w}^i \mid \partial_{\kappa^+}(e) = (ai)\} \subset ((\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa)} \times \mathbb{A}^d)$*,*
3. *a hyperplane arrangement*

$$\mathcal{A}_{St(a)} := (P_e \in \mathbb{A}^d \mid e \in St(a))$$

where

$$P_e := \{\mathbf{z}^a \in \mathbb{A}^d \mid q^{ai} = q(\mathbf{z}^a - \mathbf{z}^i, \mathbf{w}^a - \mathbf{w}^i) = 0 \text{ and } (ai) = \partial_{\kappa^+}(e)\},$$

when $\mathbf{b} \notin (Z_\kappa \times \mathbb{A}^d) \cup \bigcup_{e\in St(a))} \{\mathbf{w}^a = \mathbf{w}^i\}$*.*
Moreover, for any $I \subset St(a)$ *with* $|I| \leq d$*, the subarrangements* $\mathcal{A}_I = (P_e \mid e \in I)$ *satisfy*

$$P_I := \bigcap_{e\in I} P_e = \mathbb{A}^{d-|I|}.$$

In other words, they are in general position.

Proof. The statement directly follows from the ideals defining of the irreducible quadrics $\mathcal{H}^e$:

(1) If $\mathbf{b} \in (Z_\kappa \times \mathbb{A}^d) \subset ((\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa)} \times \mathbb{A}^d)$, then the equation

$$\prod_{\substack{e\in Edg(\kappa^+)\\ \partial_{\kappa^+}(e)=(ij)}} q^{ij} = \left(\prod_{\substack{e\in Edg(\kappa)\\ \partial_{\kappa}(e)=(ij)}} q^{ij}\right) \cdot \left(\prod_{\substack{e\in St(a)\\ \partial_{\kappa^+}(e)=(ai)}} q^{ai}\right) = 0 \cdot \left(\prod_{\substack{e\in St(a)\\ \partial_{\kappa^+}(e)=(ai)}} q^{ai}\right) = 0$$

is satisfied for any $\mathbf{z}^a$ in the fibre $\mathbb{A}^d = \pi^{-1}_{\kappa^+}(\mathbf{b})$ of (3.1). Therefore, all $\mathbf{z}^a \in \mathbb{A}^d$ must be in Z_{κ^+}.

(2) If $\mathbf{b} \in \{\mathbf{w}^a = \mathbf{w}^i\}$ for an edge $e \in St(a)$ with $\partial_{\kappa^+}(e) = (ai)$, then the equation $q^{ai} = (\mathbf{z}^a - \mathbf{z}^i) \cdot (\mathbf{w}^a - \mathbf{w}^i) = (\mathbf{z}^a - \mathbf{z}^i) \cdot 0 = 0$ is satisfied for any $\mathbf{z}^a$ in the fibre $\mathbb{A}^d = \pi^{-1}_{\kappa^+}(\mathbf{b})$ of (3.1). Therefore, all $\mathbf{z}^a \in \mathbb{A}^d$ must be in $\mathcal{H}^e \subset Z_{\kappa^+}$ in such a case.

(3) In all other cases, the intersections of $\mathcal{H}^e$ for $e \in St(a)$ with the fibres $\mathbb{A}^d = \pi^{-1}_{\kappa^+}(\mathbf{b})$ of the forgetful morphism (3.1) are simply defined by the equations $q^{ai} = (\mathbf{z}^a - \mathbf{z}^i) \cdot (\mathbf{w}^a - \mathbf{w}^i) = 0$ where $\partial_{\kappa^+}(e) = (ai)$. Therefore, the fibre $(\bigcup_{e\in St(a)} \mathcal{H}^e) \cap \pi^{-1}_{\kappa^+}(\mathbf{b})$ of (3.2) over $\mathbf{b}$ is the hyperplane arrangement as stated. Note that, $q^{ai} = 0$ cannot be the hyperplane infinity, since that hyperplane is given by the equation $z^a_0 = 0$.

We only need to show that the subarrangements $\mathcal{A}_I = (P_e \mid e \in I)$ is in general position for all $I \subset St(a)$ with $|I| \leq d$.

As we consider the compliment of cases examined in (1) and (2), we can simply set

$$P_e = \{\mathbf{z}^a \mid (\mathbf{z}^a - \mathbf{z}^i) \perp (\mathbf{w}^a - \mathbf{w}^i)\}.$$

Note that, if a subarrangement $\mathcal{A}_I$ of $\mathcal{A}_{St(a)}$ is in general position, then the vectors $\mathbf{v}^i = \mathbf{w}^a - \mathbf{w}^i$ that are normal to the affine hyperplanes $P_e = \{q^{ai} = (\mathbf{z}^a - \mathbf{z}^i) \cdot (\mathbf{w}^a - \mathbf{w}^i) = 0 \mid \partial_{\Gamma^+}(e) = (ai)\}$ for $e \in I$ must span a $|I|$-dimensional vector space when $|I| \leq d$, or simply d-dimensional vector space when $|I| \geq d$. We prove the statement by induction on the cardinality of $I \subset St(a)$:

First step, $|I| = 2$ case: Let $I = \{e_i, e_j\}$ and $\partial_{\kappa^+}(e_*) = (a*)$. If the statement does not hold for the pair P_{e_i}, P_{e_j}, then the affine part of these hyperplanes must be parallel. Hence their normal vectors satisfy

$$(\mathbf{w}^a - \mathbf{w}^i) = \lambda(\mathbf{w}^a - \mathbf{w}^j) \tag{3.3}$$

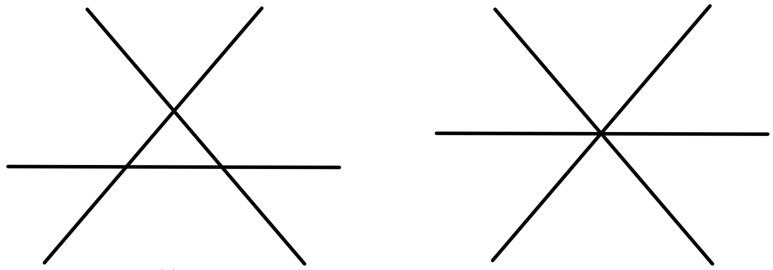

FIGURE 2. Arrangements of 3 hyperplanes in $\mathbb{A}^2$: (I) In general positions, (II) In almost general (but not in general) position.

for a nonzero λ. In this case, we have

$$\begin{array}{ccc} (\mathbf{z}^a - \mathbf{z}^i) \perp (\mathbf{w}^a - \mathbf{w}^i) \ \& & \Longrightarrow & (\mathbf{z}^a - \mathbf{z}^i) \perp (\mathbf{w}^a - \mathbf{w}^i) \ \& \\ (\mathbf{z}^a - \mathbf{z}^j) \perp (\mathbf{w}^a - \mathbf{w}^j) & \text{due to(3.3)} & (\mathbf{z}^a - \mathbf{z}^j) \perp \lambda(\mathbf{w}^a - \mathbf{w}^i) \end{array} \tag{3.4}$$

which implies that $(\mathbf{z}^a - \mathbf{z}^j) - (\mathbf{z}^a - \mathbf{z}^i) = (\mathbf{z}^i - \mathbf{z}^j) \perp (\mathbf{w}^a - \mathbf{w}^i)$. After interchanging indices i and j, the same argument implies $(\mathbf{z}^i - \mathbf{z}^j) \perp (\mathbf{w}^a - \mathbf{w}^j)$. We conclude that

$$\begin{array}{l} (\mathbf{z}^i - \mathbf{z}^j) \perp (\mathbf{w}^a - \mathbf{w}^i) \\ (\mathbf{z}^i - \mathbf{z}^j) \perp (\mathbf{w}^a - \mathbf{w}^j) \end{array} \quad \Longrightarrow \quad (\mathbf{z}^i - \mathbf{z}^j) \perp (\mathbf{w}^i - \mathbf{w}^j) = (\mathbf{w}^a - \mathbf{w}^j) - (\mathbf{w}^a - \mathbf{w}^i).$$

In other words, the normal vectors $(\mathbf{w}^a - \mathbf{w}^i)$ and $(\mathbf{w}^a - \mathbf{w}^i)$ can be parallel only when $\mathbf{b} \in \mathcal{H}^e \times \mathbb{A}^d \subset Z_\kappa \times \mathbb{A}^d$. That contradicts with our initial assumption on $\mathbf{b}$.

Next, we assume that the statement holds for $I = \{e_{i_1}, \ldots, e_{i_k}\}$ with $|I| < d$ but not for $J = I \cup \{e_{i_{k+1}}\} \subset St(a)$. Then, the sets of normal vectors $\mathbf{v}^{i_1}, \ldots, \mathbf{v}^{i_{k-1}}, \mathbf{v}^{i_k}$ and $\mathbf{v}^{i_1}, \ldots, \mathbf{v}^{i_{k-1}}, \mathbf{v}^{i_{k+1}}$ must span the same vector space. This can be satisfied if and only if

$$(\mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_{k-1}}) \wedge \mathbf{v}^{i_k} = \lambda \cdot (\mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_{k-1}}) \wedge \mathbf{v}^{i_{k+1}}$$

for a nonzero λ. This equality implies that $\mathbf{v}^{i_k} = \lambda \cdot \mathbf{v}^{i_{k+1}}$, which again contradicts with the assumption that $\mathbf{b} \notin Z_\kappa \times \mathbb{A}^d$. Therefore, the arrangement $\mathcal{A}_J = (P_e \mid e \in J)$ with $|J| \leq d$ must be also be in general position and, can be characterized in terms of intersections as stated. □

3.3. **Configuration space of hyperplane arrangements in "almost general position".** The case (3) in Lemma 3.3 suggests us that the space

$$\mathcal{M}^{St(a)} := ((\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa)} \times \mathbb{A}^d) \setminus ((Z_\kappa \times \mathbb{A}^d) \cup \bigcup_{e \in St(a)} \{\mathbf{w}^i = \mathbf{w}^a \mid (ai) = \partial_\kappa(e)\})$$

can be thought as a configuration space of the hyperplane arrangements satisfying certain conditions, i.e., it parameterizes the arrangements $\mathcal{A}_{St(a)} = (P_e \mid e \in St(a))$ of hyperplanes in $\mathbb{A}^d$ labeled by the index set $St(a)$ such that the intersections

$$P_I := \bigcap_{e \in I} P_e = \mathbb{A}^{d-|I|} \quad \text{for each} \quad I \subset St(a) \quad \text{with} \quad |I| \leq d. \tag{3.5}$$

We call these hyperplane arrangements are in *almost general position* as they can be put into general position by using the parallel translations of (at most $|St(a)| - d$) hyperplanes.

3.3.1. *Degeneration types of hyperplane arrangements in almost general position.* Matroids and intersection posets provide general setups for encoding the degeneration types of the hyperplane arrangements, see, for instance [1, 28]. As we only need to consider the degenerations of hyperplane arrangements of very particular type, we provide a simpler setup below.

Here, we note that a part of the conditions imposed by (3.5) on the *intersection poset* $L(\mathcal{A}_{St(a)})$ is implicit.

Lemma 3.4. *Let $\mathcal{A}_{St(a)}$ (resp. $\mathcal{A}'_{St(a)}$) be a hyperplane arrangement in (resp. almost) general position. Then, the map*

$$L(\mathcal{A}_{St(a)}) \to L(\mathcal{A}'_{St(a)}) : P_I \mapsto P'_I \tag{3.6}$$

is injective, and the complement of its image $\{P'_J \mid J \in \Theta'\}$ is indexed by the set

$$\Theta' := \{J \subset St(a) \mid \ |J| > d \ \& \ P'_J = \{point\}\} \subset L(\mathcal{A}'_{St(a)}) \tag{3.7}$$

of deepest intersections.

We call the possible index sets (3.7) as the *degeneration types* of the hyperplane arrangements in almost general position. If $\Theta' = \emptyset$, then $L(\mathcal{A}_{St(a)}) = L(\mathcal{A}'_{St(a)})$ and the hyperplane arrangement $\mathcal{A}'_{St(a)}$ is also in general position. If $\Theta' \neq \emptyset$, we call $\mathcal{A}'_{St(a)}$ a *degenerate* hyperplane arrangement.

Proof. (of Lemma 3.4) For the subsets J with $|J| \leq d$, the condition (3.5) is explicit and implies that the intersections P_J, P'_J are elements respectively in $\mathcal{A}_{St(a)}$ and $\mathcal{A}'_{St(a)}$. Hence, the map (3.6) is bijection for the index set $\{J \subset St(a) \mid |J| \leq d\}$.

Since $\mathcal{A}_{St(a)}$ is in general position, the deeper intersections $P_J = \emptyset$ for $J \in L(\mathcal{A}_{St(a)})$ with $|J| > d$, and we conclude that the difference between the intersection posets of these hyperplane arrangements can be given by the following index set

$$\{J \subset St(a) \mid |J| > d \ \& \ P'_J \neq \emptyset\}$$

However, such intersections $P'_J \in L(\mathcal{A}'_{St(a)})$ are not arbitrary and being in almost general position (3.5) implies certain implicit conditions on P'_J. For instance, while the triple intersections in Figure 2 in $\mathbb{A}^2$ provides arrangements in almost general position, a similar triple intersection in $\mathbb{A}^3$ does not. More precisely, there may exists hyperplanes arrangement in almost general position in $\mathbb{A}^d$ with more then d hyperplanes intersect at the same point. However, note also that, an intersection along a higher dimensional subspace for $|J| > d$ is prohibited by the condition (3.5): If there exists $P'_J = \mathbb{A}^k, k > 0$ for a subset $|J| > d$, then there must be a subset $K \subset J$ with $d \geq |K| > d - k$, such that, the corresponding subarrangement $(P_e \mid e \in K)$ of hyperplanes violate (3.5), i.e., P'_J can only be a single point as stated. □

The following stratification of the configuration space $\mathcal{M}^{St(a)}$ is a tautology:

Proposition 3.5. (1) *For any given degeneration type Θ, there is a quasi-projective subvariety $S_\Theta \subset \mathcal{M}^{St(a)}$ parameterizing the hyperplane arrangements in almost general position with the degeneration type Θ.*
(2) *The configuration space $\mathcal{M}^{St(a)}$ is stratified by these pairwise disjoint subvarities S_Θ.*

The rest of this paper examines certain geometric properties of the very particular case of $|St(a)| \leq d$, that will be used in §4 to prove that $\mathcal{M}^{St(a)}$ define objects in the category of mixed Tate motives in these cases.

3.4. **Forgetful morphism.** Consider the trivial fibration

$$
\begin{array}{c}
(\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)}\times\mathbb{A}^d \xrightarrow{\ \psi_b\ } (\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)\setminus\{b\}}\times\mathbb{A}^d \\
(\mathbf{z}^k,\mathbf{w}^k\mid k\in Ver(\kappa))\times\mathbf{w}^a \longmapsto (\mathbf{z}^k,\mathbf{w}^k\mid k\in Ver(\kappa)\setminus\{b\})\times\mathbf{w}^a
\end{array}
\tag{3.8}
$$

whose fibres are $\mathbb{A}^d\times\mathbb{A}^d=\{(\mathbf{z}^b,\mathbf{w}^b)\}$. The restriction of ψ_b to the configuration space $\mathcal{M}^{St(a)}$ of hyperplane arrangements in almost general position provides us *the forgetful morphism*

$$
\begin{array}{rcl}
\psi_b:\mathcal{M}^{St(a)} & \to & \mathcal{M}^{St(a)\setminus\{f\}}\\
(P_e\mid e\in St(a)) & \mapsto & (P_e\mid e\in St(a)\setminus\{f\})
\end{array}
\tag{3.9}
$$

where $\{f\}=St(a)\cap St(b)$, i.e., $\partial_\kappa(f)=(ab)$. The image $\psi_b(P_e\mid e\in St(a))$ of the hyperplane arrangements $(P_e\mid e\in St(a))$ is obtained by forgetting the hyperplane P_f.

3.4.1. *Fibres of the forgetful morphism when* $|St(a)|\le d$. We decompose the forgetful morphism by using the following projections

$$
\begin{array}{ccccc}
(\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)}\times\mathbb{A}^d & \xrightarrow{\rho_b} & (\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)\setminus\{b\}}\times\mathbb{A}^d\times\mathbb{A}^d & \xrightarrow{\phi_b} & (\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)\setminus\{b\}}\times\mathbb{A}^d\\
\uparrow & & \uparrow & & \uparrow\\
\mathcal{M}^{St(a)} & \longrightarrow & \mathcal{N}^{st(a)} & \longrightarrow & \mathcal{M}^{St(a)\setminus\{f\}}
\end{array}
\tag{3.10}
$$

Here, the morphism ρ_b forgets $\mathbf{z}^b$, that is in fact the product

$$
\rho_b:=\pi_\kappa\times\mathrm{id}:((\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)})\times\mathbb{A}^d\to((\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)\setminus\{b\}}\times\mathbb{A}^d)\times\mathbb{A}^d
$$

of the projection defined in (3.1) with the identity morphism of the last factor $\mathbb{A}^d=\{\mathbf{w}^a\}$. The morphism ϕ_b forgets $\mathbf{w}^b$. The restriction of the composition $\phi_b\circ\rho_b$ onto the configuration space $\mathcal{M}^{St(a)}\subset(\mathbb{A}^d\times\mathbb{A}^d)^{Ver(\kappa)}\times\mathbb{A}^d$ gives the forgetful morphism ψ_b defined in (3.9).

Step 1. Fibers of ϕ_b. Let $|Ver(\kappa)|\le d$. Consider a trivial bundle

$$
\mathcal{M}^{St(a)\setminus\{f\}}\times\mathbb{A}^d\to\mathcal{M}^{St(a)\setminus\{f\}}
$$

as the restriction of ϕ_b in (3.10).

For any $I\subset St(a)\setminus\{f\}$, there is a rank-$|I|$ subbundle $\mathcal{W}_I$ whose fibers over $(\mathbf{z}^k,\mathbf{w}^k\mid k\in St(a)\setminus\{f\})\times\mathbf{w}^a\in\mathcal{M}^{St(a)\setminus\{f\}}$ consists of the space $\mathbb{A}^{|I|}$ which is the vector space spanned by $(\mathbf{w}^a-\mathbf{w}^k)$ for $k\in I$, that are the normals to the hyperplanes P_e in $\mathcal{A}_{St(a)\setminus\{f\}}$.

Lemma 3.6. *The bundle $\mathcal{W}_I$ over the variety $\mathcal{M}^{St(a)\setminus\{f\}}$ is trivial for all $I\subset St(a)\setminus\{f\}$.*

Proof. Above description of $\mathcal{W}_I$ means precisely that the map

$$
\begin{array}{rcl}
\mathcal{M}^{St(a)\setminus\{f\}}\times\mathbb{K}^{|I|} & \to & \mathcal{W}_I\\
((P_e\mid e\in St(a)\setminus\{f\})\ ,\ (c_k\mid k\in I)) & \mapsto & ((P_e\mid e\in St(a)\setminus\{f\})\ ,\ (\sum_{k\in I}c_k(\mathbf{w}^a-\mathbf{w}^k)))
\end{array}
$$

is an isomorphism and provides the trivialization that is needed. □

The bundle $\mathcal{W}_I$ parameterizes the pairs $(\mathcal{A}_{St(a)\setminus\{f\}},\mathbf{w}^b-\mathbf{w}^a)$ where the normal $\mathbf{w}^b-\mathbf{w}^a$ of forgetten hyperplane P_f lies in the vector space spanned by the normal vectors $(\mathbf{w}^a-\mathbf{w}^k)$ for $k\in I$. Such P_f's should be in the complement of the configuration space due to (3.5). Therefore, we will be interested in its complement below.

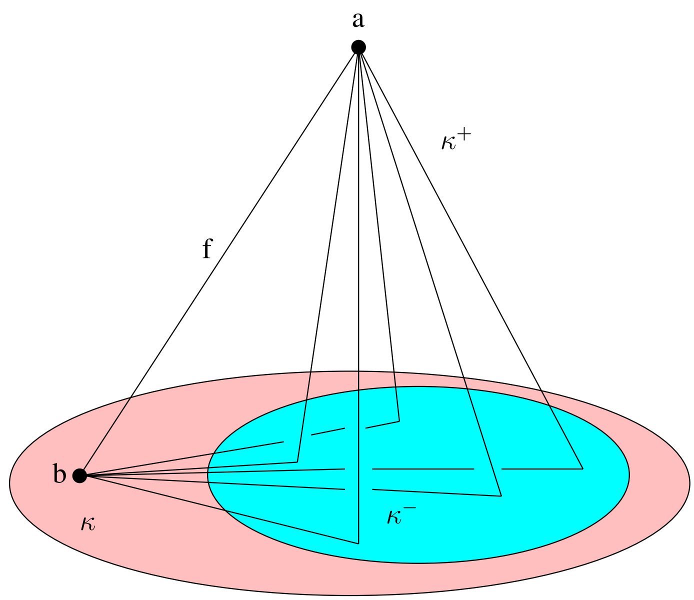

FIGURE 3. The complete graphs κ^+, κ, and κ^-.

Step 2. Fibers of ρ_b. The condition (3.5) implies that the image of $\mathcal{M}^{St(a)}$ under ρ_b is contained in the complement

$$\begin{aligned} \mathcal{N}^{St(a)} &:= (\mathcal{M}^{St(a)\setminus\{f\}} \times \mathbb{A}^d) \setminus \mathcal{W}^{St(a)} \\ &= \{(\mathcal{A}_{St(a)\setminus\{f\}}, \mathbf{w}^b - \mathbf{w}^a) \mid \mathbf{w}^b - \mathbf{w}^a \notin \langle \mathbf{w}^k - \mathbf{w}^a : k \in I \ \&\ |I| \leq d \rangle\} \end{aligned} \tag{3.11}$$

of the union $\mathcal{W}^{St(a)} := \bigcup_I \mathcal{W}_I$. However, it is not clear whether the image of this morphism covers $\mathcal{N}^{St(a)}$. In the following, we observe that the fibres of ρ_i are complements of hyperplane arrangements hence non-empty.

Lemma 3.7. *The fibre $\rho_i^{-1}(\mathbf{n})$ over a point $\mathbf{n} := (\mathcal{A}_{St(a)\setminus\{f\}}, \mathbf{w}^b - \mathbf{w}^a) \in \mathcal{N}^{St(a)}$ is the complement of a hyperplane arrangement*

$$\mathcal{B}_{St(b)} = (H_e \in \mathbb{P}^d \mid e \in (St(b) \setminus \{f\}) \subset Ver(\kappa))$$

where

$$H_e := \{\mathbf{z}^b \mid q^{bj} = (\mathbf{z}^b - \mathbf{z}^j) \cdot (\mathbf{w}^b - \mathbf{w}^j) = 0 \text{ and } (bj) = \partial_\kappa(e)\}.$$

Moreover, for any $I \subset St(b)$ (since we assumed $|I| \leq |St(b)| \leq d$), the subarrangements $\mathcal{B}_I = (H_e \mid e \in I)$ are in general position, i.e.,

$$H_I := \bigcap_{e \in I} H_e = \mathbb{A}^{d-|I|}.$$

Proof. This statement is just an iteration of the case (3) in Lemma 3.3 where κ, κ^- is a pair of complete graphs such that κ^- is obtained from obtained from κ by removing its vertex b and its star $St(b)$, see Figure 3.

All we need to make sure is that the elements $\mathbf{n} \in \mathcal{N}^{St(a)}$ is in the complement of the $Z_{\kappa^-} \times \mathbb{A}^d$ (which simply follows from the fact that $\in \mathcal{N}^{St(a)}$ fibres over $\mathcal{M}^{St(a)\setminus\{f\}}$ which is in the complement of $Z_{\kappa^-} \times \mathbb{A}^d$) and $\bigcup_{e \in St(b)} \{\mathbf{w}^i = \mathbf{w}^a\}$ (that is guaranteed as this locus lies inside $\mathcal{W}^{St(a)}$ (see, Step 1 above)). $\square$

Consider the trivial bundle $\mathcal{N}^{St(a)} \times \mathbb{A}^d \to \mathcal{N}^{St(a)}$ as the restriction of of ρ_b in (3.10). For any $I \subset St(a) \setminus \{f\}$, there is a rank-$|I|$ subbundle $\mathcal{R}_I$ whose fibers over $(\mathcal{A}_{St(a)\setminus\{f\}}, \mathbf{w}^b - \mathbf{w}^a)$ is the intersection $R_I := \bigcap_{e\in I} H_e = \mathbb{A}^{d-|I|}$ of the hyperplanes in Lemma 3.7.

Lemma 3.8. *Let $\mathcal{R}_I \to \mathcal{N}^{St(a)}$ are trivial for all $I \subset St(a) \setminus \{f\}$.*

The proof this Lemma is same as Lemma 3.6.

4. Motive of the Feynman quadric

In this section, we use the results from §3.3.1 and §3.4 to prove that the motive of configuration space $\mathcal{M}^{St(a)}$ is mixed Tate when $|St(a)| \leq d$. We then prove that the Feynman quadrics also give mixed Tate motives in corresponding cases via §3.2.1.

4.0.2. *Assumptions for the induction.* We prove our main theorem by induction on the number vertices of the complete graphs. We assume that the Feynman quadrics Z_κ define mixed Tate motives for all $|Ver(\kappa)| \leq n$. This assumption implies that the motives of configuration spaces $\mathcal{M}^{St(a)}$ are also mixed Tate for all such κ with $a \in Ver(\kappa)$ (for details, see Case (2) in the proof of Main Theorem). These assumptions are already verified for the initial step of the induction, i.e., 1-edge graph, in §2.

We will use the projections in §3.2.1 and §3.4.1 to examine the motive of the Feynman quadric Z_{κ^+} for a bigger graph κ^+.

4.1. **The motive of the configuration space $\mathcal{M}^{St(a)}$: The case of $St(a) \leq d$.**

In this particular case, we will use the forgetful morphism (3.9) and its detailed analysis in §3.4.1 to prove that the configuration spaces define objects in the category of mixed Tate motives.

In §3.4.1, we show that we need to consider the complement of certain stratified spaces to examine the motive of the configuration space $\mathcal{M}^{St(a)}$. The following Lemma will be useful for our purpose.

Lemma 4.1. *Let $\{X_i, i \in I\}$ be the set of irreducible components of a scheme X such that the motives $\mathfrak{m}^c(X_J)$ of the intersections*

$$X_J := \bigcap_{j\in J} X_j$$

are mixed Tate for all $J \subseteq I$. Then, the motive $\mathfrak{m}^c(X)$ is also mixed Tate.

Proof. For inclusions

$$\bigcup_{|J'|=k+1} X_{J'} \hookrightarrow \bigcup_{|J|=k} X_J,$$

we consider the following canonical distinguished triangle

$$\mathfrak{m}^c(\textstyle\bigcup_{|J'|=k+1} X_{J'}) \longrightarrow \mathfrak{m}^c(\bigcup_{|J|=k} X_J) \longrightarrow \mathfrak{m}^c(\bigcup_{|J|=k} X_J \setminus \bigcup_{J'\supset J} X_{J'}) \longrightarrow \mathfrak{m}^c(\bigcup_{|J'|=k+1} X_{J'})[1].$$

If we assume that the motive $\mathfrak{m}^c(\bigcup_{|J|=k} X_J \setminus \bigcup_{J'\supset J} X_{J'})$ of the complement is mixed Tate for all k, then we can claim that the motive of $X = \bigcup_i X_i$ is mixed Tate by induction on k: The case $k = |I|$ simply follows from the assumption that $\mathfrak{m}^c(X_I)$ is mixed Tate. As we assume that $\mathfrak{m}^c(\bigcup_{|J|=k} X_J \setminus \bigcup_{J'\supset J} X_{J'})$ is mixed Tate, and know that $\mathfrak{m}^c(\bigcup_{|J'|=k+1} X_{J'})$ is mixed Tate from the previous step, the motive $\bigcup_{|J|=k} X_J$ must be mixed Tate due to the above distinguished triangle. The final step $k = 1$ gives us the motive $\mathfrak{m}^c(\bigcup_i X_i)$ of X.

It remains to prove our assumption, that $\mathfrak{m}^c(\bigcup_{|J|=k} X_J \setminus \bigcup_{J'\supset J} X_{J'})$ is mixed Tate for all k. We simplify the problem by observing that

$$\mathfrak{m}^c(\bigcup_{|J|=k} X_J \setminus \bigcup_{J'\supset J} X_{J'}) = \bigoplus_{|J|=k} \mathfrak{m}^c(X_J \setminus \bigcup_{J'\supset J} X_{J'}).$$

Notice that $\mathfrak{m}^c(\bigcup_{J'\supset J} X_{J'})$ is mixed Tate due to induction hypothesis. Therefore, we can employ the distinguished triangle for the inclusion $\bigcup_{J'\supset J} X_{J'} \hookrightarrow X_J$

$$\mathfrak{m}^c(\bigcup_{J'\supset J} X_{J'}) \longrightarrow \mathfrak{m}^c(X_J) \longrightarrow \mathfrak{m}^c(X_J \setminus \bigcup_{J'\supset J} X_{J'}) \longrightarrow \mathfrak{m}^c(\bigcup_{J'} X_{J'})[1].$$

to complete the proof. □

Lemma 4.2. *The motive of $\mathcal{N}^{St(a)}$ lies in the category of mixed Tate motives.*

Proof. From §1.2.3 of [4] we know that the motive $\mathfrak{m}(X \times \mathbb{A}^k) = \mathfrak{m}(X)[-k](2k)$ is obtained from $\mathfrak{m}(X)$ by Tate twists and shifts, hence the bundles $\mathcal{W}_I$ discussed in Lemma 3.6 are in the subcategory of mixed Tate motives inside the Voevodsky's category. Lemma 4.1 implies that their union $\mathcal{W}^{St(a)}$ also must be a mixed Tate motives. Therefore, we conclude that its complement $\mathcal{N}^{St(a)}$ inside a mixed Tate motive $\mathcal{M}^{St(a)\setminus\{f\}} \times \mathbb{A}^d$ (due to the induction assumption) is also a mixed Tate motive. □

The exact same line of arguments applies to the motive of the configuration space $\mathcal{M}^{St(a)}$:

Proposition 4.3. *If $|St(a)| \leq d$, then the configuration space $\mathcal{M}^{St(a)}$ defines an object in the category of mixed Tate motives.*

Proof. Due to Lemma 3.7, the configuration space $\mathcal{M}^{St(a)}$ is the complement of the union $\bigcup \mathcal{R}_I$ inside $\mathcal{N}^{St(a)} \times \mathbb{A}^d$. According to Lemmata 3.8 and 4.2, the strata $\mathcal{R}_I$ are trivial $\mathbb{A}^{|I|}$-fibrations over a mixed Tate base $\mathcal{N}^{St(a)}$. Hence, each stratum $\mathcal{R}_I$ defines a mixed Tate motive. Therefore, the motive of their union $\bigcup \mathcal{H}_I$ is also mixed Tate due to Lemma 4.1. The ambient space $\mathcal{N}^{St(a)} \times \mathbb{P}^d$ is mixed Tate due to Lemma 4.2, so is its complement $\mathcal{M}^{St(a)}$. □

4.2. **The motive of the Feynman quadric.** Let κ^+ be a complete graph with $|Ver(\kappa)| = n+1 \leq d+1$.

Main Theorem: *The Feynman quadric Z_{κ^+} defines an object in the category of mixed Tate motives.*

Proof. We use the projection

$$\pi_{\kappa^+} : Z_{\kappa^+} \quad \rightarrow \quad (\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa)} \times \mathbb{A}^d$$

which was studied in Lemma 3.3. Remember that we assume in §4.0.2 that $\mathfrak{m}^c(Z_\kappa)$ is mixed Tate motives to implement the induction on the number of vertices.

According to Lemma 3.3, the Feynman quadric Z_{κ^+} is a union of three pairwise disjoint pieces:

(1) A trivial fibration $\mathcal{A} \to Z_\kappa \times \mathbb{A}^d$ with fibres $\mathbb{A}^d$,
(2) A trivial fibration $\mathcal{B} \to \bigcup_{e_i \in St(a)} \{\mathbf{w}^a = \mathbf{w}^i \mid \partial_{\kappa^+}(e_i) = (ai)\}$ with fibres $\mathbb{A}^d$,
(3) The "universal" family over the configuration space $\mathcal{M}^{St(a)}$ of hyperplane arrangements.

Case (1). It can be shown that the motive $\mathfrak{m}^c(\mathcal{A})$ is mixed Tate via §1.2.3 of [4].

Case (2). To be able use the same arguments for trivial fibrations, one needs to check the motive of the mutual intersections of the diagonals $\{\mathbf{w}^a = \mathbf{w}^i\}$ and their intersections with $Z_\kappa \times \mathbb{A}^d$ in $(\mathbb{A}^d \times \mathbb{A}^d)^{Ver(\kappa)} \times \mathbb{A}^d$. However, the intersections $\{\mathbf{w}^a = \mathbf{w}^i\} \cap \{\mathbf{w}^a = \mathbf{w}^j\} = \{\mathbf{w}^a = \mathbf{w}^i = \mathbf{w}^i\} \subset \mathcal{H}^{ij} \times A^d$ are contained in $Z_\kappa \times \mathbb{A}^d$. Therefore, we only need to the check the motive of pairwise disjoint subspaces $\{\mathbf{w}^a = \mathbf{w}^i\} \setminus ((Z_\kappa \times \mathbb{A}^d) \cap \{\mathbf{w}^a = \mathbf{w}^i\})$.

Note that $(Z_\kappa \times \mathbb{A}^d) \cap \{\mathbf{w}^a = \mathbf{w}^i\}$ is isomorphic to Z_κ as this intersection as be given as the graph of the map

$$Z_\kappa \hookrightarrow Z_\kappa \times \mathbb{A}^d$$

$$(\mathbf{z}^m; \mathbf{w}^m \mid m \in Ver(\kappa)) \longmapsto (\mathbf{z}^m; \mathbf{w}^m \mid m \in Ver(\kappa)) \times \mathbf{w}^i,$$

which shows that $\mathfrak{m}^c((Z_\kappa \times \mathbb{A}^d) \cap \{\mathbf{w}^a = \mathbf{w}^i\})$ and, therefore, the motives $\mathfrak{m}^c(\{\mathbf{w}^a = \mathbf{w}^i\} \setminus ((Z_\kappa \times \mathbb{A}^d) \cap \{\mathbf{w}^a = \mathbf{w}^i\}))$ of the compliments are mixed Tate motives, so are the locally trivial $\mathbb{A}^d$-bundles $\mathcal{B}$ over them.

Case(3). The remaining part of Z_{κ^+} after Case (1) and (2) is the subspace $\pi_{\kappa^+}^{-1}(\mathcal{M}^{St(a)}) \cap Z_{\kappa^+}$. This subspace is stratified by the trivial fibration $\mathcal{R}_I$ as in Lemma 3.8. Following the same steps in Proposition 4.3, we show that the remaining part, the universal family $\pi_{\kappa^+}^{-1}(\mathcal{M}^{St(a)}) \cap Z_{\kappa^+}$ over the configuration space $\mathcal{M}^{St(a)}$ also defines an object in the category of mixed Tate motives.

Finally, we patch these three pieces together via the distinguished triangles and show that the Feynman quadrics $\bigcup_{e\in Edg(\kappa^+)} \mathcal{H}^e$ indeed defines mixed Tate motives as the same was true for one-edge graph, see Proposition 2.9 . □

5. Corollaries and ramifications

5.1. **Graph hypersurfaces vs Feynman quadrics.** The *graph polynomial* of Γ is given as

$$\Psi_\Gamma := \sum_T \prod_{e \notin T} \alpha_e$$

where α_e's are the variables associated to the edges of Γ and the summation runs over the set of spanning trees $T \subset \Gamma$. We note that the graph hypersurfaces $X_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{A}^{Edg(\Gamma)}$ do not depend on the dimension of the spacetime, hence the dimension of the corresponding momentum space. On the other hand, we can choose the dimension high enough so that the residues of the form (1.6) for can be interpreted in terms of the periods of a mixed Tate motive, the motive of the Feynman quadric Z_{κ_n}.

The contrast between our main results and the results on graph hypersurface (see [9, 10, 16] for counter examples and [24] for a general detailed account on the subject) draws our attention to two main unknown knowns:

5.1.1. *Transferring between the position and the momentum space is not as direct as hoped.* The Fourier transform provides an isomorphism between the state spaces (i.e., the spaces of square integrable functions) in the momentum and position spaces. However, this isomorphism utilizes the cut-off function, that is likely to prohibit us from transferring the algebraic therefore motivic structures directly.

5.1.2. *The dependence on regularization scheme.* The obstacle due to (the transcendental nature of) the Fourier transform however can be overcomed by directly considering the Feynman quadric in momentum space setting. Our recent computation in [15] shows that momentum space Feynman quadrics also define mixed Tate motives. This concludes that the motives arising in momentum space depends on the choice of the regularization procedures.

The parametric formulation of Feynman integrals uses an integral presentation of propagators and the Fubini theorem to change the orders of (divergent) integrals. These results indicate that the integral presentation of propagator and introducing Schwinger parameters are transcendental in some sense as they replace mixed Tate motives, the Feynman quadrics, with non-mixed Tate ones, the graph hypersurfaces.

The potential problems of changing the orders of divergent integrals is implicitly noted in [5]. Bloch, Esnault and Kreimer considered the locus of the divergence of the momentum space propagator as the zeros of the smooth quadric

$$\{x_1^2 + \cdots + x_d^2 = 0\} \subset \mathbb{A}^d.$$

The cohomology of the union of these momentum space quadrics as well as their periods have been consider in §10 of [5] in case of logarithmically divergent primitive graphs, i.e., where the period integral is convergent and Schwinger trick cannot cause above mentioned problems.

In addition to these, there is a recent observation on mixed Tate motives which may also play a role in Feynman integrals:

5.1.3. *Tate motives are more elusive than one anticipates.* A very recent paper [6] observes an unexpected property of the mixed Tate motives: The class of the affine line is a zero divisor in the Grothendieck ring of algebraic varieties. This result is counter intuitive and indicates that the mixed Tate motives are in fact more sophisticated than they are generally depicted.

5.2. **Feynman quadrics for general Feynman graphs.** The main construction of this paper fails at a number places for graphs other than the complete graphs: Firstly, if we remove the restriction $|Ver(\Gamma)| \leq d+1$, we need to consider hyperplane arrangements in almost general position, see §3.3.1. For more general Feynman graphs, the arrangements having parallel hyperplanes appear in the picture as the corresponding case in Lemma 3.3 uses the fact that each pair of vertices connected by an edge to establish the non-parallelity. The same statements are also true for graph with external structures.

These cases require more elaborate study of the motives of the stratification of the respective configuration spaces. We plan to write these details in a separate paper. The result remain the same however the motives of the Feynman quadrics in such cases fail to be unramified over $Spec(\mathbb{Z})$ due to trivial reasons: The arrangements containing more than $(d+1)$ hyperplanes almost never unramified, and we need to consider the spaces of them.

5.3. **The signature doesn't change anything.** The constructions in this paper focuses on Euclidean case, the metric with signature $(+,+,\dots,+)$. However, if we change the signature, we only need to replace our simple Feynman quadric (2.1) with

$$q'(\mathbf{z},\mathbf{w}) = z_1 \cdot w_1 + \cdots + z_k \cdot w_k - z_{k+1} \cdot w_{k+1} - \cdots - z_d \cdot w_d = 0.$$

The rest of the statements and the proofs remain the same.

5.4. **Other geometries and motives for position space Feynman integrals.** While the main problem seems well defined at the first glance, it hides an ambiguity between the lines. As a period, the integral (1.1) must be an integral over a semialgebraic set. However, as defined in (1.2), there is no canonical choice among the complexifications. In fact, there any infinitely many complexifications of the spacetime X such that restriction of the complexified propagator $G(\mathbf{x})$ to its real part would provide the propagator on X.

As a simplest example, one can start with any given complexification of the spacetime X and blow it up along subvarities having no real points. Such blowups do not change the real locus of $X^{|Ver(\Gamma)|}$ and the propagator, therefore, provide different complexifications. However, there are more interesting cases in which the complexification remains the same. Here is a sample of real structures on $\mathbb{A}^{2d}(\mathbb{C})$ that each lead to quite different loci of divergences despite that their restriction to real locus $\mathbb{A}^{2d}(\mathbb{R})$ are the same.

	Real structure on $\mathbb{A}^{2d}(\mathbb{C})$	Propagator	Locus of divergence
(I)	$(z_1,\dots,z_{2d}) \mapsto (\bar{z}_1,\dots,\bar{z}_{2d})$	$\frac{1}{(\sum_{i=1}^d \epsilon_i (z_i)^2)^{d-1}}$	$\left\{\sum_{i=1}^d \epsilon_i (z_i)^2 = 0\right\}$,
(II)	$(z_1,\dots,z_{2d}) \mapsto (\bar{z}_1,\dots,\bar{z}_{2d})$	$\frac{1}{(\sum_{i=1}^d \lvert z_i\rvert^2)^{d-1}}$	$\{z_1 = \cdots = z_{2d} = 0\}$,
(III)	$(z_1,\dots,z_{2d}) \mapsto (\bar{z}_{d+1},\dots,\bar{z}_{2d},\bar{z}_1,\dots,\bar{z}_d)$	$\frac{1}{(\sum_{i=1}^d \epsilon_i (z_i \cdot z_{d+i}))^{d-1}}$	$\left\{\sum_{i=1}^d \epsilon_i (z_i \cdot z_{d+i}) = 0\right\}$.

where $(\epsilon_i = \pm 1)$ is the signature of the metric.

The loci of divergences are all algebraic varieties each having distinct geometric properties: While the loci of divergence in (I) and (II) are smooth, the locus of divergence in (III) has a singularity at the origin. In the case (I) and (III), they are hypersurfaces however it is a codimension d subvariety in the case of (II). Finally, being mixed Tate motive is by definition in the case of (II), the same property is quite nontrivial in the case (III) as we have seen in this paper.

5.4.1. *Feynman integrals as configuration space integrals.* In a series of paper [12, 13, 14], the Feynman integrals are studied as the homological pairings in the configuration spaces of points defined via the propagator in (II). It is quite easy to observe that these configuration spaces of points can be nicely stratified and desingularized, and they all define mixed Tate motives. However, the propagator defined in (II) is a real valued distribution and is not algebraic, i.e., the corresponding Feynman integrals cannot manifest themselves as periods directly.

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