Representation of Gaussian Isotropic Spin Random Fields

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Abstract

We develop a technique for the construction of random fields on algebraic structures. We deal with two general situations: random fields on homogeneous spaces of a compact group and in the spin line bundles of the 2-sphere. In particular, every complex Gaussian isotropic spin random field can be represented in this way. Our construction extends P. Lévy's original idea for the spherical Brownian motion.

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1 Introduction

In recent years the investigation of random fields on algebraic structures has received a renewed interest from a theoretical point of view ([17]), but mainly motivated by applications concerning the modeling of the Cosmic Microwave Background data (see [1], [7], [13] e.g. and also the book [14]). Actually one of the features of this radiation, the *temperature*, is well modeled as a single realization of a random field on the sphere $\mathbb{S}^2$.

Moreover thanks to the ESA satellite Planck mission, new data concerning the *polarization* of the CMB will soon be available and the modeling of this quantity has led quite naturally to the investigation of spin random fields on $\mathbb{S}^2$, a subject that has already received much attention (see [8],[10], [12] and again [14] chapter 12 e.g.).

The object of this paper is the investigation of Gaussian isotropic random fields or, more precisely random sections, in the homogeneous line bundles of $\mathbb{S}^2$. In this direction we first investigate the simpler situation of random fields on the homogeneous space $\mathscr{X} = G/K$ of a compact group G. Starting from P. Lévy's construction of his "mouvement brownien fonction d'un point de la sphère de Riemann" in [11], we prove that to every square integrable bi-K-invariant function $f : G \to \mathbb{R}$ a real Gaussian isotropic random field on $\mathscr{X}$ can be associated and also that every real Gaussian isotropic random field on $\mathscr{X}$ can be obtained in this way.

Then, turning to our main object, we prove first that every random section of a homogeneous vector bundle on $\mathscr{X}$ is actually "equivalent" to a random field on the group G, enjoying a specific pathwise invariance property. This fact is of great importance as it reduces the investigation of random sections of the homogeneous vector bundle to a much simpler situation and is our key tool for the results that follow.

We are then able to give a method for constructing Gaussian isotropic random sections of the homogeneous line bundles of $\mathbb{S}^2$ and prove that every complex Gaussian isotropic random section can actually be obtained in this way. More precisely given $s \in \mathbb{Z}$, we prove that to every function $f : SO(3) \to \mathbb{C}$ which is bi-s-associated, i.e. that transforms under both the right and left action of the isotropy group $K \simeq SO(2)$ according to its s-th linear character, one can associate a Gaussian

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isotropic random section of the s-homogeneous line bundle and also that every complex Gaussian isotropic random section is represented in this way. In some sense this extends the representation result for Gaussian isotropic random fields on homogeneous spaces described above: a bi-K-invariant function being associated to the 0-th character and a random field on $\mathbb{S}^2$ being also a random section of the 0-homogeneous line bundle.

In [11] P. Lévy proves the existence on the spheres $\mathbb{S}^m$, $m \geq 1$, of a Gaussian random field T such that $T_x - T_y$ is normally distributed with variance $d(x, y)$, d denoting the distance on $\mathbb{S}^m$. The existence of such a random field on a more general Riemannian manifold has been the object of a certain number of papers ([2], [6], [19] e.g.). We have given some thought about what a P. Lévy random section should be. We think that our treatment should be very useful in this direction but we have to leave this as an open question.

The plan is as follows. In §2 we recall general facts concerning Fourier expansions on the homogeneous space $\mathscr{X}$ of a compact group G. In §3 we introduce the topic of isotropic random fields on $\mathscr{X}$ and their relationship with positive definite functions enjoying certain invariance properties. These facts are the basis for the representation results for real Gaussian isotropic random fields on $\mathscr{X}$ which are obtained in §4.

In §5 we introduce the subject of random sections of homogeneous vector bundles on $\mathscr{X}$ and in particular in §6 we consider the case $\mathscr{X} = \mathbb{S}^2$. We were much inspired here by the approach of [12], but the introduction of pullback random fields considerably simplifies the understanding of the notions that are involved.

In §7 we extend the construction of §4 as described above and prove the representation result for complex Gaussian isotropic random sections of the homogeneous line bundles of $\mathbb{S}^2$. When dealing with such random sections different approaches are available in the literature. They are equivalent but we did not find a formal proof of this equivalence. So §8 is devoted to the comparison with other constructions ([8], [10], [12], [16]). In particular we show that the construction of spin random fields on $\mathbb{S}^2$ of [8] is actually equivalent to ours, the difference being that their point of view is based on an accurate description of local charts and transition functions, instead of our more global perspective.

2 Fourier expansions

Throughout this paper $\mathscr{X}$ denotes the (compact) homogeneous space of a topological *compact* group G. Therefore G acts transitively on $\mathscr{X}$ with an action $x \mapsto gx, g \in G$. $\mathscr{B}(\mathscr{X})$, $\mathscr{B}(G)$ stand for the Borel σ-fields of $\mathscr{X}$ and G respectively. We shall denote dg the Haar measure of G and dx the corresponding G-invariant measure on $\mathscr{X}$ and assume, unless explicitly stated, that both these measures have total mass equal to 1. Actually at some places (mainly in §6, §7 and §8) we consider for $\mathbb{S}^2$ the total mass 4π in order to be consistent with the existing literature. We shall write $L^2(G)$ for $L^2(G, dg)$ and similarly $L^2(\mathscr{X})$ for $L^2(\mathscr{X}, dx)$. Unless otherwise stated the L^2-spaces are spaces of *complex valued* square integrable functions.

We fix once forever a point $x_0 \in \mathscr{X}$ and denote K the isotropy group of x_0, i.e. the subgroup of G of the elements g such that $gx_0 = x_0$, so that $\mathscr{X} \cong G/K$. In the case $G = SO(3)$, $\mathscr{X} = \mathbb{S}^2$, x_0 will be the north pole, as usual, and $K \cong SO(2)$.

Given a function $f : \mathscr{X} \to \mathbb{C}$ we define its *pullback* as

$$\widetilde{f}(g) := f(gx_0) \tag{2.1}$$

which is a $G \to \mathbb{C}$ right-K-invariant function, i.e. constant on left cosets of K. By the integration rule of image measures it holds

$$\int_{\mathscr{X}} f(x)\, dx = \int_G \widetilde{f}(g)\, dg\ , \tag{2.2}$$

whenever one of the integrals has sense.

We denote L the *left regular representation* of G on $L^2(G)$ given by $L_g f(h) := f(g^{-1}h)$ for $g \in G$ and $f \in L^2(G)$, and $\widehat{G}$ the *dual* of G, i.e., the set of the equivalence classes of irreducible unitary representations of G. As G is assumed to be compact, $\widehat{G}$ is at most countable.

For every $\sigma \in \widehat{G}$ let (D^σ, H_σ) be a representative element where the (unitary) operator D^σ acts irreducibly on the complex (finite dimensional) Hilbert space H_σ. We denote $\langle \cdot, \cdot \rangle$ the inner product of H_σ and $\dim \sigma = \dim H_\sigma$.

Given $f \in L^2(G)$, for every $\sigma \in \widehat{G}$ we define

$$\widehat{f}(\sigma) := \sqrt{\dim \sigma} \int_G f(g) D^\sigma(g^{-1})\, dg \tag{2.3}$$

which is a linear endomorphism of H_σ. It is immediate that, denoting $*$ the convolution on G,

$$\widehat{f_1 * f_2}(\sigma) = \frac{1}{\sqrt{\dim \sigma}}\, \widehat{f}_2(\sigma) \widehat{f}_1(\sigma)\ . \tag{2.4}$$

The Peter-Weyl Theorem (see [14] or [18] e.g.) can be stated as

$$f(g) = \sum_{\sigma \in \widehat{G}} \sqrt{\dim \sigma}\, \mathrm{tr}\big(\widehat{f}(\sigma) D^\sigma(g)\big)\ , \tag{2.5}$$

the convergence of the series taking place in $L^2(G)$. Fix any orthonormal basis $v_1, v_2, \dots, v_{\dim \sigma}$ of H_σ and denote $D^\sigma_{ij}(g) := \langle D^\sigma(g) v_j, v_i \rangle$ the (i,j)-th coefficient of the matrix representation of $D^\sigma(g)$ with respect to this basis, then the matrix representation of $\widehat{f}(\sigma)$ has entries

$$\begin{aligned} \widehat{f}(\sigma)_{i,j} = \langle \widehat{f}(\sigma) v_j, v_i \rangle = \sqrt{\dim \sigma} \int_G f(g) \langle D^\sigma(g^{-1}) v_j, v_i \rangle\, dg = \\ = \sqrt{\dim \sigma} \int_G f(g) D^\sigma_{i,j}(g^{-1})\, dg = \sqrt{\dim \sigma} \int_G f(g) \overline{D^\sigma_{j,i}(g)}\, dg \end{aligned}$$

and the Peter-Weyl Theorem (2.5) becomes

$$f(g) = \sum_{\sigma \in \widehat{G}} \sqrt{\dim \sigma} \sum_{i,j=1}^{\dim \sigma} \widehat{f}(\sigma)_{j,i} D^\sigma_{i,j}(g)\ , \tag{2.6}$$

the above series converging in $L^2(G)$. Let $L^2_\sigma(G) \subset L^2(G)$ be the isotypical subspace of $\sigma \in \widehat{G}$, i.e. the subspace generated by the functions $D^\sigma_{i,j}$; it is a G-module that can be decomposed into the orthogonal direct sum of $\dim \sigma$ irreducible and equivalent G-modules $(L^2_{\sigma,j}(G))_{j=1,\dots,\dim \sigma}$ where each $L^2_{\sigma,j}(G)$ is spanned by the functions $D^\sigma_{i,j}$ for $i = 1, \dots, \dim \sigma$, loosely speaking by the j-th column of the matrix D^σ.

If we denote

$$f^\sigma(g) = \sqrt{\dim \sigma}\, \mathrm{tr}(\widehat{f}(\sigma) D^\sigma(g)) \tag{2.7}$$

the component (i.e. the projection) of f in $L^2_\sigma(G)$, then (2.5) and (2.6) become

$$f(g) = \sum_{\sigma \in \widehat{G}} f^\sigma(g) \tag{2.8}$$

or equivalently the Peter-Weyl Theorem can be stated as

$$L^2(G) = \bigoplus_{\sigma \in \widehat{G}} \overset{\dim \sigma}{\underset{j=1}{\oplus}} L^2_{\sigma,j}(G)\ , \tag{2.9}$$

the direct sums being orthogonal.

The Fourier expansion of functions $f \in L^2(\mathscr{X})$ can be obtained easily just remarking that, thanks to (2.2), their pullbacks $\widetilde{f}$ belong to $L^2(G)$ and form a G-invariant closed subspace of $L^2(G)$. We can therefore associate to $f \in L^2(\mathscr{X})$ the family of operators $\big(\widehat{\widetilde{f}}(\sigma)\big)_{\sigma \in \widehat{G}}$. Let $H_{\sigma,0}$ denote the subspace of H_σ (possibly reduced to $\{0\}$) formed by the vectors that remain fixed under the action of K, i.e. for every $k \in K, v \in H_{\sigma,0}$ $D^\sigma(k)v = v$. Right-K-invariance implies that the image of $\widehat{\widetilde{f}}(\sigma)$ is contained in $H_{\sigma,0}$:

$$\begin{aligned}\widehat{\widetilde{f}}(\sigma) &= \sqrt{\dim\sigma}\int_G \widetilde{f}(g)D^\sigma(g^{-1})\,dg = \sqrt{\dim\sigma}\int_G \widetilde{f}(gk)D^\sigma(g^{-1})\,dg = \\ &= \sqrt{\dim\sigma}\int_G \widetilde{f}(h)D^\sigma(kh^{-1})\,dh = D^\sigma(k)\sqrt{\dim\sigma}\int_G \widetilde{f}(h)D^\sigma(h^{-1})\,dh = D^\sigma(k)\widehat{\widetilde{f}}(\sigma)\ .\end{aligned} \tag{2.10}$$

Let us denote by $P_{\sigma,0}$ the projection of H_σ onto $H_{\sigma,0}$, so that $\widehat{\widetilde{f}}(\sigma) = P_{\sigma,0}\widehat{\widetilde{f}}(\sigma)$, and $\widehat{G}_0$ the set of irreducible unitary representations of G whose restriction to K contains the trivial representation. If $\sigma \in \widehat{G}_0$ let us consider a basis of H_σ such that the elements $\{v_{p+1},\dots,v_{\dim\sigma}\}$, for some integer $p \ge 0$, span $H_{\sigma,0}$. Then the first p rows of the representative matrix of $\widehat{\widetilde{f}}(\sigma)$ in this basis contain only zeros. Actually, by (2.10) and $P_{\sigma,0}$ being self-adjoint, for $i \le p$

$$\widehat{\widetilde{f}}_{i,j}(\sigma) = \langle \widehat{\widetilde{f}}(\sigma)v_j, v_i\rangle = \langle P_{\sigma,0}\widehat{\widetilde{f}}(\sigma)v_j, v_i\rangle = \langle \widehat{\widetilde{f}}(\sigma)v_j, P_{\sigma,0}v_i\rangle = 0\ .$$

Recall that a function $f \in L^2(G)$ is *bi-K-invariant* if for every $g \in G, k_1, k_2 \in K$

$$f(k_1 g k_2) = f(g)\ . \tag{2.11}$$

Condition (2.11) immediately entails that, for every $k_1, k_2 \in K$, $\sigma \in \widehat{G}$,

$$\widehat{f}(\sigma) = D^\sigma(k_1)\widehat{f}(\sigma)D^\sigma(k_2)$$

and it is immediate that a function $f \in L^2(G)$ is bi-K-invariant if and only if for every $\sigma \in \hat{G}$

$$\widehat{f}(\sigma) = P_{\sigma,0}\widehat{f}(\sigma)P_{\sigma,0}\ . \tag{2.12}$$

Now we briefly focus on the case $\mathscr{X} = \mathbb{S}^2$ under the action of $G = SO(3)$ recalling basic facts we will need in the sequel (see [5], [14] e.g. for further details). The isotropy group $K \cong SO(2)$ of the north pole is abelian, therefore its unitary irreducible representations are unitarily equivalent to its linear characters which we shall denote $\chi_s, s \in \mathbb{Z}$, throughout the whole paper.

A complete set of unitary irreducible matrix representations of $SO(3)$ is given by the so-called Wigner's D matrices $\{D^\ell, \ell \ge 0\}$, where each $D^\ell(g)$ has dimension $(2\ell+1)\times(2\ell+1)$ and acts on a representative space that we shall denote H_ℓ. The restriction to K of each D^ℓ being unitarily equivalent to the direct sum of the representations χ_m, $m = -\ell,\dots,\ell$, we can suppose $v_{-\ell}, v_{-\ell+1},\dots,v_\ell$ to be an orthonormal basis for H_ℓ such that for every $m : |m| \le \ell$

$$D^\ell(k)v_m = \chi_m(k)v_m\ , \qquad k \in K\ . \tag{2.13}$$

Let $D^\ell_{m,n} = \langle D^\ell v_n, v_m\rangle$ be the (m,n)-th entry of D^ℓ with respect to the basis fixed above. It follows from (2.13) that for every $g \in SO(3), k_1, k_2 \in K$,

$$D^\ell_{m,n}(k_1 g k_2) = \chi_m(k_1)D^\ell_{m,n}(g)\chi_n(k_2)\ . \tag{2.14}$$

The functions on $SO(3)$ $\{g \to D^\ell_{m,n}(g),\ \ell \ge 0, m,n = -\ell,\dots,\ell\}$ are usually called Wigner's D functions.

Given $f \in L^2(SO(3))$, its ℓ-th Fourier coefficient defined in (2.3) is

$$\widehat{f}(\ell) := \sqrt{2\ell+1}\int_{SO(3)} f(g)D^\ell(g^{-1})\,dg \tag{2.15}$$

and (2.6) becomes

$$f(g) = \sum_{\ell\ge0}\sqrt{2\ell+1}\sum_{m.n=-\ell}^{\ell}\widehat{f}(\ell)_{n,m}D^\ell_{m,n}(g)\ . \tag{2.16}$$

If $\widetilde{f}$ is the pullback of $f \in L^2(\mathbb{S}^2)$, (2.13) entails that for every $\ell \geq 0$

$$\widehat{\widetilde{f}}(\ell)_{n,m} \neq 0 \iff n = 0 \ .$$

Moreover if f is left-K-invariant, then

$$\widehat{\widetilde{f}}(\ell)_{n,m} \neq 0 \iff n, m = 0 \ .$$

In other words, an orthogonal basis for the space of the square integrable right-K-invariant functions on $SO(3)$ is given by the central columns of the matrices D^ℓ, $\ell \geq 0$. Furthermore the subspace of the bi-K-invariant functions is spanned by the central functions $D^\ell_{0,0}(\cdot)$, $\ell \geq 0$, which are also *real-valued*.

The important role of the other columns of Wigner's D matrices will appear in §6.

For every $\ell \geq 0, m = -\ell \dots, \ell$, let

$$Y_{\ell,m}(x) := \sqrt{\frac{2\ell+1}{4\pi}}\, \overline{D^\ell_{m,0}(g_x)} \ , \qquad x \in \mathbb{S}^2 \ , \tag{2.17}$$

where g_x is any rotation mapping the north pole of the sphere to x. This is a good definition thanks to the invariance of each $D^\ell_{m,0}(\cdot)$ under the right action of K. The functions in (2.17) form an orthonormal basis of the space $L^2(\mathbb{S}^2)$ considering the sphere with total mass equal to 4π. They are usually known as *spherical harmonics*.

By the previous discussion the Fourier expansion of a left-K-invariant function $f \in L^2(\mathbb{S}^2)$ is

$$f = \sum_\ell \beta_\ell Y_{\ell,0} \ , \tag{2.18}$$

where $\beta_\ell := \int_{\mathbb{S}^2} f(x) Y_{\ell,0}(x)\, dx$. The functions $Y_{\ell,0}$, $\ell \geq 0$ are called *central spherical harmonics*.

3 Isotropic random fields and positive definite functions

Let $T = (T_x)_{x\in\mathscr{X}}$ be a random field on the G-homogeneous space $\mathscr{X}$, i.e. a collection of complex-valued r.v.'s defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, such that the map $(\omega, x) \mapsto T_x(\omega)$ is $\mathscr{F} \otimes \mathscr{B}(\mathscr{X})$-measurable.

T is said to be *a.s. continuous* if the functions $\mathscr{X} \ni x \mapsto T_x$ are continuous a.s. T is said to be *second order* if $T_x \in L^2(P)$ for every $x \in \mathscr{X}$. T is *a.s. square integrable* if

$$\int_{\mathscr{X}} |T_x|^2\, dx < +\infty, \qquad a.s. \ , \tag{3.1}$$

i.e. if the function $x \mapsto T_x$ belongs to $L^2(\mathscr{X})$ a.s. In this case, for every $f \in L^2(\mathscr{X})$, we can consider the integral

$$T(f) := \int_{\mathscr{X}} T_x \overline{f(x)}\, dx$$

which defines a r.v. on $(\Omega, \mathscr{F}, \mathbb{P})$. For every $g \in G$ let T^g be the *rotated field* defined as

$$T^g_x := T_{gx}, \qquad x \in \mathscr{X} \ .$$

Definition 3.1. *An a.s. square integrable random field T on the homogeneous space $\mathscr{X}$ is said to be (strict sense) G-*invariant *or* isotropic *if and only if the joint laws of*

$$(T(f_1), \dots, T(f_m)) \qquad \text{and} \qquad (T(L_g f_1), \dots, T(L_g f_m)) = (T^g(f_1), \dots, T^g(f_m)) \tag{3.2}$$

coincide for every $g \in G$ and $f_1, f_2, \dots, f_m \in L^2(\mathscr{X})$.

This definition is somehow different from the one usually considered in the literature, where the requirement is the equality of the finite dimensional distributions, i.e. that the random vectors

$$(T_{x_1},\dots,T_{x_m}) \qquad \text{and} \qquad (T_{gx_1},\dots,T_{gx_m}) \tag{3.3}$$

have the same law for every choice of $g\in G$ and $x_1,\dots,x_m\in\mathscr{X}$. Remark that (3.2) implies (3.3) (see [15]) and that, conversely, by standard approximation arguments (3.3) implies (3.2) if T is continuous.

To every a.s. square integrable random field T on the group G we can associate the set of operator-valued r.v.'s $(\widehat{T}(\sigma))_{\sigma\in\widehat{G}}$ defined "pathwise" as

$$\widehat{T}(\sigma)=\sqrt{\dim\sigma}\int_G T_g D^\sigma(g^{-1})\,dg\ . \tag{3.4}$$

From (2.5) therefore

$$T_g=\sum_{\sigma\in\widehat{G}}\sqrt{\dim\sigma}\,\mathrm{tr}(\widehat{T}(\sigma)D^\sigma(g))\ , \tag{3.5}$$

where the convergence takes place in $L^2(G)$ a.s.

The following statement points out a remarkable property that is enjoyed by every isotropic and second order random field (see [15]).

Proposition 3.2. *Every a.s. square integrable, isotropic and second order random field T on the homogeneous space $\mathscr{X}$ of the compact group G is* mean square continuous, *i.e.*

$$\lim_{y\to x}\mathbb{E}[|T_y-T_x|^2]=0\ . \tag{3.6}$$

Assume that T is (i) a.s. square integrable, (ii) isotropic and (iii) second order. Assume also that T is centered, which can be done without loss of generality. The *covariance kernel* $R:\mathscr{X}\times\mathscr{X}\to\mathbb{C}$ of T is defined as

$$R(x,y)=\mathrm{Cov}\,(T_x,T_y)=\mathbb{E}[T_x\overline{T_y}]\ .$$

As T is isotropic, R is G-invariant, i.e. $R(gx,gy)=R(x,y)$ for every $g\in G$, and also continuous thanks to Proposition 3.2. To R we can associate the function $\phi:G\to\mathbb{C}$

$$\phi(g):=R(gx_0,x_0)\ . \tag{3.7}$$

Such a function ϕ is

- continuous, as a consequence of the continuity of R.
- Bi-K-invariant: for every $k_1,k_2\in K$ and $g\in G$ we have, using the invariance of R and the fact that $k_ix_0=x_0, i=1,2$,

$$\phi(k_1gk_2)=R(k_1gk_2x_0,x_0)=R(k_1gx_0,x_0)=R(gx_0,k_1^{-1}x_0)=R(gx_0,x_0)=\phi(g)\ .$$

- Positive definite: as R is a positive definite kernel, for every $g_1,\dots,g_m\in G$, $\xi_1,\dots,\xi_m\in\mathbb{C}$,

$$\sum_{i,j}\phi(g_i^{-1}g_j)\overline{\xi_i}\xi_j=\sum_{i,j}R(g_i^{-1}g_jx_0,x_0)\overline{\xi_i}\xi_j=\sum_{i,j}R(g_jx_0,g_ix_0)\overline{\xi_i}\xi_j\geq 0\ . \tag{3.8}$$

By standard approximation arguments (3.8) is equivalent to

$$\int_G\int_G\phi(h^{-1}g)f(h)\overline{f(g)}\,dg\,dh\geq 0 \tag{3.9}$$

for every continuous function f. Finally ϕ determines the covariance kernel R: if $g_xx_0=x, g_yx_0=y$, then

$$R(x,y)=R(g_xx_0,g_yx_0)=R(g_y^{-1}g_xx_0,x_0)=\phi(g_y^{-1}g_x)\ .$$

For a function ζ on G let

$$\breve{\zeta}(g):=\overline{\zeta(g^{-1})}\ .$$

Recall that every positive definite function ϕ on G satisfies (see [18] p.123 e.g.)

$$\breve{\phi}(g)=\phi(g)\ . \tag{3.10}$$

Remark 3.3. If $\zeta \in L^2(G)$, then for every $\sigma \in \widehat{G}$ we have $\widehat{\check{\zeta}}(\sigma) = \widehat{\zeta}(\sigma)^*$. Actually,

$$\widehat{\check{\zeta}}(\sigma) = \sqrt{\dim \sigma} \int_G \overline{\zeta(g^{-1})} D^\sigma(g^{-1})\, dg = \sqrt{\dim \sigma} \int_G \overline{\zeta(g)} D^\sigma(g)\, dg =$$
$$= \sqrt{\dim \sigma} \int_G \big(\zeta(g) D^\sigma(g^{-1})\big)^*\, dg = \widehat{\zeta}(\sigma)^* \ .$$

□

The following proposition states some (not really unexpected) properties of positive definite functions that we shall need later.

Proposition 3.4. *Let ϕ be a continuous positive definite function and $\sigma \in \widehat{G}$.*
a) The operator coefficient $\widehat{\phi}(\sigma) : H_\sigma \to H_\sigma$ as defined in (2.3) is Hermitian positive definite.
b) Let $\phi^\sigma : G \to \mathbb{C}$ be the σ component of ϕ defined in (2.7). Then ϕ^σ is also positive definite.

Proof. a) For a fixed a basis $v_1, \dots, v_{\dim \sigma}$ of H_σ, we have by the invariance of the Haar measure

$$\langle \widehat{\phi}(\sigma) v, v\rangle = \sqrt{\dim \sigma} \int_G \phi(g) \langle D^\sigma(g^{-1}) v, v\rangle\, dg = \sqrt{\dim \sigma} \int_G \int_G \phi(h^{-1}g) \langle D^\sigma(g^{-1}h) v, v\rangle\, dg\, dh =$$
$$= \sqrt{\dim \sigma} \int_G \int_G \phi(h^{-1}g) \langle D^\sigma(h) v, D^\sigma(g) v\rangle\, dg\, dh =$$
$$= \sqrt{\dim \sigma} \int_G \int_G \phi(h^{-1}g) \sum_k (D^\sigma(h) v)_k \overline{(D^\sigma(g) v)}_k\, dg\, dh =$$
$$= \sqrt{\dim \sigma} \sum_k \int_G \int_G \phi(h^{-1}g) f_k(h) \overline{f_k(g)}\, dg\, dh \ge 0 \ ,$$

where, for every k, $f_k(g) := (D^\sigma(g) v)_k$ and we can conclude thanks to (3.9).

b) By the Peter-Weyl theorem (2.8)

$$\phi = \sum_{\sigma \in \widehat{G}} \phi^\sigma \tag{3.11}$$

in $L^2(G)$. Let $f \in L^2_\sigma(G)$ as in (3.9) and recall that f is a continuous function. We have

$$\int_G \int_G \phi(h^{-1}g) f(h) \overline{f(g)}\, dg\, dh = \sum_{\sigma' \in \widehat{G}} \int_G \int_G \phi^{\sigma'}(h^{-1}g) f(h) \overline{f(g)}\, dg\, dh =$$
$$= \int_G \sum_{\sigma' \in \widehat{G}} \underbrace{\int_G \phi^{\sigma'}(h^{-1}g) f(h)\, dh}_{= f * \phi^{\sigma'}(g)} \overline{f(g)}\, dg = \int_G \int_G \phi^\sigma(h^{-1}g) f(h) \overline{f(g)}\, dg\, dh$$

as $f * \phi^{\sigma'} \ne 0$ if and only if $\sigma' = \sigma$. Therefore for every $\sigma \in \widehat{G}$ and $f \in L^2_\sigma(G)$

$$\int_G \int_G \phi^\sigma(h^{-1}g) f(h) \overline{f(g)}\, dg\, dh = \int_G \int_G \phi(h^{-1}g) f(h) \overline{f(g)}\, dg\, dh \ge 0 \ . \tag{3.12}$$

Let now $f \in L^2(G)$ and $f = \sum_{\sigma'} f^{\sigma'}$ be its Fourier series. The same argument as above gives

$$\int_G \int_G \phi^\sigma(h^{-1}g) f(h) \overline{f(g)}\, dg\, dh = \int_G \int_G \phi^\sigma(h^{-1}g) f^\sigma(h) \overline{f^\sigma(g)}\, dg\, dh \ge 0$$

so that ϕ^σ is a positive definite function. □

Another important property enjoyed by positive definite functions on G is shown in the following classical theorem (see [6], Theorem 3.20 p.151).

Theorem 3.5. *Let ζ be a continuous positive definite function on G and denote ζ^σ the component of ζ on the σ-isotypical subspace $L^2_\sigma(G)$. Then*

$$\sum_{\sigma\in\widehat{G}} \sqrt{\dim\sigma}\,\operatorname{tr}\widehat{\zeta}(\sigma) < +\infty \tag{3.13}$$

and the Fourier series

$$\zeta = \sum_{\sigma\in\widehat{G}} \zeta^\sigma$$

converges uniformly on G.

We shall need the following "square root" theorem in the proof of the representation formula of Gaussian isotropic random fields on $\mathscr{X}$.

Theorem 3.6. *Let ϕ be a bi-K-invariant positive definite continuous function on G. Then there exists a bi-K-invariant function $f \in L^2(G)$ such that $\phi = f * \breve{f}$. Moreover, if ϕ is real valued then f also can be chosen to be real valued.*

Proof. For every $\sigma \in \widehat{G}$, $\widehat{\phi}(\sigma)$ is Hermitian positive definite. Therefore there exist matrices $\Lambda(\sigma)$ such that $\Lambda(\sigma)\Lambda(\sigma)^* = \widehat{\phi}(\sigma)$. Let

$$f = \sum_{\sigma\in\widehat{G}} \underbrace{\sqrt{\dim\sigma}\,\operatorname{tr}\big(\Lambda(\sigma)D^\sigma\big)}_{=f^\sigma}\ .$$

This actually defines a function $f \in L^2(G)$ as it is easy to see that

$$\|f^\sigma\|_2^2 = \sum_{i,j=1}^{\dim\sigma} (\Lambda(\sigma)_{ij})^2 = \operatorname{tr}(\Lambda(\sigma)\Lambda(\sigma)^*) = \operatorname{tr}(\widehat{\phi}(\sigma))$$

so that

$$\|f\|_2^2 = \sum_{\sigma\in\widehat{G}} \|f^\sigma\|_2^2 = \sum_{\sigma\in\widehat{G}} \operatorname{tr}(\widehat{\phi}(\sigma)) < +\infty$$

thanks to (3.13). By Remark 3.3 and (2.4), we have

$$\phi = f * \breve{f}\ .$$

Finally the matrix $\Lambda(\sigma)$ can be chosen to be Hermitian and with this choice f is bi-K-invariant as the relation (2.12) $\widehat{f}(\sigma) = P_{\sigma,0}\widehat{f}(\sigma)P_{\sigma,0}$ still holds.

The last statement follows from Proposition 9.1 in the Appendix.

□

Note that the decomposition of Theorem 3.6 is not unique, as the Hermitian square root of the positive definite operator $\widehat{\phi}(\sigma)$ is not unique itself (see Remark 4.1 below for explicit examples).

4 Construction of Gaussian isotropic random fields

In this section we develop a method of constructing isotropic Gaussian random fields on $\mathscr{X}$. It is pretty much inspired by P. Lévy's white noise construction of his spherical Brownian motion ([11]). Let $(X_n)_n$ be a sequence of i.i.d. $N(0,1)$-distributed r.v.'s on some probability space $(\Omega,\mathscr{F},\mathbb{P})$ and denote by $\mathscr{H} \subset L^2(\mathbb{P})$ the real Hilbert space generated by $(X_n)_n$. Let $(e_n)_n$ be an orthonormal basis of $L^2_{\mathbb{R}}(\mathscr{X})$. We define an isometry $S : L^2_{\mathbb{R}}(\mathscr{X}) \to \mathscr{H}$ by

$$L^2_{\mathbb{R}}(\mathscr{X}) \ni \sum_k \alpha_k e_k \ \leftrightarrow\ \sum_k \alpha_k X_k \in \mathscr{H}\ .$$

It is easy to extend S to an isometry on $L^2(\mathscr{X})$, indeed if $f \in L^2(\mathscr{X})$, then $f = f_1 + if_2$, with $f_1, f_2 \in L^2_{\mathbb{R}}(\mathscr{X})$, hence just set $S(f) = S(f_1) + iS(f_2)$. Such an isometry respects the real character of the function $f \in L^2(\mathscr{X})$ (i.e. if f is real then $S(f)$ is a real r.v.).

Let f be a left K-invariant function in $L^2(\mathscr{X})$. We then define a random field $(T^f_x)_{x\in\mathscr{X}}$ associated to f as follows: set $T^f_{x_0} = S(f)$ and, for every $x \in \mathscr{X}$,

$$T^f_x = S(L_g f)\ , \tag{4.1}$$

where $g \in G$ is such that $gx_0 = x$ (L still denotes the left regular action of G). This is a good definition: in fact if also $\widetilde{g} \in G$ is such that $\widetilde{g}x_0 = x$, then $\widetilde{g} = gk$ for some $k \in K$ and therefore $L_{\widetilde{g}} f(x) = f(k^{-1}g^{-1}x) = f(g^{-1}x) = L_g f(x)$ so that

$$S(L_{\widetilde{g}} f) = S(L_g f)\ .$$

The random field T^f is mean square integrable, i.e.

$$\mathbb{E}\Big[\int_{\mathscr{X}} |T^f_x|^2\, dx\Big] < +\infty\ .$$

Actually, if g_x is any element of G such that $g_x x_0 = x$ (chosen in some measurable way), then, as $\mathbb{E}[|T^f_x|^2] = \mathbb{E}[|S(L_{g_x} f)|]^2 = \|L_{g_x} f\|^2_{L^2(\mathscr{X})} = \|f\|^2_{L^2(\mathscr{X})}$, we have $\mathbb{E}\int_{\mathscr{X}} |T^f_x|^2\, dx = \|f\|^2_{L^2(\mathscr{X})}$. T^f is a centered and *complex-valued Gaussian* random field. Let us now check that T^f is isotropic. Recall that the law of a complex-valued Gaussian random vector $Z = (Z_1, Z_2)$ is completely characterized by its mean value $\mathbb{E}[Z]$, its covariance matrix $\mathbb{E}[(Z - \mathbb{E}[Z])\,(Z - \mathbb{E}[Z])^*]$ and the *pseudocovariance* or *relation matrix* $\mathbb{E}[(Z - \mathbb{E}[Z])\,(Z - \mathbb{E}[Z])']$. We have

(i) as S is an isometry

$$\mathbb{E}[T^f_{gx}\overline{T^f_{gy}}] = \mathbb{E}[S(L_{gg_x} f)\overline{S(L_{gg_y} f)}] = \langle L_{gg_x} f, L_{gg_y} f\rangle_{L^2(\mathscr{X})} = \langle L_{g_x} f, L_{g_y} f\rangle_{L^2(\mathscr{X})} = \mathbb{E}[T^f_x\overline{T^f_y}]\ .$$

(ii) Moreover, as complex conjugation commutes both with S and the left regular representation of G,

$$\mathbb{E}[T^f_{gx}T^f_{gy}] = \mathbb{E}[S(L_{gg_x} f)\overline{S(L_{gg_y} \overline{f})}] = \langle L_{gg_x} f, L_{gg_y} \overline{f}\rangle_{L^2(\mathscr{X})} = \langle L_{g_x} f, L_{g_y} \overline{f}\rangle_{L^2(\mathscr{X})} = \mathbb{E}[T^f_x T^f_y]\ .$$

Therefore T^f is isotropic because it has the same covariance and relation kernels as the rotated field $(T^f)^g$ for every $g \in G$.

If $R^f(x, y) = \mathbb{E}[T^f_x\overline{T^f_y}]$ denotes its covariance kernel, then the associated positive definite function $\phi^f(g) := R(gx_0, x_0)$ satisfies

$$\begin{aligned} \phi^f(g) &= \mathbb{E}[S(L_g f)\overline{S(f)}] = \langle L_g f, f\rangle = \\ &= \int_G \widetilde{f}(g^{-1}h)\overline{\widetilde{f}(h)}\, dh = \int_G \widetilde{f}(g^{-1}h)\breve{\overline{\widetilde{f}}}(h^{-1})\, dh = \widetilde{f} * \breve{\overline{\widetilde{f}}}(g^{-1})\ , \end{aligned} \tag{4.2}$$

where $\widetilde{f}$ is the pullback on G of f and the convolution $*$ is in G. Moreover the relation function of T^f $\zeta^f(g) := \mathbb{E}[T^f_{gx_0}T^f_{x_0}]$ satisfies

$$\zeta^f(g) = \mathbb{E}[S(L_g f)S(f)] = \langle L_g f, \overline{f}\rangle\ . \tag{4.3}$$

One may ask whether every a.s. square integrable, isotropic, complex-valued Gaussian centered random field on $\mathscr{X}$ can be obtained with this construction: the answer is *no* in general. It is however positive if we consider *real* isotropic Gaussian random fields (see Theorem 4.3 below). Before considering the case of a general homogeneous space $\mathscr{X}$, let us look first at the case of the sphere, where things are particularly simple.

Remark 4.1. (Representation of real Gaussian isotropic random fields on $\mathbb{S}^2$) If $\mathscr{X} = \mathbb{S}^2$ under the action of $SO(3)$, every isotropic, *real* Gaussian and centered random field is of the form (4.1) for some left-K-invariant function $f : \mathbb{S}^2 \to \mathbb{R}$. Indeed let us consider on $L^2(\mathbb{S}^2)$ the Fourier basis $Y_{\ell,m}$,

$\ell = 1, 2, \ldots,\ m = -\ell, \ldots, \ell$, given by the spherical harmonics (2.17). Every continuous positive definite left-K-invariant function ϕ on $\mathbb{S}^2$ has a Fourier expansion of the form (2.18)

$$\phi = \sum_{\ell \geq 0} \alpha_\ell Y_{\ell,0}\ , \tag{4.4}$$

where (Proposition 3.4) $\alpha_\ell \geq 0$ and

$$\sum_{\ell \geq 0} \sqrt{2\ell+1}\, \alpha_\ell < +\infty$$

(Theorem 3.5). The $Y_{\ell,0}$'s being real, the function ϕ in (4.4) is *real*, so that, by (3.10), $\phi(g) = \phi(g^{-1})$ (in this remark and in the next example we identify functions on $\mathbb{S}^2$ with their pullbacks on $SO(3)$ for simplicity of notations).

If ϕ is the positive definite left-K-invariant function associated to T, then, keeping in mind that $Y_{\ell,0} * Y_{\ell',0} = (2\ell+1)^{-1/2} Y_{\ell,0}\delta_{\ell,\ell'}$, a square root f of ϕ is given by

$$f = \sum_{\ell \geq 0} \beta_\ell\, Y_{\ell,0}\ , \tag{4.5}$$

where β_ℓ is a complex number such that

$$\frac{|\beta_\ell|^2}{\sqrt{2\ell+1}} = \sqrt{\alpha_\ell}\ .$$

Therefore there exist infinitely many real functions f such that $\phi(g) = \phi(g^{-1}) = f * \breve{f}(g)$, corresponding to the choices $\beta_\ell = \pm((2\ell+1)\alpha_\ell)^{1/4}$. For each of these, the random field T^f has the same distribution as T, being real and having the same associated positive definite function.

$\square$

Example 4.2. (P.Lévy's spherical Brownian field). Let us choose as a particular instance of the previous construction $f = c1_H$, where H is the half-sphere centered at the north pole x_0 of $\mathbb{S}^2$ and c is some constant to be chosen later.

Still denoting by S a white noise on $\mathbb{S}^2$, from (4.1) we have

$$T^f_x = cS(1_{H_x})\ , \tag{4.6}$$

where 1_{H_x} is the half-sphere centered at $x \in \mathbb{S}^2$. Now, let $x, y \in \mathbb{S}^2$ and denote by $d(x,y) = \theta$ their distance, then, S being an isometry,

$$\mathrm{Var}(T^f_x - T^f_y) = c^2 \|1_{H_x \triangle H_y}\|^2\ . \tag{4.7}$$

The symmetric difference $H_x \triangle H_y$ is formed by the union of two wedges whose total surface is equal to $\frac{\theta}{\pi}$ (recall that we consider the surface of $\mathbb{S}^2$ normalized with total mass $= 1$). Therefore, choosing $c = \sqrt{\pi}$, we have

$$\mathrm{Var}(T^f_x - T^f_y) = d(x,y) \tag{4.8}$$

and furthermore $\mathrm{Var}(T^f_x) = \frac{\pi}{2}$. Thus

$$\mathrm{Cov}\,(T^f_x, T^f_y) = \tfrac{1}{2}\big(\mathrm{Var}(T^f_x) + \mathrm{Var}(T^f_y) - \mathrm{Var}(T^f_x - T^f_y)\big) = \tfrac{\pi}{2} - \tfrac{1}{2} d(x,y)\ . \tag{4.9}$$

Note that the positive definiteness of (4.9) implies that the distance d is a Schoenberg restricted negative definite kernel on $\mathbb{S}^2$. The random field W

$$W_x := T^f_x - T^f_{x_0} \tag{4.10}$$

is *P.Lévy's spherical Brownian field*, as its covariance kernel is

$$\mathrm{Cov}\,(W_x, W_y) = \tfrac{1}{2}\left(d(x,x_0) + d(y,x_0) - d(x,y)\right)\ . \tag{4.11}$$

In particular the kernel at the r.h.s. of (4.11) is positive definite (see also [6]). Let us compute the expansion into spherical harmonics of the positive definite function ϕ associated to the random field T^f and to f. We have $\phi(x) = \frac{\pi}{2} - \frac{1}{2}\, d(x, x_0)$, i.e. $\phi(x) = \frac{\pi}{2} - \frac{1}{2}\,\vartheta$ in spherical coordinates, ϑ being the colatitude of x, whereas $Y_{\ell,0}(x) = \sqrt{2\ell+1}\, P_\ell(\cos\vartheta)$ where P_ℓ is the ℓ-th Legendre polynomial. This formula for the central spherical harmonics differs slightly from the usual one, as we consider the total measure of $\mathbb{S}^2$ to be $= 1$. Then, recalling the normalized measure of the sphere is $\frac{1}{4\pi}\,\sin\vartheta\, d\vartheta\, d\phi$ and that $Y_{\ell,0}$ is orthogonal to the constants

$$\int_{\mathbb{S}^2} \phi(x) Y_{\ell,0}(x)\, dx = -\frac{1}{4}\sqrt{2\ell+1}\int_0^\pi \vartheta P_\ell(\cos\theta)\sin\vartheta\, d\vartheta = -\frac{1}{4}\sqrt{2\ell+1}\int_{-1}^1 \arccos t\, P_\ell(t)\, dt =$$
$$= \frac{1}{4}\sqrt{2\ell+1}\int_{-1}^1 \arcsin t\, P_\ell(t)\, dt = \frac{1}{4}\sqrt{2\ell+1}\, c_\ell\ ,$$

where

$$c_\ell = \pi\Big\{\frac{3\cdot 5\cdots(\ell-2)}{2\cdot 4\cdots(\ell+1))}\Big\}^2 \qquad \ell = 1, 3, \dots$$

and $c_\ell = 0$ for ℓ even (see [20], p. 325). As for the function $f = \sqrt{\pi}\, 1_H$, we have

$$\int_{\mathbb{S}^2} f(x) Y_{\ell,0}(x)\, dx = \frac{\sqrt{\pi}}{2}\sqrt{2\ell+1}\int_0^{\pi/2} P_\ell(\cos\vartheta)\sin\vartheta\, d\vartheta = \frac{\sqrt{\pi}}{2}\sqrt{2\ell+1}\int_0^1 P_\ell(t)\, dt\ .$$

The r.h.s. can be computed using Rodrigues formula for the Legendre polynomials (see again [20], p. 297) giving that it vanishes for ℓ even and equal to

$$(-1)^{m+1}\frac{\sqrt{\pi}}{2}\sqrt{2\ell+1}\,\frac{(2m)!\binom{2m+1}{m}}{2^{2m+1}(2m+1)!} \tag{4.12}$$

for $\ell = 2m+1$. Details of this computation are given in Remark 9.2 in the Appendix. Simplifying the factorials the previous expression becomes

$$(-1)^m\frac{\sqrt{\pi}}{2}\sqrt{2\ell+1}\,\frac{(2m)!}{2^{2m+1}m!(m+1)!} = (-1)^m\frac{\sqrt{\pi}}{2}\sqrt{2\ell+1}\,\frac{3\cdots(2m-1)}{2\cdots(2m+2)} = (-1)^m\frac{1}{2}\sqrt{2\ell+1}\sqrt{c_\ell}\ .$$

Therefore the choice $f = \sqrt{\pi}\, 1_H$ corresponds to taking alternating signs when taking the square roots. Note that the choice $f' = \sum_\ell \beta_\ell Y_{\ell,0}$ with $\beta_\ell = \frac{1}{2}\sqrt{2\ell+1}\sqrt{c_\ell}$ would have given a function diverging at the north pole x_0. Actually it is elementary to check that the series $\sum_\ell (2\ell+1)\sqrt{c_\ell}$ diverges so that f' cannot be continuous by Theorem 3.5.

□

The result of Remark 4.1 concerning $\mathbb{S}^2$ can be extended to the case of a general homogeneous space.

Theorem 4.3. *Let $\mathscr{X}$ be the homogeneous space of a compact group G and let T be an a.s. square integrable isotropic Gaussian* real *random field on $\mathscr{X}$. Then there exists a left-K-invariant function $f \in L^2(\mathscr{X})$ such that T^f has the same distribution as T.*

Proof. Let ϕ be the invariant positive definite function associated to T. Thanks to (4.2) it is sufficient to prove that there exists a *real* K-invariant function $f \in L^2(\mathscr{X})$ such that $\phi(g) = \widetilde{f} * \breve{\widetilde{f}}(g^{-1})$. Keeping in mind that $\phi(g) = \phi(g^{-1})$, as ϕ is real and thanks to (3.10), this follows from Theorem 3.6.

□

As remarked above f is not unique.

Recall that a complex valued Gaussian r.v. $Z = X + iY$ is said to be *complex Gaussian* if the r.v.'s X, Y are jointly Gaussian, are independent and have the same variance. A $\mathbb{C}^m$-valued r.v. $Z = (Z_1, \dots, Z_m)$ is said to be complex Gaussian if the r.v. $\alpha_1 Z_1 + \dots + \alpha_m Z_m$ is complex Gaussian for every choice of $\alpha_1, \dots, \alpha_m \in \mathbb{C}$.

Definition 4.4. *An a.s. square integrable random field T on $\mathscr{X}$ is said to be* complex Gaussian *if and only if the complex valued r.v.'s*

$$\int_{\mathscr{X}} T_x f(x)\,dx$$

are complex Gaussian for every choice of $f \in L^2(\mathscr{X})$.

Complex Gaussian random fields will play an important role in the next sections. By now let us remark that, in general, it is not possible to obtain a complex Gaussian random field by the procedure (4.1).

Proposition 4.5. *Let $\zeta(x,y) = \mathbb{E}[T_x T_y]$ be the relation kernel of a centered complex Gaussian random field T. Then $\zeta \equiv 0$.*

Proof. It easy to check that a centered complex valued r.v. Z is complex Gaussian if and only if $\mathbb{E}[Z^2] = 0$. As for every $f \in L^2(\mathscr{X})$

$$\int_{\mathscr{X}}\int_{\mathscr{X}} \zeta(x,y) f(x) f(y)\,dxdy = \mathbb{E}\Big[\Big(\int_{\mathscr{X}} T_x f(x)\,dx\Big)^2\Big] = 0\ ,$$

it is easy to derive that $\zeta \equiv 0$. $\square$

Going back to the situation of Remark 4.1, the relation function ζ of the random field T^f is easily found to be

$$\zeta^f = \sum_{\ell \geq 0} \beta_\ell^2\, Y_{\ell,0}\ . \tag{4.13}$$

and cannot vanish unless $f \equiv 0$ and T^f vanishes itself. Therefore no isotropic complex Gaussian random field on the sphere can be obtained by the construction (4.1).

5 Random sections of vector bundles

We now investigate the case of Gaussian isotropic spin random fields on $\mathbb{S}^2$, with the aim of extending the representation result of Theorem 4.3. These models have received recently much attention (see [10], [12] or [14]), being motivated by the modeling of CMB data. Actually our point of view begins from [12].

We consider first the case of a general vector bundle. Let $\xi = (E, p, B)$ be a finite-dimensional *complex vector bundle* on the topological space B, which is called the *base space*. The surjective map

$$p : E \longrightarrow B \tag{5.1}$$

is the *bundle projection*, $p^{-1}(x)$, $x \in B$ is the *fiber* above x. Let us denote $\mathscr{B}(B)$ the Borel σ-field of B. A section of ξ is a map $u : B \to E$ such that $p \circ u = id_B$. As E is itself a topological space, we can speak of continuous sections.

We suppose from now on that every fibre $p^{-1}(x)$ carries an inner product and a measure μ is given on the base space. Hence we can consider square integrable sections, as those such that

$$\int_B \langle u(x), u(x)\rangle_{p^{-1}(x)}\, d\mu(x) < +\infty$$

and define the corresponding L^2 space accordingly.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 5.1. *A* random section *T of the vector bundle ξ is a collection of E-valued random variables $(T_x)_{x\in B}$ indexed by the elements of the base space B such that the map $\Omega \times B \ni (\omega, x) \mapsto T_x(\omega)$ is $\mathscr{F} \otimes \mathscr{B}(B)$-measurable and, for every ω, the path*

$$B \ni x \to T_x(\omega) \in E$$

is a section of ξ, i.e. $p \circ T_\cdot(\omega) = id_B$.

Continuity of a random section T is easily defined by requiring that for every $\omega \in \Omega$ the map $x \mapsto T_x$ is a continuous section of ξ. Similarly a.s. continuity is defined. A random section T of ξ is a.s. square integrable if the map $x \mapsto \|T_x(\omega)\|^2_{p^{-1}(x)}$ is a.s. integrable, it is second order if $\mathbb{E}[\|T_x\|^2_{p^{-1}(x)}] < +\infty$ for every $x \in B$ and mean square integrable if

$$\mathbb{E}\Big[\int_B \|T_x\|^2_{p^{-1}(x)}\, d\mu(x)\Big] < +\infty \ .$$

As already remarked in [12], in defining the notion of mean square continuity for a random section, the naive approach

$$\lim_{y \to x} \mathbb{E}[\|T_x - T_y\|^2] = 0$$

is not immediately meaningful as T_x and T_y belong to different (even if possibly isomorphic) spaces (i.e. the fibers). A similar difficulty arises for the definition of strict sense invariance w.r.t. the action of a topological group on the bundle. We shall investigate these points below.

A case of particular interest to us are the homogeneous (or twisted) vector bundles. Let G be a compact group, K a closed subgroup and $\mathscr{X} = G/K$. Given an irreducible unitary representation τ of K on the complex (finite-dimensional) Hilbert space H, K acts on the Cartesian product $G \times H$ by the action

$$k(g, z) := (gk, \tau(k^{-1})z) \ .$$

Let $G \times_\tau H = \{\theta(g, z) : (g, z) \in G \times H\}$ denote the quotient space of the orbits $\theta(g, z) = \{(gk, \tau(k^{-1})z) : k \in K\}$ under the above action. G acts on $G \times_\tau H$ by

$$h\theta(g, z) := \theta(hg, z) \ . \tag{5.2}$$

The map $G \times H \to \mathscr{X} : (g, z) \to gK$ is constant on the orbits $\theta(g, z)$ and induces the projection

$$G \times_\tau H \ni \theta(g, z) \ \stackrel{\pi_\tau}{\to} \ gK \in \mathscr{X}$$

which is a continuous G-equivariant map. $\xi_\tau = (G \times_\tau H, \pi_\tau, \mathscr{X})$ is a G-vector bundle: it is the *homogeneous vector bundle associated to the representation* τ. The fiber $\pi_\tau^{-1}(x)$ is isomorphic to H for every $x \in \mathscr{X}$ (see [4]). More precisely, for $x \in \mathscr{X}$ the fiber $\pi_\tau^{-1}(x)$ is the set of elements of the form $\theta(g, z)$ such that $gK = x$. We define the scalar product of two such elements as

$$\langle \theta(g, z), \theta(g, w) \rangle_{\pi_\tau^{-1}(x)} = \langle z, w \rangle_H \tag{5.3}$$

for some fixed $g \in G$ such that $gK = x$, as it is immediate that this definition does not depend on the choice of such a g. Given a function $f : G \to H$ satisfying

$$f(gk) = \tau(k^{-1})f(g) \ , \tag{5.4}$$

then to it we can associate the section of ξ_τ

$$u(x) = u(gK) = \theta(g, f(g)) \tag{5.5}$$

as again this is a good definition, not depending of the choice of g in the coset gK. The functions f satisfying to (5.4) are called right K-covariant functions of type τ (*functions of type* τ from now on).

More interestingly, also the converse is true.

Proposition 5.2. *Given a section u of ξ_τ, there exists a unique function f of type τ on G such that $u(x) = \theta(g, f(g))$ where $gK = x$. Moreover u is continuous if and only if $f : G \to H$ is continuous.*

Proof. Let $(g_x)_{x \in \mathscr{X}}$ be a measurable selection such that $g_xK = x$ for every $x \in \mathscr{X}$. If $u(x) = \theta(g_x, z)$, then define $f(g_x) := z$ and extend the definition to the elements of the coset g_xK by $f(g_xk) := \tau(k^{-1})z$; it is easy to check that such a f is of type τ, satisfies (5.5) and is the unique function of type τ with this property.

Note that the continuity of f is equivalent to the continuity of the map

$$F : g \in G \to (g, f(g)) \in G \times H \ . \tag{5.6}$$

Denote $pr_1 : G \to \mathscr{X}$ the canonical projection onto the quotient space $\mathscr{X}$ and $pr_2 : G \times H \to G \times_\tau H$ the canonical projection onto the quotient space $G \times_\tau H$. It is immediate that

$$pr_2 \circ F = u \circ pr_1 \ .$$

Therefore F is continuous if and only if u is continuous, the projections pr_1 and pr_2 being continuous open mappings.

□

We shall again call f the *pullback* of u. Remark that, given two sections u_1, u_2 of ξ_τ and their respective pullbacks f_1, f_2, we have

$$\langle u_1, u_2 \rangle := \int_{\mathscr{X}} \langle u_1(x), u_2(x) \rangle_{\pi_\tau^{-1}(x)} \, dx = \int_G \langle f_1(g), f_2(g) \rangle_H \, dg \tag{5.7}$$

so that $u \longleftrightarrow f$ is an isometry between the space $L^2(\xi_\tau)$ of the square integrable sections of ξ_τ and the space $L^2_\tau(G)$ of the square integrable functions of type τ.

The left regular action of G on $L^2_\tau(G)$ (also called the *representation of G induced by τ*)

$$L_h f(g) := f(h^{-1} g)$$

can be equivalently realized on $L^2(\xi_\tau)$ by

$$U_h u(x) = h u(h^{-1} x) \ . \tag{5.8}$$

We have

$$U_h u(gK) = h u(h^{-1} gK) = h\theta(h^{-1} g, f(h^{-1} g)) = \theta(g, f(h^{-1} g)) = \theta(g, L_h f(g))$$

so that, thanks to the uniqueness of the pullback function:

Proposition 5.3. *If f is the pullback function of the section u then $L_h f$ is the pullback of the section $U_h u$.*

Let $T = (T_x)_{x \in \mathscr{X}}$ be a random section of the homogeneous vector bundle ξ_τ. As, for fixed ω, $x \mapsto T_x(\omega)$ is a section of ξ_τ, by Proposition 5.2 there exists a unique function $g \mapsto X_g(\omega)$ of type τ such that $T_{gK}(\omega) = \theta(g, X_g(\omega))$. We refer to the random field $X = (X_g)_{g \in G}$ as the *pullback random field of T*. It is a random field on G of type τ, i.e. $X_{gk}(\omega) = \tau(k^{-1}) X_g(\omega)$ for each ω. Conversely every random field X on G of type τ uniquely defines a random section of ξ_τ whose pullback random field is X. It is immediate that

Proposition 5.4. *Let T be a random section of ξ_τ.*
a) T is a.s. square integrable if and only if its pullback random field X is a.s. square integrable.
b) T is second order if and only if its pullback random field X is second order.
c) T is a.s. continuous if and only if its pullback random field X is a.s. continuous.

Proposition 5.4 introduces the fact that many properties of random sections of the homogeneous bundle can be stated or investigated through corresponding properties of their pullbacks, which are just ordinary random fields to whom the results of previous sections can be applied. A first instance is the following definition.

Definition 5.5. *The random section T of the homogeneous vector bundle ξ_τ is said to be* mean square continuous *if its pullback random field X is mean square continuous, i.e.,*

$$\lim_{h \to g} \mathbb{E}[\|X_h - X_g\|^2_H] = 0 \ . \tag{5.9}$$

Recalling Definition 3.1, we state now the definition of strict-sense invariance. Let T be an a.s. square integrable random section of ξ_τ. For every $g \in G$, the "rotated" random section T^g is defined as

$$T^g_x(\cdot) := g^{-1} T_{gx}(\cdot) \tag{5.10}$$

which is still an a.s. square integrable random section of ξ_τ. For any square integrable section u of ξ_τ, let

$$T(u) := \int_{\mathscr{X}} \langle T_x, u(x) \rangle_{\pi^{-1}(x)}\, dx\ . \tag{5.11}$$

Definition 5.6. *Let T be an a.s. square integrable random section of the homogeneous vector bundle ξ_τ. It is said to be* (strict-sense) G-invariant *or* isotropic *if and only if for every choice of square integrable sections $u_1, u_2, \dots, u_m$ of ξ_τ, the random vectors*

$$\big(T(u_1), \dots, T(u_m)\big) \quad \textit{and} \quad \big(T^g(u_1), \dots, T^g(u_m)\big) = \big(T(U_g u_1), \dots, T(U_g u_m)\big) \tag{5.12}$$

have the same law for every $g \in G$.

Proposition 5.7. *Let T be an a.s. square integrable random section of ξ_τ and let X be its pullback random field on G. Then X is isotropic if and only if T is an isotropic random section.*

Proof. Let us denote $X(f) := \int_G \langle X_g, f(g) \rangle_H\, dg$. Thanks to Proposition 5.3 the equality in law (5.12) is equivalent to the requirement that for every choice of square integrable functions $f_1, f_2, \dots, f_m$ of type τ (i.e. the pullbacks of corresponding sections of ξ_τ), the random vectors

$$(X(f_1), \dots, X(f_m)) \quad \text{and} \quad (X(L_g f_1), \dots, X(L_g f_m)) \tag{5.13}$$

have the same law for every $g \in G$. As $L^2_\tau(G)$ is a closed subspace of $L^2(G)$ and is invariant under the left regular representation of G, every square integrable function $f : G \to H$ can be written as the sum $f^{(1)} + f^{(2)}$ with $f^{(1)} \in L^2_\tau(G)$, $f^{(2)} \in L^2_\tau(G)^\perp$. As the paths of the random field X are of type τ we have $X(f^{(2)}) = X(L_h f^{(2)}) = 0$ for every $h \in G$ so that

$$X(f) = X(f^{(1)}) \quad \text{and} \quad X(L_h f) = X(L_h f^{(1)})\ . \tag{5.14}$$

Therefore (5.13) implies that, for every choice $f_1, f_2, \dots, f_m$ of square integrable H-valued functions on G, the random vectors

$$(X(L_g f_1), \dots, X(L_g f_m)) \quad \text{and} \quad (X(f_1), \dots, X(f_m)) \tag{5.15}$$

have the same law for every $h \in G$ so that the pullback random field X is a strict-sense isotropic random field on G.

□

As a consequence of Proposition 3.2 (see also [15]) we have

Corollary 5.8. *Every a.s. square integrable, second order and isotropic random section T of the homogeneous vector bundle ξ_τ is mean square continuous.*

In order to make a comparison with the pullback approach developed above, we briefly recall the approach to the theory of random fields in vector bundles introduced by Malyarenko in [12]. The main tool is the scalar random field associated to the random section T of $\xi = (E, p, B)$. More precisely, it is the complex-valued random field T^{sc} indexed by the elements $\eta \in E$ given by

$$T^{sc}_\eta := \langle \eta, T_b \rangle_{p^{-1}(b)},\ b \in B, \eta \in p^{-1}(b)\ . \tag{5.16}$$

T^{sc} is a scalar random field on E and we can give the definition that T is mean square continuous if and only if T^{sc} is mean square continuous, i.e., if the map

$$E \ni \eta \mapsto T^{sc}_\eta \in L^2_{\mathbb{C}}(\mathbb{P}) \tag{5.17}$$

is continuous. Given a topological group G acting with a continuous action $(g,x)\mapsto gx, g\in G$ on the base space B, an action of G on E is called associated if its restriction to any fiber $p^{-1}(x)$ is a linear isometry between $p^{-1}(x)$ and $p^{-1}(gx)$. In our case of interest, i.e. the homogeneous vector bundles $\xi_\tau=(G\times_\tau H,\pi_\tau,\mathscr{X})$, we can consider the action defined in (5.2) which is obviously associated. We can now define that T is strict sense G-invariant w.r.t. the action of G on B if the finite-dimensional distributions of T^{sc} are invariant under the associated action (5.2). In the next statement we prove the equivalence of the two approaches.

Proposition 5.9. *The square integrable random section T of the homogeneous bundle ξ_τ is mean square continuous (i.e. its pullback random field on G is mean square continuous) if and only if the associated scalar random field T^{sc} is mean square continuous. Moreover if T is a.s. continuous then it is isotropic if and only if the associated scalar random field T^{sc} is G-invariant.*

Proof. Let X be the pullback random field of T. Consider the scalar random field on $G\times H$ defined as $X^{sc}_{(g,z)}:=\langle z,X_g\rangle_H$. Let us denote pr the projection $G\times H\to G\times_\tau H$: keeping in mind (5.3) we have

$$T^{sc}\circ pr = X^{sc}\ , \tag{5.18}$$

i.e.

$$T^{sc}_{\theta(g,z)}(\omega)=X^{sc}_{(g,z)}(\omega)$$

for every $(g,z)\in G\times H$, $\omega\in\Omega$. Therefore the map $G\times_\tau H\ni\theta(g,z)\mapsto T^{sc}_{\theta(g,z)}\in L^2_{\mathbb{C}}(P)$ is continuous if and only if the map $G\times H\ni(g,z)\mapsto X^{sc}_{(g,z)}\in L^2_{\mathbb{C}}(P)$ is continuous, the projection pr being open and continuous. Let us show that the continuity of the latter map is equivalent to the mean square continuity of the pullback random field X, which will allow to conclude. The proof of this equivalence is inspired by the one of a similar statement in [12], §2.2.

Actually consider an orthonormal basis $\{v_1,v_2,\dots,v_{\dim\tau}\}$ of H, and denote $X^i=\langle X,v_i\rangle$ the i-th component of X w.r.t. the above basis. Assume that the map $G\times H\ni(g,z)\mapsto X^{sc}_{(g,z)}\in L^2_{\mathbb{C}}(P)$ is continuous, then the random field on G

$$g\mapsto X^{sc}_{(g,v_i)}=\overline{X^i_g}$$

is continuous for every $i=1,\dots,\dim\tau$. As $\mathbb{E}[|\overline{X^i_g}-\overline{X^i_h}|^2]=\mathbb{E}[|X^i_g-X^i_h|^2]$,

$$\lim_{h\to g}\mathbb{E}[\|X_g-X_h\|^2_H]=\lim_{h\to g}\sum_{i=1}^{\dim\tau}\mathbb{E}[|X^i_g-X^i_h|^2]=0\ .$$

Suppose that the pullback random field X is mean square continuous. Then for each $i=1,\dots,\dim\tau$

$$0\le\limsup_{h\to g}\mathbb{E}[|X^i_g-X^i_h|^2]\le\lim_{h\to g}\mathbb{E}[\|X_g-X_h\|^2_H]=0$$

so that the maps $G\ni g\mapsto X^i_g\in L^2_{\mathbb{C}}(\mathbb{P})$ are continuous. Therefore

$$\lim_{(h,w)\to(g,z)}\mathbb{E}[|X^{sc}_{(h,w)}-X^{sc}_{(g,z)}|^2]\le 2\sum_{i=1}^{\dim\tau}\lim_{(h,w)\to(g,z)}\mathbb{E}[|w_iX^i_h-z_iX^i_g|^2]=0\ ,$$

a_i denoting the i-th component of $a\in H$.

Assume that T is a.s. continuous and let us show that it is isotropic if and only if the associated scalar random field T^{sc} is G-invariant. Note first that, by (5.18) and $(T^{sc})^h=(X^{sc})^h\circ pr$ for any $h\in G$, T^{sc} is G-invariant if and only if X^{sc} is G-invariant. Actually if the random fields X^{sc} and its rotated $(X^{sc})^h$ have the same law, then $T^{sc}\overset{law}{=}X^{sc}$ and vice versa. Now recalling the definition of X^{sc}, it is obvious that X^{sc} is G-invariant if and only if X is isotropic.

□

6 Random sections of the homogeneous line bundles on $\mathbb{S}^2$

We now concentrate on the case of the homogeneous line bundles on $\mathscr{X} = \mathbb{S}^2$ with $G = SO(3)$ and $K \cong SO(2)$. For every character χ_s of K, $s \in \mathbb{Z}$, let ξ_s be the corresponding homogeneous vector bundle on $\mathbb{S}^2$, as explained in the previous section. Given the action of K on $SO(3) \times \mathbb{C}$: $k(g, z) = (gk, \chi_s(k^{-1})z)$, $k \in K$, let $\mathscr{E}_s := SO(3) \times_s \mathbb{C}$ be the space of the orbits $\mathscr{E}_s = \{\theta(g, z), (g, z) \in G \times \mathbb{C}\}$ where $\theta(g, z) = \{(gk, \chi_s(k^{-1})z); k \in K\}$. If $\pi_s : \mathscr{E}_s \ni \theta(g, z) \to gK \in \mathbb{S}^2$, $\xi_s = (\mathscr{E}_s, \pi_s, \mathbb{S}^2)$ is an homogeneous line bundle (each fiber $\pi_s^{-1}(x)$ is isomorphic to $\mathbb{C}$ as a vector space).

The space $L^2(\xi_s)$ of the square integrable sections of ξ_s is therefore isomorphic to the space $L^2_s(SO(3))$ of the square integrable *functions of type* s, i.e. such that, for every $g \in G$ and $k \in K$,

$$f(gk) = \chi_s(k^{-1})f(g) = \overline{\chi_s(k)}f(g) \ . \tag{6.1}$$

Let us investigate the Fourier expansion of a function of type s.

Proposition 6.1. *Every function of type s is an infinite linear combination of the functions appearing in the $(-s)$-columns of Wigner's D matrices D^ℓ, $\ell \geq |s|$. In particular functions of type s and type s' are orthogonal if $s \neq s'$.*

Proof. For every $\ell \geq |s|$, let $\widehat{f}(\ell)$ be as in (2.15). We have, for every $k \in K$,

$$\begin{aligned} \widehat{f}(\ell) = \sqrt{2\ell+1}\int_{SO(3)} f(g)D^\ell(g^{-1})\,dg = \sqrt{2\ell+1}\,\chi_s(k)\int_{SO(3)} f(gk)D^\ell(g^{-1})\,dg = \\ = \sqrt{2\ell+1}\,\chi_s(k)\int_{SO(3)} f(g)D^\ell(kg^{-1})\,dg = \sqrt{2\ell+1}\,\chi_s(k)D^\ell(k)\int_{SO(3)} f(g)D^\ell(g^{-1})\,dg = \\ = \chi_s(k)D^\ell(k)\widehat{f}(\ell) \ , \end{aligned} \tag{6.2}$$

i.e. the image of $\widehat{f}(\ell)$ is contained in the subspace $H_\ell^{(-s)} \subset H_\ell$ of the vectors such that $D^\ell(k)v = \chi_{-s}(k)v$ for every $k \in K$. In particular $\widehat{f}(\ell) \neq 0$ only if $\ell \geq |s|$, as for every ℓ the restriction to K of the representation D^ℓ is unitarily equivalent to the direct sum of the representations χ_m, $m = -\ell, \dots, \ell$ as recalled at the end of §2.

Let $\ell \geq |s|$ and $v_{-\ell}, v_{-\ell+1}, \dots, v_\ell$ be the orthonormal basis of H_ℓ as in (2.13), in other words v_m spans H_ℓ^m, the one-dimensional subspace of H_ℓ formed by the vectors that transform under the action of K according to the representation χ_m. It is immediate that

$$\widehat{f}(\ell)_{i,j} = \langle \widehat{f}(\ell)v_j, v_i\rangle = 0 \ , \tag{6.3}$$

unless $i = -s$. Thus the Fourier coefficients of f vanish but those corresponding to the column $(-s)$ of the matrix representation D^ℓ and the Peter-Weyl expansion (2.16) of f becomes, in $L^2(SO(3))$,

$$f = \sum_{\ell \geq |s|} \sqrt{2\ell+1}\sum_{m=-\ell}^{\ell} \widehat{f}(\ell)_{-s,m} D^\ell_{m,-s} \ . \tag{6.4}$$

□

We introduced the spherical harmonics in (2.17) from the entries $D^\ell_{m,0}$ of the central columns of Wigner's D matrices. Analogously to the case of $s = 0$, for any $s \in \mathbb{Z}$ we define for $\ell \geq |s|, m = -\ell, \dots, \ell$

$${}_{-s}Y_{\ell,m}(x) := \theta\Big(g, \sqrt{\frac{2\ell+1}{4\pi}}\,\overline{D^\ell_{m,s}(g)}\Big) \ , \qquad x = gK \in \mathbb{S}^2 \ . \tag{6.5}$$

${}_{-s}Y_{\ell,m}$ is a section of ξ_s whose pullback function (up to a factor) is $g \mapsto D^\ell_{-m,-s}(g)$ (recall the relation $\overline{D^\ell_{m,s}(g)} = (-1)^{m-s}D^\ell_{-m,-s}(g)$, see [14] p. 55 e.g.). Therefore thanks to Proposition 6.1 the sections ${}_{-s}Y_{\ell,m}$, $\ell \geq |s|, m = -\ell, \dots, \ell$, form an *orthonormal* basis of $L^2(\xi_s)$. Actually recalling (5.3) and (considering the total mass equal to 4π on the sphere and to 1 on $SO(3)$)

$$\int_{\mathbb{S}^2} {}_{-s}Y_{\ell,m}\,\overline{{}_{-s}Y_{\ell',m'}}\,dx = 4\pi\int_{SO(3)} \sqrt{\frac{2\ell+1}{4\pi}}\,\overline{D^\ell_{m,s}(g)}\,\sqrt{\frac{2\ell'+1}{4\pi}}\,D^{\ell'}_{m',s}(g)\,dg = \delta^\ell_{\ell'}\delta^m_{m'} \ .$$

The sections ${}_{-s}Y_{\ell,m}$, $\ell \geq |s|, m = -\ell, \dots, \ell$ in (6.5) are called *spin $-s$ spherical harmonics*. Recall that the spaces $L^2_s(SO(3))$ and $L^2(\xi_s)$ are isometric through the identification $u \longleftrightarrow f$ between a section u and its pullback f and the definition of the scalar product on $L^2(\xi_s)$ in (5.7). Proposition (6.1) can be otherwise stated as
Every square integrable section u of the homogeneous line bundle $\xi_s = (\mathscr{E}_s, \pi_s, \mathbb{S}^2)$ admits a Fourier expansion in terms of spin $-s$ spherical harmonics of the form

$$u(x) = \sum_{\ell \geq |s|} \sum_{m=-\ell}^{\ell} u_{\ell,m}\ {}_{-s}Y_{\ell,m}(x)\ , \tag{6.6}$$

where $u_{\ell,m} := \langle u, {}_{-s}Y_{\ell,m}\rangle_2$, the above series converging in $L^2(\xi_s)$.

In particular we have the relation

$$u_{\ell,m} = \int_{\mathbb{S}^2} u(x)\, {}_{-s}Y_{\ell,m}(x)\, dx = 4\pi \int_{SO(3)} f(g) \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{m,s}(g)\, dg =$$
$$(-1)^{s-m}\sqrt{4\pi(2\ell+1)} \int_{SO(3)} f(g) \overline{D^{\ell}_{-m,-s}(g)}\, dg = (-1)^{s-m}\sqrt{4\pi}\, \widehat{f}(\ell)_{-s,-m}\ .$$

Definition 6.2. *Let $s \in \mathbb{Z}$. A square integrable function f on $SO(3)$ is said to be* bi-s-associated *if for every $g \in SO(3), k_1, k_2 \in K$,*

$$f(k_1 g k_2) = \chi_s(k_1) f(g) \chi_s(k_2)\ . \tag{6.7}$$

Of course for $s = 0$ bi-0-associated is equivalent to bi-K-invariant. We are particularly interested in bi-s-associated functions as explained in the remark below.

Remark 6.3. Let X be an isotropic random field of type s on $SO(3)$. Then its associate positive definite function ϕ is bi-$(-s)$-associated. Actually, assuming for simplicity that X is centered, as $\phi(g) = \mathbb{E}[X_g \overline{X_e}]$, we have, using invariance on k_1 and type s property on k_2,

$$\phi(k_1 g k_2) = \mathbb{E}[X_{k_1 g k_2} \overline{X_e}] = \mathbb{E}[X_{g k_2} \overline{X_{k_1^{-1}}}] = \chi_s(k_1^{-1}) \mathbb{E}[X_g \overline{X_e}] \chi_s(k_2^{-1}) = \chi_{-s}(k_1) \phi(g) \chi_{-s}(k_2)\ .$$

□

Let us investigate the Fourier expansion of a bi-s-associated function f: note first that a bi-s-associated function is also a function of type $(-s)$, so that $\widehat{f}(\ell) \neq 0$ only if $\ell \geq |s|$ as above and all its rows vanish but for the s-th. A repetition of the computation leading to (6.2) gives easily that

$$\widehat{f}(\ell) = \chi_{-s}(k) \widehat{f}(\ell) D^{\ell}(k)$$

so that the only non-vanishing entry of the matrix $\widehat{f}(\ell)$ is the (s, s)-th.

Therefore the Fourier expansion of a bi-s-associated function ϕ is

$$f = \sum_{\ell \geq |s|} \sqrt{2\ell+1}\, \alpha_\ell D^{\ell}_{s,s}\ , \tag{6.8}$$

where we have set $\alpha_\ell = \widehat{f}(\ell)_{s,s}$.

Now let T be an a.s. square integrable random section of the line bundle ξ_s and X its pullback random field. Recalling that X is a random function of type s and its sample paths are a.s. square integrable, we easily obtain the stochastic Fourier expansion of X applying (6.4) to the functions $g \mapsto X_g(\omega)$. Actually define, for every $\ell \geq |s|$, the random operator

$$\widehat{X}(\ell) = \sqrt{2\ell+1} \int_{SO(3)} X_g D^{\ell}(g^{-1})\, dg\ . \tag{6.9}$$

The basis of H_ℓ being fixed as above and recalling (6.4), we obtain, a.s. in $L^2(SO(3))$,

$$X_g = \sum_{\ell \geq |s|} \sqrt{2\ell+1} \sum_{m=-\ell}^{\ell} \widehat{X}(\ell)_{-s,m} D^{\ell}_{m,-s}(g)\ . \tag{6.10}$$

If T is isotropic, then by Definition 5.6 its pullback random field X is also isotropic in the sense of Definition 3.1. The following is a consequence of well known general properties of the random coefficients of invariant random fields (see [3] Theorem 3.2 or [12] Theorem 2).

Proposition 6.4. *Let $s \in \mathbb{Z}$ and $\xi_s = (\mathscr{E}_s, \pi_s, \mathbb{S}^2)$ be the homogeneous line bundle on $\mathbb{S}^2$ induced by the s-th linear character χ_s of $SO(2)$. Let T be a random section of ξ_s and X its pullback random field. If T is second order and strict-sense isotropic, then the Fourier coefficients $X(\ell)_{-s,m}$ of X in its stochastic expansion (6.10) are pairwise orthogonal and the variance, c_ℓ, of $\widehat{X}(\ell)_{-s,m}$ does not depend on m. Moreover $\mathbb{E}[\widehat{X}(\ell)_{-s,m}] = 0$ unless $\ell = 0, s = 0$.*

For the random field X of Proposition 6.4 we have immediately

$$\mathbb{E}[|X_g|^2] = \sum_{\ell \ge |s|} (2\ell+1) c_\ell < +\infty \ . \tag{6.11}$$

The convergence of the series above is also a consequence of Theorem 3.5, as the positive definite function ϕ associated to X is given by

$$\phi(g) = \mathbb{E}[X_g \overline{X_e}] = \sum_{\ell \ge |s|} (2\ell+1) c_\ell \sum_{m=-\ell}^{\ell} D^\ell_{m,-s}(g) \overline{D^\ell_{m,-s}(e)} =$$

$$= \sum_{\ell \ge |s|} (2\ell+1) c_\ell \sum_{m=-\ell}^{\ell} D^\ell_{m,-s}(g) D^\ell_{-s,m}(e) = \sum_{\ell \ge |s|} (2\ell+1) c_\ell D^\ell_{-s,-s}(g) \ .$$

Remark 6.5. Let X be a type s random field on $SO(3)$ with $s \neq 0$. Then the relation $X_{gk} = \chi_s(k^{-1}) X_g$ implies that X cannot be real (unless it is vanishing). If in addition it was Gaussian, then, the identity in law between X_g and $X_{gk} = \chi_s(k^{-1}) X_g$ would imply that, for every $g \in G$, X_g is a complex Gaussian r.v.

7 Construction of Gaussian isotropic spin random fields

We now give an extension of the construction of §4 and prove that every complex Gaussian random section of a homogeneous line bundle on $\mathbb{S}^2$ can be obtained in this way, a result much similar to Theorem 4.3. Let $s \in \mathbb{Z}$ and let ξ_s be the homogeneous line bundle associated to the representation χ_s.

Let $(X_n)_n$ be a sequence of i.i.d. standard Gaussian r.v.'s on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $\mathscr{H} \subset L^2(\Omega, \mathscr{F}, \mathbb{P})$ the *complex* Hilbert space generated by $(X_n)_n$. Let $(e_n)_n$ be an orthonormal basis of $L^2(SO(3))$ and define an isometry S between $L^2(SO(3))$ and $\mathscr{H}$ by

$$L^2(SO(3)) \ni \sum_k \alpha_k e_k \ \to \ \sum_k \alpha_k X_k \in \mathscr{H} \ .$$

Let $f \in L^2(SO(3))$, we define a random field X^f on $SO(3)$ by

$$X^f_g = S(L_g f) \ , \tag{7.1}$$

L denoting as usual the left regular representation.

Proposition 7.1. *If f is a square integrable bi-s-associated function on $SO(3)$, then X^f defined in (7.1) is a second order, square integrable Gaussian isotropic random field of type s. Moreover it is complex Gaussian.*

Proof. It is immediate that X^f is second order as $\mathbb{E}[|X^f_g|^2] = \|L_g f\|_2^2 = \|f\|_2^2$. It is of type s as for every $g \in SO(3)$ and $k \in K$,

$$X^f_{gk} = S(L_{gk} f) = \chi_s(k^{-1}) S(L_g f) = \chi_s(k^{-1}) X^f_g \ .$$

Let us prove strict-sense invariance. Actually, S being an isometry, for every $h \in SO(3)$

$$\mathbb{E}[X^f_{hg}\overline{X^f_{hg'}}] = \mathbb{E}[S(L_{hg}f)\overline{S(L_{hg'}f)}] = \langle L_{hg}f, L_{hg'}f\rangle_2 = \langle L_g f, L_{g'}f\rangle_2 = \mathbb{E}[X^f_g\overline{X^f_{g'}}]\ .$$

Therefore the random fields X^f and its rotated $(X^f)^h$ have the same covariance kernel. Let us prove that they also have the same relation function. Actually we have, for every $g, g' \in SO(3)$,

$$\mathbb{E}[X^f_g X^f_{g'}] = \mathbb{E}[S(L_{hg}f)S(L_{hg'}f)] = \langle L_{hg}f, \overline{L_{hg'}f}\rangle_2 = \langle L_g f, \overline{L_{g'}f}\rangle_2 = 0 \tag{7.2}$$

as the function $\overline{L_{g'}f}$ is bi-$(-s)$-associated and therefore of type s and orthogonal to $L_g f$ which is of type $-s$ (orthogonality of functions of type s and $-s$ is a consequence of Proposition 6.1).

In order to prove that X^f is complex Gaussian we must show that for every $\psi \in L^2(SO(3))$, the r.v.

$$Z = \int_{SO(3)} X^f_g \psi(g)\,dg$$

is complex Gaussian. As Z is Gaussian by construction we must just prove that $\mathbb{E}[Z^2] = 0$. But as, thanks to (7.2), $\mathbb{E}[X^f_g X^f_{g'}] = 0$

$$\mathbb{E}[Z^2] = \mathbb{E}\Big[\int_{SO(3)}\int_{SO(3)} X^f_g X^f_h \psi(g)\psi(h)\,dgdh\Big] = \int_{SO(3)}\int_{SO(3)} \mathbb{E}[X^f_g X^f_h]\psi(g)\psi(h)\,dgdh = 0\ .$$

□

Let us investigate the stochastic Fourier expansion of X^f. Let us consider first the random field X^ℓ associated to $f = D^\ell_{s,s}$. Recall first that the r.v. $Z = S(D^\ell_{s,s})$ has variance $\mathbb{E}[|Z|^2] = \|D^\ell_{s,s}\|_2^2 = (2\ell+1)^{-1}$ and that $\overline{D^\ell_{m,s}} = (-1)^{m-s}D^\ell_{-m,-s}$. Therefore

$$X^\ell_g = S(L_g D^\ell_{s,s}) = \sum_{m=-\ell}^{\ell} S(D^\ell_{m,s})D^\ell_{s,m}(g^{-1}) =$$

$$= \sum_{m=-\ell}^{\ell} S(D^\ell_{m,s})\overline{D^\ell_{m,s}(g)} = \sum_{m=-\ell}^{\ell} S(D^\ell_{m,s})(-1)^{m-s}D^\ell_{-m,-s}(g)\ .$$

Therefore the r.v.'s

$$a_{\ell,m} = \sqrt{2\ell+1}\,S(D^\ell_{m,s})(-1)^{m-s}$$

are complex Gaussian, independent and with variance $\mathbb{E}[|a_{\ell,m}|^2] = 1$ and we have the expansion

$$X^\ell_g = \frac{1}{\sqrt{2\ell+1}}\sum_{m=-\ell}^{\ell} a_{\ell,m}D^\ell_{m,-s}(g)\ . \tag{7.3}$$

Note that the coefficients $a_{\ell m}$ are independent complex Gaussian r.v.'s. This is a difference with respect to the case $s = 0$, where in the case of a real random field, the coefficients $a_{\ell,m}$ and $a_{\ell,-m}$ were not independent. Recall that random fields of type $s \neq 0$ on $SO(3)$ cannot be real.

In general, for a square integrable bi-s-associated function f

$$f = \sum_{\ell \ge |s|} \sqrt{2\ell+1}\,\alpha_\ell D^\ell_{s,s} \tag{7.4}$$

with

$$\|f\|_2^2 = \sum_{\ell \ge |s|} |\alpha_\ell|^2 < +\infty\ ,$$

the Gaussian random field X^f has the expansion

$$X^f_g = \sum_{\ell \ge |s|} \alpha_\ell \sum_{m=\ell}^{\ell} a_{\ell,m}D^\ell_{m,-s}(g)\ , \tag{7.5}$$

where $(a_{\ell,m})_{\ell,m}$ are independent complex Gaussian r.v.'s with zero mean and unit variance.

The associated positive definite function of X^f, $\phi^f(g) := \mathbb{E}[X^f_g \overline{X^f_e}]$ is bi-$(-s)$-associated (Remark 6.3) and continuous (Theorem 3.5) and, by (4.2), is related to f by

$$\phi^f = f * \breve{f}(g^{-1}) \ .$$

This allows to derive its Fourier expansion:

$$\phi^f(g) = f*\breve{f}(g^{-1}) = \int_{SO(3)} f(h)\overline{f(gh)}\,dh = \sum_{\ell,\ell'\ge|s|} \sqrt{2\ell+1}\,\sqrt{2\ell'+1}\,\alpha_\ell\overline{\alpha_{\ell'}} \int_{SO(3)} D^\ell_{s,s}(h)\overline{D^{\ell'}_{s,s}(gh)}\,dh =$$

$$= \sum_{\ell,\ell'\ge|s|} \sqrt{2\ell+1}\,\sqrt{2\ell'+1}\,\alpha_\ell\overline{\alpha_{\ell'}} \sum_{j=-\ell}^{\ell} \underbrace{\Big(\int_{SO(3)} D^\ell_{s,s}(h)\overline{D^{\ell'}_{j,s}(h)}\,dh\Big)}_{=\frac{1}{2\ell+1}\delta_{\ell,\ell'}\delta_{s,j}} \overline{D^\ell_{s,j}(g)} =$$

$$= \sum_{\ell\ge|s|} |\alpha_\ell|^2 D^\ell_{-s,-s}(g) \ .$$

Note that in accordance with Theorem 3.5, as $|D^\ell_{-s,-s}(g)| \le D^\ell_{-s,-s}(e) = 1$, the above series converges uniformly.

Conversely, it is immediate that, given a continuous positive definite bi-$(-s)$-associated function ϕ, whose expansion is

$$\phi^f(g) = \sum_{\ell\ge|s|} |\alpha_\ell|^2 D^\ell_{-s,-s}(g) \ ,$$

by choosing

$$f(g) = \sum_{\ell\ge|s|} \sqrt{2\ell+1}\,\beta_\ell D^\ell_{-s,-s}(g)$$

with $|\beta_\ell| = \sqrt{\alpha_\ell}$, there exist a square integrable bi-s-associated function f as in (7.4) such that $\phi(g) = f * \breve{f}(g^{-1})$. Therefore, for every random field X of type s on $SO(3)$ there exists a square integrable bi-s-associated function f such that X and X^f coincide in law. Such a function f is not unique.

From X^f we can define a random section T^f of the homogeneous line bundle ξ_s by

$$T^f_x := \theta(g, X^f_g) \ , \tag{7.6}$$

where $x = gK \in \mathbb{S}^2$. Now, as for the case $s=0$ that was treated in §4, it is natural to ask whether every Gaussian isotropic section of ξ_s can be obtained in this way.

Theorem 7.2. *Let $s \in \mathbb{Z}\setminus\{0\}$. For every square integrable, isotropic, (complex) Gaussian random section T of the homogeneous s-spin line bundle ξ_s, there exists a square integrable and bi-s-associated function f on $SO(3)$ such that*

$$T^f \ \stackrel{law}{=} \ T \ . \tag{7.7}$$

Such a function f is not unique.

Proof. Let X be the pullback random field (of type s) of T. X is of course mean square continuous. Let R be its covariance kernel. The function $\phi(g) := R(g,e)$ is continuous, positive definite and bi-$(-s)$-associated, therefore has the expansion

$$\phi = \sum_{\ell\ge|s|} \sqrt{2\ell+1}\,\beta_\ell D^\ell_{-s,-s} \ , \tag{7.8}$$

where $\beta_\ell = \sqrt{2\ell+1}\,\int_{SO(3)} \phi(g)\overline{D^\ell_{-s,-s}(g)}\,dg \ge 0$. Furthermore, by Theorem 3.5, the series in (7.8) converges uniformly, i.e.

$$\sum_{\ell\ge|s|} \sqrt{2\ell+1}\,\beta_\ell < +\infty \ .$$

Now set $f := \sum_{\ell\ge|s|}(2\ell+1)\sqrt{\beta_\ell}D^\ell_{s,s}$. Actually, $f \in L^2_s(SO(3))$ as $\|f\|^2_{L^2(SO(3))} = \sum_{\ell\ge|s|}(2\ell+1)\beta_\ell < +\infty$ so that it is bi-s-associated.

Note that every function f of the form $f = \sum_{\ell\ge|s|}(2\ell+1)\alpha_\ell D^\ell_{s,s}$ where α_ℓ is such that $\alpha^2_\ell = \beta_\ell$ satisfies (7.7) (and clearly every function f such that $\phi(g) = f * \breve{f}(g^{-1})$ is of this form).

$\square$

8 The connection with classical spin theory

There are different approaches to the theory of random sections of homogeneous line bundles on $\mathbb{S}^2$ (see [8], [10], [12], [16] e.g.). In this section we compare them, taking into account, besides the one outlined in §6, the classical Newman and Penrose spin theory ([16]) later formulated in a more mathematical framework by Geller and Marinucci ([8]).

Let us first recall some basic notions about vector bundles. From now on $s \in \mathbb{Z}$. We shall state them concerning the complex line bundle $\xi_s = (\mathscr{E}_s, \pi_s, \mathbb{S}^2)$ even if they can be immediately extended to more general situations. An atlas of ξ_s (see [9] e.g.) can be defined as follows. Let $U \subset \mathbb{S}^2$ be an open set and Ψ a diffeomorphism between U and an open set of $\mathbb{R}^2$. A chart Φ of ξ_s over U is an isomorphism

$$\Phi : \pi_s^{-1}(U) \longrightarrow \Psi(U) \times \mathbb{C} \ , \tag{8.1}$$

whose restriction to every fiber $\pi_s^{-1}(x)$ is a linear isomorphism $\leftrightarrow \mathbb{C}$. An atlas of ξ_s is a family $(U_j, \Phi_j)_{j\in J}$ such that Φ_j is a chart of ξ_s over U_j and the family $(U_j)_{j\in J}$ covers $\mathbb{S}^2$.

Given an atlas $(U_j, \Phi_j)_{j\in J}$, For each pair $i, j \in J$ there exists a unique map (see [9] Prop. 2.2) $\lambda_{i,j} : U_i \cap U_j \longrightarrow \mathbb{C} \setminus 0$ such that for $x \in U_i \cap U_j$, $z \in \mathbb{C}$,

$$\Phi_i^{-1}(\Psi_i(x), z) = \Phi_j^{-1}(\Psi_j(x), \lambda_{i,j}(x)z) \ . \tag{8.2}$$

The map $\lambda_{i,j}$ is called the *transition function* from the chart (U_j, Φ_j) to the chart (U_i, Φ_i). Transition functions satisfy the cocycle conditions, i.e. for every $i, j, l \in J$

$$\begin{array}{lll} \lambda_{j,j} = 1 & \text{on} & U_j \ , \\ \lambda_{j,i} = \lambda_{i,j}^{-1} & \text{on} & U_i \cap U_j \ , \\ \lambda_{l,i}\lambda_{i,j} = \lambda_{l,j} & \text{on} & U_i \cap U_j \cap U_l \ . \end{array}$$

Recall that we denote $K \cong SO(2)$ the isotropy group of the north pole as in §6, §7, so that $\mathbb{S}^2 \cong SO(3)/K$. We show now that an atlas of the line bundle ξ_s is given as soon as we specify

a) an atlas $(U_j, \Psi_j)_{j\in J}$ of the manifold $\mathbb{S}^2$,

b) for every $j \in J$ a family $(g^j_x)_{x\in U_j}$ of representative elements $g^j_x \in G$ with $g^j_x K = x$.

More precisely, let $(g^j_x)_{x\in U_j}$ be as in b) such that $x \mapsto g^j_x$ is smooth for each $j \in J$. Let $\eta \in \pi_s^{-1}(U_j) \subset \mathscr{E}_s$ and $x := \pi_s(\eta) \in U_j$, therefore $\eta = \theta(g^j_x, z)$, for a unique $z \in \mathbb{C}$. Define the chart Φ_j of ξ_s over U_j as

$$\Phi_j(\eta) = (\Psi_j(x), z) \ . \tag{8.3}$$

Transition functions of this atlas are easily determined. If $\eta \in \xi_s$ is such that $x = \pi_s(\eta) \in U_i \cap U_j$, then $\Phi_j(\eta) = (\Psi_j(x), z_j)$, $\Phi_i(\eta) = (\Psi_i(x), z_i)$. As $g^i_x K = g^j_x K$, there exists a unique $k = k_{i,j}(x) \in K$ such that $g^j_x = g^i_x k$, so that $\eta = \theta(g^i_x, z_i) = \theta(g^j_x, z_j) = \theta(g^i_x k, z_j) = \theta(g^i_x, \chi_s(k)z_j)$ which implies $z_i = \chi_s(k)z_j$. Therefore

$$\lambda_{i,j}(x) = \chi_s(k) \ . \tag{8.4}$$

The spin s concept was introduced by Newman and Penrose in [16]: *a quantity u defined on $\mathbb{S}^2$ has spin weight s if, whenever a tangent vector ρ at any point x on the sphere transforms under coordinate change by $\rho' = e^{i\psi}\rho$, then the quantity at this point x transforms by $u' = e^{is\psi}u$.* Recently, Geller and Marinucci in [8] have put this notion in a more mathematical framework modeling such a u as a section of a complex line bundle on $\mathbb{S}^2$ and they describe this line bundle by giving charts and fixing transition functions to express the transformation laws under changes of coordinates.

More precisely, they define an atlas of $\mathbb{S}^2$ as follows. They consider the open covering $(U_R)_{R\in SO(3)}$ of $\mathbb{S}^2$ given by

$$U_e := \mathbb{S}^2 \setminus \{x_0, x_1\} \qquad \text{and} \qquad U_R := RU_e \ , \tag{8.5}$$

where x_0 =the north pole (as usual), x_1 =the south pole. On U_e they consider the usual spherical coordinates (ϑ, φ), ϑ =colatitude, φ =longitude and on any U_R the "rotated" coordinates (ϑ_R, φ_R) in such a way that x in U_e and Rx in U_R have the same coordinates.

The transition functions are defined as follows. For each $x \in U_R$, let $\rho_R(x)$ denote the unit tangent vector at x, tangent to the circle $\vartheta_R = const$ and pointing to the direction of increasing φ_R. If $x \in U_{R_1} \cap U_{R_2}$, let $\psi_{R_2,R_1}(x)$ denote the (oriented) angle from $\rho_{R_1}(x)$ to $\rho_{R_2}(x)$. They prove that the quantity

$$e^{is\psi_{R_2,R_1}(x)} \tag{8.6}$$

satisfies the cocycle relations (8.3) so that this defines a unique (up to isomorphism) structure of complex line bundle on $\mathbb{S}^2$ having (8.6) as transition functions at x (see [9] Th. 3.2).

We shall prove that this spin s line bundle is the same as the homogeneous line bundle $\xi_{-s} = (\mathscr{E}_{-s}, \pi_{-s}, \mathbb{S}^2)$. To this aim we have just to check that, for a suitable choice of the atlas $(U_R, \Phi_R)_{R\in SO(3)}$ of ξ_{-s} of the type described in a), b) above, the transition functions (8.4) and (8.6) are the same. Essentially we have to determine the family $(g_x^R)_{R\in SO(3), x\in U_R}$ as in b).

Recall first that every rotation $R \in SO(3)$ can be realized as a composition of three rotations: (i) a rotation by an angle γ_R around the z axis, (ii) a rotation by an angle β_R around the y axis and (iii) a rotation by an angle α_R around the z axis (the so called z-y-z convention), $(\alpha_R,\ \beta_R,\ \gamma_R)$ are the *Euler angles* of R. Therefore the rotation R acts on the north pole x_0 of $\mathbb{S}^2$ as mapping x_0 to the new location on $\mathbb{S}^2$ whose spherical coordinates are (β_R, α_R) after rotating the tangent plane at x_0 by an angle γ_R. In each coset $\mathbb{S}^2 \ni x = gK$ let us choose the element $g_x \in SO(3)$ as the rotation such that $g_x x_0 = x$ and having its third Euler angle γ_{g_x} equal to 0. Of course if $x \neq x_0, x_1$, such g_x is unique.

Consider the atlas $(U_R, \Psi_R)_{R\in SO(3)}$ of $\mathbb{S}^2$ defined as follows. Set the charts as

$$\Psi_e(x) := (\beta_{g_x}, \alpha_{g_x}) \ , \qquad x \in U_e \ , \tag{8.7}$$

$$\Psi_R(x) := \Psi_e(R^{-1}x) \ , \qquad x \in U_R \ . \tag{8.8}$$

Note that for each R, $\Psi_R(x)$ coincides with the "rotated" coordinates (ϑ_R, φ_R) of x. Let us choose now the family $(g_x^R)_{x\in U_R, R\in SO(3)}$. For $x \in U_e$ choose $g_x^e := g_x$ and for $x \in U_R$

$$g_x^R := R g_{R^{-1}x} \ . \tag{8.9}$$

Therefore the corresponding atlas $(U_R, \Phi_R)_{R\in SO(3)}$ of ξ_s is given, for $\eta \in \pi_s^{-1}(U_R)$, by

$$\Phi_R(\eta) = (\Psi_R(x), z) \ , \tag{8.10}$$

where $x := \pi_s(\eta) \in U_R$ and z is such that $\eta = \theta(g_x^R, z)$. Moreover for $R_1, R_2 \in SO(3)$, $x \in U_{R_1} \cap U_{R_2}$ we have

$$k_{R_2,R_1}(x) = (g_{R_2^{-1}x})^{-1} R_2^{-1} R_1 g_{R_1^{-1}x} \tag{8.11}$$

and the transition function from the chart (U_{R_1}, Φ_{R_1}) to the chart (U_{R_2}, Φ_{R_2}) at x is given by (8.4)

$$\lambda^{(-s)}_{R_2,R_1}(x) := \chi_s(k) \ . \tag{8.12}$$

From now on let us denote $\omega_{R_2,R_1}(x)$ the rotation angle of $k_{R_2,R_1}(x)$. Note that, with this choice of the family $(g_x^R)_{x\in U_R, R\in SO(3)}$, $\omega_{R_2,R_1}(x)$ is the third Euler angle of the rotation $R_2^{-1}R_1 g_{R_1^{-1}x}$.

Remark 8.1. Note that we have

$$R^{-1} g_x = g_{R^{-1}x} \ ,$$

i.e. $g_x^R = g_x$, in any of the following two situations

a) R is a rotation around the north-south axis (i.e. not changing the latitude of the points of $\mathbb{S}^2$).

b) The rotation axis of R is orthogonal to the plane $[x_0, x]$ (i.e. changes the colatitude of x leaving its longitude unchanged).

Note that if each of the rotations R_1, R_2 are of type a) or of type b), then

$$k_{R_2,R_1}(x) = g^{-1}_{R_2^{-1}x} R_2^{-1} R_1 g_{R_1^{-1}x} = (R_2 g_{R_2^{-1}x})^{-1} R_1 g_{R_1^{-1}x} = g_x^{-1} g_x = \text{the identity}$$

and in this case the rotation angle of $k_{R_2,R_1}(x)$ coincides with the angle $-\psi_{R_2,R_1}(x)$, as neither R_1 nor R_2 change the orientation of the tangent plane at x.

Another situation in which the rotation k can be easily computed appears when R_1 is the identity and R_2 is a rotation of an angle γ around an axis passing through x. Actually

$$k_{R_2,e}(x) = g_x^{-1} R_2^{-1} g_x \tag{8.13}$$

which, by conjugation, turns out to be a rotation of the angle $-\gamma$ around the north-south axis. In this case also it is immediate that the rotation angle $\omega_{R_2,R_1}(x)$ coincides with $-\psi_{R_2,R_1}(x)$.

□

The following relations will be useful in the sequel, setting $y_1 = R_1^{-1}x$, $y_2 = R_2^{-1}x$,

$$\begin{aligned} k_{R_2,R_1}(x) &= g^{-1}_{R_2^{-1}x} R_2^{-1} R_1 g_{R_1^{-1}x} = g^{-1}_{R_2^{-1}R_1 y_1} R_2^{-1} R_1 g_{y_1} = k_{R_1^{-1}R_2,e}(R_1^{-1}x)\ , \\ k_{R_2,R_1}(x) &= g^{-1}_{R_2^{-1}x} R_2^{-1} R_1 g_{R_1^{-1}x} = g_{y_2}^{-1} R_2^{-1} R_1 g_{R_1^{-1}R_2 y_2} = k_{e,R_2^{-1}R_1}(R_2^{-1}x)\ . \end{aligned} \tag{8.14}$$

We have already shown in Remark 8.1 that $\omega_{R_2,R_1}(x) = -\psi_{R_2,R_1}(x)$ in two particular situations: rotations that move $y_1 = R_1^{-1}x$ to $y_2 = R_2^{-1}x$ without turning the tangent plane and rotations that turn the tangent plane without moving the point. In the next statement, by combining these two particular cases, we prove that actually they coincide always.

Lemma 8.2. *Let $x \in U_{R_1} \cap U_{R_2}$, then $\omega_{R_2,R_1}(x) = -\psi_{R_2,R_1}(x)$.*

Proof. The matrix $R_2^{-1}R_1$ can be decomposed as $R_2^{-1}R_1 = EW$ where W is the product of a rotation around an axis that is orthogonal to the plane $[x_0, y_1]$ bringing y_1 to a point having the same colatitude as y_2 and of a rotation around the north-south axis taking this point to y_2. By Remark 8.1 we have $W g_{y_1} = g_{W y_1} = g_{y_2}$. E instead is a rotation around an axis passing by y_2 itself.

We have then, thanks to (8.13) and (8.14)

$$k_{R_2,R_1}(x) = k_{R_1^{-1}R_2,e}(R_1^{-1}x) = k_{W^{-1}E^{-1},e}(y_1) = g^{-1}_{EWy_1} EW g_{y_1} = g^{-1}_{y_2} E g_{y_2} = k_{E^{-1},e}(y_2)\ .$$

By the previous discussion, $\omega_{E^{-1},e}(y_2) = -\psi_{E^{-1},e}(y_2)$. To finish the proof it is enough to show that

$$\psi_{R_2,R_1}(x) = \psi_{E^{-1},e}(y_2)\ . \tag{8.15}$$

Let us denote $\rho(x) = \rho_e(x)$ the tangent vector at x which is parallel to the curve $\vartheta = const$ and pointing in the direction of increasing φ. Then in coordinates

$$\rho(x) = \frac{1}{\sqrt{x_1^2 + x_2^2}}\ \bigl(-x_2, x_1, 0\bigr)$$

and the action of R is given by ([8],§3) $\rho_R(x) = R\rho(R^{-1}x)$. As $W\rho(y_1) = \rho(y_2)$ (W does not change the orientation of the tangent plane),

$$\begin{aligned} \langle \rho_{R_2}(x), \rho_{R_1}(x)\rangle &= \langle R_2\rho(R_2^{-1}x), R_1\rho(R_1^{-1}x)\rangle = \langle R_1^{-1}R_2\rho(R_2^{-1}x), \rho(R_1^{-1}x)\rangle = \\ &= \langle W^{-1}E^{-1}\rho(EWR_1^{-1}x), \rho(W^{-1}E^{-1}R_2^{-1}x)\rangle = \langle E^{-1}\rho(Ey_2), W\rho(W^{-1}y_2)\rangle = \\ &= \langle E^{-1}\rho(y_2)), W\rho(y_1)\rangle = \langle E^{-1}\rho(y_2)), \rho(y_2)\rangle\ , \end{aligned}$$

so that the oriented angle $\psi_{R_2,R_1}(x)$ between $\rho_{R_2}(x)$ and $\rho_{R_1}(x)$ is actually the rotation angle of E^{-1}.

□

9 Appendix

Proposition 9.1. *Let ϕ be a real positive definite function on a compact group G, then there exists a real function f such that $\phi = f * \check{f}$.*

Proof. Let

$$\phi(g) = \sum_{\sigma \in \widehat{G}} \phi^\sigma(g) = \sum_{\sigma \in \widehat{G}} \sqrt{\dim \sigma}\ \mathrm{tr}(\widehat{\phi}(\sigma) D^\sigma(g))$$

be the Peter-Weyl decomposition of ϕ into isotypical components. We know that the Hermitian matrices $\widehat{\phi}(\sigma)$ are positive definite, so that there exist square roots $\widehat{\phi}(\sigma)^{1/2}$ i.e. matrices such that $\widehat{\phi}(\sigma)^{1/2}\widehat{\phi}(\sigma)^{1/2^*} = \widehat{\phi}(\sigma)$ and the functions

$$f(g) = \sum_{\sigma \in \widehat{G}} \sqrt{\dim \sigma}\ \mathrm{tr}(\widehat{\phi}(\sigma)^{1/2} D^\sigma(g))$$

are such that $\phi = f * \check{f}$. We need to prove that these square roots can be chosen in such a way that f is also real. Recall that a representation of a compact group G can be classified as being of real, complex or quaternionic type (see [4], p. 93 e.g. for details).

a) If σ is of real type then there exists a conjugation J of $H_\sigma \subset L^2(G)$ such that $J^2 = 1$. A conjugation is a G-equivariant antilinear endomorphism. It is well known that in this case one can choose a basis $v_1, \dots, v_{d_\sigma}$ of H_σ formed of "real" vectors, i.e. such that $Jv_i = v_i$. It is then immediate that the representative matrix D^σ of the action of G on H_σ is real. Actually, as J is equivariant and $Jv_i = v_i$,

$$D^\sigma_{ij}(g) = \langle gv_j, v_i \rangle = \overline{\langle Jgv_j, Jv_i \rangle} = \overline{\langle gv_j, v_i \rangle} = \overline{D^\sigma_{ij}(g)}\ .$$

With this choice of the basis, the matrix $\widehat{\phi}(\sigma)$ is real and also $\widehat{\phi}(\sigma)^{1/2}$ can be chosen to be real and $g \mapsto \sqrt{\dim \sigma}\ \mathrm{tr}(\widehat{\phi}(\sigma)^{1/2} D^\sigma(g))$ turns out to be real itself.

b) If σ is of complex type, then it is not isomorphic to its dual representation σ^*. As $D^{\sigma^*}(g) := D^\sigma(g^{-1})^t = \overline{D^\sigma(g)}$ and ϕ is real-valued, we have

$$\widehat{\phi}(\sigma^*) = \overline{\widehat{\phi}(\sigma)}\ ,$$

so that we can choose $\widehat{\phi}(\sigma^*)^{1/2} = \overline{\widehat{\phi}(\sigma)^{1/2}}$ and, as σ and σ^* have the same dimension, the function

$$g \mapsto \sqrt{\dim \sigma}\, \mathrm{tr}(\widehat{\phi}(\sigma)^{1/2} D^\sigma(g)) + \sqrt{\dim \sigma^*}\, \mathrm{tr}(\widehat{\phi}(\sigma^*)^{1/2} D^{\sigma^*}(g))$$

turns out to be real.

c) If σ is quaternionic, let J be the corresponding conjugation. It is immediate that the vectors v and Jv are orthogonal and from this it follows that $\dim \sigma = 2k$ and that there exists an orthogonal basis for H_σ of the form

$$v_1, \dots, v_k, w_1 = J(v_1), \dots, w_k = J(v_k)\ . \tag{9.1}$$

In such a basis the representation matrix of any linear transformation $U : H_\sigma \to H_\sigma$ which commutes with J has the form

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \tag{9.2}$$

and in particular $D^\sigma(g)$ takes the form

$$D^\sigma(g) = \begin{pmatrix} A(g) & B(g) \\ -\overline{B(g)} & \overline{A(g)} \end{pmatrix}\ . \tag{9.3}$$

By (9.3) we have also, ϕ being real valued,

$$\widehat{\phi}(\sigma) = \begin{pmatrix} \int_G \phi(g) A(g^{-1})\, dg & \int_G \phi(g) B(g^{-1})\, dg \\ -\int_G \phi(g)\overline{B(g^{-1})}\, dg & \int_G \phi(g)\overline{A(g^{-1})}\, dg \end{pmatrix} := \begin{pmatrix} \phi_A & \phi_B \\ -\overline{\phi_B} & \overline{\phi_A} \end{pmatrix}\ . \tag{9.4}$$

More interestingly, if ϕ is any function such that, with respect to the basis above, $\widehat{\phi}(\sigma)$ is of the form (9.4), then the corresponding component ϕ^σ is necessarily a real valued function: actually

$$\phi^\sigma(g) = \mathrm{tr}(\widehat{\phi}(\sigma)D^\sigma(g)) =$$
$$= \mathrm{tr}\big(\phi_A A(g) - \phi_B \overline{B(g)} - \overline{\phi_B} B(g) + \overline{\phi_A}\, \overline{A(g)}\big) = \mathrm{tr}\big(\phi_A A(g) + \overline{\phi_A A(g)}\big) - \mathrm{tr}\big(\phi_B \overline{B(g)} + \overline{\phi_B \overline{B(g)}}\big) \ .$$

We now prove that the Hermitian square root, U say, of $\widehat{\phi}(\sigma)$ is of the form (9.4). Actually note that $\widehat{\phi}(\sigma)$ is self-adjoint, so that it can be diagonalized and all its eigenvalues are real (and positive by Proposition 3.4 a)). Let λ be an eigenvalue and v a corresponding eigenvector. Then, as

$$\widehat{\phi}(\sigma)Jv = J\widehat{\phi}(\sigma) = J\lambda v = \lambda Jv \ ,$$

Jv is also an eigenvector associated to λ. Therefore there exists a basis as in (9.1) that is formed of eigenvectors, i.e. of the form $v_1, \dots, v_k, w_1, \dots, w_k$ with $Jv_j = w_j$ and v_j and w_j associated to the same positive eigenvalue λ_j. In this basis $\widehat{\phi}(\sigma)$ is of course diagonal with the (positive) eigenvalues on the diagonal. Its Hermitian square root U is also diagonal, with the square roots of the eigenvalues on the diagonal. Therefore U is also the form (9.4) and the corresponding function $\psi(g) = \mathrm{tr}(UD(g))$ is real valued and such that $\psi * \breve{\psi} = \phi^\sigma$.

Remark 9.2. Rodrigues formula for the Legendre polynomials states that

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\,\frac{d^\ell}{dx^\ell}\,(x^2-1)^\ell \ .$$

Therefore

$$\int_0^1 P_\ell(x)\,dx = \frac{1}{2^\ell \ell!}\,\frac{d^{\ell-1}}{dx^{\ell-1}}\,(x^2-1)^\ell\Big|_0^1 \ . \tag{9.5}$$

The primitive vanishes at 1, as the polynomial $(x^2-1)^\ell$ has a zero of order ℓ at $x=1$ and all its derivatives up to the order $\ell-1$ vanish at $x=1$. In order to compute the primitive at 0 we make the binomial expansion of $(x^2-1)^\ell$ and take the result of the $(\ell-1)$-th derivative of the term of order $\ell-1$ of the expansion. This is actually the term of order 0 of the primitive. If ℓ is even then $\ell-1$ is odd so that this term of order $\ell-1$ does not exist (in the expansion only even powers of x can appear). If $\ell = 2m+1$, then the term of order $\ell - 1 = 2m$ in the expansion is

$$(-1)^m \binom{2m+1}{m} z^{2m}$$

and the result of the integral in (9.5) is actually, as given in (4.12),

$$(-1)^{m+1}\,\frac{(2m)!}{2^{2m+1}(2m+1)!}\binom{2m+1}{m} \ .$$

□

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