Optimal Berry-Esseen bounds on the Poisson space

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Abstract

We establish new lower bounds for the normal approximation in the Wasserstein distance of random variables that are functionals of a Poisson measure. Our results generalize previous findings by Nourdin and Peccati (2012, 2015) and Biermé, Bonami, Nourdin and Peccati (2013), involving random variables living on a Gaussian space. Applications are given to optimal Berry-Esseen bounds for edge counting in random geometric graphs.

Keywords: Berry-Esseen Bounds; Limit Theorems; Optimal Rates; Poisson Space; Random Graphs; Stein's Method; U-statistics.
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Contents

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1 Introduction

1.1 Overview

Let $(Z, \mathscr{Z})$ be a Borel space endowed with a σ-finite non-atomic measure μ, and let $\widehat{\eta}$ be a compensated Poisson random measure on the state space $(Z, \mathscr{Z})$, with non-atomic and σ-finite control measure μ (for the rest of the paper, we assume that all random objects are defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$). Consider a sequence of centered random variables $F_n = F_n(\widehat{\eta})$, $n \geq 1$ and assume that, as $n \to \infty$, $\mathrm{Var}(F_n) \to 1$ and F_n converges in distribution to a standard Gaussian random variable. In recent years (see e.g. [2, 6, 7, 8, 10, 16, 21, 23]) several new techniques – based on the interaction between Stein's method [4] and Malliavin calculus [9] – have been introduced, allowing one to find explicit Berry-Esseen bounds of the type

$$d(F_n, N) \leq \varphi(n), \quad n \geq 1, \tag{1.1}$$

where d is some appropriate distance between the laws of F_n and N, and $\{\varphi(n) : n \geq 1\}$ is an explicit and strictly positive numerical sequence converging to 0. The aim of this paper is to find some general sufficient conditions, ensuring that the rate of convergence induced by $\varphi(n)$ in (1.1) is *optimal*, whenever d equals the 1-Wasserstein distance d_W, that is:

$$d(F_n, N) = d_W(F_n, N) := \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}[h(F_n)] - \mathbb{E}[h(N)]|, \tag{1.2}$$

with $\mathrm{Lip}(a)$ indicating the set of a-Lipschitz mappings on $\mathbb{R}$ $(a > 0)$. As usual, the rate of convergence induced by $\varphi(n)$ is said to be optimal if there exists a constant $c \in (0, 1)$ (independent of n) such that, for n large enough,

$$\frac{d_W(F_n, N)}{\varphi(n)} \in (c, 1]. \tag{1.3}$$

As demonstrated below, our findings generalize to the framework of random point measures some previous findings (see [1, 3, 13, 15]) for random variables living on a Gaussian space. Several important differences between the Poisson and the Gaussian settings will be highlighted as our analysis unfolds. Important new applications U-statistics, in particular to edge-counting in random geometric graphs, are discussed in Section 4.

1.2 Main abstract result (and some preliminaries)

Let the above assumptions and notation prevail. The following elements are needed for the subsequent discussion, and will be formally introduced and discussed in Section 2.2:

- For every $z \in Z$ and any functional $F = F(\widehat{\eta})$, the *difference* (or *add-one cost operator*) $D_z F(\widehat{\eta}) = F(\widehat{\eta} + \delta_z) - F(\widehat{\eta})$. For reasons that are clarified below, we shall write $F \in \mathrm{dom}\, D$, whenever $\mathbb{E} \int_Z (D_s F)^2 \mu(\mathrm{d}s) < \infty$.
- The symbol L^{-1} denotes the *pseudo-inverse of the generator of the Ornstein-Uhlenbeck semigroup* on the Poisson space.

We also denote by $N \sim \mathscr{N}(0, 1)$ a standard Gaussian random variable with mean zero and variance one. It will be also necessary to consider the family

$$\mathscr{F}_W := \{f : \mathbb{R} \to \mathbb{R} : \|f'\|_\infty \leq 1 \,\mathrm{and}\, f' \in \mathrm{Lip}(2)\},$$

whereas the notation $\mathscr{F}_0$ indicates the subset of $\mathscr{F}_W$ that is composed of twice continuously differentiable functions such that $\|f'\|_\infty \leq 1$ and $\|f''\|_\infty \leq 2$.

For any two sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ of non-negative real numbers, the notation $a_n \sim b_n$ indicates that $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$.

The next theorem is the main theoretical achievement of the present paper.

Theorem 1.1. *Let $\{F_n : n \geq 1\}$ be a sequence of square-integrable functionals of $\widehat{\eta}$, such that $\mathbb{E}(F_n) = 0$, and $F_n \in \operatorname{dom} D$. Let $\{\varphi(n) : n \geq 1\}$ be a numerical sequence such that $\varphi(n) \geq \varphi_1(n) + \varphi_2(n)$, where*

$$\varphi_1(n) := \sqrt{\mathbb{E}\left(1 - \langle DF_n, -DL^{-1}F_n \rangle_{L^2(\mu)}\right)^2}, \tag{1.4}$$

$$\varphi_2(n) := \mathbb{E} \int_Z (D_z F_n)^2 \times |D_z L^{-1} F_n| \mu(\mathrm{d}z). \tag{1.5}$$

(I) *For every n, one has the estimate $d_W(F_n, N) \leq \varphi(n)$.*

(II) *Fix $f \in \mathscr{F}_0$, set $R_n^f(z) := \int_0^1 f''(F_n + (1-u)D_z F_n) u \, \mathrm{d}u$ for any $z \in Z$, and assume moreover that the following asymptotic conditions are in order :*

(i) (a) *$\varphi(n)$ is finite for every n;* (b) *$\varphi(n) \to 0$ as $n \to \infty$; and* (c) *there exists $m \geq 1$ such that $\varphi(n) > 0$ for all $n \geq m$.*

(ii) *For $\mu(\mathrm{d}z)$-almost every $z \in Z$, the sequence $D_z F_n$ converges in probability towards zero.*

(iii) *There exist a centered two dimensional Gaussian random vector (N_1, N_2) with $\mathbb{E}(N_1^2) = \mathbb{E}(N_2^2) = 1$, and $\mathbb{E}(N_1 \times N_2) = \rho$, and moreover a real number $\alpha \geq 0$ such that*

$$\left(F_n, \frac{1 - \langle DF_n, -DL^{-1}F_n \rangle_{L^2(\mu)}}{\varphi(n)} \right) \overset{law}{\to} (N_1, \alpha N_2).$$

(iv) *There exists a sequence $\{u_n : n \geq 1\}$ of deterministic and non-negative measurable functions such that $\int_Z u_n(z)\mu(\mathrm{d}z)/\varphi(n) \to \beta < \infty$, and moreover*

$$\frac{1}{\varphi(n)} \left\{ \int_Z (D_z F_n)^2 \times (-D_z L^{-1} F_n) \times R_n^f(z) \mu(\mathrm{d}z) - \int_Z u_n(z) \times R_n^f(z) \mu(\mathrm{d}z) \right\} \overset{L^1(\Omega)}{\longrightarrow} 0,$$

and $\sup_n \varphi(n)^{-(1+\epsilon)} \int_Z u_n(z)^{1+\epsilon} \mu(\mathrm{d}z) < \infty$, for some $\epsilon > 0$.

Then, as $n \to \infty$, we have

$$\frac{\mathbb{E}\left(f'(F_n) - F_n f(F_n)\right)}{\varphi(n)} \to \left\{ \frac{\beta}{2} + \rho\,\alpha \right\} \mathbb{E}(f''(N)).$$

(III) *If Assumptions* (i)–(iv) *at Point* **(II)** *are verified and $\rho\alpha \neq \frac{\beta}{2}$, then the rate of convergence induced by $\varphi(n)$ is optimal, in the sense of* (1.3).

Remark 1.1. It is interesting to observe that Assumptions **(II)**-(ii) and **(II)**-(iv) in the statement of Theorem 1.1 do not have any counterpart in the results on Wiener space obtained in [13]. To see this, let X denote a isonormal Gaussian process over a real separable Hilbert space $\mathfrak{H}$, and assume that $\{F_n : n \geq 1\}$ is a sequence of smooth functionals (in the sense of Malliavin differentiability) of X — for example, each element F_n is a finite sum of multiple Wiener integrals. Assume that $\mathbb{E}(F_n^2) = 1$, and write

$$\varphi(n) := \sqrt{\mathbb{E}(1 - \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}})^2}.$$

Assume that $\varphi(n) > 0$ for all n and also that, as $n \to \infty$, $\varphi(n) \to 0$ and the two dimensional random vector $\left(F_n, \frac{1 - \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}}}{\varphi(n)} \right)$ converges in distribution to a centered two dimensional Gaussian vector (N_1, N_2), such that $\mathbb{E}(N_1^2) = \mathbb{E}(N_2^2) = 1$ and $\mathbb{E}(N_1 N_2) = \rho \neq 0$. Then, the results of [13] imply that, for any function $f \in \mathscr{F}_W$,

$$\frac{\mathbb{E}\left(f'(F_n) - F_n f(F_n)\right)}{\varphi(n)} \to \rho \mathbb{E}(f''(N)), \tag{1.6}$$

where, as before, $N \sim \mathscr{N}(0,1)$. This implies in particular that the sequence $\varphi(n)$ determines an optimal rate of convergence, in the sense of (1.3). Also, on a Gaussian space one has that relation (1.6) extends to functions of the type f_x, where f_x is the solution of the Stein's equation associated with the indicator function $\mathbf{1}_{\{\cdot \leq x\}}$ (see Section 2.3 below): in this case the limiting value equals $\frac{\rho}{3}(x^2-1)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$.

2 Preliminaries

2.1 Poisson measures and chaos

As before, $(Z, \mathscr{Z}, \mu)$ indicates a Borel measure space such that Z is a Borel space and μ is a σ-finite and non-atomic Borel measure. We define the class $\mathscr{Z}_\mu$ as $\mathscr{Z}_\mu = \{B \in \mathscr{Z} : \mu(B) < \infty\}$. The symbol $\widehat{\eta} = \{\widehat{\eta}(B) : B \in \mathscr{Z}_\mu\}$ indicates a *compensated Poisson random measure* on $(Z, \mathscr{Z})$ with control μ. This means that $\widehat{\eta}$ is a collection of random variables defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, indexed by the elements of $\mathscr{Z}_\mu$, and such that: (i) for every $B, C \in \mathscr{Z}_\mu$ such that $B \cap C = \varnothing$, $\widehat{\eta}(B)$ and $\widehat{\eta}(C)$ are independent, (ii) for every $B \in \mathscr{Z}_\mu$,$\widehat{\eta}(B)$ has a centered Poisson distribution with parameter $\mu(B)$. Note that properties (i)-(ii) imply, in particular, that $\widehat{\eta}$ is an *independently scattered* (or *completely random*) measure. Without loss of generality, we may assume that $\mathscr{F} = \sigma(\widehat{\eta})$, and write $L^2(\mathbb{P}) := L^2(\Omega, \mathscr{F}, \mathbb{P})$. See e.g. [9, 17] for details on the notions evoked above.

Fix $n \geq 1$. We denote by $L^2(\mu^n)$ the space of real valued functions on Z^n that are square-integrable with respect to μ^n, and we write $L^2_s(\mu^n)$ to indicate the subspace of $L^2(\mu^n)$ composed of symmetric functions. We also write $L^2(\mu) = L^2(\mu^1) = L^2_s(\mu^1)$. For every $f \in L^2_s(\mu^n)$, we denote by $I_n(f)$ the *multiple Wiener-Itô integral* of order n, of f with respect to $\widehat{\eta}$. Observe that, for every $m, n \geq 1$, $f \in L^2_s(\mu^n)$ and $g \in L^2_s(\mu^m)$, one has the isometric formula (see e.g. [17]):

$$\mathbb{E}[I_n(f)I_m(g)] = n! \langle f, g\rangle_{L^2(\mu^n)} \mathbf{1}_{(n=m)}. \tag{2.1}$$

The Hilbert space of random variables of the type $I_n(f)$, where $n \geq 1$ and $f \in L^2_s(\mu^n)$ is called the nth *Wiener chaos* associated with $\widehat{\eta}$. We also use the following standard notation: $I_1(f) = \widehat{\eta}(f)$, $f \in L^2(\mu)$; $I_0(c) = c$, $c \in \mathbb{R}$. The following proposition, whose content is known as the *chaotic representation property* of $\widehat{\eta}$, is one of the crucial results used in this paper. See e.g. [17].

Proposition 2.1 (Chaotic decomposition)**.** *Every random variable $F \in L^2(\mathscr{F}, \mathbb{P}) = L^2(\mathbb{P})$ admits a (unique) chaotic decomposition of the type*

$$F = \mathbb{E}(F) + \sum_{n\geq 1}^{\infty} I_n(f_n), \tag{2.2}$$

where the series converges in L^2 and, for each $n \geq 1$, the kernel f_n is an element of $L^2_s(\mu^n)$.

2.2 Malliavin operators

We recall that the space $L^2(\mathbb{P}; L^2(\mu)) \simeq L^2(\Omega \times Z, \mathscr{F} \otimes \mathscr{Z}, \mathbb{P} \otimes \mu)$ is the space of the measurable random functions $u : \Omega \times Z \to \mathbb{R}$ such that

$$\mathbb{E}\left[\int_Z u_z^2 \, \mu(\mathrm{d}z)\right] < \infty.$$

In what follows, given $f \in L^2_s(\mu^q)$ ($q \geq 2$) and $z \in Z$, we write $f(z, \cdot)$ to indicate the function on Z^{q-1} given by $(z_1, ..., z_{q-1}) \to f(z, z_1, ..., z_{q-1})$.

(a) *The derivative operator D.* The derivative operator, denoted by D, transforms random variables into random functions. Formally, the domain of D, written dom$D$, is the set of those random variables $F \in L^2(\mathbb{P})$ admitting a chaotic decomposition (2.2) such that

$$\sum_{n\geq 1} nn! \, \|f_n\|^2_{L^2(\mu^n)} < \infty. \tag{2.3}$$

If F verifies (2.3) (that is, if $F \in \text{dom}D$), then the random function $z \to D_zF$ is given by

$$D_zF = \sum_{n\geq 1} nI_{n-1}(f(z,\cdot)), \quad z \in Z. \tag{2.4}$$

Plainly $DF \in L^2(\mathbb{P}; L^2(\mu))$, for every $F \in \text{dom}\, D$. For every random variable of the form $F = F(\widehat{\eta})$ and for every $z \in Z$, we write $F_z = F_z(\widehat{\eta}) = F(\widehat{\eta} + \delta_z)$. The following fundamental result combines classic findings from [11] and [18, Lemma 3.1].

Lemma 2.1. *For every $F \in L^2(\mathbb{P})$ one has that F is in* dom D *if and only if the mapping $z \mapsto (F_z - F)$ is an element of $L^2(\mathbb{P}; L^2(\mu))$. Moreover, in this case one has that $D_zF = F_z - F$ almost everywhere* $\mathrm{d}\mu \otimes \mathrm{d}\mathbb{P}$.

(b) *The Ornstein-Uhlenbeck generator L.* The domain of the *Ornstein-Uhlenbeck generator* (see e.g. [9, 11])), written dom L, is given by those $F \in L^2(\mathbb{P})$ such that their chaotic expansion (2.2) verifies

$$\sum_{n\geq 1} n^2 n! \, \|f_n\|^2_{L^2(\mu^n)} < \infty.$$

If $F \in \text{dom}L$, then the random variable LF is given by

$$LF = -\sum_{n\geq 1} nI_n(f_n). \tag{2.5}$$

We will also write L^{-1} to denote the pseudo-inverse of L. Note that $\mathbb{E}(LF) = 0$, by definition. The following result is a direct consequence of the definitions of D and L, and involves the adjoint δ of D (with respect to the space $L^2(\mathbb{P}; L^2(\mu))$ — see e.g. [9, 11] for a proof).

Lemma 2.2. *For every $F \in \text{dom}L$, one has that $F \in \text{dom}D$ and DF belongs to the domain to the adjoint δ of D. Moreover,*

$$\delta DF = -LF. \tag{2.6}$$

2.3 Some estimates based on Stein's method

We shall now present some estimates based on the use of Stein's method for the one-dimensional normal approximation. We refer the reader to the two monographs [4, 14] for a detailed presentation of the subject. Let F be a random variable and let $N \sim \mathscr{N}(0,1)$, and consider a real-valued function $h : \mathbb{R} \to \mathbb{R}$ such that the expectation $\mathbb{E}[h(X)]$ is well-defined. We recall that the *Stein equation* associated with h and F is classically given by

$$h(u) - \mathbb{E}[h(F)] = f'(u) - uf(u), \quad u \in \mathbb{R}. \tag{2.7}$$

A solution to (2.7) is a function f depending on h which is Lebesgue a.e.-differentiable, and such that there exists a version of f' verifying (2.7) for every $x \in \mathbb{R}$. The following lemma gathers together some fundamental relations. Recall the notation $\mathscr{F}_W$ and $\mathscr{F}_0$ introduced in Section 1.2.

Lemma 2.3. (i) *If $h \in \text{Lip}(1)$, then (2.7) has a solution f_h that is an element of $\mathscr{F}_W$.*

(ii) *If h is twice continuously differentiable and $\|h'\|_\infty, \|h''\|_\infty \leq 1$, then (2.7) has a solution f_h that is an element of $\mathscr{F}_0$.*

(iii) *Let F be an integrable random variable. Then,*

$$d_W(F,N) = \sup_{h\in \mathrm{Lip}(1)} \left|\mathbb{E}[f_h(F)F - f_h'(F)]\right| \leq \sup_{f\in\mathscr{F}_W} \left|\mathbb{E}[f(F)F - f'(F)]\right|.$$

(iv) *If, in addition, F is a centered element of* $\mathrm{dom}\, D$*, then*

$$d_W(F,N) \quad \leq \quad \sqrt{\mathbb{E}\left(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\right)^2} \qquad (2.8)$$
$$+\mathbb{E}\int_Z (D_zF)^2 \times |D_zL^{-1}F|\mu(\mathrm{d}z).$$

Both estimates at Point (i) and (ii) follow e.g. from [5, Theorem 1.1]. Point (iii) is an immediate consequence of the definition of d_W, as well as of the content of Point (i) and Point (ii) in the statement. Finally, Point (iv) corresponds to the main estimate established in [16].

3 Proof of Theorem 1.1

We start with a general lemma.

Lemma 3.1. *Let F be such that $\mathbb{E}(F) = 0$ and $F \in \mathrm{dom}D$. Assume that $N \sim \mathscr{N}(0,1)$. For any $f \in \mathscr{F}_0$, and $z \in Z$ we set*

$$R^f(z) := \int_0^1 f''\left(F + (1-u)D_zF\right) u\,\mathrm{d}u. \qquad (3.1)$$

Then,

$$\mathbb{E}\left(f'(F) - Ff(F)\right) = \mathbb{E}\Big(f'(F)\left(1 - \langle DF, -DL^{-1}F\rangle_{L(\mu)}\right)\Big)$$
$$+\,\mathbb{E}\int_Z (D_zF)^2 \times (-D_zL^{-1}F) \times R^f(z)\mu(\mathrm{d}z).$$

Proof. Using Lemma 2.2 and the characterization of δ as the adjoint of D, we deduce that, for any $f \in \mathscr{F}_0$

$$\begin{aligned}\mathbb{E}(Ff(F)) &= \mathbb{E}(LL^{-1}Ff(F)) = \mathbb{E}(\delta(-DL^{-1}F)f(F)) \\ &= \mathbb{E}(\langle Df(F), -DL^{-1}F\rangle_{L^2(\mu)}).\end{aligned} \qquad (3.2)$$

In view of Lemma 2.1) and of a standard application of Taylor formula, one immediately infers that

$$\begin{aligned}D_zf(F) := f(F_z) - f(F) &= f'(F)D_zF + \int_F^{F_z} f''(u)(F_z - u)\mathrm{d}u \\ &= f'(F)D_zF + (D_zF)^2 \times \int_0^1 f''(F + (1-u)D_zF)u\,\mathrm{d}u.\end{aligned} \qquad (3.3)$$

Plugging (3.3) into (3.2), we deduce the desired conclusion. □

Proof of Theorem 1.1. Part **(I)** is a direct consequence of Lemma 2.3-(iv). To prove Point **(II)**, we fix $f \in \mathscr{F}_0$, and use Lemma 3.1 to deduce that

$$\begin{aligned}\frac{\mathbb{E}\left(f'(F_n) - F_nf(F_n)\right)}{\varphi(n)} &= \mathbb{E}\left(f'(F_n) \times \frac{1 - \langle DF_n, -DL^{-1}F_n\rangle_{L^2(\mu)}}{\varphi(n)}\right) \\ &+ \frac{1}{\varphi(n)}\mathbb{E}\int_Z (D_zF_n)^2 \times (-D_zL^{-1}F_n) \times R_n^f(z)\mu(\mathrm{d}z) \\ &:= I_{1,n} + I_{2,n}.\end{aligned}$$

Assumption (iii) implies that

$$\sup_{n\geq 1}\mathbb{E}\left(\frac{1-\langle DF_n,-DL^{-1}F_n\rangle_{L^2(\mu)}}{\varphi(n)}\right)^2<+\infty.$$

Since $\|f'\|_\infty\leq 1$ by assumption, we infer that the class

$$\left\{f'(F_n)\times\frac{1-\langle DF_n,-DL^{-1}F_n\rangle_{L^2(\mu)}}{\varphi(n)}:n\geq 1\right\}$$

is uniformly integrable. Assumption (iii) implies therefore that, as $n\to\infty$,

$$I_{1,n}\to\mathbb{E}\left(f'(N_1)\times\alpha N_2\right)=\rho\times\alpha\,\mathbb{E}f''(N).$$

To deal with the term $I_{2,n}$, first note that for each $z\in Z$, Assumptions (ii) and (iii) and Slutsky Theorem imply that, for any $u\in(0,1)$,

$$F_n+(1-u)D_zF_n\overset{law}{\to}N.$$

Therefore, using the fact that $\|f''\|_\infty\leq 2$, and by a direct application of the dominated convergence Theorem, we infer that

$$\mathbb{E}R_n^f(z)\to\int_0^1\mathbb{E}(f''(N))u\mathrm{d}u=\frac{1}{2}\mathbb{E}f''(N). \tag{3.4}$$

At this point, Assumption (iv) and the triangle inequality immediately imply that, in order to obtain the desired conclusion, it is sufficient to prove that, as $n\to\infty$,

$$\frac{1}{\varphi(n)}\int_Z u_n(z)\times\left\{\mathbb{E}R_n^f(z)-\frac{1}{2}\mathbb{E}f''(N)\right\}\mu(\mathrm{d}z)\to 0. \tag{3.5}$$

To show (3.5), it is enough to prove that the function integrated on the right-hand side is uniformly integrable: this is straightforward, since $|R_n^f(z)-\frac{1}{2}\mathbb{E}f''(N)|\leq 2$, and of the fact that the sequence $n\mapsto\varphi(n)^{-(1+\epsilon)}\int_Z u_n(z)^{1+\epsilon}\mu(\mathrm{d}z)$ is bounded. In view of the first equality in Lemma 2.3-(iii), to prove the remaining Point **(III)** in the statement, it is enough to show that there exists a function h such that $\|h'\|_\infty$, $\|h''\|_\infty\leq 1$, and $\mathbb{E}[f_h''(N)]\neq 0$. Selecting $h(x)=\sin x$, we deduce from [1, formula (5.2)] that $\mathbb{E}f_h''(N)=3^{-1}\mathbb{E}[\sin(N)H_3(N)]=e^{-1/2}>0$, thus concluding the proof. □

4 Applications to U-statistics

4.1 Preliminaries

In this section we shall present some important application of our theoretical finding. We introduce the concept of a *U-statistics* associated with the Poisson random measure η. Those are the most natural examples of elements in $L^2(\mathbb{P})$ and having a finite Wiener-Itô expansion.

Definition 4.1. *Fix $k\geq 1$. A random variable F is called a Ustatistics of order k, based on a Poisson random measure η with control measure μ, if there exists a symmetric kernel $h\in L_s^1(\mu^k)$ such that*

$$F=\sum_{\boldsymbol{x}\in\eta_{\neq}^k}h(\boldsymbol{x}), \tag{4.1}$$

where the symbol $\eta_{\neq}^k$ indicates the class of all k-dimensional vectors $\boldsymbol{x}=(x_1,\cdots,x_k)$ such that $x_i\in\eta$ and $x_i\neq x_j$ for every $1\leq i\neq j\leq k$.

The following crucial fact on Poisson U-statistics is taken from [21, Lemma 3.5 and Theorem 3.6]. We shall also adopt the special notation $L^{p,p'}(Z^k,\mu^k) = L^p(Z^k,\mu^k) \cap L^{p'}(Z^k,\mu^k)$ for $p,p' \geq 1$.

Proposition 1. *Consider a kernel $h \in L^1_s(\mu^k)$ such that the corresponding U-statistic F given by* (4.10) *is square-integrable. Then, h is necessarily square-integrable, and F admits a representation of the form*

$$F = \mathbb{E}(F) + \sum_{i=1}^{\infty} I_i(g_i),$$

where

$$g_i(\boldsymbol{x}_i) := h_i(\boldsymbol{x}_i) = \binom{k}{i} \int_{Z^{k-i}} h(\boldsymbol{x}_i, \boldsymbol{x}_{k-i})\, \mathrm{d}(\mu^{k-i}), \quad \boldsymbol{x}_i \in Z_i, \tag{4.2}$$

for $1 \leq i \leq k$, and $g_i = 0$ for $i > k$. In particular, $h = g_k$, and the projection $g_i \in L^{1,2}_s(\mu^i)$ for each $1 \leq i \leq k$.

Now, let X be a compact subset of $\mathbb{R}^d$ endowed with the Lebesgue measure l, and let M be a locally compact space, that we shall call the *mark space*, endowed with a probability measure ν. We shall assume that X contains the origin in its interior, is symmetric, that is: $X = -X$, and that the boundary of X is negligible with respect to Lebesgue measure. We set $Z = X \times M$, and we endow such a product space with the measure $\mu = l \otimes \nu$ on $\mathbb{R}^d$. Define $\mu_n = n\mu$, and let η_n be a Poisson measure on Z with control measure μ_n. As before, $h \in L^{1,2}_s(Z^k,\mu^k)$ is such that each functional $F_n := \sum_{\mathbf{x}\in\eta^k_{\neq}} h(\mathbf{x})$, $n \geq 1$ is a square-integrable U-statistic. In the terminology of [21] each F_n is a *geometric U-statistic.* Note that, according to Proposition 1, we have

$$F_n = \mathbb{E}(F_n) + \sum_{i=1}^{k} F_{i,n} := \mathbb{E}(F_n) + \sum_{i=1}^{k} I_i(g_{i,n}),$$

where

$$g_{i,n}(\mathbf{x}_i) = n^{k-i}\binom{k}{i} \int_{Z^{k-i}} h(\mathbf{x}_i, \mathbf{x}_{k-i}) \mathrm{d}\mu^{k-i} := n^{k-i} h_i(\mathbf{x}_i),\ \mathbf{x}_i \in Z^i, \tag{4.3}$$

and each multiple integral is realized with respect to compensated Poisson random measure $\hat{\eta}_n = \eta_n - n\mu$. Also, since X is a compact set of $\mathbb{R}^d$ and ν is a probability measure, one has that, for every n, $\mu_n(Z) = nl(X)\nu(M) < \infty$ and $\eta_n(Z)$ is a Poisson random variable of parameter $\mu_n(Z)$.

4.2 Geometric U-statistics of order 2

Our main result of this section is given in below. Note that items (a) and (b) in the forthcoming theorem are a consequence of [8, Theorem 7.3], as well [21].

Theorem 4.1. *Assume that the kernels $h_{1,n}$ and $h_{2,n}$ are given by the RHS of* (4.3)*.*

(a) *If $\|h_1\|_{L^2(\mu)} > 0$, then as $n \to \infty$, one has the exact asymptotic*

$$\mathrm{Var}(F_n) \sim \mathrm{Var}(F_{1,n}) \sim \|h_1\|^2_{L^2(\mu)}\, n^3.$$

(b) *Define*

$$\widetilde{F}_n := \frac{F_n - \mathbb{E}F_n}{\sqrt{\mathrm{Var}(F_n)}}, \quad n \geq 1.$$

Set $\tilde{h}_{1,n} = (\mathrm{Var}(F_{1,n}))^{-\frac{1}{2}}\, h_{1,n}$, and $\widetilde{\varphi}(n) := \|\tilde{h}_{1,n}\|^3_{L^3(\mu_n)} = \|h_1\|^3_{L^3(\mu)} \times \|h_1\|^{-3}_{L^2(\mu)} \times n^{-\frac{1}{2}}$. Let $N \sim \mathscr{N}(0,1)$. Then, there exists a constant $C \in (0,\infty)$, independent of n, such that

$$d_W(\widetilde{F}_n, N) \leq C\, \widetilde{\varphi}(n). \tag{4.4}$$

(c) *Denote*

$$\alpha^2_{h_1,h_2} := \frac{9\|h_1\|^2_{L^2(\mu)}\,\|h_1 \star_1^1 h_2\|^2_{L^2(\mu)}}{\|h_1\|^6_{L^3(\mu)}}, \quad \textit{and} \tag{4.5}$$

$$\rho_{h_1,h_2} := -\frac{3\langle h_1, h_1 \star_1^1 h_2\rangle_{L^2(\mu)}}{\|h_1\|^3_{L^3(\mu)}}. \tag{4.6}$$

If moreover $h(x_1,x_2) \geq 0$, for $(x_1,x_2) \in Z^2$ a.e. μ^2, and also $\alpha_{h_1,h_2} \times \rho_{h_1,h_2} \neq -1/2$, then there exists a constant $0 < c < C$ such that, for n large enough,

$$c\,\widetilde{\varphi}(n) \leq d_W(\widetilde{F}_n, N) \leq C\,\widetilde{\varphi}(n),$$

and the rate of convergence induced by the sequence $\varphi(n) := C\,\widetilde{\varphi}(n) = Cn^{-1/2}$ is therefore optimal.

Before giving a proof of Theorem 4.1, we present three following lemmas.

Lemma 4.1. *Let $h \in L^{1,2}_s(Z^2,\mu^2)$ be a non-negative kernel, and F_n be the corresponding geometric U-statistic of order 2 given by (4.1) based on Poisson random measure η_n. Assume that $\|h_1\| \neq 0$, where the kernels h_1 and h_2 are given by (4.3). Denote*

$$\tilde{F}_n = \frac{F_n - \mathbb{E}(F_n)}{\sqrt{Var(F_n)}} := \frac{1}{\sqrt{Var(F_n)}}\left\{I_1(g_{1,n}) + I_2(g_{2,n})\right\}.$$

Then, for any function $f \in \mathscr{F}_0$, and every $z \in Z$, as $n \to \infty$, we have

$$\mathbb{E}\int_0^1 f''(\tilde{F}_n + (1-u)D_z\tilde{F}_n)\mathrm{d}u \to \mathbb{E}(f''(N)).$$

Proof. According to [8, Theorem 7.3], under assumption $\|h_1\| \neq 0$, we know that the sequence $\tilde{F}_n$ converges in distribution to $N \sim \mathscr{N}(0,1)$. Moreover $\mathrm{Var}(F_n) \sim n^3$, and $\mathbb{E}(I_2(g_{2,n})^2) = 2\|g_{2,n}\|^2_{L^2(\mu_n^2)} \sim n^2$. Fix $z \in Z$. Then by definition of Malliavin derivative, we have

$$D_z\tilde{F}_n = \frac{1}{\sqrt{\mathrm{Var}(F_n)}}\left\{g_{1,n}(z) + 2I_1(g_{2,n}(z,\cdot))\right\}.$$

Clearly, using representation of kernels $g_{1,n} = nh_1(z)$, and $g_{2,n} = h_2$, we obtain that $\mathrm{Var}(F_n)^{-\frac{1}{2}} \times g_{1,n}(z) = O_z(n^{-\frac{1}{2}}) \to 0$, as $n \to \infty$. For the other term, using the isometry for Poisson multiple integrals, and the representation $g_{2,n} = h_2$ (in particular $g_{2,n}$ does not depend on n), we obtain that

$$\mathbb{E}\Big(\frac{I_1(g_{2,n}(z,\cdot))}{\sqrt{\mathrm{Var}(F_n)}}\Big)^2 = \frac{\int_Z g^2_{2,n}(z,x)\mu_n(\mathrm{d}x)}{\mathrm{Var}(F_n)} = O_z(n^{-2}) \to 0.$$

Hence, $D_z\tilde{F}_n$ converges in distribution to zero, and therefore using Slutsky Theorem, for any $u \in (0,1)$, we conclude that the sequence $\tilde{F}_n + (1-u)D_z\tilde{F}_n$ converges in distribution to $N \sim \mathscr{N}(0,1)$. Now, since $\|f''\|_\infty \leq 2$, a direct application of dominated convergence theorem completes the proof. ☐

Lemma 4.2. *Let all notations and assumptions of Lemma 4.1 prevail. Denote*

$$\alpha^2_{h_1,h_2} := \frac{9\|h_1\|^2_{L^2(\mu)}\,\|h_1 \star_1^1 h_2\|^2_{L^2(\mu)}}{\|h_1\|^6_{L^3(\mu)}}, \quad \textit{and} \tag{4.7}$$

$$\rho_{h_1,h_2} := -\frac{3\langle h_1, h_1 \star_1^1 h_2\rangle_{L^2(\mu)}}{\|h_1\|^3_{L^3(\mu)}}. \tag{4.8}$$

Set $\tilde{g}_{1,n} = (\,Var(F_{1,n}))^{-\frac{1}{2}}\, g_{1,n}$, and $\varphi(n) := \|\tilde{g}_{1,n}\|^3_{L^3(\mu_n)}$. Assume that $\alpha^2_{h_1,h_2} \neq 0$. Then, as $n \to \infty$, two dimensional random vector

$$\left(\tilde{F}_n, \frac{1 - \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\mu_n)}}{\varphi(n)} \right)$$

converges in distribution to two dimensional random vector (N_1, N_2), where $N_1 \sim \mathscr{N}(0,1)$, $N_2 \sim \mathscr{N}(0, \alpha^2_{h_1,h_2})$, and moreover $\mathbb{E}(N_1 \times N_2) = \rho_{h_1,h_2}$.

Proof. First note that the random variables $\tilde{F}_n$ take the form

$$\tilde{F}_n = \frac{1}{\sqrt{\text{Var}(F_n)}} \left\{ I_1(g_{1,n}) + I_2(g_{2,n}) \right\},$$

where the kernels $g_{1,n}$ and $g_{2,n}$ are given by (4.3). Recall that under assumption $\|h_1\| \neq 0$, according to [8, Theorem 7.3], we know that sequence $\tilde{F}_n$ converges in distribution to $N \sim \mathscr{N}(0,1)$. Moreover, representation (4.3) yields that $\varphi(n) = \|h_1\|^3_{L^3(\mu)} \times \|h_1\|^{-3}_{L^2(\mu)}\, n^{-\frac{1}{2}}$. Now, fix $z \in Z$. Then

$$\begin{aligned} D_z\tilde{F}_n &= \frac{1}{\sqrt{\text{Var}(F_n)}} \left\{ g_{1,n}(z) + 2I_1(g_{2,n}(z,\cdot)) \right\}, \quad \text{and} \\ -D_zL^{-1}\tilde{F}_n &= \frac{1}{\sqrt{\text{Var}(F_n)}} \left\{ g_{1,n}(z) + I_1(g_{2,n}(z,\cdot)) \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\mu_n)} &= \frac{1}{\text{Var}(F_n)} \Big\{ \int_Z g^2_{1,n}(z)\mu_n(\mathrm{d}z) \\ &+ 3\int_Z g_{1,n}(z) I_1(g_{2,n}(z,\cdot))\mu_n(\mathrm{d}z) + 2\int_Z I_1^2(g_{2,n}(z,\cdot))\mu_n(\mathrm{d}z) \Big\}. \end{aligned}$$

Using multiplication formula for multiple Poisson integrals, one can deduce that

$$\begin{aligned} X_n &:= \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\mu_n)} - 1 \\ &= \frac{1}{\text{Var}(F_n)} \Big\{ 3I_1(g_{1,n} \star_1^1 g_{2,n}) + 2I_1(g_{2,n} \star_2^1 g_{2,n}) + 2I_2(g_{2,n} \star_1^1 g_{2,n}) \Big\}. \end{aligned}$$

On the other hand, we know that $\text{Var}(F_n) \sim n^3 \times \|h_1\|^2_{L^2(\mu)}$, as $n \to \infty$. Also, the isometry property of Poisson multiple integrals tells us that $\mathbb{E}(I_1(g_{1,n} \star_1^1 g_{2,n})^2) = \|g_{1,n} \star_1^1 g_{2,n}\|^2_{L^2(\mu_n)} = n^5 \times \|h_1 \star_1^1 h_2\|^2_{L^2(\mu)}$. Hence $\mathbb{E}(X_n^2) \sim n^{-1}$, as $n \to \infty$. Let denote $\frac{X_n}{\varphi(n)} := X_{1,n} + X_{2,n} + X_{3,n}$, where

$$\mathbb{E}(X^2_{2,n}) := \mathbb{E}\left(\frac{I_1(2g_{2,n} \star_2^1 g_{2,n})}{\varphi(n)\text{Var}(F_n)} \right)^2 = O(n^{-2}) \to 0.$$

And similarly,

$$\mathbb{E}(X^2_{3,n}) := \mathbb{E}\left(\frac{I_2(2g_{2,n} \star_1^1 g_{2,n})}{\varphi(n)\text{Var}(F_n)} \right)^2 = O(n^{-1}) \to 0.$$

Hence, in particular these observations imply that sequences $X_{2,n}$ and $X_{3,n}$ converge in distribution to 0. Also,

$$\mathbb{E}(X_{1,n}^2) := \mathbb{E}\left(\frac{I_1(3g_{1,n}\star_1^1 g_{2,n})}{\varphi(n)\mathrm{Var}(F_n)}\right)^2 = \frac{9\|h_1\star_1^1 h_2\|^2_{L^2(\mu)}\times n^5}{\varphi(n)^2\mathrm{Var}(F_n)^2}$$

$$\text{as}\overset{n\to\infty}{\longrightarrow} \frac{9\|h_1\|^2_{L^2(\mu)}\|h_1\star_1^1 h_2\|^2_{L^2(\mu)}}{\|h_1\|^6_{L^3(\mu)}} = \alpha^2_{h_1,h_2}\neq 0.$$

Moreover, as $n\to\infty$:

$$\left\|\frac{g_{1,n}\star_1^1 g_{2,n}}{\varphi(n)\mathrm{Var}(F_n)}\right\|^3_{L^3(\mu_n)} \sim n^{-\frac{1}{2}}\to 0.$$

Hence, according to [16, Corollary 3.4], sequence $X_{1,n}$ converges in distribution to $\mathscr{N}(0,\sigma^2_{h_1,h_2})$. Now, [19, Corollary 3.4] also implies that two dimensional random vector

$$\Big(\frac{I_1(g_{1,n})}{\sqrt{\mathrm{Var}(F_n)}}, \frac{I_1(3g_{1,n}\star_1^1 g_{2,n})}{\varphi(n)\mathrm{Var}(F_n)}\Big) \overset{\text{law}}{\longrightarrow} (N_1,N_2),$$

where $N_1\sim\mathscr{N}(0,1)$, $N_2\sim\mathscr{N}(0,\alpha^2_{h_1,h_2})$, and moreover

$$\mathbb{E}\Big(\frac{I_1(g_{1,n})}{\sqrt{\mathrm{Var}(F_n)}}\times\frac{I_1(3g_{1,n}\star_1^1 g_{2,n})}{\varphi(n)\mathrm{Var}(F_n)}\Big) = \frac{3}{\varphi(n)\mathrm{Var}^{\frac{3}{2}}(F_n)}\langle g_{1,n}, g_{1,n}\star_1^1 g_{2,n}\rangle_{L^2(\mu_n)}$$

$$\overset{n\to\infty}{\longrightarrow} \frac{3\langle h_1, h_1\star_1^1 h_2\rangle_{L^2(\mu)}}{\|h_1\|^3_{L^3(\mu)}} = -\rho_{h_1,h_2} = -\mathbb{E}(N_1N_2).$$

Now, an application of multi-dimensional version of Slutsky Theorem completes the proof. □

Lemma 4.3. *Let all notations and assumptions of Lemmas 4.1 and 4.2 prevail. Assume that $\rho_{h_1,h_2}\neq-\frac{1}{2}$, and $N\sim\mathscr{N}(0,1)$. Then, for any function $f\in\mathscr{F}_0$, as $n\to\infty$, we have*

$$\frac{\mathbb{E}\left(f'(\tilde F_n)-\tilde F_n f(\tilde F_n)\right)}{\varphi(n)} \to \left\{\frac{1}{2}+\rho_{h_1,h_2}\right\}\mathbb{E}(f''(N)).$$

Proof. Recall that according to [8, Theorem 7.3], assumption $\|h_1\|\neq 0$ implies that $d_W(\tilde F_n,\mathscr{N}(0,1))\le Cn^{-1/2}$ for some constant C independent of n. Let $f\in\mathscr{F}_0$. Then according to Lemma 3.1, we have

$$\frac{\mathbb{E}\left(f'(\tilde F_n)-\tilde F_n f(\tilde F_n)\right)}{\varphi(n)} = \mathbb{E}\left(f'(\tilde F_n)\times\frac{1-\langle D\tilde F_n,-DL^{-1}\tilde F_n\rangle_{L^2(\mu_n)}}{\varphi(n)}\right)$$

$$+\frac{1}{2\varphi(n)}\mathbb{E}\int_Z (D_z\tilde F_n)^2\times(-D_zL^{-1}\tilde F_n)\times R_n^f(z)\mu_n(\mathrm{d}z).$$

Note that $\|f'\|_\infty\le 1$. Also, proof of Lemma 4.2 reveals that for some constant M

$$\sup_{n\ge1}\mathbb{E}\left(\frac{1-\langle D\tilde F_n,-DL^{-1}\tilde F_n\rangle_{L^2(\mu_n)}}{\varphi(n)}\right)^2 \le M<\infty.$$

Hence, the sequence

$$\Big\{f'(\tilde F_n)\times\frac{1-\langle D\tilde F_n,-DL^{-1}\tilde F_n\rangle_{L^2(\mu_n)}}{\varphi(n)}\Big\}_{n\ge1}$$

is uniformly integrable. Therefore, using Lemma 4.2, one can deduce as $n \to \infty$ that

$$\begin{aligned} \mathbb{E}\Big(f'(\tilde{F}_n) \times \frac{1 - \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\mu_n)}}{\varphi(n)} \Big) &\to \mathbb{E}(f'(N_1)N_2) \\ &= \rho_{h_1,h_2} \mathbb{E}(f''(N)). \end{aligned}$$

Hence, we are left to show that as $n \to \infty$, we have

$$\frac{1}{2\varphi(n)} \mathbb{E} \int_Z (D_z\tilde{F}_n)^2 \times (-D_zL^{-1}\tilde{F}_n) \times R_n^f(z) \mu_n(\mathrm{d}z) \to \frac{1}{2}\mathbb{E}(f''(N)).$$

First, note that

$$\begin{aligned} (D_z\tilde{F}_n)^2 \times (-D_zL^{-1}\tilde{F}_n) = \frac{1}{(\mathrm{Var}(F_n))^{\frac{3}{2}}} \Big\{ & g_{1,n}^3(z) + 5g_{1,n}^2(z) \times I_1(g_{2,n}(z,\cdot)) \\ & + 8g_{1,n}(z) \times I_1(g_{2,n}(z,\cdot))^2 \Big\}. \end{aligned}$$

As a result,

$$\begin{aligned} &\frac{1}{2\varphi(n)} \mathbb{E} \int_Z (D_z\tilde{F}_n)^2 \times (-D_zL^{-1}\tilde{F}_n) \times R_n^f(z)\, \mu_n(\mathrm{d}z) \\ &= \frac{(\mathrm{Var}(F_n))^{-\frac{3}{2}}}{2\varphi(n)} \Big\{ \mathbb{E} \int_Z g_{1,n}^3(z) \times R_n^f(z)\, \mu_n(\mathrm{d}z) \\ &+ 5\mathbb{E} \int_Z g_{1,n}^2(z)\, I_1(g_{2,n}(z,\cdot)) \times R_n^f(z)\, \mu_n(\mathrm{d}z) \\ &+ 8\mathbb{E} \int_Z g_{1,n}(z) \times I_1(g_{2,n}(z,\cdot))^2 \times R_n^f(z)\, \mu_n(\mathrm{d}z) \Big\} \\ &:= B_{1,n} + B_{2,n} + B_{3,n}. \end{aligned}$$

Now, using the facts that $\mathrm{Var}(F_n) \sim \mathrm{Var}(F_{1,n})$, and Lemma 4.1, for the convergence $B_{1,n} \to \frac{1}{2}\mathbb{E}(f''(N))$, it is enough to show that the sequence

$$\left\{ \frac{(\mathrm{Var}(F_n))^{-\frac{3}{2}}}{2\varphi(n)} \int_Z g_{1,n}^3(z) \left(R_n^f(z) - \frac{1}{2}\mathbb{E}(f''(N)) \right) \mu_n(\mathrm{d}z) \right\}_{n \geq 1}$$

is uniformly integrable. However, this is clear because of the fact $|R_n^f(z) - \frac{1}{2}\mathbb{E}(f''(N))| \leq 2$, and moreover for any $\epsilon > 0$, we have

$$\int_Z \Big(\frac{g_{1,n}^3(z)}{\varphi(n) \times (\mathrm{Var}(F_n))^{\frac{3}{2}}} \Big)^{1+\epsilon} \mu_n(\mathrm{d}z) = O(n^{-\epsilon}).$$

For the term $B_{2,n}$, using the fact $|R_n^f| \leq 1$, and Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned} \mathbb{E}\left(I_1(g_{2,n}(z,\cdot)) \times R_n^f(z) \right) &\leq \sqrt{\mathbb{E}(I_1(g_{2,n}(z,\cdot))^2)} \\ &= C_z n^{\frac{1}{2}} \end{aligned}$$

where $C_z = \sqrt{\|h_2(z,\cdot)\|^2_{L^2(\mu)}} > 0$. Therefore $B_{2,n} = O(n^{-\frac{1}{2}}) \to 0$, as $n \to \infty$. For the last term $B_{3,n}$, we again use $|R_n^f| \leq 1$ and isometry for Poisson multiple integrals to obtain that

$$\begin{aligned} |B_{3,n}| &\leq \frac{(\mathrm{Var}(F_n))^{-\frac{3}{2}}}{2\varphi(n)} \int_Z g_{1,n}(z) \left(\int_Z g_{2,n}^2(z,x)\mu_n(\mathrm{d}x) \right) \mu_n(\mathrm{d}z) \\ &= \frac{(\mathrm{Var}(F_n))^{-\frac{3}{2}}}{2\varphi(n)} \times n^3 \times \int_Z h_1(z) \left(\int_Z h_2^2(z,x)\mu(\mathrm{d}x) \right) \mu(\mathrm{d}z). \end{aligned}$$

Now, since $\varphi(n) \sim (\mathrm{Var}(F_n))^{-\frac{3}{2}} \times n^4$, we deduce that in fact $B_{3,n} = O(n^{-1}) \to 0$, as $n \to \infty$. This completes the proof. □

Proof of Theorem 4.1. One just needs to apply Theorem 1.1, item **(III)**. □

4.3 Edge-counting in random geometric graphs

In this section, we shall apply our main results to the following situation:

- $\widehat{\eta}$ is a compensated Poisson measure on the product space $(\mathbb{R}_+ \times Z, \mathscr{B}(\mathbb{R}_+) \otimes \mathcal{Z})$ (where $(Z, \mathcal{Z})$ is a Borel space) with control measure given by

$$\nu := \ell \times \mu, \tag{4.9}$$

  with $\ell(\mathrm{d}x) = \mathrm{d}x$ equal to the Lebesgue measure and μ equal to a σ-finite Borel measure with no atoms.
- For every $n \geq 1$, we set $\widehat{\eta}_n$ to be the Poisson measure on $(Z, \mathcal{Z})$ given by the mapping $A \mapsto \widehat{\eta}_n(A) := \widehat{\eta}([0, n] \times A)$ $(A \in \mathcal{Z}_\mu)$, in such a way that $\widehat{\eta}_n$ is a Poisson measure on $(Z, \mathcal{Z})$, with intensity $\mu_n := n \times \mu$.
- For every n, the random variable F_n is a U-statistic of order 2 with respect to the Poisson measure $\eta_n := \widehat{\eta}_n + \mu_n$, in the sense of the following definition.

Definition 4.2. A random variable F is called a *U-statistic* of order 2, based on the Poisson random measure η_n defined above, if there exists a kernel $h \in L^1_s(\mu^2)$ (that is, h is symmetric and in $L^1_s(\mu^2)$, such that

$$F = \sum_{(x_1, x_2) \in \eta^2_{n,\neq}} h(x_1, x_2), \tag{4.10}$$

where the symbol $\eta^2_{n,\neq}$ indicates the class of all 2-dimensional vectors (x_1, x_2) such that x_i is in the support of η_n $(i = 1, 2)$ and $x_1 \neq x_2$.

We recall that, according to the general results proved in [21, Lemma 3.5 and Theorem 3.6], one has that, if a random variable F as in (4.10) is square-integrable, then necessarily $h \in L^2_s(\mu^2)$, and F admits a representation of the form

$$F = \mathbb{E}(F) + F_1 + F_2 := \mathbb{E}(F) + I_1(h_1) + I_2(h_2), \tag{4.11}$$

where I_1 and I_2 indicate (multiple) Wiener-Itô integrals of order 1 and 2, respectively, with respect to $\widehat{\eta}$, and

$$\begin{aligned} h_1(t, z) &:= 2\mathbf{1}_{[0,n]}(t) \int_Z h(a, z)\, \mu_n(\mathrm{d}a) \qquad (4.12)\\ &= 2\mathbf{1}_{[0,n]}(t) \int_{\mathbb{R}_+ \times Z} \mathbf{1}_{[0,n]}(s) h(a, z)\, \nu(\mathrm{d}s, \mathrm{d}a) \\ &= 2\mathbf{1}_{[0,n]}(t) n \int_Z h(a, z)\, \mu(\mathrm{d}a) \in L^2(\mu), \\ h_2((t_1, x_1), (t_2, x_2)) &:= \mathbf{1}_{[0,n]^2}(t_1, t_2) h(x_1, x_2), \qquad (4.13) \end{aligned}$$

where ν is defined in (4.9). Let the framework and notation of Section 4.1 prevail, set $Z = \mathbb{R}^d$, and assume that μ is a probability measure that is absolutely continuous with respect to the Lebesgue measure, with a density f that is bounded and everywhere continuous. It is a standard result that, in this case, the non-compensated Poisson measure η_n has the same distribution as the point process

$$A \mapsto \sum_{i=1}^{N_n} \delta_{Y_i}(A), \quad A \in \mathscr{B}(\mathbb{R}^d),$$

where δ_y indicates the Dirac mass at y, $\{Y_i : i \geq 1\}$ is a sequence of i.i.d. random variables with distribution μ, and N_n is an independent Poisson random variable with mean n. Throughout this section, we consider a sequence $\{t_n : n \geq 1\}$ of strictly positive numbers decreasing to zero, and consider the sequence of kernels $\{h_n : n \geq 1\}$ given by

$$h_n : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ : (x_1, x_2) \mapsto h_n(x_1, x_2) := \frac{1}{2}\mathbf{1}_{\{0<\|x_1-x_2\|\leq t_n\}},$$

where, here and for the rest of the section, $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^d$. Then, it is easily seen that, for every n, the U-statistic

$$F_n := \sum_{(x_1,x_2)\in\eta^2_{n,\neq}} h_n(x_1, x_2), \tag{4.14}$$

equals the number of edges in the random geometric graph (V_n, E_n) where the set of vertices V_n is given by the points in the support of η_n, and $\{x, y\} \in E_n$ if and only if $0 < \|x - y\| \leq t_n$ (in particular, no loops are allowed).

We will now state and prove the main achievement of the section, refining several limit theorems for edge-counting one can find in the literature (see e.g. [2, 7, 8, 20, 21, 22] and the references therein). Observe that, quite remarkably, the conclusion of the forthcoming Theorem 4.2 is independent of the specific form of the density f.For every d, we denote by κ_d the volume of the ball with unit radius in $\mathbb{R}^d$.

Theorem 4.2. *Assume that $nt_n^d \to \infty$, as $n \to \infty$.*

(a) *As $n \to \infty$, one has the exact asymptotics*

$$\mathbb{E}(F_n) \sim \frac{\kappa_d\, n^2\, t_n^d}{2} \int_{\mathbb{R}^d} f(x)^2 \mathrm{d}x, \quad \mathrm{Var}(F_n) \sim \frac{\kappa_d^2\, n^3 (t_n^d)^2}{4} \int_{\mathbb{R}^d} f(x)^3 \mathrm{d}x. \tag{4.15}$$

(b) *Define*

$$\tilde{F}_n := \frac{F_n - \mathbb{E}F_n}{\sqrt{\mathrm{Var}(F_n)}}, \quad n \geq 1,$$

and let $N \sim \mathscr{N}(0, 1)$. Then, there exists a constant $C \in (0, \infty)$, independent of n, such that

$$d_W(\tilde{F}_n, N) \leq \varphi(n) := Cn^{-1/2}. \tag{4.16}$$

(c) *If moreover $n\,(t_n^d)^3 \to \infty$, then there exists a constant $0 < c < C$ such that, for n large enough,*

$$c\,n^{-1/2} \leq d_W(\tilde{F}_n, N) \leq C\,n^{-1/2},$$

and the rate of convergence induced by the sequence $\varphi(n) = C\,n^{-1/2}$ is therefore optimal.

Proof. The two asymptotic results at Point (a) follow from [20, Proposition 3.1] and [20, formula (3.23)], respectively. Point (b) is a special case of the general estimates proved in [8, Theorem 3.3]. In order to prove Point (c) it is therefore sufficient to show that the sequence $\tilde{F}_n$ verifies Assumptions (ii), (iii) and (iv) of Theorem 1.1-**(II)**, with respect to the control measure ν defined in (4.9), and with values of α, β and ρ such that $\alpha\,\rho \neq \beta/2$. First of all, in view of (4.11), one has that, a.e. $\mathrm{d}t \otimes \mu(\mathrm{d}z)$,

$$D_{t,z}\tilde{F}_n = \frac{1}{\sqrt{\mathrm{Var}(F_n)}} \left\{h_{1,n}(t, z) + 2I_1(h_{2,n}(t, z, \cdot))\right\},$$

where the kernels $h_{1,n}$ and $h_{2,n}$ are obtained from (4.3) and (4.13), by taking $h = h_n$. Since $h_{1,n}(t, z) = 2\,\mathbf{1}_{[0,n]}(t)n \int_{\mathbb{R}^d} h(z, a)\mu(\mathrm{d}a)$, we obtain that $\mathrm{Var}(F_n)^{-\frac{1}{2}} \times h_{1,n}(t, z) =$

$O((t_n^{2d} n)^{-\frac{1}{2}}) \to 0$, as $n \to \infty$. Also, using the isometric properties of Poisson multiple integrals,

$$\mathbb{E}\Big(\frac{I_1(h_{2,n}((t,z),\cdot))}{\sqrt{\mathrm{Var}(F_n)}}\Big)^2 \leq \frac{n\int_{\mathbb{R}^d} h_n(z,x)\mu(\mathrm{d}x)}{\mathrm{Var}(F_n)} = O((nt_n^d)^{-2}) \to 0.$$

It follows that $D_{t,z}\tilde{F}_n$ converges in probability to zero for $\mathrm{d}\nu$-almost every $(t,z) \in \mathbb{R}_+ \times \mathbb{R}^d$, and Assumption (ii) of Theorem 1.1-**(II)** is therefore verified. In order to show that Assumption (iii) in Theorem 1.1-**(II)** also holds, we need to introduce three (standard) auxiliary kernels:

$$\begin{aligned}
h_{2,n} \star_2^1 h_{2,n}(t,x) &:= \mathbf{1}_{[0,n]}(t) n \int_{\mathbb{R}^d} h_n^2(x,a)\mu(\mathrm{d}a) \\
h_{2,n} \star_1^1 h_{2,n}((t,x),(s,y)) &:= \mathbf{1}_{[0,n]}(t)\mathbf{1}_{[0,n]}(s) n \int_{\mathbb{R}^d} h_n(x,a)h_n(y,a)\mu(\mathrm{d}a) \\
h_{1,n} \star_1^1 h_{2,n}(t,x) &:= 2\,\mathbf{1}_{[0,n]}(t) n^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_n(a,y)h_n(x,y)\mu(\mathrm{d}a)\mu(\mathrm{d}x).
\end{aligned}$$

The following asymptotic relations (for $n \to \infty$) can be routinely deduced from the calculations contained in [8, Proof of Theorem 3.3] (recall that the symbol '$\sim$' indicates an exact asymptotic relation, and observe moreover that the constant C is the same appearing in (4.16)):

$$\begin{aligned}
\|h_{2,n} \star_2^1 h_{2,n}\|^2_{L^2(\nu)} &= O(n^3(t_n^d)^2) && (4.17)\\
\|h_{2,n} \star_1^1 h_{2,n}\|^2_{L^2(\nu^2)} &= O(n^4(t_n^d)^3) && (4.18)\\
\|h_{1,n} \star_1^1 h_{2,n}\|^2_{L^2(\nu)} &\sim \frac{\kappa_d^4 n^5 (t_n^d)^4}{4}\int_{\mathbb{R}^d} f(x)^5 \mathrm{d}x && (4.19)\\
\langle h_{1,n}, h_{1,n} \star_1^1 h_{2,n}\rangle_{L^2(\nu)} &\sim \frac{\kappa_d^3 n^4 (t_n^d)^3}{2}\int_{\mathbb{R}^d} f(x)^4 \mathrm{d}x && (4.20)\\
\varphi(n)\mathrm{Var}(F_n) &\sim C\frac{\kappa_d^2 n^{5/2} (t_n^d)^2}{4}\int_{\mathbb{R}^d} f(x)^3 \mathrm{d}x && (4.21)\\
\|h_{1,n} \star_1^1 h_{2,n}\|^3_{L^3(\nu)} &= O(n^7(t_n^d)^6) && (4.22)\\
\|h_{1,n}\|^3_{L^3(\nu)} &\sim \kappa_d^3 n^4 (t_n^d)^3 \int_{\mathbb{R}^d} f(x)^4 \mathrm{d}x && (4.23)\\
\|h_{1,n}\|^4_{L^4(\nu)} &= O(n^5(t_n^d)^4) && (4.24)\\
\|h_{2,n}\|^2_{L^2(\nu)} &= O(n^2(t_n^d)). && (4.25)
\end{aligned}$$

Using the fact that, by definition, $L^{-1}Y = -q^{-1}Y$ for every random variable Y living in the qth Wiener chaos of $\widehat{\eta}$, we deduce that (using the control measure ν defined in (4.9))

$$\begin{aligned}
\langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n\rangle_{L^2(\nu)} &= \frac{1}{\mathrm{Var}(F_n)}\Big\{\int_{\mathbb{R}_+\times\mathbb{R}^d} h_{1,n}^2(t,z)\nu(\mathrm{d}t,\mathrm{d}z) \\
&+ 3\int_{\mathbb{R}_+\times\mathbb{R}^d} h_{1,n}(t,z)I_1(h_{2,n}((t,z),\cdot))\nu(\mathrm{d}t,\mathrm{d}z) + 2\int_{\mathbb{R}_+\times\mathbb{R}^d} I_1^2(h_{2,n}((t,z),\cdot))\nu(\mathrm{d}t,\mathrm{d}z)\Big\}.
\end{aligned}$$

Using a standard multiplication formula for multiple Poisson integrals (see e.g. [17, Section 6.5]) on the of the previous equation, one deduces that

$$\begin{aligned}
&\frac{\langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n\rangle_{L^2(\nu)} - 1}{\varphi(n)} \\
&= \frac{1}{\varphi(n)\mathrm{Var}(F_n)}\Big\{3I_1(h_{1,n} \star_1^1 h_{2,n}) + 2I_1(h_{2,n} \star_2^1 h_{2,n}) + 2I_2(h_{2,n} \star_1^1 h_{2,n})\Big\} \\
&:= X_{1,n} + X_{2,n} + X_{3,n}.
\end{aligned}$$

Now, in view of (4.17), (4.18) and (4.21), one has that, as $n \to \infty$,

$$\mathbb{E}(X_{2,n}^2) = O((nt_n^d)^{-2}) \to 0, \quad \text{and} \quad \mathbb{E}(X_{3,n}^2) = O((nt_n^d)^{-1}) \to 0.$$

Also, (4.19) yields that

$$\mathbb{E}(X_{1,n}^2) \longrightarrow \frac{9 \int_{\mathbb{R}^d} f(x)^5 \mathrm{d}x}{C^2 (\int_{\mathbb{R}^d} f(x)^3 \mathrm{d}x)^2} := \alpha^2 > 0.$$

Finally, in view of (4.22) and (4.23),

$$\Big\| \frac{h_{1,n} \star_1^1 h_{2,n}}{\varphi(n)\mathrm{Var}(F_n)} \Big\|_{L^3(\nu)}^3, \Big\| \frac{h_{1,n}}{\varphi(n)\mathrm{Var}(F_n)} \Big\|_{L^3(\nu)}^3 = O(n^{-\frac{1}{2}}) \to 0, \quad \text{and}$$

A standard application of [19, Corollary 3.4] now implies that, as $n \to \infty$

$$\Big(\frac{I_1(h_{1,n})}{\sqrt{\mathrm{Var}(F_n)}}, -\frac{I_1(3h_{1,n} \star_1^1 h_{2,n})}{\varphi(n)\mathrm{Var}(F_n)} \Big) \overset{\text{law}}{\to} (Z_1, Z_2), \tag{4.26}$$

where $Z_1 \sim \mathscr{N}(0,1)$ and $Z_2 \sim \mathscr{N}(0, \alpha^2)$ are two jointly Gaussian random variables such that

$$\rho' := \mathbb{E}(Z_1 Z_2) = -\lim_n \frac{3}{\varphi(n)\mathrm{Var}^{\frac{3}{2}}(F_n)} \langle h_{1,n}, h_{1,n} \star_1^1 h_{2,n} \rangle_{L^2(\nu)} = -\frac{12}{C} \frac{\int_{\mathbb{R}^d} f(x)^4 \mathrm{d}x}{\left(\int_{\mathbb{R}^d} f(x)^3 \mathrm{d}x\right)^{3/2}}.$$

Now, since relation (4.25) implies that $\mathrm{Var}(F_n)^{-1/2} I_2(h_{2,n})$ converges to zero in probability, we deduce that the sequence

$$\left(\tilde{F}_n, \frac{1 - \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n \rangle_{L^2(\nu)}}{\varphi(n)} \right), \quad n \geq 1,$$

converges necessarily to the same limit as the one appearing on the RHS of (4.26). We therefore conclude that Assumption (iii) in Theorem 1.1-**(II)** is verified with α defined as above, and $\rho := \rho'/\alpha$. To conclude the proof, we will now show that Assumption (iv) in Theorem 1.1-**(II)** is satisfied for

$$u_n = h_{1,n}^3 \times \mathrm{Var}(F_n)^{-3/2} \quad \text{and} \quad \beta = \frac{8}{C} \frac{\int_{\mathbb{R}^d} f(x)^4 \mathrm{d}x}{\left(\int_{\mathbb{R}^d} f(x)^3 \mathrm{d}x\right)^{3/2}}. \tag{4.27}$$

To see this, we use again a product formula for multiple stochastic integrals to infer that

$$\begin{aligned}
&\frac{1}{\varphi(n)} \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} (D_{t,z}\tilde{F}_n)^2 \times (-D_{t,z}L^{-1}\tilde{F}_n) \times R_n^f(t,z)\, \nu(\mathrm{d}t, \mathrm{d}z) \\
&= \frac{(\mathrm{Var}(F_n))^{-\frac{3}{2}}}{\varphi(n)} \Big\{ \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}^3(t,z) \times R_n^f(t,z)\, \nu(\mathrm{d}t, \mathrm{d}z) \\
&+ 5\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}^2(t,z)\, I_1(h_{2,n}((t,z),\cdot)) \times R_n^f(t,z)\, \nu(\mathrm{d}t, \mathrm{d}z) \\
&+ 8\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}(t,z) \times I_1(h_{2,n}((t,z),\cdot))^2 \times R_n^f(t,z)\, \nu(\mathrm{d}t, \mathrm{d}z) \Big\} \\
&:= B_{1,n} + B_{2,n} + B_{3,n}.
\end{aligned}$$

Since relations (4.23) and (4.24) imply that, as $n \to \infty$ and using the notation (4.27),

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} \frac{u_n(t,z)}{\varphi(n)} \nu(\mathrm{d}t, \mathrm{d}z) \to \beta, \text{ and } \int_{\mathbb{R}_+ \times \mathbb{R}^d} \left(\frac{u_n(t,z)}{\varphi(n)} \right)^{4/3} \nu(\mathrm{d}t, \mathrm{d}z) = O(n^{-1/3}),$$

the proof is concluded once we show that $\mathbb{E}B_{2,n}$, $\mathbb{E}B_{3,n} \to 0$. In order to deal with $B_{2,n}$, we use the fact that $|R_n^f| \leq 1$, together with the Cauchy-Schwarz and Jensen inequalities and the isometric properties of multiple integrals, to deduce that

$$\begin{aligned}
&\left|\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}^d} h_{1,n}^2(t,z)\, I_1(h_{2,n}((t,z),\cdot)) \times R_n^f(t,z)\, \nu(\mathrm{d}t, \mathrm{d}z)\right| \\
&\leq n^{7/2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h_n(z,a)\mu(\mathrm{d}a)\right)^2 \sqrt{\int_{\mathbb{R}^d} h_n^2(z,a)\mu(\mathrm{d}a)}\, \mu(\mathrm{d}z) \\
&\leq n^{7/2} \sqrt{\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h_n(z,a)\mu(\mathrm{d}a)\right)^2 \left(\int_{\mathbb{R}^d} h_n^2(z,a)\mu(\mathrm{d}a)\right) \mu(\mathrm{d}z)} \\
&= O(n^{7/2}(t_n^d)^{3/2}).
\end{aligned}$$

Since we work under the assumption that $n(t_n^d)^3 \to \infty$, this yields that, as $n \to \infty$, $\mathbb{E}B_{2,n} = O(n^{-1/2}(t_n^d)^{-3/2}) \to 0$. Analogous computations yield that $\mathbb{E}B_{3,n} = O((n\, t_n^d)^{-1}) \to 0$, and this concludes the proof. □

5 Connection with generalized Edgeworth expansions

In this section, we briefly explain the link with generalized Edgeworth expansion. For more details, we refer the reader to [3], [12, Chapter 5] and [14, Appendix A]. Let X_1 and X_2 be two real–valued random variables, with finite moments up to some order $M \in \{1, 2, \cdots\} \cup \{+\infty\}$. For $i = 1, 2$, write $\{\kappa_p(X_i) : p = 1, \cdots, M\}$ for the associated sequence of cumulants. Define the *formal cumulants* associated with (X_1, X_2) as

$$\kappa_p^f(X_1, X_2) := \kappa_p(X_1) - \kappa_p(X_2),\ p = 1, \cdots, M.$$

The *formal moments* associated with the sequence $\{\kappa_p(X_1, X_2)\}$, denoted by $\{m_p^f(X_1, X_2)\}$ are recursively defined as $m_0^f(X_1, X_2) = 0$ and for $p = 1, \cdots, M$:

$$m_p^f(X_1, X_2) := \sum_{q=0}^{p-1} \binom{p-1}{q} \kappa_{q+1}^f(X_1, X_2) \times m_{p-q-1}^f(X_1, X_2).$$

Example 5.1. *In general, we have that*

$$m_1^f(X_1, X_2) = \kappa_1^f(X_1, X_2), \tag{5.1}$$
$$m_2^f(X_1, X_2) = \kappa_1^f(X_1, X_2)^2 + \kappa_2^f(X_1, X_2), \tag{5.2}$$
$$m_3^f(X_1, X_2) = \kappa_1^f(X_1, X_2)^3 + 3\kappa_1^f(X_1, X_2)\, \kappa_2^f(X_1, X_2) + \kappa_3^f(X_1, X_2). \tag{5.3}$$

In particular, if $\kappa_p(X_1) = \kappa_p(X_2)$ for $p = 1, 2$, then $m_p^f(X_1, X_2) = 0$ for $p = 1, 2$. Moreover, if $\kappa_3(X_2) = 0$, then $m_3^f(X_1, X_2) = \kappa_3(X_1)$.

Definition 5.1. *Let h be a Mth differentiable function with bounded derivatives up to order M. We define the generalized Mth Edgeworth expansion $\mathcal{E}_M(X_1, X_2, h)$ of $\mathbb{E}(h(X_1))$ around $\mathbb{E}(h(X_2))$ by*

$$\mathcal{E}_M(X_1, X_2, h) := \mathbb{E}(h(X_2)) + \sum_{p=1}^{M} \frac{m_p^f(X_1, X_2)}{p!} \mathbb{E}(h^{(p)}(X_2)). \tag{5.4}$$

In particular, if $X_2 = N \sim \mathscr{N}(0,1)$, and X_1 is a random variable such that $\kappa_1(X_1) = 0$ and $\kappa_2(X_1) = 1$, then we obtain the generalized third-order Edgeworth expansion of $\mathbb{E}(h(X_1))$ around $\mathbb{E}(h(N))$ as

$$\begin{aligned}\mathcal{E}_3(X_1, N, h) &= \mathbb{E}(h(N)) + \frac{\kappa_3(X_1)}{3!}\mathbb{E}(h^{(3)}(N))\\ &= \mathbb{E}(h(N)) + \frac{\kappa_3(X_1)}{3!}\mathbb{E}(h(N)\,H_3(N)),\end{aligned}$$

where H_p stands for Hermite polynomial of order p defined as:

$$H_p(x) = (-1)^p e^{\frac{x^2}{2}} \frac{d^p}{dx^p}(e^{-\frac{x^2}{2}}).$$

For example $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$ and $H_3(x) = x^3 - 3x$. We also need two following properties of Hermite polynomials:

$$(i)\, H_p' = pH_{p-1} \quad \text{and} \quad (ii)\, xH_p(x) = H_{p+1}(x) + pH_{p-1}(x). \tag{5.5}$$

We continue with the following general lemma.

Lemma 5.1. *Let $N \sim \mathscr{N}(0,1)$. Assume that $h \in C_b^1$, i.e continuously differentiable with bounded derivative. Denote by f_h the corresponding solution to the Stein's equation* (2.7) *associated to h. Then the function f_h is twice differentiable, and moreover for any $p \geq 2$*

$$\mathbb{E}\left(h(N)\,H_{p+1}(N)\right) = -(p+1)\,\mathbb{E}\left(H_{p-2}(N) f_h''(N)\right).$$

In particular, for $p = 2$, we obtain $\mathbb{E}(h(N)\,H_3(N)) = -3\,\mathbb{E}(f_h''(N))$.

Proof. The fact that the solution f_h is twice differentiable with bounded derivatives is indeed Lemma 2.3, item (ii). Now, since f_h is the solution of Stein's equation (2.7), we have

$$f_h''(u) - f_h(u) - u\, f_h'(u) = h'(u). \tag{5.6}$$

Multiply both sides of (5.6) by second Hermite polynomial H_p, and compute mathematical expectation evaluated at $N \sim \mathscr{N}(0,1)$, and using property (ii) in (5.5), we obtain

$$\begin{aligned}\mathbb{E}(H_p(N) f_h''(N)) - \mathbb{E}(H_p(N) f_h(N)) - \mathbb{E}(H_{p+1}(N) f_h'(N))\\ - p\mathbb{E}(H_{p-1}(N) f_h'(N)) = \mathbb{E}(H_p(N) h'(N)).\end{aligned}$$

Now, using property (i) in (5.5), and integration by parts formula [14, Theorem 2.9.1], we infer that in fact

$$\begin{aligned}\mathbb{E}(H_{p+1}(N) f_h'(N)) &= \mathbb{E}(H_p(N) f_h''(N)),\\ \mathbb{E}(H_p(N) f_h(N)) &= \mathbb{E}(H_{p-2} f_h''(N)), \text{ and}\\ \mathbb{E}(H_p(N) h'(N)) &= \mathbb{E}(H_{p+1}(N) h(N)).\end{aligned}$$

Now the claim follows immediately. □

Lemma 5.2. *Let all notations and assumptions of Lemmas 4.1 and 4.2 prevail. As before, we denote $X_n := \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n\rangle_{L^2(\mu_n)} - 1$. Then, as $n \to \infty$, the following asymptotic relations take place:*

(1) $\mathbb{E}(\tilde{F}_n \times X_n) \sim \frac{3\langle h_1, h_1 \star_1^1 h_2\rangle_{L^2(\mu)}}{\|h_1\|^3_{L^2(\mu)}}\, n^{-\frac{1}{2}}$,

(2) $\kappa_3(\tilde{F}_n) \sim \frac{\langle h_1, h_1 \star_1^0 h_1\rangle_{L^2(\mu)} + 6\langle h_2, h_1 \star_0^0 h_1\rangle_{L^2(\mu^2)}}{\|h_1\|^3_{L^2(\mu)}}\, n^{-\frac{1}{2}}$,

(3) $\kappa_4(\tilde{F}_n) \sim C_{h_1,h_2} n^{-1}$*, for some constant* C_{h_1,h_2} *depending only on the norms and inner products of kernels* h_1 *and* h_2*.*

Proof. The claims are direct consequences of straightforward computations based on multiplication formula for multiple Poisson integrals together with the isometry property. □

Remark 5.1. Unlike Wiener structure (see [14, Proposition 9.4.1]), for Poisson multiple integrals seems it is impossible to express the conditions in Lemma 4.2 in terms of cumulants $\kappa_3(\tilde{F}_n)$ and $\kappa_4(\tilde{F}_n)$. However in virtue of Lemma 5.2, one can easily deduce that for some constant C_{h_1,h_2} (depending only on kernels h_1 and h_2) that, as $n \to \infty$:

$$\mathbb{E}\left(\tilde{F}_n \times \frac{1 - \langle D\tilde{F}_n, -DL^{-1}\tilde{F}_n\rangle_{L^2(\mu_n)}}{\varphi(n)}\right) \sim C_{h_1,h_2} \frac{\kappa_3(\tilde{F}_n)}{\sqrt{\kappa_4(\tilde{F}_n)}}.$$

Theorem 5.1. *Let all notations and assumptions of Lemmas 4.1 and 4.2 prevail. Then, as* $n \to \infty$*,*

$$\frac{\kappa_3(\tilde{F}_n)}{2\varphi(n)} \to \frac{1}{2} - \rho_{h_1,h_2}. \tag{5.7}$$

Moreover, for any function $g \in C_b^3$*, we have, as* $n \to \infty$

$$\frac{\mathbb{E}(g(\tilde{F}_n)) - \mathcal{E}_3(\tilde{F}_n, N, g)}{\varphi(n)} \to \mathbb{E}(f_g''(N)).$$

Proof. Using the fact that $\varphi(n) = \|h_1\|^3_{L^3(\mu)} \times \|h_1\|^{-3}_{L^2(\mu)} n^{-\frac{1}{2}}$, Lemma 5.2, item (ii), and $\langle h_1, h_1 \star_0^0 h_1\rangle_{L^2(\mu)} = \|h_1\|^3_{L^3(\mu)}$, we infer that, as $n \to \infty$:

$$\frac{\kappa_3(\tilde{F}_n)}{\varphi(n)} \to 1 + \frac{6\langle h_2, h_1 \star_0^0 h_1\rangle_{L^2(\mu^2)}}{\|h_1\|^3_{L^3(\mu)}}.$$

Now, taking into account that $\langle h_2, h_1 \star_0^0 h_1\rangle_{L^2(\mu^2)} = \langle h_1, h_1 \star_1^1 h_2\rangle_{L^2(\mu)}$, the convergence implication (5.7) follows at once. For the second part, using Lemma 5.1, we infer that

$$\begin{aligned}
&\varphi(n)^{-1}\Big(\mathbb{E}(g(\tilde{F}_n)) - \mathcal{E}_3(\tilde{F}_n, N, g)\Big) \\
&\quad= \frac{\mathbb{E}(g(\tilde{F}_n)) - \mathbb{E}(g(N))}{\varphi(n)} - \frac{\kappa_3(\tilde{F}_n)}{3!\,\varphi(n)}\mathbb{E}(g(N)H_3(N)) \\
&\quad= \frac{\mathbb{E}(f_g'(\tilde{F}_n) - \tilde{F}_n f_g(\tilde{F}_n))}{\varphi(n)} + \frac{\kappa_3(\tilde{F}_n)}{2\varphi(n)}\mathbb{E}(f_g''(N)) \\
&\quad:= A_{1,n} + A_{2,n}.
\end{aligned}$$

Now, according to Lemma 4.3, we have $A_{1,n} \to \{\frac{1}{2} + \rho_{h_1,h_2}\}\mathbb{E}(f_g''(N))$, and claim follows. □

Remark 5.2. It is worth to mention that in Wiener structure, it is well known that (see [14, Proposition 9.3.1]), replacing $\mathbb{E}(g(F_n)) - \mathbb{E}(g(N))$ with $\mathbb{E}(g(F_n)) - \mathcal{E}_3(F_n, N, g)$ actually increases the rate of convergence to zero. However, Theorem 5.1 reveals that this is not the case for sequence $\tilde{F}_n$ of geometric U-statistcs of order two. A decisive reason to explain this phenomenon is existence of an extra term (in fact the second term in RHS of 2.8) in the upper bound for the Wasserstein distance for normal approximation of Poisson functionals compare to Wiener functionals.

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