Central and non-central limit theorems for weighted power variations of fractional Brownian motion

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Abstract: In this paper, we prove some central and non-central limit theorems for renormalized weighted power variations of order $q \geq 2$ of the fractional Brownian motion with Hurst parameter $H \in (0,1)$, where q is an integer. The central limit holds for $\frac{1}{2q} < H \leq 1 - \frac{1}{2q}$, the limit being a conditionally Gaussian distribution. If $H < \frac{1}{2q}$ we show the convergence in L^2 to a limit which only depends on the fractional Brownian motion, and if $H > 1 - \frac{1}{2q}$ we show the convergence in L^2 to a stochastic integral with respect to the Hermite process of order q.

Key words: fractional Brownian motion, central limit theorem, non-central limit theorem, Hermite process.

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1 Introduction

The study of single path behavior of stochastic processes is often based on the study of their power variations, and there exists a very extensive literature on the subject. Recall that, a real $q > 0$ being given, the q-power variation of a stochastic process X, with respect to a subdivision $\pi_n = \{0 = t_{n,0} < t_{n,1} < \ldots < t_{n,\kappa(n)} = 1\}$ of $[0,1]$, is defined to be the sum

$$\sum_{k=1}^{\kappa(n)} |X_{t_{n,k}} - X_{t_{n,k-1}}|^q.$$

For simplicity, consider from now on the case where $t_{n,k} = k2^{-n}$ for $n \in \{1,2,3,\ldots\}$ and $k \in \{0,\ldots,2^n\}$. In the present paper we wish to point out some interesting phenomena when $X = B$ is a fractional Brownian motion of Hurst index $H \in (0,1)$, and when $q \geq 2$ is an integer. In fact, we will also drop the absolute value (when q is odd) and we will introduce

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some weights. More precisely, we will consider

$$\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})(\Delta B_{k2^{-n}})^q, \quad q \in \{2,3,4,\ldots\}, \tag{1.1}$$

where the function $f:\mathbb{R}\to\mathbb{R}$ is assumed to be smooth enough and where $\Delta B_{k2^{-n}}$ denotes, here and in all the paper, the increment $B_{k2^{-n}} - B_{(k-1)2^{-n}}$.

The analysis of the asymptotic behavior of quantities of type (1.1) is motivated, for instance, by the study of the exact rates of convergence of some approximation schemes of scalar stochastic differential equations driven by B (see [7], [12] and [13]) besides, of course, the traditional applications of quadratic variations to parameter estimation problems.

Now, let us recall some known results concerning q-power variations (for $q = 2,3,4,\ldots$), which are today more or less classical. First, assume that the Hurst index is $H = \frac{1}{2}$, that is B is a standard Brownian motion. Let μ_q denote the qth moment of a standard Gaussian random variable $G \sim \mathscr{N}(0,1)$. By the scaling property of the Brownian motion and using the central limit theorem, it is immediate that, as $n\to\infty$:

$$2^{-n/2}\sum_{k=1}^{2^n}\left[(2^{n/2}\Delta B_{k2^{-n}})^q - \mu_q\right] \xrightarrow{\text{Law}} \mathscr{N}(0, \mu_{2q}-\mu_q^2). \tag{1.2}$$

When weights are introduced, an interesting phenomenon appears: instead of Gaussian random variables, we rather obtain mixing random variables as limit in (1.2). Indeed, when q is even and $f:\mathbb{R}\to\mathbb{R}$ is continuous and has polynomial growth, it is a very particular case of a more general result by Jacod [10] (see also Section 2 in Nourdin and Peccati [16] for related results) that we have, as $n\to\infty$:

$$2^{-n/2}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\left[(2^{n/2}\Delta B_{k2^{-n}})^q - \mu_q\right] \xrightarrow{\text{Law}} \sqrt{\mu_{2q}-\mu_q^2}\int_0^1 f(B_s)dW_s. \tag{1.3}$$

Here, W denotes another standard Brownian motion, independent of B. When q is odd, still for $f:\mathbb{R}\to\mathbb{R}$ continuous with polynomial growth, we have, this time, as $n\to\infty$:

$$2^{-n/2}\sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})(2^{n/2}\Delta B_{k2^{-n}})^q \xrightarrow{\text{Law}} \int_0^1 f(B_s)\big(\sqrt{\mu_{2q}-\mu_{q+1}^2}\,dW_s + \mu_{q+1}\,dB_s\big), \tag{1.4}$$

see for instance [16].

Secondly, assume that $H \neq \frac{1}{2}$, that is the case where the fractional Brownian motion B has not independent increments anymore. Then (1.2) has been extended by Breuer and Major [1], Dobrushin and Major [5], Giraitis and Surgailis [6] or Taqqu [21]. Precisely, five cases are considered, according to the evenness of q and the value of H:

- if q is even and if $H \in (0,\frac{3}{4})$, as $n\to\infty$,

$$2^{-n/2}\sum_{k=1}^{2^n}\left[(2^{nH}\Delta B_{k2^{-n}})^q - \mu_q\right] \xrightarrow{\text{Law}} \mathscr{N}(0, \widetilde{\sigma}^2_{H,q}). \tag{1.5}$$

- if q is even and if $H = \frac{3}{4}$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} 2^{-n/2} \sum_{k=1}^{2^n} \left[(2^{\frac{3}{4}n} \Delta B_{k2^{-n}})^q - \mu_q \right] \overset{\text{Law}}{\longrightarrow} \mathscr{N}(0, \widetilde{\sigma}^2_{\frac{3}{4},q}). \tag{1.6}$$

- if q is even and if $H \in (\frac{3}{4}, 1)$, as $n \to \infty$,

$$2^{n-2nH} \sum_{k=1}^{2^n} \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] \overset{\text{Law}}{\longrightarrow} \text{“Hermite r.v.”}. \tag{1.7}$$

- if q is odd and if $H \in (0, \frac{1}{2}]$, as $n \to \infty$,

$$2^{-n/2} \sum_{k=1}^{2^n} (2^{nH} \Delta B_{k2^{-n}})^q \overset{\text{Law}}{\longrightarrow} \mathscr{N}(0, \widetilde{\sigma}^2_{H,q}). \tag{1.8}$$

- if q is odd and if $H \in (\frac{1}{2}, 1)$, as $n \to \infty$,

$$2^{-nH} \sum_{k=1}^{2^n} (2^{nH} \Delta B_{k2^{-n}})^q \overset{\text{Law}}{\longrightarrow} \mathscr{N}(0, \widetilde{\sigma}^2_{H,q}). \tag{1.9}$$

Here, $\widetilde{\sigma}_{H,q} > 0$ denote some constant depending only on H and q. The term “Hermite r.v.” denotes a random variable whose distribution is the same as that of $Z^{(2)}$ at time one, for $Z^{(2)}$ defined in Definition 7 below.

Now, let us proceed with the results concerning the *weighted* power variations in the case where $H \neq \frac{1}{2}$. Consider the following condition on a function $f : \mathbb{R} \to \mathbb{R}$, where $q \geq 2$ is an integer:

$(\mathbf{H}_q)$ *f belongs to* $\mathscr{C}^{2q}$ *and, for any* $p \in (0, \infty)$ *and* $0 \leq i \leq 2q$: $\sup_{t \in [0,1]} E\left\{ |f^{(i)}(B_t)|^p \right\} < \infty$.

Suppose that f satisfies $(\mathbf{H}_q)$. If q is even and $H \in (\frac{1}{2}, \frac{3}{4})$, then by Theorem 2 in León and Ludeña [11] (see also Corcuera *et al* [4] for related results on the asymptotic behavior of the p-variation of stochastic integrals with respect to B) we have, as $n \to \infty$:

$$2^{-n/2} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] \overset{\text{Law}}{\longrightarrow} \widetilde{\sigma}_{H,q} \int_0^1 f(B_s) dW_s, \tag{1.10}$$

where, once again, W denotes a standard Brownian motion independent of B while $\widetilde{\sigma}_{H,q}$ is the constant appearing in (1.5). Thus, (1.10) shows for (1.1) a similar behavior to that observed in the standard Brownian case, compare with (1.3). In contradistinction, the asymptotic behavior of (1.1) can be completely different of (1.3) or (1.10) for other values of H. The first result in this direction has been observed by Gradinaru *et al* [9]. Namely, if $q \geq 3$ is odd and $H \in (0, \frac{1}{2})$, we have, as $n \to \infty$:

$$2^{nH-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})(2^{nH} \Delta B_{k2^{-n}})^q \xrightarrow{L^2} -\frac{\mu_{q+1}}{2} \int_0^1 f'(B_s) ds. \tag{1.11}$$

Also, when $q = 2$ and $H \in (0, \frac{1}{4})$, Nourdin [14] proved that we have, as $n \to \infty$:

$$2^{2Hn-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \big[(2^{nH} \Delta B_{k2^{-n}})^2 - 1\big] \xrightarrow{L^2} \frac{1}{4} \int_0^1 f''(B_s) ds. \tag{1.12}$$

In view of (1.3), (1.4), (1.10), (1.11) and (1.12), we observe that the asymptotic behaviors of the power variations of fractional Brownian motion (1.1) can be really different, depending on the values of q and H. The aim of the present paper is to investigate what happens in the whole generality with respect to q and H. Our main tool is the Malliavin calculus that appeared, in several recent papers, to be very useful in the study of the power variations for stochastic processes. As we will see, the Hermite polynomials play a crucial role in this analysis. In the sequel, for an integer $q \geq 2$, we write H_q for the Hermite polynomial with degree q defined by

$$H_q(x) = \frac{(-1)^q}{q!} e^{\frac{x^2}{2}} \frac{d^q}{dx^q} \left(e^{-\frac{x^2}{2}} \right),$$

and we consider, when $f : \mathbb{R} \to \mathbb{R}$ is a deterministic function, the sequence of *weighted Hermite variation of order* q defined by

$$V_n^{(q)}(f) := \sum_{k=1}^{2^n} f\left(B_{(k-1)2^{-n}}\right) \, H_q\big(2^{nH} \Delta B_{k2^{-n}}\big). \tag{1.13}$$

The following is the main result of this paper.

Theorem 1 *Fix an integer $q \geq 2$, and suppose that f satisfies $(\mathbf{H}_q)$.*

1. *Assume that $0 < H < \frac{1}{2q}$. Then, as $n \to \infty$, it holds*

$$2^{nqH-n} V_n^{(q)}(f) \xrightarrow{L^2} \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B_s) ds. \tag{1.14}$$

2. *Assume that $\frac{1}{2q} < H < 1 - \frac{1}{2q}$. Then, as $n \to \infty$, it holds*

$$\big(B, 2^{-n/2} V_n^{(q)}(f)\big) \xrightarrow{\text{Law}} \big(B, \sigma_{H,q} \int_0^1 f(B_s) dW_s\big), \tag{1.15}$$

where W is a standard Brownian motion independent of B and

$$\sigma_{H,q} = \sqrt{\frac{1}{2^q q!} \sum_{r \in \mathbb{Z}} \big(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}\big)^q}. \tag{1.16}$$

3. *Assume that* $H = 1 - \frac{1}{2q}$*. Then, as* $n \to \infty$*, it holds*

$$\big(B, \frac{1}{\sqrt{n}} 2^{-n/2} V_n^{(q)}(f)\big) \overset{\text{Law}}{\longrightarrow} \big(B, \sigma_{1-1/(2q),q} \int_0^1 f(B_s) dW_s\big), \tag{1.17}$$

where W *is a standard Brownian motion independent of* B *and*

$$\sigma_{1-1/(2q),q} = \frac{2\log 2}{q!}(1 - \frac{1}{2q})^q (1 - \frac{1}{q})^q. \tag{1.18}$$

4. *Assume that* $H > 1 - \frac{1}{2q}$*. Then, as* $n \to \infty$*, it holds*

$$2^{nq(1-H)-n} V_n^{(q)}(f) \overset{L^2}{\longrightarrow} \int_0^1 f(B_s) dZ_s^{(q)}, \tag{1.19}$$

where $Z^{(q)}$ *denotes the Hermite process of order* q *introduced in Definition 7 below.*

Remark 1. When $q = 1$, we have $V_n^{(1)}(f) = 2^{-nH} \sum_{k=1}^{2^n} f\left(B_{(k-1)2^{-n}}\right) \Delta B_{k2^{-n}}$. For $H = \frac{1}{2}$, $2^{nH} V_n^{(1)}(f)$ converges in L^2 to the Itô stochastic integral $\int_0^1 f(B_s)dB_s$. For $H > \frac{1}{2}$, $2^{nH} V_n^{(1)}(f)$ converges in L^2 and almost surely to the Young integral $\int_0^1 f(B_s)dB_s$. For $H < \frac{1}{2}$, $2^{3nH-n} V_n^{(1)}(f)$ converges in L^2 to $-\frac{1}{2}\int_0^1 f'(B_s)ds$.

Remark 2. In the critical case $H = \frac{1}{2q}$ $(q \geq 2)$, we conjecture the following asymptotic behavior: as $n \to \infty$,

$$\big(B, 2^{-n/2} V_n^{(q)}(f)\big) \overset{\text{Law}}{\longrightarrow} \big(B, \sigma_{1/(2q),q} \int_0^1 f(B_s) dW_s + \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B_s) ds\big), \tag{1.20}$$

for W a standard Brownian motion independent of B and $\sigma_{1/(2q),q}$ the constant defined by (1.16). Actually, (1.20) for $q = 2$ and $H = \frac{1}{4}$ has been proved in [2, 15, 17] after that the first draft of the current paper have been submitted. The reader is also referred to [16] for the study of the weighted variations associated with iterated Brownian motion, which is a non-Gaussian self-similar process of order $\frac{1}{4}$.

When H is between $\frac{1}{4}$ and $\frac{3}{4}$, one can refine point **2** of Theorem 1 as follows:

Proposition 2 *Let* $q \geq 2$ *be an integer,* $f : \mathbb{R} \to \mathbb{R}$ *be a function such that* $(\mathbf{H}_q)$ *holds and assume that* $H \in (\frac{1}{4}, \frac{3}{4})$*. Then*

$$\Big(B, 2^{-n/2} V_n^{(2)}(f), \ldots, 2^{-n/2} V_n^{(q)}(f)\Big) \overset{\text{Law}}{\longrightarrow} \left(B, \sigma_{H,2} \int_0^1 f(B_s) dW_s^{(2)}, \ldots, \sigma_{H,q} \int_0^1 f(B_s) dW_s^{(q)}\right), \tag{1.21}$$

where $(W^{(2)}, \ldots, W^{(q)})$ *is a* $(q-1)$*-dimensional standard Brownian motion independent of* B *and the* $\sigma_{H,p}$*'s,* $2 \leq p \leq q$*, are given by (1.16).*

Theorem 1 together with Proposition 2 allow to complete the missing cases in the understanding of the asymptotic behavior of weighted *power* variations of fractional Brownian motion:

Corollary 3 *Let $q \geq 2$ be an integer, and $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(\mathbf{H}_q)$ holds. Then, as $n \to \infty$:*

1. *When $H > \frac{1}{2}$ and q is odd,*

$$2^{-nH} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})(2^{nH} \Delta B_{k2^{-n}})^q \xrightarrow{L^2} q\mu_{q-1} \int_0^1 f(B_s)dB_s = q\mu_{q-1} \int_0^{B_1} f(x)dx. \tag{1.22}$$

2. *When $H < \frac{1}{4}$ and q is even,*

$$2^{2nH-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q\big] \xrightarrow{L^2} \frac{1}{4}\binom{q}{2}\mu_{q-2} \int_0^1 f''(B_s)ds. \tag{1.23}$$

   (We recover (1.12) by choosing $q = 2$).

3. *When $H = \frac{1}{4}$ and q is even,*

$$\left(B, 2^{-n/2} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[(2^{n/4} \Delta B_{k2^{-n}})^q - \mu_q\big]\right) \xrightarrow{\text{Law}} \left(B, \frac{1}{4}\binom{q}{2}\mu_{q-2} \int_0^1 f''(B_s)ds + \widetilde{\sigma}_{1/4,q} \int_0^1 f(B_s)dW_s\right), \tag{1.24}$$

   where W is a standard Brownian motion independent of B and $\widetilde{\sigma}_{1/4,q}$ is the constant given by (1.26) just below.

4. *When $\frac{1}{4} < H < \frac{3}{4}$ and q is even,*

$$\left(B, 2^{-n/2} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})\big[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q\big]\right) \xrightarrow{\text{Law}} \left(B, \widetilde{\sigma}_{H,q} \int_0^1 f(B_s)dW_s\right), \tag{1.25}$$

   for W a standard Brownian motion independent of B and

$$\widetilde{\sigma}_{H,q} = \sqrt{\sum_{p=2}^{q} p! \binom{q}{p}^2 \mu_{q-p}^2 \, 2^{-p} \sum_{r \in \mathbb{Z}} \left(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}\right)^p}. \tag{1.26}$$

5. *When $H = \frac{3}{4}$ and q is even,*

$$\left(B, \frac{1}{\sqrt{n}} 2^{-n/2} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \big[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q\big]\right) \xrightarrow{\text{Law}} \left(B, \widetilde{\sigma}_{\frac{3}{4},q} \int_0^1 f(B_s) dW_s\right), \tag{1.27}$$

for W a standard Brownian motion independent of B and

$$\widetilde{\sigma}_{\frac{3}{4},q} = \sqrt{\sum_{p=2}^{q} 2 \log 2 \, p! \binom{q}{p}^2 \mu_{q-p}^2 \big(1 - \frac{1}{2q}\big)^q \big(1 - \frac{1}{q}\big)^q}.$$

6. *When $H > \frac{3}{4}$ and q is even,*

$$2^{n-2Hn} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \big[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q\big] \xrightarrow{L^2} 2\mu_{q-2} \binom{q}{2} \int_0^1 f(B_s) dZ_s^{(2)}, \tag{1.28}$$

for $Z^{(2)}$ the Hermite process introduced in Definition 7.

Finally, we can also give a new proof of the following result, stated and proved by Gradinaru *et al.* [8] and Cheridito and Nualart [3] in a *continuous* setting:

Theorem 4 *Assume that $H > \frac{1}{6}$, and that $f : \mathbb{R} \to \mathbb{R}$ verifies* ($\mathbf{H_6}$). *Then the limit in probability, as $n \to \infty$, of the symmetric Riemann sums*

$$\frac{1}{2} \sum_{k=1}^{2^n} \big(f'(B_{k2^{-n}}) + f'(B_{(k-1)2^{-n}})\big) \Delta B_{k2^{-n}} \tag{1.29}$$

exists and is given by $f(B_1) - f(0)$.

Remark 3 When $H \leq \frac{1}{6}$, quantity (1.29) does not converge in probability in general. As a counterexample, one can consider the case where $f(x) = x^3$, see Gradinaru *et al.* [8] or Cheridito and Nualart [3].

2 Preliminaries and notation

We briefly recall some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [19] for further details. Let $B = (B_t)_{t \in [0,1]}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. That is, B is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{A}, P)$, with the covariance function

$$R_H(t, s) = E(B_t B_s) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t - s|^{2H}\big), \quad s, t \in [0, 1].$$

We suppose that $\mathcal{A}$ is the sigma-field generated by B. Let $\mathscr{E}$ be the set of step functions on $[0,T]$, and $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1$ associated with B. We will denote this isometry by $\varphi \mapsto B(\varphi)$.

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables, *i.e.* of the form

$$F = \phi(B_{t_1}, \ldots, B_{t_m})$$

where $m \geq 1$, $\phi : \mathbb{R}^m \to \mathbb{R} \in \mathscr{C}_b^\infty$ and $0 \leq t_1 < \ldots < t_m \leq 1$. The derivative of F with respect to B is the element of $L^2(\Omega, \mathfrak{H})$ defined by

$$D_s F \;=\; \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(B_{t_1}, \ldots, B_{t_m}) \mathbf{1}_{[0,t_i]}(s), \quad s \in [0,1].$$

In particular $D_s B_t = \mathbf{1}_{[0,t]}(s)$. For any integer $k \geq 1$, we denote by $\mathbb{D}^{k,2}$ the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{k,2}^2 \;=\; E(F^2) + \sum_{j=1}^k E\left[\|D^j F\|^2_{\mathfrak{H}^{\otimes j}}\right].$$

The Malliavin derivative D satisfies the chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{C}_b^1$ and if $(F_i)_{i=1,\ldots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \ldots, F_n) \in \mathbb{D}^{1,2}$ and we have

$$D\,\varphi(F_1, \ldots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \ldots, F_n) DF_i.$$

We also have the following formula, which can easily be proved by induction on q. Let $\varphi, \psi \in \mathscr{C}_b^q$ $(q \geq 1)$, and fix $0 \leq u < v \leq 1$ and $0 \leq s < t \leq 1$. Then $\varphi(B_t - B_s)\psi(B_v - B_u) \in \mathbb{D}^{q,2}$ and

$$D^q\big(\varphi(B_t - B_s)\psi(B_v - B_u)\big) = \sum_{a=0}^q \binom{q}{a} \varphi^{(a)}(B_t - B_s)\psi^{(q-a)}(B_v - B_u)\mathbf{1}_{[s,t]}^{\otimes a} \widetilde{\otimes} \mathbf{1}_{[u,v]}^{\otimes (q-a)}, \quad (2.30)$$

where $\widetilde{\otimes}$ means the symmetric tensor product.

The divergence operator I is the adjoint of the derivative operator D. If a random variable $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of the divergence operator, that is, if it satisfies

$$|E\langle DF, u\rangle_{\mathfrak{H}}| \leq c_u \sqrt{E(F^2)} \quad \text{for any } F \in \mathscr{S},$$

then $I(u)$ is defined by the duality relationship

$$E\big(FI(u)\big) = E\big(\langle DF, u\rangle_{\mathfrak{H}}\big),$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega, \mathcal{A}, P)$ generated by the random variables $\{H_n(B(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot n}$ (equipped with the modified norm $\|\cdot\|_{\mathfrak{H}^{\odot n}} = \frac{1}{\sqrt{n!}} \|\cdot\|_{\mathfrak{H}^{\otimes n}}$) and $\mathcal{H}_n$. For $H = \frac{1}{2}$, I_n coincides with the multiple Wiener-Itô integral of order n. The following duality formula holds

$$E(F I_n(h)) = E\left(\langle D^n F, h \rangle_{\mathfrak{H}^{\otimes n}}\right), \tag{2.31}$$

for any element $h \in \mathfrak{H}^{\odot n}$ and any random variable $F \in \mathbb{D}^{n,2}$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$, for every $r = 0, \ldots, n \wedge m$, the contraction of f and g of order r is the element of $\mathfrak{H}^{\otimes(n+m-2r)}$ defined by

$$f \otimes_r g = \sum_{k_1, \ldots, k_r = 1}^{\infty} \langle f, e_{k_1} \otimes \ldots \otimes e_{k_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{k_1} \otimes \ldots \otimes e_{k_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

Notice that $f \otimes_r g$ is not necessarily symmetric: we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(n+m-2r)}$. We have the following product formula: if $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$ then

$$I_n(f) I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f \widetilde{\otimes}_r g). \tag{2.32}$$

We recall the following simple formula for any $s < t$ and $u < v$:

$$E((B_t - B_s)(B_v - B_u)) = \frac{1}{2}\left(|t-v|^{2H} + |s-u|^{2H} - |t-u|^{2H} - |s-v|^{2H}\right). \tag{2.33}$$

We will also need the following lemmas:

Lemma 5 *1. Let $s < t$ belong to $[0,1]$. Then, if $H < 1/2$, one has*

$$\left|E\big(B_u(B_t - B_s)\big)\right| \leq (t-s)^{2H} \tag{2.34}$$

for all $u \in [0,1]$.

2. For all $H \in (0,1)$,

$$\sum_{k,l=1}^{2^n} \left|E\left(B_{(k-1)2^{-n}} \Delta B_{l2^{-n}}\right)\right| = O(2^n). \tag{2.35}$$

3. For any $r \geq 1$, we have, if $H < 1 - \frac{1}{2r}$,

$$\sum_{k,l=1}^{2^n} \left|E\left(\Delta B_{k2^{-n}} \Delta B_{l2^{-n}}\right)\right|^r = O(2^{n-2rHn}). \tag{2.36}$$

4. *For any $r \geq 1$, we have, if $H = 1 - \frac{1}{2r}$,*

$$\sum_{k,l=1}^{2^n} |E(\Delta B_{k2^{-n}} \Delta B_{l2^{-n}})|^r = O(n 2^{2n-2rn}). \tag{2.37}$$

Proof: To prove inequality (2.34), we just write

$$E(B_u(B_t - B_s)) = \frac{1}{2}(t^{2H} - s^{2H}) + \frac{1}{2}\left(|s-u|^{2H} - |t-u|^{2H}\right),$$

and observe that we have $|b^{2H} - a^{2H}| \leq |b-a|^{2H}$ for any $a, b \in [0,1]$, because $H < \frac{1}{2}$. To show (2.35) using (2.33), we write

$$\begin{aligned}\sum_{k,l=1}^{2^n} \left|E\left(B_{(k-1)2^{-n}} \Delta B_{l2^{-n}}\right)\right| &= 2^{-2Hn-1} \sum_{k,l=1}^{2^n} \left||l-1|^{2H} - l^{2H} - |l-k+1|^{2H} + |l-k|^{2H}\right| \\ &\leq C 2^n,\end{aligned}$$

the last bound coming from a telescoping sum argument. Finally, to show (2.36) and (2.37), we write

$$\begin{aligned}\sum_{k,l=1}^{2^n} |E(\Delta B_{k2^{-n}} \Delta B_{l2^{-n}})|^r &= 2^{-2nrH-r} \sum_{k,l=1}^{2^n} \left||k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H}\right|^r \\ &\leq 2^{n-2nrH-r} \sum_{p=-\infty}^{\infty} \left||p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H}\right|^r,\end{aligned}$$

and observe that, since the function $\left||p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H}\right|$ behaves as $C_H p^{2H-2}$ for large p, the series in the right-hand side is convergent because $H < 1 - \frac{1}{2r}$. In the critical case $H = 1 - \frac{1}{2r}$, this series is divergent, and

$$\sum_{p=-2^n}^{2^n} \left||p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H}\right|^r$$

behaves as a constant time n. ∎

Lemma 6 *Assume that $H > \frac{1}{2}$.*

1. *Let $s < t$ belong to $[0,1]$. Then*

$$\left|E\big(B_u(B_t - B_s)\big)\right| \leq 2H(t-s) \tag{2.38}$$

for all $u \in [0,1]$.

2. Assume that $H > 1 - \frac{1}{2l}$ for some $l \geq 1$. Let $u < v$ and $s < t$ belong to $[0,1]$. Then

$$|E(B_u - B_v)(B_t - B_s)| \leq H(2H-1)\left(\frac{2}{2Hl+1-2l}\right)^{\frac{1}{l}}(u-v)^{\frac{l-1}{l}}(t-s). \qquad (2.39)$$

3. Assume that $H > 1 - \frac{1}{2l}$ for some $l \geq 1$. Then

$$\sum_{i,j=1}^{2^n} \left|E\big(\Delta B_{i2^{-n}}\, \Delta B_{j2^{-n}}\big)\right|^l = O(2^{2n-2ln}). \qquad (2.40)$$

Proof: We have

$$E\big(B_u(B_t - B_s)\big) = \frac{1}{2}\big(t^{2H} - s^{2H}\big) + \frac{1}{2}\big(|s-u|^{2H} - |t-u|^{2H}\big).$$

But, when $0 \leq a < b \leq 1$:

$$b^{2H} - a^{2H} = 2H\int_0^{b-a}(u+a)^{2H-1}du \leq 2H\, b^{2H-1}\,(b-a) \leq 2H(b-a).$$

Thus, $|b^{2H} - a^{2H}| \leq 2H|b-a|$ and the first point follows.

Concerning the second point, using Hölder inequality, we can write

$$\begin{aligned}
|E(B_u - B_v)(B_t - B_s)| &= H(2H-1)\int_u^v\int_s^t |y-x|^{2H-2}dydx \\
&\leq H(2H-1)|u-v|^{\frac{l-1}{l}}\left(\int_0^1\left(\int_s^t |y-x|^{2H-2}dy\right)^l dx\right)^{\frac{1}{l}} \\
&\leq H(2H-1)|u-v|^{\frac{l-1}{l}}|t-s|^{\frac{l-1}{l}}\left(\int_0^1\int_s^t |y-x|^{(2H-2)l}dydx\right)^{\frac{1}{l}}.
\end{aligned}$$

Denote by $H' = 1 + (H-1)l$ and observe that $H' > \frac{1}{2}$ (because $H > 1 - \frac{1}{2l}$). Since $2H' - 2 = (2H-2)l$, we can write

$$H'(2H'-1)\int_0^1\int_s^t |y-x|^{(2H-2)l}dydx = E\left|B_1^{H'}(B_t^{H'} - B_s^{H'})\right| \leq 2H'|t-s|$$

by the first point of this lemma. This gives the desired bound.

We prove now the third point. We have

$$\begin{aligned}
\sum_{i,j=1}^{2^n}\left|E\big(\Delta B_{i2^{-n}}\,\Delta B_{j2^{-n}}\big)\right|^l &= 2^{-2Hnl-l}\sum_{i,j=1}^{2^n}\left||i-j+1|^{2H} + |i-j-1|^{2H} - 2|i-j|^{2H}\right|^l \\
&\leq 2^{n-2Hnl+1-l}\sum_{k=-2^n+1}^{2^n-1}\left||k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\right|^l
\end{aligned}$$

and the function $|k+1|^{2H}+|k-1|^{2H}-2|k|^{2H}$ behaves as $|k|^{2H-2}$ for large k. As a consequence, since $H>1-\frac{1}{2l}$, the sum

$$\sum_{k=-2^n+1}^{2^n-1} \left||k+1|^{2H}+|k-1|^{2H}-2|k|^{2H}\right|^l$$

behaves as $2^{(2H-2)ln+n}$ and the third point follows. ■

Now, let us introduce the Hermite process of order $q\geq 2$ appearing in (1.19). Fix $H>1/2$ and $t\in[0,1]$. The sequence $\big(\varphi_n(t)\big)_{n\geq 1}$, defined as

$$\varphi_n(t)=2^{nq-n}\frac{1}{q!}\sum_{j=1}^{[2^nt]}\mathbf{1}_{[(j-1)2^{-n},j2^{-n}]}^{\otimes q},$$

is a Cauchy sequence in the space $\mathfrak{H}^{\otimes q}$. Indeed, since $H>1/2$, we have

$$\langle \mathbf{1}_{[a,b]},\mathbf{1}_{[u,v]}\rangle_{\mathfrak{H}}=E\big((B_b-B_a)(B_v-B_u)\big)=H(2H-1)\int_a^b\int_u^v|s-s'|^{2H-2}dsds',$$

so that, for any $m\geq n$

$$\langle\varphi_n(t),\varphi_m(t)\rangle_{\mathfrak{H}^{\otimes q}}=\frac{H^q(2H-1)^q}{q!^2}2^{nq+mq-n-m}\sum_{j=1}^{[2^mt]}\sum_{k=1}^{[2^nt]}\left(\int_{(j-1)2^{-m}}^{j2^{-m}}\int_{(k-1)2^{-n}}^{k2^{-n}}|s-s'|^{2H-2}dsds'\right)^q.$$

Hence

$$\lim_{m,n\to\infty}\langle\varphi_n(t),\varphi_m(t)\rangle_{\mathfrak{H}^{\otimes q}}=\frac{H^q(2H-1)^q}{q!^2}\int_0^t\int_0^t|s-s'|^{(2H-2)q}dsds'=c_{q,H}t^{(2H-2)q+2},$$

where $c_{q,H}=\frac{H^q(2H-1)^q}{q!^2(Hq-q+1)(2Hq-2q+1)}$. Let us denote by $\mu_t^{(q)}$ the limit in $\mathfrak{H}^{\otimes q}$ of the sequence of functions $\varphi_n(t)$. For any $f\in\mathfrak{H}^{\otimes q}$, we have

$$\begin{aligned}
\langle\varphi_n(t),f\rangle_{\mathfrak{H}^{\otimes q}} &= 2^{nq-n}\frac{1}{q!}\sum_{j=1}^{[2^nt]}\langle\mathbf{1}_{[(j-1)2^{-n},j2^{-n}]}^{\otimes q},f\rangle_{\mathfrak{H}^{\otimes q}}\\
&= 2^{nq-n}\frac{1}{q!}H^q(2H-1)^q\sum_{j=1}^{[2^nt]}\int_0^1 ds_1\int_{(j-1)2^{-n}}^{j2^{-n}}ds_1'|s_1-s_1'|^{2H-2}\ldots\\
&\qquad\times\int_0^1 ds_q\int_{(j-1)2^{-n}}^{j2^{-n}}ds_q'|s_q-s_q'|^{2H-2}f(s_1,\ldots,s_q)\\
&\underset{n\to\infty}{\longrightarrow} \frac{1}{q!}H^q(2H-1)^q\int_0^t ds'\int_{[0,1]^q}ds_1\ldots ds_q|s_1-s'|^{2H-2}\ldots|s_q-s'|^{2H-2}f(s_1,\ldots,s_q)\\
&= \langle\mu_t^{(q)},f\rangle_{\mathfrak{H}^{\otimes q}}.
\end{aligned}$$

Definition 7 *Fix $q \geq 2$ and $H > 1/2$. The Hermite process $Z^{(q)} = (Z_t^{(q)})_{t\in[0,1]}$ of order q is defined by $Z_t^{(q)} = I_q(\mu_t^{(q)})$ for $t \in [0,1]$.*

Let $Z_n^{(q)}$ be the process defined by $Z_n^{(q)}(t) = I_q(\varphi_n(t))$ for $t \in [0,1]$. By construction, it is clear that $Z_n^{(q)}(t) \xrightarrow{L^2} Z^{(q)}(t)$ as $n \to \infty$, for all fixed $t \in [0,1]$. On the other hand, it follows, from Taqqu [21] and Dobrushin and Major [5], that $Z_n^{(q)}$ converges in law to the "standard" and historical qth Hermite process, defined through its moving average representation as a multiple integral with respect to a Wiener process with time horizon $\mathbb{R}$. In particular, the process introduced in Definition 7 has the same finite dimensional distributions as the historical Hermite process.

Let us finally mention that it can be easily seen that $Z^{(q)}$ is $q(H-1)+1$ self-similar, has stationary increments and admits moments of all orders. Moreover, it has Hölder continuous paths of order strictly less than $q(H-1)+1$. For further results, we refer to Tudor [22].

3 Proof of the main results

In this section we will provide the proofs of the main results. For notational convenience, from now on, we write $\varepsilon_{(k-1)2^{-n}}$ (resp. $\delta_{k2^{-n}}$) instead of $\mathbf{1}_{[0,(k-1)2^{-n}]}$ (resp. $\mathbf{1}_{[(k-1)2^{-n},k2^{-n}]}$). The following proposition provides information on the asymptotic behavior of $E\left(V_n^{(q)}(f)^2\right)$, as n tends to infinity, for $H \leq 1 - \frac{1}{2q}$.

Proposition 8 *Fix an integer $q \geq 2$. Suppose that f satisfies $(\mathbf{H}_q)$. Then, if $H \leq \frac{1}{2q}$, then*

$$E\left(V_n^{(q)}(f)^2\right) = O(2^{n(-2Hq+2)}). \tag{3.41}$$

If $\frac{1}{2q} \leq H < 1 - \frac{1}{2q}$, then

$$E\left(V_n^{(q)}(f)^2\right) = O(2^n). \tag{3.42}$$

Finally, if $H = 1 - \frac{1}{2q}$, then

$$E\left(V_n^{(q)}(f)^2\right) = O(n2^n). \tag{3.43}$$

Proof. Using the relation between Hermite polynomials and multiple stochastic integrals, we have $H_q\left(2^{nH}\Delta B_{k2^{-n}}\right) = \frac{1}{q!}2^{qnH} I_q\left(\delta_{k2^{-n}}^{\otimes q}\right)$. In this way we obtain

$$\begin{aligned}
& E\left(V_n^{(q)}(f)^2\right) \\
= \; & \sum_{k,l=1}^{2^n} E\left\{f(B_{(k-1)2^{-n}})\, f(B_{(l-1)2^{-n}})\, H_q\left(2^{nH}\Delta B_{k2^{-n}}\right)\, H_q\left(2^{nH}\Delta B_{l2^{-n}}\right)\right\} \\
= \; & \frac{1}{q!^2}\, 2^{2Hqn} \sum_{k,l=1}^{2^n} E\left\{f(B_{(k-1)2^{-n}})\, f(B_{(l-1)2^{-n}})\, I_q\left(\delta_{k2^{-n}}^{\otimes q}\right) I_q\left(\delta_{l2^{-n}}^{\otimes q}\right)\right\}.
\end{aligned}$$

Now we apply the product formula (2.32) for multiple stochastic integrals and the duality relationship (2.31) between the multiple stochastic integral I_N and the iterated derivative operator D^N, obtaining

$$\begin{aligned}
& E\left(V_n^{(q)}(f)^2\right) \\
= & \frac{2^{2Hqn}}{q!^2}\sum_{k,l=1}^{2^n}\sum_{r=0}^{q} r!\binom{q}{r}^2 \\
& \times E\left\{f(B_{(k-1)2^{-n}})\,f(B_{(l-1)2^{-n}})\,I_{2q-2r}\left(\delta_{k2^{-n}}^{\otimes q-r}\widetilde{\otimes}\delta_{l2^{-n}}^{\otimes q-r}\right)\right\}\langle\delta_{k2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}^r \\
= & 2^{2Hqn}\sum_{k,l=1}^{2^n}\sum_{r=0}^{q}\frac{1}{r!(q-r)!^2} \\
& \times E\left\{\left\langle D^{2q-2r}\left(f(B_{(k-1)2^{-n}})\,f(B_{(l-1)2^{-n}})\right),\delta_{k2^{-n}}^{\otimes q-r}\widetilde{\otimes}\delta_{l2^{-n}}^{\otimes q-r}\right\rangle_{\mathfrak{H}^{\otimes(2q-2r)}}\right\}\langle\delta_{k2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}^r,
\end{aligned}$$

where $\widetilde{\otimes}$ denotes the symmetrization of the tensor product. By (2.30), the derivative of the product $D^{2q-2r}\left(f(B_{(k-1)2^{-n}})\,f(B_{(l-1)2^{-n}})\right)$ is equal to a sum of derivatives:

$$\begin{aligned}
D^{2q-2r}\left(f(B_{(k-1)2^{-n}})\,f(B_{(l-1)2^{-n}})\right) = & \sum_{a+b=2q-2r} f^{(a)}(B_{(k-1)2^{-n}})\,f^{(b)}(B_{(l-1)2^{-n}}) \\
& \times\frac{(2q-2r)!}{a!b!}\left(\varepsilon_{(k-1)2^{-n}}^{\otimes a}\widetilde{\otimes}\varepsilon_{(l-1)2^{-n}}^{\otimes b}\right).
\end{aligned}$$

We make the decomposition

$$E\left(V_n^{(q)}(f)^2\right) = A_n + B_n + C_n + D_n, \tag{3.44}$$

where

$$A_n = \frac{2^{2Hqn}}{q!^2}\sum_{k,l=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\,f^{(q)}(B_{(l-1)2^{-n}})\right\}\langle\varepsilon_{(k-1)2^{-n}},\delta_{k2^{-n}}\rangle^q\ \langle\varepsilon_{(l-1)2^{-n}},\delta_{l2^{-n}}\rangle^q,$$

$$\begin{aligned}
B_n = & 2^{2Hqn}\sum_{\substack{c+d+e+f=2q\\ d+e\geq 1}}\sum_{k,l=1}^{2^n} E\left\{f^{(q)}(B_{(k-1)2^{-n}})\,f^{(q)}(B_{(l-1)2^{-n}})\right\}\alpha(c,d,e,f) \\
& \times\langle\varepsilon_{(k-1)2^{-n}},\delta_{k2^{-n}}\rangle_{\mathfrak{H}}^c\langle\varepsilon_{(k-1)2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}^d\ \langle\varepsilon_{(l-1)2^{-n}},\delta_{k2^{-n}}\rangle_{\mathfrak{H}}^e\ \langle\varepsilon_{(l-1)2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}^f, \\
C_n = & 2^{2Hqn}\sum_{\substack{a+b=2q\\ (a,b)\neq(q,q)}}\sum_{k,l=1}^{2^n} E\left\{f^{(a)}(B_{(k-1)2^{-n}})\,f^{(b)}(B_{(l-1)2^{-n}})\right\}\frac{(2q)!}{q!^2a!b!} \\
& \times\langle\varepsilon_{(k-1)2^{-n}}^{\otimes a}\widetilde{\otimes}\varepsilon_{(l-1)2^{-n}}^{\otimes b},\delta_{k2^{-n}}^{\otimes q}\widetilde{\otimes}\delta_{l2^{-n}}^{\otimes q}\rangle_{\mathfrak{H}^{\otimes(2q)}},
\end{aligned}$$

and

$$D_n = 2^{2Hqn}\sum_{r=1}^{q}\sum_{a+b=2q-2r}\sum_{k,l=1}^{2^n} E\left\{f^{(a)}(B_{(k-1)2^{-n}})\,f^{(b)}(B_{(l-1)2^{-n}})\right\}\frac{(2q-2r)!}{r!(q-r)!^2a!b!}$$
$$\times\langle\varepsilon_{(k-1)2^{-n}}^{\otimes a}\widetilde{\otimes}\varepsilon_{(l-1)2^{-n}}^{\otimes b},\delta_{k2^{-n}}^{\otimes q-r}\widetilde{\otimes}\delta_{l2^{-n}}^{\otimes q-r}\rangle_{\mathfrak{H}^{\otimes(2q-2r)}}\,\langle\delta_{k2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}^{r},$$

for some combinatorial constants $\alpha(c,d,e,f)$. That is, A_n and B_n contain all the terms with $r=0$ and $(a,b)=(q,q)$; C_n contains the terms with $r=0$ and $(a,b)\neq(q,q)$; and D_n contains the remaining terms.

For any integer $r\geq 1$, we set

$$\alpha_n = \sup_{k,l=1,\dots,2^n}\left|\langle\varepsilon_{(k-1)2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}\right|, \tag{3.45}$$

$$\beta_{r,n} = \sum_{k,l=1}^{2^n}\left|\langle\delta_{k2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}\right|^{r}, \tag{3.46}$$

$$\gamma_n = \sum_{k,l=1}^{2^n}\left|\langle\varepsilon_{(k-1)2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}\right|. \tag{3.47}$$

Then, under assumption $(\mathbf{H_q})$, we have the following estimates:

$$|A_n| \leq C2^{2Hqn+2n}(\alpha_n)^{2q},$$
$$|B_n|+|C_n| \leq C2^{2Hqn}(\alpha_n)^{2q-1}\gamma_n,$$
$$|D_n| \leq C2^{2Hqn}\sum_{r=1}^{q}(\alpha_n)^{2q-2r}\beta_{r,n},$$

where C is a constant depending only on q and the function f. Notice that the second inequality follows from the fact that when $(a,b)\neq(q,q)$, or $(a,b)=(q,q)$ and $c+d+e+f=2q$ with $d\geq 1$ or $e\geq 1$, there will be at least a factor of the form $\langle\varepsilon_{(k-1)2^{-n}},\delta_{l2^{-n}}\rangle_{\mathfrak{H}}$ in the expression of B_n or C_n.

In the case $H<\frac{1}{2}$, we have by (2.34) that $\alpha_n\leq 2^{-2nH}$, by (2.36) that $\beta_{r,n}\leq C2^{n-2rHn}$, and by (2.35) that $\gamma_n\leq C2^n$. As a consequence, we obtain

$$|A_n| \leq C2^{n(-2Hq+2)}, \tag{3.48}$$
$$|B_n|+|C_n| \leq C2^{n(-2Hq+2H+1)}, \tag{3.49}$$
$$|D_n| \leq C\sum_{r=1}^{q}2^{n(-2(q-r)H+1)}, \tag{3.50}$$

which implies the estimates (3.41) and (3.42).

In the case $\frac{1}{2} \leq H < 1 - \frac{1}{2q}$, we have by (2.38) that $\alpha_n \leq C 2^{-n}$, by (2.36) that $\beta_{r,n} \leq C 2^{n-2rHn}$, and by (2.35) that $\gamma_n \leq C 2^n$. As a consequence, we obtain

$$\begin{aligned} |A_n| + |B_n| + |C_n| &\leq C 2^{n(2q(H-1)+2)}, \\ |D_n| &\leq C \sum_{r=1}^{q} 2^{n((2q-2r)(H-1)+1)}, \end{aligned}$$

which also implies (3.42).

Finally, if $H = 1 - \frac{1}{2q}$, we have by (2.38) that $\alpha_n \leq C 2^{-n}$, by (2.37) that $\beta_{r,n} \leq C n 2^{2n-2rn}$, and by (2.35) that $\gamma_n \leq C 2^n$. As a consequence, we obtain

$$\begin{aligned} |A_n| + |B_n| + |C_n| &\leq C 2^n, \\ |D_n| &\leq C \sum_{r=1}^{q} n 2^{n\frac{r}{q}}, \end{aligned}$$

which implies (3.43). ■

3.1 Proof of Theorem 1 in the case $0 < H < \frac{1}{2q}$

In this subsection we are going to prove the first point of Theorem 1. The proof will be done in three steps. Set $V_{1,n}^{(q)}(f) = 2^{n(qH-1)} V_n^{(q)}(f)$. We first study the asymptotic behavior of $E\left(V_{1,n}^{(q)}(f)^2\right)$, using Proposition 8.

Step 1. The decomposition (3.44) leads to

$$E\left(V_{1,n}^{(q)}(f)^2\right) = 2^{2n(qH-1)} \left(A_n + B_n + C_n + D_n\right).$$

From the estimate (3.49) we obtain $2^{2n(qH-1)} \left(|B_n| + |C_n|\right) \leq C 2^{n(2H-1)}$, which converges to zero as n goes to infinity since $H < \frac{1}{2q} < \frac{1}{2}$. On the other hand (3.50) yields

$$2^{2n(qH-1)} |D_n| \leq C \sum_{r=1}^{q} 2^{n(2rH-1)},$$

which tends to zero as n goes to infinity since $2rH - 1 \leq 2qH - 1 < 0$ for all $r = 1, \ldots, q$.

In order to handle the term A_n, we make use of the following estimate, which follows from (2.34) and (2.33):

$$\begin{aligned} & \left| \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^q - \left(-\frac{2^{-2Hn}}{2} \right)^q \right| \\ = & \left| \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}} + \frac{2^{-2Hn}}{2} \right| \left| \sum_{s=0}^{q-1} \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^s \left(-\frac{2^{-2Hn}}{2} \right)^{q-1-s} \right| \\ \leq & \ C \left(k^{2H} - (k-1)^{2H} \right) 2^{-2Hqn}. \end{aligned} \tag{3.51}$$

Thus,

$$\left| \frac{2^{4Hqn-2n}}{q!^2} \sum_{k,l=1}^{2^n} E\left\{ f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}}) \right\} \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^{q} \, \langle \varepsilon_{(l-1)2^{-n}}, \delta_{l2^{-n}} \rangle_{\mathfrak{H}}^{q} - \frac{2^{-2n-2q}}{q!^2} \sum_{k,l=1}^{2^n} E\left\{ f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}}) \right\} \right| \leq C 2^{2Hn-n},$$

which implies, as $n \to \infty$:

$$E\big(V_{1,n}^{(q)}(f)^2\big) = \frac{2^{-2n-2q}}{q!^2} \sum_{k,l=1}^{2^n} E\left\{ f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}}) \right\} + o(1). \tag{3.52}$$

Step 2: We need the asymptotic behavior of the double product

$$J_n := E\left(V_{1,n}^{(q)}(f) \times 2^{-n} \sum_{l=1}^{2^n} f^{(q)}(B_{(l-1)2^{-n}}) \right).$$

Using the same arguments as in Step 1 we obtain

$$\begin{aligned}
J_n &= 2^{Hqn-2n} \sum_{k,l=1}^{2^n} E\left\{ f(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}})\, H_q\big(2^{nH} \Delta B_{k2^{-n}}\big) \right\} \\
&= \frac{1}{q!} 2^{2Hqn-2n} \sum_{k,l=1}^{2^n} E\left\{ f(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}})\, I_q\big(\delta_{k2^{-n}}^{\otimes q}\big) \right\} \\
&= \frac{1}{q!} 2^{2Hqn-2n} \sum_{k,l=1}^{2^n} E\left\{ \left\langle D^q\big(f(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}})\big), \delta_{k2^{-n}}^{\otimes q} \right\rangle_{\mathcal{H}^{\otimes q}} \right\} \\
&= 2^{2Hqn-2n} \sum_{k,l=1}^{2^n} \sum_{a=0}^{q} \frac{1}{a!(q-a)!} E\left\{ f^{(a)}(B_{(k-1)2^{-n}})\, f^{(2q-a)}(B_{(l-1)2^{-n}}) \right\} \\
&\quad \times \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^{a} \, \langle \varepsilon_{(l-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^{q-a}.
\end{aligned}$$

It turns out that only the term with $a = q$ will contribute to the limit as n tends to infinity. For this reason we make the decomposition

$$J_n = 2^{2Hqn-2n} \sum_{k,l=1}^{2^n} \frac{1}{q!} E\left\{ f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}}) \right\} \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^{q} + S_n,$$

where

$$S_n = 2^{2Hqn-2n} \sum_{k,l=1}^{2^n} \langle \varepsilon_{(l-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}} \sum_{a=0}^{q-1} \frac{1}{a!(q-a)!} E\left\{ f^{(a)}(B_{(k-1)2^{-n}})\, f^{(2q-a)}(B_{(l-1)2^{-n}}) \right\}$$
$$\times \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^{a} \, \langle \varepsilon_{(l-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^{q-a-1}.$$

By (2.34) and (2.35), we have

$$|S_n| \leq C 2^{2Hn-n},$$

which tends to zero as n goes to infinity. Moreover, by (3.51), we have

$$\left| \frac{2^{2Hqn-2n}}{q!} \sum_{k,l=1}^{2^n} E\left\{ f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}}) \right\} \langle \varepsilon_{(k-1)2^{-n}}, \delta_{k2^{-n}} \rangle_{\mathfrak{H}}^q \right.$$
$$\left. -(-1)^q \frac{2^{-2n-q}}{q!} \sum_{k,l=1}^{2^n} E\left\{ f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}}) \right\} \right| \leq C\, 2^{2Hn-n},$$

which also tends to zero as n goes to infinity. Thus, finally, as $n \to \infty$:

$$J_n = (-1)^q \frac{2^{-2n-q}}{q!} \sum_{k,l=1}^{2^n} E\left\{ f^{(q)}(B_{(k-1)2^{-n}})\, f^{(q)}(B_{(l-1)2^{-n}}) \right\} + o(1). \tag{3.53}$$

Step 3: By combining (3.52) and (3.53), we obtain that

$$E\left| V_{1,n}^{(q)}(f) - \frac{(-1)^q}{2^q q!} 2^{-n} \sum_{k=1}^{2^n} f^{(q)}(B_{(k-1)2^{-n}}) \right|^2 = o(1),$$

as $n \to \infty$. Thus, the proof of the first point of Theorem 1 is done using a Riemann sum argument. ■

3.2 Proof of Theorem 1 in the case $H > 1 - \frac{1}{2q}$: the weighted non-central limit theorem

We prove here that the sequence $V_{3,n}(f)$, given by

$$V_{3,n}^{(q)}(f) = 2^{n(1-H)q-n}\, V_n^{(q)}(f) = 2^{qn-n} \frac{1}{q!} \sum_{k=1}^{2^n} f\left(B_{(k-1)2^{-n}}\right) I_q\left(\delta_{k2^{-n}}^{\otimes q}\right),$$

converges in L^2 as $n \to \infty$ to the pathwise integral $\int_0^1 f(B_s) dZ_s^{(q)}$ with respect to the Hermite process of order q introduced in Definition 7.

Observe first that, by construction of $Z^{(q)}$ (precisely, see the discussion before Definition 7 in Section 2), the desired result is in order when the function f is identically one. More precisely:

Lemma 9 *For each fixed $t \in [0,1]$, the sequence $2^{qn-n} \frac{1}{q!} \sum_{k=1}^{[2^n t]} I_q\left(\delta_{k2^{-n}}^{\otimes q}\right)$ converges in L^2 to the Hermite random variable $Z_t^{(q)}$.*

Now, consider the case of a general function f. We fix two integers $m \geq n$, and decompose the sequence $V_{3,m}^{(q)}(f)$ as follows:

$$V_{3,m}^{(q)}(f) = A^{(m,n)} + B^{(m,n)},$$

where

$$A^{(m,n)} = \frac{1}{q!} 2^{m(q-1)} \sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right) \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} I_q\left(\delta_{i2^{-m}}^{\otimes q}\right),$$

and

$$B^{(m,n)} = \frac{1}{q!}\, 2^{m(q-1)} \sum_{j=1}^{2^n} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} \Delta_{i,j}^{m,n} f(B)\, I_q\left(\delta_{i2^{-m}}^{\otimes q}\right),$$

with the notation $\Delta_{i,j}^{m,n} f(B) = f(B_{(i-1)2^{-m}}) - f(B_{(j-1)2^{-n}})$. We shall study $A^{(m,n)}$ and $B^{(m,n)}$ separately.

Study of $A^{(m,n)}$. When n is fixed, Lemma 9 yields that the random vector

$$\left(\frac{1}{q!} 2^{m(q-1)} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} I_q\left(\delta_{i2^{-m}}^{\otimes q}\right); j = 1, \ldots, 2^n\right)$$

converges in L^2, as $m \to \infty$, to the vector

$$\left(Z_{j2^{-n}}^{(q)} - Z_{(j-1)2^{-n}}^{(q)}; j = 1, \ldots, 2^n\right).$$

Then, as $m \to \infty$, $A^{(m,n)} \xrightarrow{L^2} A^{(\infty,n)}$, where

$$A^{(\infty,n)} := \sum_{j=1}^{2^n} f(B_{(j-1)2^{-n}}) \left(Z_{j2^{-n}}^{(q)} - Z_{(j-1)2^{-n}}^{(q)}\right).$$

Finally, we claim that when n tends to infinity, $A^{(\infty,n)}$ converges in L^2 to $\int_0^1 f(B_s)\, dZ_s^{(q)}$. Indeed, observe that the stochastic integral $\int_0^1 f(B_s)\, dZ_s^{(q)}$ is a pathwise Young integral. So, to get the convergence in L^2 it suffices to show that the sequence $A^{(\infty,n)}$ is bounded in L^p for some $p \geq 2$. The integral $\int_0^1 f(B_s)\, dZ_s^{(q)}$ has moments of all orders, because for all $p \geq 2$

$$E\left[\sup_{0 \leq s < t \leq 1} \left(\frac{\left|Z_t^{(q)} - Z_s^{(q)}\right|}{|t-s|^{\gamma}}\right)^p\right] < \infty$$

and

$$E\left[\sup_{0 \leq s < t \leq 1} \left(\frac{|B_t - B_s|}{|t-s|^{\beta}}\right)^p\right] < \infty,$$

if $\gamma < q(H-1)+1$ and $\beta < H$. On the other hand, Young's inequality implies

$$\left| A^{(\infty,n)} - \int_0^1 f(B_s)\, dZ_s^{(q)} \right| \le c_{\rho,\nu} \mathrm{Var}_\rho\big(f(B)\big) \mathrm{Var}_\nu\big(Z^{(q)}\big),$$

where Var_ρ denotes the variation of order ρ, and with $\rho, \nu > 1$ such that $\frac{1}{\rho} + \frac{1}{\nu} > 1$. Choosing $\rho > \frac{1}{H}$ and $\nu > \frac{1}{q(H-1)+1}$, the result follows.

This proves that, by letting m and then n go to infinity, $A^{(m,n)}$ converges in L^2 to $\int_0^1 f(B_s)\, dZ_s^{(q)}$.

Study of the term $B^{(m,n)}$: We prove that

$$\lim_{n\to\infty} \sup_m E \left| B^{(m,n)} \right|^2 = 0. \tag{3.54}$$

We have, using the product formula (2.32) for multiple stochastic integrals,

$$\begin{aligned} E\left|B^{(m,n)}\right|^2 &= 2^{2m(q-1)} \sum_{j=1}^{2^n} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} \sum_{j'=1}^{2^n} \sum_{i'=(j'-1)2^{m-n}+1}^{j'2^{m-n}} \sum_{l=0}^{q} \frac{l!}{q!^2} \binom{q}{l}^2 \\ &\quad \times b_l^{(m,n)} \langle \delta_{i2^{-m}}, \delta_{i'2^{-m}} \rangle_{\mathfrak{H}}^l, \end{aligned} \tag{3.55}$$

where

$$b_l^{(m,n)} = E\left(\Delta_{i,j}^{m,n} f(B) \Delta_{i',j'}^{m,n} f(B) I_{2(q-l)} \left(\delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right) \right). \tag{3.56}$$

By (2.31) and (2.30), we obtain that $b_l^{(m,n)}$ is equal to

$$\begin{aligned} & E \left\langle D^{2(q-l)} \left(\Delta_{i,j}^{m,n} f(B) \Delta_{i',j'}^{m,n} f(B) \right), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}} \\ = & \sum_{a=0}^{2q-2l} \binom{2q-2l}{a} \Big\langle E \Big(\Big(f^{(a)}(B_{(i-1)2^{-m}}) \varepsilon_{(i-1)2^{-m}}^{\otimes a} - f^{(a)}(B_{(j-1)2^{-n}}) \varepsilon_{(j-1)2^{-n}}^{\otimes a} \Big) \widetilde{\otimes} \\ & \Big(f^{(2q-2l-a)}(B_{(i'-1)2^{-m}}) \varepsilon_{(i'-1)2^{-m}}^{\otimes b} - f^{(2q-2l-a)}(B_{(j'-1)2^{-n}}) \varepsilon_{(j'-1)2^{-m}}^{\otimes b} \Big) \Big), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \Big\rangle_{\mathfrak{H}^{\otimes 2(q-l)}}. \end{aligned}$$

The term in (3.55) corresponding to $l = q$ can be estimated by

$$\frac{1}{q!} 2^{2m(q-1)} \sup_{|x-y| \le 2^{-n}} E \left| f(B_x) - f(B_y) \right|^2 \beta_{q,m},$$

where $\beta_{q,m}$ has been introduced in (3.46). So it converges to zero as n tends to infinity, uniformly in m, because, by (2.40) and using that $H > 1 - \frac{1}{2q}$, we have

$$\sup_m 2^{2m(q-1)} \beta_{q,m} < \infty.$$

In order to handle the terms with $0 \le l \le q-1$, we make the decomposition

$$\left|b_l^{(m,n)}\right| \le \sum_{a=0}^{2q-2l} \binom{2q-2l}{a} \sum_{h=1}^{4} B_h, \tag{3.57}$$

where

$$\begin{aligned}
B_1 &= E\left|\Delta_{i,j}^{m,n} f(B) \Delta_{i',j'}^{m,n} f(B)\right| \left\langle \varepsilon_{(i-1)2^{-m}}^{\otimes a} \widetilde{\otimes} \varepsilon_{(i'-1)2^{-m}}^{\otimes(2q-2l-a)}, \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}}, \\
B_2 &= E\left|f^{(a)}(B_{(j-1)2^{-n}}) \Delta_{i',j'}^{m,n} f(B)\right| \\
&\quad \times \left\langle \left(\varepsilon_{(i-1)2^{-m}}^{\otimes a} - \varepsilon_{(j-1)2^{-n}}^{\otimes a}\right) \widetilde{\otimes} \varepsilon_{(i'-1)2^{-m}}^{\otimes(2q-2l-a)}, \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}}, \\
B_3 &= E\left|\Delta_{i,j}^{m,n} f(B) f^{(2q-2l-a)}(B_{(j'-1)2^{-n}})\right| \\
&\quad \times \left\langle \varepsilon_{(i-1)2^{-m}}^{\otimes a} \widetilde{\otimes} \left(\varepsilon_{(i'-1)2^{-m}}^{\otimes(2q-2l-a)} - \varepsilon_{(j'-1)2^{-n}}^{\otimes(2q-2l-a)}\right), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}}, \\
B_4 &= E\left|f^{(a)}(B_{(j-1)2^{-n}}) f^{(2q-2l-a)}(B_{(j'-1)2^{-n}})\right| \\
&\quad \times \left\langle \left(\varepsilon_{(i-1)2^{-m}}^{\otimes a} - \varepsilon_{(j-1)2^{-n}}^{\otimes a}\right) \widetilde{\otimes} \left(\varepsilon_{(i'-1)2^{-m}}^{\otimes(2q-2l-a)} - \varepsilon_{(j'-1)2^{-n}}^{\otimes(2q-2l-a)}\right), \delta_{i2^{-m}}^{\otimes(q-l)} \widetilde{\otimes} \delta_{i'2^{-m}}^{\otimes(q-l)} \right\rangle_{\mathfrak{H}^{\otimes 2(q-l)}}.
\end{aligned} \tag{3.58}$$

By using (2.38) and the conditions imposed on the function f, one can bound the terms B_1, B_2 and B_3 as follows:

$$|B_1| \le c(q,f,H) \sup_{|x-y| \le \frac{1}{2^n}, 0 \le a \le 2q} E\left|f^{(a)}(B_x) - f^{(a)}(B_y)\right|^2 2^{-2m(q-l)},$$

$$|B_2| + |B_3| \le c(q,f,H) \sup_{|x-y| \le \frac{1}{2^n}, 0 \le a \le 2q} E\left|f^{(2q-2l-a)}(B_x) - f^{(2q-2l-a)}(B_y)\right| 2^{-2m(q-l)},$$

and, by using (2.39), we obtain that

$$|B_4| \le c(q,f,H) 2^{-n\frac{q-1}{q} - 2m(q-l)}.$$

By setting

$$R_n = \frac{1}{q!} \sup_{|x-y| \le 2^{-n}} E\left|f(B_x) - f(B_y)\right|^2 \sup_m 2^{2m(q-1)} \beta_{q,m},$$

we can finally write, by the estimate (2.40),

$$\begin{aligned}
&E\left|B^{(m,n)}\right|^2 \\
\leq\ & R_n + c(H,f,q)2^{2m(q-1)}\left(\sup_{|x-y|\leq\frac{1}{2^n},0\leq a\leq 2q}\left|f^{(2q-2l-a)}(B_x)-f^{(2q-2l-a)}(B_y)\right|+(2^{-n})^{\frac{q-1}{q}}\right) \\
&\times\sum_{j=1}^{2^n}\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}}\sum_{j'=1}^{2^n}\sum_{i'=(j'-1)2^{m-n}+1}^{j'2^{m-n}}\sum_{l=0}^{q-1}2^{-2m(q-l)}\langle\delta_{i2^{-m}},\delta_{i'2^{-m}}\rangle_{\mathfrak{H}}^l \\
\leq\ & R_n + c(H,f,q)2^{2m(q-1)}\left(\sup_{|x-y|\leq\frac{1}{2^n},0\leq a\leq 2q}\left|f^{(2q-2l-a)}(B_x)-f^{(2q-2l-a)}(B_y)\right|+(2^{-n})^{\frac{q-1}{q}}\right) \\
&\times\sum_{l=0}^{q-1}2^{-2m(q-l)}\sum_{i,j=0}^{2^m}\langle\delta_{i2^{-m}},\delta_{i'2^{-m}}\rangle_{\mathfrak{H}}^l \\
\leq\ & R_n + c(H,f,q)\left(\sup_{|x-y|\leq\frac{1}{2^n},0\leq a\leq 2q}\left|f^{(2q-2l-a)}(B_x)-f^{(2q-2l-a)}(B_y)\right|+(2^{-n})^{\frac{q-1}{q}}\right)
\end{aligned}$$

and this converges to zero due to the continuity of B and since $q>1$. ■

3.3 Proof of Theorem 1 in the case $\frac{1}{2q} < H \leq 1-\frac{1}{2q}$: the weighted central limit theorem

Suppose first that $\frac{1}{2q} < H < 1-\frac{1}{2q}$. We study the convergence in law of the sequence $V_{2,n}^{(q)}(f) = 2^{-\frac{n}{2}}V_n^{(q)}(f)$. We fix two integers $m\geq n$, and decompose this sequence as follows:

$$V_{2,m}^{(q)}(f) = A^{(m,n)} + B^{(m,n)},$$

where

$$A^{(m,n)} = 2^{-\frac{m}{2}}\sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right)\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} H_q\left(2^{mH}\Delta B_{i2^{-m}}\right),$$

and

$$B^{(m,n)} = \frac{1}{q!}2^{m(Hq-\frac{1}{2})}\sum_{j=1}^{2^n}\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}}\Delta_{i,j}^{m,n}f(B)I_q\left(\delta_{i2^{-m}}^{\otimes q}\right),$$

and where as before we make use of the notation $\Delta_{i,j}^{m,n}f(B) = f(B_{(i-1)2^{-m}}) - f(B_{(j-1)2^{-n}})$.

Let us first consider the term $A^{(m,n)}$. From Theorem 1 in Breuer and Major [1], and taking into account that $H < 1-\frac{1}{2q}$, it follows that the random vector

$$\left(B, 2^{-\frac{m}{2}}\sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} H_q(2^{mH}\Delta B_{i2^{-m}});\quad j=1,\ldots,2^n\right)$$

converges in law, as $m \to \infty$, to

$$\left(B, \sigma_{H,q} \Delta W_{j2^{-n}}; \quad j = 1, \ldots, 2^n\right)$$

where $\sigma_{H,q}$ is the constant defined by (1.16) and W is a standard Brownian motion independent of B (the independence is a consequence of the central limit theorem for multiple stochastic integrals proved in Peccati and Tudor [20]). Since

$$\sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right) \Delta W_{j2^{-n}}$$

converges in L^2 as $n \to \infty$ to the Itô integral $\int_0^1 f(B_s) dW_s$ we conclude that, by letting $m \to \infty$ and then $n \to \infty$, we have

$$\left(B, A^{(m,n)}\right) \xrightarrow{\text{Law}} \left(B, \sigma_{H,q} \int_0^1 f(B_s) dW_s\right).$$

Then it suffices to show that

$$\lim_{n\to\infty} \sup_{m\to\infty} E\left|B^{(m,n)}\right|^2 = 0. \tag{3.59}$$

We have, as in (3.55),

$$\begin{aligned} E\left|B^{(m,n)}\right|^2 &= 2^{m(2Hq-1)} \sum_{j=1}^{2^n} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} \sum_{j'=1}^{2^n} \sum_{i'=(j'-1)2^{m-n}+1}^{j'2^{m-n}} \sum_{l=0}^{q} \frac{l!}{q!^2} \binom{q}{l}^2 \\ &\quad \times b_l^{(m,n)} \langle \delta_{i2^{-m}}, \delta_{i'2^{-m}} \rangle_{\mathfrak{H}}^l, \end{aligned} \tag{3.60}$$

where $b_l^{(m,n)}$ has been defined in (3.56). The term in (3.60) corresponding to $l = q$ can be estimated by

$$\frac{1}{q!} 2^{m(2Hq-1)} \sup_{|x-y|\le 2^{-n}} E\left|f(B_x) - f(B_y)\right|^2 \beta_{q,m},$$

which converges to zero as n tends to infinity, uniformly in m, because by (2.36) and using that $H < 1 - \frac{1}{2q}$, we have

$$\sup_m 2^{m(2Hq-1)} \beta_{q,m} < \infty.$$

In order to handle the terms with $0 \le l \le q-1$, we will distinguish two different cases, depending on the value of H.

Case $H < 1/2$. Suppose $0 \le l \le q-1$. By (2.35), we can majorize $b_l^{(m,n)}$ as follows:

$$|b_l^{(m,n)}| \le C 2^{-4Hm(q-l)}.$$

As a consequence, applying again (2.36), the corresponding term in (3.60) is bounded by

$$C 2^{m(2Hq-1)} 2^{-4Hm(q-l)} \beta_{l,m} \le C 2^{2mH(l-q)},$$

which converges to zero as m tends to infinity because $l < q$.

Case $H > 1/2$. Suppose $0 \le l \le q-1$. By (2.38), we get the estimate

$$|b_l^{(m,n)}| \le C 2^{-2m(q-l)}.$$

As a consequence, applying again (2.36), the corresponding term in (3.60) is bounded by

$$C 2^{m(2Hq-1)} 2^{-2m(q-l)} \beta_{l,m}.$$

If $H < 1 - \frac{1}{2l}$, applying (2.36), this is bounded by $C 2^{m(2H(q-l)-2(q-l))}$, which converges to zero as m tends to infinity because $H < 1$ and $l < q$. In the case $H = 1 - \frac{1}{2l}$, applying (2.37), we get the estimate $Cm 2^{m(2H(q-l)-2(q-l))}$, which converges to zero as m tends to infinity because $H < 1$ and $l < q$. In the case $H > 1 - \frac{1}{2l}$, we apply (2.38) and we get the estimate $C 2^{m(2H2+1-2q)}$, which converges to zero as m tends to infinity because $H < 1 - \frac{1}{2q}$.

The proof in the case $H = 1 - \frac{1}{2q}$ is similar. The convergence of the term $A^{(m,n)}$ is obtained by applying Theorem 1' in Breuer and Major (1983), and the convergence to zero in L^2 of the term $B^{(m,n)}$ follows the same lines as before.

3.4 Proof of Proposition 2

We proceed as in Section 3.3. For $p = 2, \dots, q$, we set $V_{2,n}^{(p)}(f) = 2^{-\frac{n}{2}} V_n^{(p)}(f)$. We fix two integers $m \ge n$, and decompose this sequence as follows:

$$V_{2,m}^{(p)}(f) = A_p^{(m,n)} + B_p^{(m,n)},$$

where

$$A_p^{(m,n)} = 2^{-\frac{m}{2}} \sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right) \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} H_p\left(2^{mH} \Delta B_{i2^{-m}}\right),$$

and

$$B_p^{(m,n)} = \frac{1}{p!} 2^{m(Hp-\frac{1}{2})} \sum_{j=1}^{2^n} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} \Delta_{i,j}^{m,n} f(B) I_p\left(\delta_{i2^{-m}}^{\otimes p}\right),$$

and where as before we make use of the notation $\Delta_{i,j}^{m,n} f(B) = f(B_{(i-1)2^{-m}}) - f(B_{(j-1)2^{-n}})$.

Let us first consider the term $A_p^{(m,n)}$. We claim that the random vector

$$\left(B, \left\{ 2^{-\frac{m}{2}} \sum_{i=(j-1)2^{m-n}+1}^{j2^{m-n}} H_p\left(2^{mH} \Delta B_{i2^{-m}}\right); \quad j = 1, \dots, 2^n \right\}_{2 \le p \le q}\right)$$

converges in law, as $m \to \infty$, to

$$\left(B, \{\sigma_{H,p} \Delta W_{j2^{-n}}^{(p)}; \quad j = 1, \dots, 2^n\}_{2 \le p \le q}\right)$$

where $(W^{(2)}, \ldots, W^{(q)})$ is a $(q-1)$-dimensional standard Brownian motion independent of B and the $\sigma_{H,p}$'s are given by (1.16). Indeed, the convergence in law of each component follows from Theorem 1 in Breuer and Major [1], taking into account that $H < \frac{3}{4} \leq 1 - \frac{1}{2q}$. The joint convergence and the fact that the processes $W^{(p)}$ for $p = 2, \ldots, q$ are independent (and also independent of B) is a direct application of the central limit theorem for multiple stochastic integrals proved in Peccati and Tudor [20].

Since, for any $p = 2, \ldots, q$, the quantity

$$\sum_{j=1}^{2^n} f\left(B_{(j-1)2^{-n}}\right) \Delta W^{(p)}_{j2^{-n}}$$

converges in L^2 as $n \to \infty$ to the Itô integral $\int_0^1 f(B_s) dW_s^{(p)}$, we conclude that, by letting $m \to \infty$ and then $n \to \infty$, we have

$$\left(B, A_2^{(m,n)}, \ldots, A_q^{(m,n)}\right) \xrightarrow{\text{Law}} \left(B, \sigma_{H,2} \int_0^1 f(B_s) dW_s^{(2)}, \ldots, \sigma_{H,q} \int_0^1 f(B_s) dW_s^{(q)}\right).$$

On the other hand, and because $H \in (\frac{1}{4}, \frac{3}{4})$ (implying that $H \in (\frac{1}{2p}, 1 - \frac{1}{2p})$), we have shown in Section 3.3 that

$$\lim_{n\to\infty} \sup_{m\to\infty} E\left|B_p^{(m,n)}\right|^2 = 0$$

for all $p = 2, \ldots, q$. This finishes the proof of Proposition 2.

3.5 Proof of Corollary 3

For any integer $q \geq 2$, we have

$$\left(2^{nH} \Delta B_{k2^{-n}}\right)^q - \mu_q = \sum_{p=1}^{q} \binom{q}{p} \mu_{q-p} 2^{Hnp} I_p(\delta_{k2^{-n}}^{\otimes p}) = \sum_{p=1}^{q} p! \binom{q}{p} \mu_{q-p} H_p\left(2^{nH} \Delta B_{k2^{-n}}\right).$$

Indeed, the pth kernel in the chaos representation of $\left(2^{nH} \Delta B_{k2^{-n}}\right)^q$ is

$$\frac{1}{p!} E(D^p \left(2^{nH} \Delta B_{k2^{-n}}\right)^q) = \binom{q}{p} 2^{nHp} \mu_{q-p} \delta_{k2^{-n}}^{\otimes p}.$$

Suppose first that q is odd and $H > \frac{1}{2}$. In this case, we have

$$2^{-nH} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}})(2^{nH} \Delta B_{k2^{-n}})^q = \sum_{p=1}^{q} p! \binom{q}{p} \mu_{q-p} 2^{-nH} V_n^{(p)}(f).$$

The term with $p = 1$ converges in L^2 to $q\mu_{q-1} \int_0^1 f(B_s) dB_s$. For $p \geq 2$, the limit in L^2 is zero. Indeed, if $H \leq 1 - \frac{1}{2p}$, then $E\left(V_n^{(p)}(f)^2\right)$ is bounded by a constant times $n2^n$ by Proposition

8. If $H > 1 - \frac{1}{2p}$, then $E\left(V_n^{(p)}(f)^2\right)$ is bounded by a constant times $2^{-n2(1-H)p+2n}$ by (1.19), with $-2(1-H)p + 2 - 2H = (1-H)(2-2p) < 0$.

Suppose now that q is even. Then

$$2^{2nH-n} \sum_{k=1}^{2^n} f(B_{(k-1)2^{-n}}) \left[(2^{nH} \Delta B_{k2^{-n}})^q - \mu_q \right] = 2^{2nH-n} \sum_{p=2}^{q} p! \binom{q}{p} \mu_{q-p} V_n^{(p)}(f) \ .$$

If $H < \frac{1}{4}$, by (1.14), one has that $2^{2nH-n} \times 2\binom{q}{2}\mu_{q-2} V_n^{(2)}(f)$ converges in L^2, as $n \to \infty$, to $\frac{1}{4}\binom{q}{2}\mu_{q-2} \int_0^1 f''(B_s)ds$. On the other hand, for $p \geq 4$, $2^{2nH-n} V_n^{(p)}(f)$ converges to zero in L^2. Indeed, if $H < \frac{1}{2p}$, then $E\left(V_n^{(p)}(f)^2\right) = O(2^{n(-2Hp+2)})$ by (3.41) with $-2Hp + 2 + 4H - 2 < 0$. If $H \geq \frac{1}{2p}$, then $E\left(V_n^{(p)}(f)^2\right) = O(2^n)$ by (3.42) with $4H - 1 < 0$. Therefore (1.23) holds.

In the case $\frac{1}{4} < H < \frac{3}{4}$, Proposition 2 implies that the vector

$$\left(B, 2^{-n/2} V_n^{(2)}(f), \ldots, 2^{-n/2} V_n^{(q)}(f) \right)$$

converges in law to

$$\left(B, \sigma_{H,2} \int_0^1 f(B_s) dW_s^{(2)}, \ldots, \sigma_{H,q} \int_0^1 f(B_s) dW_s^{(q)} \right),$$

where $(W^{(2)}, \ldots, W^{(q)})$ is a $(q-1)$-dimensional standard Brownian motion independent of B and the $\sigma_{H,p}$'s, $2 \leq p \leq q$, are given by (1.16). This implies the convergence (1.25). The proof of (1.27) is analogous (with an adequate version of Proposition 2).

The convergence (1.24) is obtained by similar arguments using the limit result (1.20) in the critical case $H = \frac{1}{4}$, $p = 2$.

Finally, consider the case $H > \frac{3}{4}$. For $p = 2$, $2^{n-2Hn} V_n^{(2)}(f)$ converges in L^2 to $\int_0^1 f(B_s) dZ_s^{(2)}$ by (1.19). If $p \geq 4$, then $2^{n-2Hn} V_n^{(p)}(f)$ converges in L^2 to zero because, again by (1.19), one has $E\left(V_n^{(p)}(f)^2\right) = O(2^{n(2-2(1-H)p)})$.

3.6 Proof of Theorem 4

We can assume $H < \frac{1}{2}$, the case where $H \geq \frac{1}{2}$ being straightforward. By a Taylor's formula, we have

$$\begin{aligned} f(B_1) &= f(0) + \frac{1}{2} \sum_{k=1}^{2^n} \left(f'(B_{k2^{-n}}) + f'(B_{(k-1)2^{-n}}) \right) \Delta B_{k2^{-n}} \\ &\quad - \frac{1}{12} \sum_{k=1}^{2^n} f^{(3)}(B_{(k-1)2^{-n}}) \left(\Delta B_{k2^{-n}} \right)^3 - \frac{1}{24} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}}) \left(\Delta B_{k2^{-n}} \right)^4 \\ &\quad - \frac{1}{80} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}}) \left(\Delta B_{k2^{-n}} \right)^5 + R_n, \end{aligned} \qquad (3.61)$$

with R_n converging towards 0 in probability as $n \to \infty$, because $H > 1/6$. We can expand the monomials x^m, $m = 2, 3, 4, 5$, in terms of the Hermite polynomials:

$$\begin{aligned} x^2 &= 2\,H_2(x) + 1 \\ x^3 &= 6\,H_3(x) + 3\,H_1(x) \\ x^4 &= 24\,H_4(x) + 12\,H_2(x) + 3 \\ x^5 &= 120\,H_5(x) + 60\,H_3(x) + 15\,H_1(x). \end{aligned}$$

In this way we obtain

$$\sum_{k=1}^{2^n} f^{(3)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^3 = 6 \times 2^{-3Hn} V_n^{(3)}(f^{(3)}) + 3 \times 2^{-2Hn} V_n^{(1)}(f^{(3)}), \quad (3.62)$$

$$\sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^4 = 24 \times 2^{-4Hn} V_n^{(4)}(f^{(4)})$$
$$+12 \times 2^{-4Hn} V_n^{(2)}(f^{(4)}) + 3 \times 2^{-4Hn} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}}), \quad (3.63)$$

$$\sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^5 = 120 \times 2^{-5Hn} V_n^{(5)}(f^{(5)})$$
$$+60 \times 2^{-5Hn} V_n^{(3)}(f^{(5)}) + 15 \times 2^{-4Hn} V_n^{(1)}(f^{(5)}). \quad (3.64)$$

By (3.42) and using that $H > \frac{1}{6}$, we have $E\left(V_n^{(3)}(f^{(3)})^2\right) \le C2^n$ and $E\left(V_n^{(3)}(f^{(5)})^2\right) \le C2^n$. As a consequence, the first summand in (3.62) and the second one in (3.64) converge to zero in L^2 as n tends to infinity. Also, by (3.42), $E\left(V_n^{(4)}(f^{(4)})^2\right) \le C2^n$ and $E\left(V_n^{(5)}(f^{(5)})^2\right) \le C2^n$. Hence, the first summand in (3.63) and the first summand in (3.64) converge to zero in L^2 as n tends to infinity. If $\frac{1}{6} < H < \frac{1}{4}$, (3.41) implies $E\left(V_n^{(2)}(f^{(4)})^2\right) \le C2^{n(-4H+2)}$, so that $2^{-4Hn}V_n^{(2)}(f^{(4)})$ converges to zero in L^2 as n tends to infinity. If $\frac{1}{4} \le H < \frac{1}{2}$, (3.42) implies $E\left(V_n^{(2)}(f)^2\right) \le C2^n$ so that $2^{-4Hn}V_n^{(2)}(f^{(4)})$ converges to zero in L^2 as n tends to infinity.

Moreover, using the following identity, valid for regular functions $h : \mathbb{R} \to \mathbb{R}$:

$$\sum_{k=1}^{2^n} h'(B_{(k-1)2^{-n}})\,\Delta B_{k2^{-n}} = h(B_1) - h(0) - \frac{1}{2}\sum_{k=1}^{2^n} h''(B_{\theta_{k2^{-n}}})\,(\Delta B_{k2^{-n}})^2$$

for some $\theta_{k2^{-n}}$ lying between $(k-1)2^{-n}$ and $k2^{-n}$, we deduce that $2^{-4Hn}V_n^{(1)}(f^{(5)})$ tends to zero, because $H > \frac{1}{6}$. In the same way, we have

$$\begin{aligned}2^{-2Hn} V_n^{(1)}(f^{(3)}) &= -\frac{1}{2}\, 2^{-2Hn} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^2 \\ &\quad -\frac{1}{6}\, 2^{-2Hn} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^3 + o(1).\end{aligned}$$

We have obtained

$$\begin{aligned}f(B_1) &= f(0) + \frac{1}{2}\sum_{k=1}^{2^n} \big(f'(B_{k2^{-n}}) + f'(B_{(k-1)2^{-n}})\big)\,\Delta B_{k2^{-n}} \\ &\quad + \frac{1}{4} \times 2^{-4Hn} \sum_{k=1}^{2^n} f^{(4)}(B_{(k-1)2^{-n}}) H_2\left(2^{nH}\Delta B_{k2^{-n}}\right) \\ &\quad - \frac{1}{24} \times 2^{-2Hn} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^3 + o(1).\end{aligned}$$

As before $2^{-4Hn} V_n^{(2)}(f^{(4)})$ converges to zero in L^2. Finally, by (1.11),

$$2^{-2Hn} \sum_{k=1}^{2^n} f^{(5)}(B_{(k-1)2^{-n}})\,(\Delta B_{k2^{-n}})^3$$

also converges to zero. This completes the proof. ■

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