A Short Note on the Bruinier-Kohnen Sign Equidistribution Conjecture and Halász' Theorem

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Abstract

In this note, we improve earlier results towards the Bruinier-Kohnen sign equidistribution conjecture for half-integral weight modular eigenforms in terms of natural density by using a consequence of Halász' Theorem. Moreover, applying a result of Serre we remove all unproved assumptions.

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By using the celebrated Sato-Tate theorem for integral weight modular eigenforms and the Shimura lift, in [4] and in [1] (together with Sara Arias-de-Reyna), we prove results related to the Bruinier-Kohnen sign equidistribution conjecture for modular eigenforms of half integral weight. In this note we improve one of our main results to a formulation in terms of natural density. Moreover, a theorem of Serre's allows us to remove all unproved assumptions.

The first improvement is due to the following application of Halász' Theorem that one of us learned from Kaisa Matomäki.

Theorem 1. *Let $g : \mathbb{N} \to \{-1,0,1\}$ be a multiplicative function. If $\sum_{p,g(p)=0} \frac{1}{p}$ converges and $\sum_{p,g(p)=-1} \frac{1}{p}$ diverges then*

$$\lim_{x\to\infty} \frac{|\{n\le x : g(n) \gtrless 0\}|}{|\{n \le x : g(n)\neq 0\}|} = \frac{1}{2}.$$

Proof. By Lemma 2.2 of [5], which is a consequence of Halász' Theorem (see [3] for details), there exists an absolute positive constant C such that

$$\sum_{n\le x} g(n) \le C \cdot x \exp\left(-\frac{1}{4}\sum_{p\le x}\frac{1-g(p)}{p}\right)$$

for all $x \ge 2$. By assumption, we have $1-g(p) \ge 0$ for all p and $1-g(p) = 2 > 1$ for any p with $g(p) = -1$. We conclude that for $x \to \infty$, $\exp\left(-\frac{1}{4}\sum_{p\le x}\frac{1-g(p)}{p}\right)$ tends to 0. Hence for the average value of g, we have $\lim_{x\to\infty} \frac{\sum_{n\le x} g(n)}{x} = 0$ and therefore

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$$\sum_{n\le x} g(n) =| \{n \le x | g(n) > 0\} | - | \{n \le x | g(n) < 0\} |= o(x).$$

Since $\sum_{p,g(p)=0} \frac{1}{p}$ converges by assumption, we conclude again by Lemma 2.2 of [5] that for $x \to \infty$,

$$\frac{\sum_{n\le x} |g(n)|}{x} = \frac{|\{n\le x|g(n)>0\}|+|\{n\le x|g(n)<0\}|}{x}$$

tends to a positive limit, hence the assertion follows immediately. □

In order to state and prove the results towards the Bruinier-Kohnen conjecture, we introduce some notation to be used throughout the note. Let $k \ge 2$ and $4|N$ be integers and χ be a quadratic Dirichlet character modulo N. We denote the space of cusp forms of weight $k + 1/2$ for the group $\Gamma_1(N)$ with character χ by $S_{k+1/2}(N, \chi)$ in the sense of Shimura, as in the main theorem in [8] on p. 458. For Hecke operators T_{p^2} for primes $p \nmid N$, let $f = \sum_{n\ge1} a(n)q^n \in S_{k+1/2}(N, \chi)$ be a non-zero cuspidal Hecke eigenform with real coefficients. For a fixed squarefree t such that $a(t) \ne 0$, denote by F_t the Shimura lift of f with respect to t. It is a cuspidal Hecke eigenform of weight $2k$ for the group $\Gamma_0(N/2)$ with trivial character. By normalising f we can and do assume $a(t) = 1$, in which case F_t is normalised.

As in our previous treatments, the following theorem, of which we only state a weak version, is in the core of our approach. Its proof is based on the Sato-Tate theorem, see [2].

Theorem 2. *[4],[1] Assume the set-up above and define the set of primes*

$$\mathbb{P}_{>0} := \{p : a(tp^2) > 0\}$$

and similarly $\mathbb{P}_{<0}$ and $\mathbb{P}_{=0}$ (depending on f and t). Then the sets $\mathbb{P}_{>0}$ and $\mathbb{P}_{<0}$ have positive natural densities and the set $\mathbb{P}_{=0}$ has natural density 0.

Due to its importance in the sequel, here we recall the following notion (Definition 2.2.1 of [1]).

Definition 3. *Let S be a set of primes. It is called* weakly regular *if there is $a \in \mathbb{R}$ (called the* Dirichlet density *of S) and a function $g(z)$ which is holomorphic on $\{\mathrm{Re}(z) > 1\}$ and continuous (in particular, finite) on $\{\mathrm{Re}(z) \ge 1\}$ such that*

$$\sum_{p\in S} \frac{1}{p^z} = a \log\left(\frac{1}{z-1}\right) + g(z).$$

The second improvement of this paper is the observation that a result of Serre's allows us to prove directly that the set $\mathbb{P}_{=0}$ is always weakly regular. This approach avoids the use of Sato-Tate equidistribution and consequently does not depend on any unproved error terms for it. It only applies to $\mathbb{P}_{=0}$ and hence does not seem to give us the weak regularity of the other sets $\mathbb{P}_{>0}$ and $\mathbb{P}_{<0}$.

Proposition 4. *Assume the setup above. Then the set $\mathbb{P}_{=0}$ is weakly regular of density zero.*

Proof. Let $F = F_t = \sum_{n=1}^{\infty} A(n)q^n$ be the Shimura lift of f with respect to t. If F has CM, then the result has been proved in Theorem 4.1.1(c) of [1]. So let us assume that F has no CM. Due to the assumption $a(t) = 1$ we have the formula

$$A(p) = a(tp^2) + \epsilon(p)p^{k-1}$$

for all primes p, where ϵ is an at most quadratic (due to the assumption that all coefficients are real) Dirichlet character of modulus $2tN^2$ (see e.g. equation (4.1) of [1]). Consequently, we have the inclusion

$$\begin{aligned}\{p < x : p \nmid 2tN, p \in \mathbb{P}_{=0}\} &= \{p < x : p \nmid 2tN, a(tp^2) = 0\}\\ &\subseteq \{p < x : p \nmid 2tN, A(p) = p^{k-1}\} \cup \{p < x : p \nmid 2tN, A(p) = -p^{k-1}\}.\end{aligned}$$

By Corollaire 1 of Théorème 15 in [7] (with $h(T) = T^{k-1}$ and $h(T) = -T^{k-1}$), it follows that

$$\#\{p < x : p \in \mathbb{P}_{=0}\} = o(\frac{x}{\log(x)^{9/8}}).$$

Consequently, Corollary 2.2.4 of [1] implies that $\mathbb{P}_{=0}$ is weakly regular of density zero. □

We now use the application of Halász' theorem and the weak regularity of $\mathbb{P}_{=0}$ to prove the equidistribution result we are after in terms of natural density. In [1] we needed regularity to achieve this goal.

Theorem 5. *Assume the setup above. Then the sets* $\{n \in \mathbb{N} | a(tn^2) > 0\}$ *and* $\{n \in \mathbb{N} | a(tn^2) < 0\}$ *have equal positive natural density, that is, both are precisely half of the natural density of the set* $\{n \in \mathbb{N} | a(tn^2) \neq 0\}$.

Proof. Let $g(n) = \begin{cases} 1 & \text{if } a(tn^2) > 0, \\ 0 & \text{if } a(tn^2) = 0, \\ -1 & \text{if } a(tn^2) < 0. \end{cases}$ Due to the relations $a(tn^2m^2)a(t) = a(tn^2)a(tm^2)$ for $gcd(n, m) = 1$ (see p. 453 of [8]), it is clear that $g(n)$ is multiplicative. Since $\mathbb{P}_{=0}$ is weakly regular of density zero by Proposition 4, it follows that $\sum_{p\in\mathbb{P}_{=0}} \frac{1}{p}$ is finite. Moreover, the fact that $\mathbb{P}_{<0}$ is of positive density implies that $\sum_{p\in\mathbb{P}_{<0}} \frac{1}{p}$ diverges. Thus the result follows from Theorem 1. □

In [1] we obtained the same conclusion under the additional assumption of the Generalised Riemann Hypothesis (GRH) because we needed to achieve the regularity of $\mathbb{P}_{=0}$, which we could derive from the very strong error term in Sato-Tate proved in [6], under the assumption of GRH.

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