GROTHENDIECK-TEICHMÜLLER AND BATALIN-VILKOVISKY

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ABSTRACT. It is proven that, for any affine supermanifold M equipped with a constant odd symplectic structure, there is a universal action (up to homotopy) of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ on the set of quantum BV structures (i. e. solutions of the quantum master equation) on M.

1. Introduction

The Grothendieck-Teichmüller group GRT_1 is a pro-unipotent group introduced by Drinfeld in [Dr]; we denote its Lie algebra by $\mathfrak{grt}_1$. It is shown in this paper that, for any affine formal $\mathbb{Z}$-graded manifold M over a field $\mathbb{K}$ equipped with a constant degree 1 symplectic structure ω, there is a universal action (up to homotopy) of the group GRT_1 on the set of quantum BV structures on M, that is, on the set of solutions, S,

$$\hbar\Delta S + \frac{1}{2}\{S,S\} = 0,$$

of the quantum master equation on M (see [Sc] for an introduction into the geometry of the BV formalism). This action is induced by, in general, homotopy non-trivial L_∞ automorphisms of the corresponding dg Lie algebra $(\mathcal{O}_M[[\hbar]], \hbar\Delta, \{\ ,\ \})$, where $\mathcal{O}_M$ is the ring of (formal) smooth functions M, and $\mathcal{O}_M[[\hbar]] := \mathcal{O}_M \otimes_{\mathbb{K}} \mathbb{K}[[\hbar]]$.

Our main technical tool is a version of the Kontsevich graph complex, $(\mathsf{GC}_2[[\hbar]], d_\hbar)$ which controls universal deformations of $(\mathcal{O}_M[[\hbar]], \hbar\Delta, \{\ ,\ \})$ in the category of L_∞ algebras. Using the main result of [Wi] we show in Sect. 2 that

$$H^0(\mathsf{GC}_2[[\hbar]], d_\hbar) \simeq \mathfrak{grt}_1.$$

In Sect. 3 we explain how to use this isomorphism of Lie algebras to define a universal homotopy action of $\mathfrak{grt}_1$ on the set of quantum BV structures on any affine odd symplectic manifold M.

2. A variant of the Kontsevich graph complex

2.1. From operads to Lie algebras. Let $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ be an operad in the category of dg vector spaces with the partial compositions $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(m+n-1)$, $1 \leq i \leq n$. Then the map

$$\begin{array}{rccc} [\ ,\ ]: & \mathsf{P}\otimes\mathsf{P} & \longrightarrow & \mathsf{P} \\ & (a\in\mathcal{P}(n), b\in\mathcal{P}(m)) & \longrightarrow & [a,b] := \sum_{i=1}^n a\circ_i b - (-1)^{|a||b|}\sum_{i=1}^m b\circ_i a \end{array}$$

makes the vector space $\mathsf{P} := \bigoplus_{n\geq 1}\mathcal{P}(n)$ into a dg Lie algebra [KM]; moreover, the bracket restricts to the subspace of invariants $\mathsf{P}^{\mathbb{S}} := \bigoplus_{n\geq 1}\mathcal{P}(n)^{\mathbb{S}_n}$ making it into a dg Lie algebras as well.

2.2. An operad of graphs and the Kontsevich graph complex. For any integers $n \geq 1$ and $l \geq 0$ we denote by $\mathsf{G}_{n,l}$ a set of graphs[1], $\{\Gamma\}$, with n vertices and l edges such that (i) the vertices of Γ are labelled by elements of $[n] := \{1,\ldots,n\}$, (ii) the set of edges, $E(\Gamma)$, is totally ordered up to an even permutations. For example, $\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \in \mathsf{G}_{2,1}$. The group $\mathbb{Z}_2$ acts freely on $\mathsf{G}_{n,l}$ by changes of the total ordering; its orbit is

[1]A *graph* Γ is, by definition, a 1-dimensional CW-complex whose 0-cells are called *vertices* and 1-dimensional cells are called *edges*. The set of vertices of Γ is denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$.

denoted by $\{\Gamma, \Gamma_{opp}\}$. Let $\mathbb{K}\langle \mathsf{G}_{n,l}\rangle$ be the vector space over a field $\mathbb{K}$ spanned by isomorphism classes, $[\Gamma]$, of elements of $\mathsf{G}_{n,l}$ modulo the relation[2] $\Gamma_{opp} = -\Gamma$, and consider a $\mathbb{Z}$-graded $\mathbb{S}_n$-module,

$$\mathsf{Gra}(n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle \mathsf{G}_{n,l}\rangle[l].$$

Note that graphs with two or more edges between any fixed pair of vertices do not contribute to $\mathsf{Gra}(n)$ so that we could have assumed right from the beginning that the sets $\mathsf{G}_{n,l}$ do not contain graphs with multiple edges. The $\mathbb{S}$-module, $\mathsf{Gra} := \{\mathsf{Gra}(n)\}_{n\geq 1}$, is naturally an operad with the operadic compositions given by

$$\begin{array}{rccc} \circ_i: & \mathsf{Gra}(n)\otimes \mathsf{Gra}(m) & \longrightarrow & \mathsf{Gra}(m+n-1) \\ & \Gamma_1\otimes\Gamma_2 & \longrightarrow & \sum_{\Gamma\in \mathsf{G}^i_{\Gamma_1,\Gamma_2}} (-1)^{\sigma_\Gamma}\Gamma \end{array}$$

where $\mathsf{G}^i_{\Gamma_1,\Gamma_2}$ is the subset of $\mathsf{G}_{n+m-1,\#E(\Gamma_1)+\#E(\Gamma_2)}$ consisting of graphs, Γ, satisfying the condition: the full subgraph of Γ spanned by the vertices labeled by the set $\{i, i+1, \ldots, i+m-1\}$ is isomorphic to Γ_2 and the quotient graph, Γ/Γ_2, obtained by contracting that subgraph to a single vertex, is isomorphic to Γ_1. The sign $(-1)^{\sigma_\Gamma}$ is determined by the equality

$$\bigwedge_{e\in E(\Gamma)} e = (-1)^{\sigma_\Gamma} \bigwedge_{e'\in E(\Gamma_1)} e' \wedge \bigwedge_{e''\in E(\Gamma_2)} e''.$$

The unique element in $\mathsf{G}_{1,0}$ serves as the unit element in the operad Gra. The associated Lie algebra of $\mathbb{S}$-invariants, $((\mathsf{Gra}\{-2\})^{\mathbb{S}}, [\ ,\ ])$ is denoted, following notations of [Wi], by fGC_2. Its elements can be understood as graphs from $\mathsf{G}_{n,l}$ but with labeling of vertices forgotten, e.g.

$$\bullet\!\!-\!\!\!-\!\!\bullet = \frac{1}{2}\left(\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} + \overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\right) \in \mathsf{fGC}_2.$$

It is easy to check that $\bullet\!\!-\!\!\!-\!\!\bullet$ is a Maurer-Cartan element in the Lie algebra fGC_2. Hence we get a dg Lie algebra,

$$(\mathsf{fGC}_2, [\ ,\ ], d := [\bullet\!\!-\!\!\!-\!\!\bullet,\ ])$$

whose dg subalgebra, GC_2, spanned by connected graphs with at least trivalent vertices is called the *Kontsevich graph complex* [Ko1]. We refer to [Wi] for a detailed explanation of why studying the dg Lie subalgebra GC_2 rather than full Lie algebra fGC_2 should be enough for most purposes. The cohomology of GC_2 was partially computed in [Wi].

2.2.1. Theorem [Wi]**.** (i) $H^0(\mathsf{GC}_2, d) \simeq \mathfrak{grt}_1$. (ii) *For any negative integer i, $H^i(\mathsf{GC}_2, d) = 0$.*

This result implies that the Grothendieck-Teichmüller group GRT_1 acts on the set of Poisson structures on an arbitrary manifold.

We shall introduce next a new graph complex which is responsible for the action of GRT_1 on the set of quantum master functions on an arbitrary odd symplectic supermanifold.

2.3. A variant of the Kontsevich graph complex.

The graph  $\in \mathsf{fGC}_2$ has degree -1 and satisfies

[ , ] = [ , $\bullet\!\!-\!\!\!-\!\!\bullet$] $= 0$.

Let $\hbar$ be a formal variable of degree 2 and consider the graph complex $\mathsf{fGC}_2[[\hbar]] := \mathsf{fGC}_2 \otimes \mathbb{K}[[\hbar]]$ with the differential

$d_\hbar := d + \hbar\Delta$, where $\Delta :=$ [ ,].

The subspace $\mathsf{GC}_2[[\hbar]] \subset \mathsf{fGC}_2[[\hbar]]$ is a subcomplex of $(\mathsf{fGC}_2[[\hbar]], d_\hbar)$.

2.3.1. Proposition. $H^0(\mathsf{GC}_2[[\hbar]], d_\hbar) \simeq \mathfrak{grt}_1$.

[2]Abusing notations we identify from now an equivalence class $[\Gamma]$ with any of its representative Γ.

Proof. Consider a decreasing filtration of $\mathsf{GC}_2[[\hbar]]$ by the powers in $\hbar$. The first term of the associated spectral sequence is

$$\mathcal{E}_1 = \bigoplus_{i\in\mathbb{Z}} \mathcal{E}_1^i, \qquad \mathcal{E}_1^i = \bigoplus_{p\geq 0} H^{i-2p}(\mathsf{GC}_2, d)\hbar^p$$

with the differential equal to $\hbar\Delta$. As $H^0(\mathsf{GC}_2, d) \simeq \mathfrak{grt}_1$ and $H^{\leq -1}(\mathsf{GC}_2, d) = 0$, we get the desired result. $\square$

2.4. Remark. Let σ be an element of $\mathfrak{grt}_1$ and let $\Gamma_\sigma^{(0)}$ be any cycle representing the cohomology class σ in the graph complex (GC_2, d). Then one can construct a cycle,

$$\Gamma_\sigma^\hbar = \Gamma_\sigma^{(0)} + \Gamma_\sigma^{(1)}\hbar + \Gamma_\sigma^{(2)}\hbar^2 + \Gamma_\sigma^{(3)}\hbar^3 + \dots, \tag{1}$$

representing the cohomology class $\sigma \in \mathfrak{grt}_1$ in the complex $(\mathsf{GC}_2[[\hbar]], d_\hbar)$ by the following induction:

1st step: As $d\Gamma_\sigma^{(0)} = 0$, we have $d(\Delta\Gamma_\sigma^{(0)}) = 0$. As $H^{-1}(\mathsf{GC}_2, d) = 0$, there exists $\Gamma_\sigma^{(1)}$ of degree -2 such that $\Delta\Gamma_\sigma^{(0)} = -d\Gamma_\sigma^{(1)}$ and hence

$$(d + \hbar\Delta)\left(\Gamma_\sigma^{(0)} + \Gamma_\sigma^{(1)}\hbar\right) = 0 \bmod O(\hbar^2).$$

n-th step: Assume we have constructed a polynomial $\sum_{i=1}^n \Gamma_\sigma^{(i)}\hbar^i$ such that

$$(d + \hbar\Delta)\sum_{i=1}^n \Gamma_\sigma^{(i)}\hbar^i = 0 \bmod O(\hbar^{n+1}).$$

Then $d(\Delta\Gamma_\sigma^{(n)}) = 0$, and, as $H^{-2n-1}(\mathsf{GC}_2, d) = 0$, there exists a graph $\Gamma_\sigma^{(n+1)}$ in GC_2 of degree $-2n-2$ such that $\Delta\Gamma_\sigma^{(n)}) = -d\Gamma_\sigma^{(n+1)}$. Hence $(d + \hbar\Delta)\sum_{i=1}^{n+1}\Gamma_\sigma^{(i)}\hbar^i = 0 \bmod O(\hbar^{n+2})$.

It is these $\hbar$-dependent additions to $\Gamma_\sigma^{(0)}$ in $\Gamma_\sigma^\hbar$ which make the action of GRT_1 on quantum master functions different from its action on Poisson structures.

3. Quantum BV structures on odd symplectic manifolds

3.1. On L_∞ automorphisms of L_∞ algebras. Let $\mathfrak{g} = \oplus_{\imath\in\mathbb{Z}}\mathfrak{g}^i$ be a $\mathbb{Z}$-graded Lie algebra admitting a bounded above complete Hausdorff filtration, $\mathfrak{g}^i = \mathfrak{g}^i_{(0)} \supset \mathfrak{g}^i_{(1)} \supset \mathfrak{g}^i_{(2)} \supset \dots$, and let $\mathcal{MC}(\mathfrak{g}) := \left\{\gamma \in \mathfrak{g}^1 \mid [\gamma,\gamma] = 0\right\}$ be the associated set of Maurer-Cartan elements. The group $G_{(1)} := \exp \mathfrak{g}^0_{(1)}$ acts naturally on $\mathcal{MC}(\mathfrak{g})$,

$$\begin{array}{ccc} G^0_{(1)} \times \mathcal{MC}(\mathfrak{g}) & \longrightarrow & \mathcal{MC}(\mathfrak{g}) \\ (e^g, \gamma) & \longrightarrow & e^{-g}\gamma e^g. \end{array}$$

We are interested in the particular case when $\mathfrak{g}$ is the Lie algebra, $CE^\bullet(V,V)$, of coderivations

$$CE^\bullet(V,V) = \left(\mathrm{Coder}(\odot^{\bullet\geq 1}(V[2])), [\ ,\ ]\right), \quad CE^\bullet(V,V)_{(m)} := \mathrm{Hom}(\odot^{\bullet\geq m+1}(V[2]), V[2]),$$

of the standard graded co-commutative coalgebra, $\odot^{\bullet\geq 1}(V[2])$, co-generated by a vector space V. Then the set $\mathcal{MC}(CE^\bullet(V,V))$ can be identified with the set of L_∞ structures[3] on the space V. Any element $\gamma \in CE^\bullet(V,V)$ defines a differential, $d_\gamma := [\gamma,\ ]$ in $CE^\bullet(V,V)$, and any element $g \in \mathsf{Ker}\, d_\gamma \cap CE^0(V,V)_{(1)}$ defines an automorphism of $\mathcal{MC}(CE^\bullet(V,V))$ which leaves the point γ invariant. Such an element e^g can be interpreted as a L_∞ automorphism of the L_∞ algebra (V,γ). Moreover, if the cohomology class of g in $H(CE^\bullet(V,V), d_\gamma)$ is non-trivial, then this L_∞ automorphism is obviously homotopy non-trivial.

[3]In our grading conventions the degree of n-th L_∞ operation on V is equal to $3-2n$.

3.2. Quantum BV manifolds. Let M be a (formal) $\mathbb{Z}$-graded manifold equipped with an odd symplectic structure ω (of degree 1). There always exist so called Darboux coordinates, $(x^a, \psi_a)_{1\leq a\leq n}$, on M such that $|\psi_a| = -|x^a| + 1$ and $\omega = \sum_a dx^a \wedge d\psi_a$. The odd symplectic structure makes, in the obvious way, the structure sheaf $\mathcal{O}_M$ into a Lie algebra with brackets, $\{\ ,\ \}$, of degree -1. A less obvious fact is that ω induces a degree -1 differential operator, Δ_ω, on the invertable sheaf of semidensities, $Ber(M)^{\frac{1}{2}}$ [Kh]. Any choice of a Darboux coordinate system on M defines an associated trivialization of the sheaf $Ber(M)^{\frac{1}{2}}$; if one denotes the associated basis section of $Ber(M)^{\frac{1}{2}}$ by $D_{x,\psi}$, then any semidensity D is of the form $f(x,\psi)D_{x,\psi}$ for some smooth function $f(x,\psi)$, and the operator Δ_ω is given by

$$\Delta_\omega\left(f(x,\psi)D_{x,\psi}\right) = \sum_{a=1}^{n} \frac{\partial^2 f}{\partial x^a \partial \psi_a} D_{x,\psi}.$$

Let $\hbar$ be a formal parameter of degree 2. A *quantum master function* on M is an $\hbar$-dependent semidensity D which satisfies the equation

$$\Delta_\omega D = 0$$

and which admits, in some Darboux coordinate system, a form

$$D = e^{\frac{S}{\hbar}} D_{x,\psi},$$

for some $S \in \mathcal{O}_M[[\hbar]]$ of total degree 2. In the literature it is this formal power series in $\hbar$ which is often called a quantum master function. Let us denote the set of all quantum master functions on M by $\mathcal{QM}(M)$. It is easy to check that the equation $\Delta_\omega D = 0$ is equivalent to the following one,

$$\hbar \Delta S + \frac{1}{2}\{S,S\} = 0, \tag{2}$$

where $\Delta := \sum_{a=1}^{n} \frac{\partial^2}{\partial x^a \partial \psi_a}$. This equation is often called the *quantum master equation*, while a triple $(M, \omega, S \in \mathcal{QM}(M))$ a *quantum BV manifold.*

Let us assume from now on that a particular Darboux coordinate system is fixed on M up to affine transformations[4].

3.3. An action of GRT_1 on quantum master functions. The vector space $V := \mathcal{O}_M[[\hbar]]$ is a dg Lie algebra with the differential $\hbar\Delta$ and the Lie brackets $\{\ ,\ \}$. These data define a Maurer-Cartan element, $\gamma_{\mathcal{QM}} := \hbar\Delta \oplus \{\ ,\ \}$ in the Lie algebra $CE^\bullet(V,V)$.

The constant odd symplectic structure on M makes V into a representation,

$$\begin{array}{rccc} \rho: & \mathsf{Gra}(n) & \longrightarrow & \mathsf{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V) \\ & \Gamma & \longrightarrow & \Phi_\Gamma \end{array}$$

of the operad $\mathsf{Gra} := \{\mathsf{Gra}(n)\}_{n\geq 1}$ as follows:

$$\Phi_\Gamma(S_1,\ldots,S_n) := \pi\left(\prod_{e\in E(\Gamma)} \Delta_e\left(S_1(x_{(1)},\psi_{(1)},\hbar)\otimes S_2(x_{(2)},\psi_{(2)},\hbar)\otimes\ldots\otimes S_n(x_{(n)},\psi_{(n)},\hbar)\right)\right)$$

where, for an edge e connecting vertices labeled by integers i and j,

$$\Delta_e = \sum_{a=1}^{n} \frac{\partial}{\partial x^a_{(i)}} \otimes \frac{\partial}{\partial \psi_{(j)a}} + \frac{\partial}{\psi_{(i)a}} \otimes \frac{\partial}{\partial x^a_{(j)}}$$

and π is the multiplication map,

$$\begin{array}{rccc} \pi: & V^{\otimes n} & \longrightarrow & V \\ & S_1\otimes S_2\otimes\ldots\otimes S_n & \longrightarrow & S_1 S_2\cdots S_n. \end{array}$$

[4]This is not a serious loss of generality as any quantum master equation can be represented in the form (2). Our action of GRT_1 on $\mathcal{QM}(M)$ depends on the choice of an affine structure on M in exactly the same way as the classical Kontsevich's formula for a universal formality map [Ko2] depends on such a choice. A choice of an appropriate affine connection on M and methods of the paper [Do] can make our formulae for the GRT_1 action invariant under the group of symplectomorphsims of (M,ω); we do not address this *globalization* issue in the present note.

This representation induces in turn a morphism of dg Lie algebras,

$$(\mathsf{GC}_2[[\hbar]], [\ ,\ ], d_h) \longrightarrow (CE^\bullet(V,V), [\ ,\ ], \delta := [\gamma_{\mathcal{QM}},\ ])\,,$$

and hence a morphism of their cohomology groups,

$$\mathfrak{grt}_1 \simeq H^0\left(\mathsf{GC}_2[[\hbar]], d_\hbar\right) \longrightarrow H^0\left(CE^\bullet(V,V), \delta\right).$$

Let σ be an arbitrary element in $\mathfrak{grt}_1$ and let $\Gamma_\sigma^\hbar$ be a cycle representing σ in the graph complex $(\mathsf{GC}_2[[\hbar]], d_\hbar)$. For any $u \in \mathbb{R}$ the adjoint action of $e^{u\Phi_{\Gamma_\sigma^\hbar}} \in G_{(1)}$ on $CE^\bullet(V,V)$ (see §**3.1**) can be interpreted as a L_∞ automorphism,

$$F^\sigma = \{F_n^\sigma : \odot^n V \longrightarrow V[2-2n]\}_{n\geq 1}\,,$$

of the dg Lie algebra $(V, \hbar\Delta, \{\ ,\ \})$ with $F_1^\sigma = \mathrm{Id}$. Hence, for any real analytic quantum master function $S \in \mathcal{QM}(M)$ and sufficiently small $u \in \mathbb{R}^+$, the series

$$S^\sigma := S + \sum_{n\geq 2} \frac{1}{n!} F_n^\sigma(S, \ldots, S)$$

if convergent, gives again a quantum master function. This is the acclaimed homotopy action of GRT_1 on $\mathcal{QM}(M)$ for any affine odd symplectic manifold M.

3.4. Remark. In QFT one often works with quantum master functions S which are formal power series in the Darboux coordinates rather than real analytic functions. In that case one should view u as a degree 0 formal parameter so that the adjoint action of $e^{u\Phi_{\Gamma_\sigma^\hbar}} \in G_{(1)}$ on $C^\bullet(V,V)[[u]]$ gives a continuous (in the adic topology) L_∞ automorphism of of the dg Lie algebra $(V[[u]], \hbar\Delta, \{\ ,\ \})$, and hence induces a transformation of master functions from $\mathcal{O}_M[[u,\hbar]]$.

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