

A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one

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Summary. We will focus – in dimension one – on the SDEs of the type $dX_t = \sigma(X_t)dB_t + b(X_t)dt$ where B is a fractional Brownian motion. Our principal aim is to describe a simple theory – from our point of view – allowing to study this SDE, and this for any $H \in (0, 1)$. We will consider several definitions of solutions and, for each of them, study conditions under which one has existence and/or uniqueness. Finally, we will examine whether or not the canonical scheme associated to our SDE converges, when the integral with respect to fBm is defined using the Russo-Vallois symmetric integral.

Key words: Stochastic differential equation; fractional Brownian motion; Russo-Vallois integrals; Newton-Cotes functional; Approximation schemes; Doss-Sussmann transformation.

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1 Introduction

The fractional Brownian motion (fBm) $B = \{B_t, t \geq 0\}$ of Hurst index $H \in (0, 1)$ is a centered and continuous Gaussian process verifying $B_0 = 0$ a.s. and

$$E[(B_t - B_s)^2] = |t - s|^{2H} \quad (1)$$

for all $s, t \geq 0$. Observe that $B^{1/2}$ is nothing but standard Brownian motion. Equality (1) implies that the trajectories of B are $(H - \varepsilon)$ -Hölder continuous, for any $\varepsilon > 0$ small enough. As the fBm is selfsimilar (of index H) and has stationary increments, it is used as a model in many fields (for example, in hydrology, economics, financial mathematics, etc.). In particular, the study of stochastic differential equations (SDEs) driven by a fBm is important in view of the applications. But, before raising the question of existence and/or uniqueness for this type of SDEs, the first difficulty is to give a meaning to the integral with respect to a fBm. It is indeed well-known that B is not a semimartingale when $H \neq 1/2$. Thus, the Itô or Stratonovich calculus does not apply to this case. There are several ways of building an integral with respect to the fBm and of obtaining a change of variables formula. Let us point out some of these contributions:

1. *Regularization or discretization techniques.* Since 1993, Russo and Vallois [31] have developed a regularization procedure, whose philosophy is similar to the discretization. They introduce forward (generalizing Itô), backward, symmetric (generalizing Stratonovich, see Definition 3 below) stochastic integrals and a generalized quadratic variation. The regularization, or discretization technique, for fBm and related processes have been performed by [12, 17, 32, 36], in the case of zero quadratic variation (corresponding to $H > 1/2$). Note also that Young integrals [35], which are often used in this case, coincide with the forward integral (but also with the backward or symmetric ones, since covariation between integrand and integrator is always zero). When the integrator has paths with finite p -variation for $p > 2$, forward and backward

integrals cannot be used. In this case, one can use some symmetric integrals introduced by Gradinaru *et al.* in [15] (see §2 below). We also refer to Errami and Russo [11] for the specific case where $H \geq 1/3$.

2. *Rough paths.* An other approach was taken by Lyons [20]. His absolutely pathwise method based on Lévy stochastic areas considers integrators having p -variation for any $p > 1$, provided one can construct a canonical geometric rough path associated with the process. We refer to the survey article of Lejay [18] for more precise statements related to this theory. Note however that the case where the integrator is a fBm with index $H > 1/4$ has been studied by Coutin and Qian [7] (see also Feyel and de La Pradelle [13]). See also Nourdin and Simon [26] for a link between the regularization technique and the rough paths theory.
3. *Malliavin calculus.* Since fBm is a Gaussian process, it is natural to use a Skorohod approach. Integration with respect to fBm has been attacked by Decreusefond and Üstünel [8] for $H > 1/2$ and it has been intensively studied since (see for instance [1, 2, 6]), even when the integrator is a more general Gaussian process. We refer to Nualart's survey article [27] for precise statements related to this theory.
4. *Wick products.* A new type of integral with zero mean defined using Wick products was introduced by Duncan, Hu and Pasik-Duncan in [10], assuming $H > 1/2$. This integral turns out to coincide with the divergence operator. In [3], Bender considers the case of arbitrary Hurst index $H \in (0, 1)$ and proves an Itô formula for generalized functionals of B .

In the sequel, we will focus – in dimension one – on SDEs of the type:

$$\begin{cases} dX_t = \sigma(X_t) dB_t + b(X_t)dt, & t \in [0, T] \\ X_0 = x_0 \in \mathbb{R} \end{cases} \quad (2)$$

where $\sigma, b : \mathbb{R} \rightarrow \mathbb{R}$ are two continuous functions and $H \in (0, 1)$. Our principal motivation is to describe a simple theory – from our point of view – allowing to study the SDE (2), for *any* $H \in (0, 1)$. It is linked to the regularization technique (see point 1 above). Moreover, we emphasize that it is already used and quoted in some research articles (see for example [4, 14, 21, 22, 24, 25, 26]). The aim of the current paper is, in particular, to clarify this approach.

The paper is organized as follows. In the second part, we will consider several definitions of solution to (2) and for each of them we will study under which condition one has existence and/or uniqueness. Finally, in the third part, we will examine whether or not the canonical scheme associated to (2) converges, when the integral with respect to fBm is defined using the Russo-Vallois symmetric integral.

2 Basic study of the SDE (2)

In the sequel, we denote by B a fBm of Hurst parameter $H \in (0, 1)$.

Definition 2.1 *Let X, Y be two real continuous processes defined on $[0, T]$. The symmetric integral (in the sense of Russo-Vallois) is defined by*

$$\int_0^T Y_u d^\circ X_u = \lim_{\varepsilon \rightarrow 0} \text{in prob} \int_0^T \frac{Y_{u+\varepsilon} + Y_u}{2} \times \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du, \quad (3)$$

provided the limit exists and with the convention that $Y_t = Y_T$ and $X_t = X_T$ when $t > T$.

Remark 2.2 *If X, Y are two continuous semimartingales then $\int_0^T Y_u d^\circ X_u$ coincides with the standard Stratonovich integral, see [31].*

Let us recall an important result for our study:

Theorem 2.3 *(see [15], p. 793). The symmetric integral $\int_0^T f(B_u) d^\circ B_u$ exists for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^5 if and only if $H \in (1/6, 1)$. In this case, we have, for any antiderivative F of f :*

$$F(B_T) = F(0) + \int_0^T f(B_u) d^\circ B_u.$$

When $H \leq 1/6$, one can consider the so-called m -order *Newton-Cotes functional*:

Definition 2.4 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (with $n \geq 1$) be a continuous function, $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ and $Y : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ be two continuous processes and $m \geq 1$ be an integer. The m -order Newton-Cotes functional of (f, Y, X) is defined by

$$\int_0^T f(Y_u) d^{\text{NC},m} X_u = \lim_{\varepsilon \rightarrow 0} \text{in prob} \int_0^T \left(\int_0^1 f(Y_u + \beta(Y_{u+\varepsilon} - Y_u)) \nu_m(d\beta) \right) \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du,$$

provided the limit exists and with the convention that $Y_t = Y_T$ and $X_t = X_T$ when $t > T$. Here, $\nu_1 = \frac{1}{2}(\delta_0 + \delta_1)$ and

$$\nu_m = \sum_{j=0}^{2(m-1)} \left(\int_0^1 \prod_{k \neq j} \frac{2(m-1)u - k}{j - k} du \right) \delta_{j/(2m-2)}, \quad m \geq 2, \quad (4)$$

δ_a being the Dirac measure at point a .

Remark 2.5 • The 1-order Newton-Cotes functional $\int_0^T f(Y_u) d^{\text{NC},1} X_u$ is nothing but the symmetric integral $\int_0^T f(Y_u) d^s X_u$ defined by (3). On the contrary, when $m > 1$, the m -order Newton-Cotes functional $\int_0^T f(Y_u) d^{\text{NC},m} X_u$ is not *a priori* a “true” integral. Indeed, its definition could be different from $\int_0^T \tilde{f}(\tilde{Y}_u) d^{\text{NC},m} X_u$ even if $f(Y) = \tilde{f}(\tilde{Y})$. This is why we call it “functional” instead of “integral”.

- The terminology “*Newton-Cotes functional*” is due to the fact that the definition of ν_m via (4) is related to the Newton-Cotes formula of numerical analysis. Indeed, ν_m is the unique discrete measure carried by the numbers $j/(2m-2)$ which coincides with Lebesgue measure on all polynomials of degree smaller than $2m-1$.

We have the following change of variable formula.

Theorem 2.6 (see [15], p. 793). Let $m \geq 1$ be an integer. The m -order Newton-Cotes functional $\int_0^T f(B_u) d^{\text{NC},m} B_u$ exists for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4m+1} if and only if $H \in (1/(4m+2), 1)$. In this case, we have, for any antiderivative F of f :

$$F(B_T) = F(0) + \int_0^T f(B_u) d^{\text{NC},m} B_u. \quad (5)$$

Remark 2.7 An immediate consequence of this result is that

$$\int_0^T f(B_u) d^{\text{NC},m} B_u = \int_0^T f(B_u) d^{\text{NC},n} B_u = F(B_T) - F(0)$$

when $m > n$, f is C^{4m+1} and $H \in (1/(4n+2), 1)$. Then, for f regular enough, it is possible to define the so-called *Newton-Cotes functional* $\int_0^T f(B_u) d^{\text{NC}} B_u$ without ambiguity by:

$$\int_0^T f(B_u) d^{\text{NC}} B_u := \int_0^T f(B_u) d^{\text{NC},n} B_u \text{ if } H \in (1/(4n+2), 1). \quad (6)$$

In the sequel, we put $n_H = \inf\{n \geq 1 : H > 1/(4n+2)\}$. An immediate consequence of (5) and (6) is that, for any $H \in (0, 1)$ and any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4n_H+1} , we have:

$$F(B_T) = F(0) + \int_0^T f(B_u) d^{\text{NC}} B_u,$$

where F is an antiderivative of f .

To specify the sense of $\int_0^t \sigma(X_s)dB_s$ in (2), it now seems natural to try and use the Newton-Cotes functional. But for the time being we are only able to consider integrands of the form $f(B)$ with $f : \mathbb{R} \rightarrow \mathbb{R}$ regular enough, see (6). That is why we first choose the following definition for a possible solution to (2):

Definition 2.8 Assume that $\sigma \in C^{4n_H+1}$ and that $b \in C^0$.

- i) Let \mathfrak{C}_1 be the class of processes $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ verifying that there exists $f : \mathbb{R} \rightarrow \mathbb{R}$ belonging to C^{4n_H+1} and such that, for every $t \in [0, T]$, $X_t = f(B_t)$ a.s.
ii) A process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a solution to (2) if:

- $X \in \mathfrak{C}_1$,
- $\forall t \in [0, T]$, $X_t = x_0 + \int_0^t \sigma(X_s)d^{\text{NC}}B_s + \int_0^t b(X_s)ds$.

Remark 2.9 Note that the first point of definition ii) allows to ensure that the integral $\int_0^t \sigma(X_s)d^{\text{NC}}B_s$ makes sense (compare with the adaptedness condition in the Itô context).

We can now state the following result.

Theorem 2.10 Let $\sigma \in C^{4n_H+1}$ be a Lipschitz function, b be a continuous function and x_0 be a real. Then equation (2) admits a solution X in the sense of Definition 2.8 if and only if b vanishes on $\mathfrak{S}(\mathbb{R})$, where \mathfrak{S} is the unique solution to $\mathfrak{S}' = \sigma \circ \mathfrak{S}$ with initial value $\mathfrak{S}(0) = x_0$. In this case, X is unique and is given by $X_t = \mathfrak{S}(B_t)$.

Remark 2.11 As a consequence of the mean value theorem, $\mathfrak{S}(\mathbb{R})$ is an interval. Moreover, it is easy to see that either \mathfrak{S} is constant or \mathfrak{S} is strictly monotone, and that $\inf \mathfrak{S}(\mathbb{R})$ and $\sup \mathfrak{S}(\mathbb{R})$ are elements of $\{\sigma = 0\} \cup \{\pm\infty\}$. In particular, if σ does not vanish, then $\mathfrak{S}(\mathbb{R}) = \mathbb{R}$ and an immediate consequence of Theorem 2.10 is that (2) admits a solution in the sense of Definition 2.8 if and only if $b \equiv 0$.

Proof of Theorem 2.10. Assume that $X_t = f(B_t)$ is a solution to (2) in the sense of Definition 2.8. Then

$$f(B_t) = x_0 + \int_0^t \sigma \circ f(B_s)d^{\text{NC}}B_s + \int_0^t b \circ f(B_s)ds = G(B_t) + \int_0^t b \circ f(B_s)ds, \quad (7)$$

where G is the antiderivative of $\sigma \circ f$ verifying $G(0) = x_0$. Set $h = f - G$ and denote by Ω^* the set of $\omega \in \Omega$ such that $t \mapsto B_t(\omega)$ is differentiable at least one point $t_0 \in [0, T]$ (it is well-known that $\mathbb{P}(\Omega^*) = 0$). If $h'(B_{t_0}(\omega)) \neq 0$ for one $(\omega, t_0) \in \Omega \times [0, T]$ then h is strictly monotone in a neighborhood of $B_{t_0}(\omega)$ and, for $|t - t_0|$ sufficiently small, one has $B_t(\omega) = h^{-1}(\int_0^t b(X_s(\omega))ds)$ and, consequently, $\omega \in \Omega^*$. Then, a.s., $h'(B_t) = 0$ for all $t \in [0, T]$, so that $h \equiv 0$. By uniqueness, one deduces $f = \mathfrak{S}$. Thus, if (2) admits a solution X in the sense of Definition 2.8, one necessarily has $X_t = \mathfrak{S}(B_t)$. Thanks to (7), one then has $b \circ \mathfrak{S}(B_t) = 0$ for all $t \in [0, T]$ a.s. and then b vanishes on $\mathfrak{S}(\mathbb{R})$. \square

Consequently, when the SDE (2) has no drift b , there is a natural solution. But what can we do when $b \neq 0$?

Denote by \mathcal{A} the set of processes $A : [0, T] \times \Omega \rightarrow \mathbb{R}$ having C^1 -trajectories and verifying $\mathbb{E}(e^{\lambda \int_0^T A_s^2 ds}) < \infty$ for at least one $\lambda > 1$.

Lemma 2.12 Let $A \in \mathcal{A}$ and $m \in \mathbb{N}^*$. Then $\int_0^T f(B_u + A_u)d^{\text{NC},m}B_u$ exists for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4m+1} if and only if $H > 1/(4m + 2)$. In this case, for any antiderivative F of f , one has:

$$F(B_T + A_T) = F(A_0) + \int_0^T f(B_u + A_u)d^{\text{NC},m}B_u + \int_0^T f(B_u + A_u)A'_u du.$$

Proof. Set $\tilde{B} = B + A$. On the one hand, using the Girsanov theorem in [28] and taking into account the assumption on A , we have that \tilde{B} is a fBm of index H under some probability \mathbb{Q} equivalent to the initial probability \mathbb{P} . On the other hand, it is easy, by going back to Definition 2.4, to prove that $\int_0^T f(B_u + A_u)d^{\text{NC},m}B_u$ exists if and only if $\int_0^T f(B_u + A_u)d^{\text{NC},m}(B_u + A_u)$ does, and in this case, one has

$$\int_0^T f(B_u + A_u)d^{\text{NC},m}(B_u + A_u) = \int_0^T f(B_u + A_u)d^{\text{NC},m}B_u + \int_0^T f(B_u + A_u)A'_u du.$$

Then, since convergence under \mathbb{Q} or under \mathbb{P} is equivalent, the conclusion of Lemma 2.12 is a direct consequence of Theorem 2.6. \square

Then, as previously, it is possible to define a functional (still called *Newton-Cotes functional*) verifying, for any $H \in (0, 1)$, for any $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C^{4n_H+1} and any process $A \in \mathcal{A}$:

$$F(B_T + A_T) = F(A_0) + \int_0^T f(B_u + A_u)d^{\text{NC}}B_u + \int_0^T f(B_u + A_u)A'_u du,$$

where F is an antiderivative of f .

Now, we can introduce an other definition of a solution to (2):

Definition 2.13 Assume that $\sigma \in C^{4n_H+1}$ and that $b \in C^0$.

- i) Let \mathfrak{C}_2 be the class of processes $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ such that there exist a function $f : \mathbb{R} \rightarrow \mathbb{R}$ in C^{4n_H+1} and a process $A \in \mathcal{A}$ such that $A_0 = 0$ and, for every $t \in [0, T]$, $X_t = f(B_t + A_t)$ a.s.
- ii) A process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a solution to (2) if:

- $X \in \mathfrak{C}_2$,
- $\forall t \in [0, T]$, $X_t = x_0 + \int_0^t \sigma(X_s)d^{\text{NC}}B_s + \int_0^t b(X_s)ds$.

Theorem 2.14 Let $\sigma \in C^{4n_H+1}$ be a Lipschitz function, b be a continuous function and x_0 be a real.

- If $\sigma(x_0) = 0$ then (2) admits a solution X in the sense of Definition 2.13 if and only if $b(x_0) = 0$. In this case, X is unique and is given by $X_t \equiv x_0$.
- If $\sigma(x_0) \neq 0$, then (2) admits a solution. If moreover $\inf_{\mathbb{R}} |\sigma| > 0$ and $b \in \text{Lip}$, this solution is unique.

Proof. Assume that $X = f(B + A)$ is a solution to (2) in the sense of Definition 2.13. Then, we have

$$f(B_t + A_t) = G(B_t + A_t) - \int_0^t \sigma(X_s)A'_s ds + \int_0^t b(X_s)ds \quad (8)$$

where G is the antiderivative of $\sigma \circ f$ verifying $G(0) = x_0$. As in the proof of Theorem 2.10, we obtain that $f = \mathfrak{S}$ where \mathfrak{S} is defined by $\mathfrak{S}' = \sigma \circ \mathfrak{S}$ with initial value $\mathfrak{S}(0) = x_0$. Thanks to (8), we deduce that, a.s., we have $b \circ \mathfrak{S}(B_t + A_t) = \sigma \circ \mathfrak{S}(B_t + A_t) A'_t$ for all $t \in [0, T]$. Consequently:

- If $\sigma(x_0) = 0$ then $\mathfrak{S} \equiv x_0$ and $b(x_0) = 0$.
- If $\sigma(x_0) \neq 0$ then \mathfrak{S} is strictly monotone and the ordinary integral equation

$$A_t = \int_0^t \frac{b \circ \mathfrak{S}}{\mathfrak{S}'}(B_s + A_s) ds$$

admits a maximal (in fact, global since we know already that A is defined on $[0, T]$) solution by Peano's theorem. If moreover $\inf_{\mathbb{R}} |\sigma| > 0$ and $b \in \text{Lip}$ then $\frac{b \circ \mathfrak{S}}{\mathfrak{S}'} = \frac{b \circ \mathfrak{S}}{\sigma \circ \mathfrak{S}} \in \text{Lip}$ and A is uniquely determined. \square

The previous theorem is not quite satisfactory because of the prominent role played by x_0 . That is why we will finally introduce a last definition for a solution to (2). We first need an analogue of Theorem 2.6 and Lemma 2.12:

Theorem 2.15 (see [23], Chapter 4). Let A be a process having C^1 -trajectories and $m \geq 1$ be an integer. If $H > 1/(2m + 1)$ then the m -order Newton-Cotes functional $\int_0^T f(B_u, A_u) d^{\text{NC}, m} B_u$ exists for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $C^{2m, 1}$. In this case, we have, for any function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ verifying $F'_b = f$:

$$F(B_T, A_T) = F(0, A_0) + \int_0^T f(B_u, A_u) d^{\text{NC}, m} B_u + \int_0^T F'_a(B_u, A_u) A'_u du.$$

Remark 2.16 • F'_a (resp. F'_b) means the derivative of F with respect to a (resp. b).

- The condition is here $H > 1/(2m + 1)$ and not $H > 1/(4m + 2)$ as in Theorem 2.6 and Lemma 2.12. Thus, for instance, if $A \in \mathcal{A}$, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is C^5 and if $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is $C^{5, 1}$ then $\int_0^T g(B_s + A_s) d^\circ B_s$ exists if (and only if) $H > 1/6$ while $\int_0^T h(B_s, A_s) d^\circ B_s$ exists *a priori* only when $H > 1/3$.
- We define $m_H = \inf\{m \geq 1 : H > 1/(2m + 1)\}$. As in the Remark 2.7, it is possible to consider, for any $H \in (0, 1)$ and without ambiguity, a functional (still called *Newton-Cotes functional*) which verifies, for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $C^{2m_H, 1}$ and any process A having C^1 -trajectories:

$$F(B_T, A_T) = F(0, A_0) + \int_0^T f(B_u, A_u) d^{\text{NC}} B_u + \int_0^T F'_a(B_u, A_u) A'_u du,$$

where F is such that $F'_b = f$.

Finally, we introduce our last definition for a solution to (2):

Definition 2.17 Assume that $\sigma \in C^{2m_H}$ and $b \in C^0$.

i) Let \mathfrak{C}_3 be the class of processes $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ verifying that there exist a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $C^{2m_H, 1}$ and a process $A : [0, T] \times \Omega \rightarrow \mathbb{R}$ having C^1 -trajectories such that $A_0 = 0$ and verifying, for every $t \in [0, T]$, $X_t = f(B_t, A_t)$ a.s.

ii) A process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a solution to (2) if:

- $X \in \mathfrak{C}_3$,
- $\forall t \in [0, T]$, $X_t = x_0 + \int_0^t \sigma(X_s) d^{\text{NC}} B_s + \int_0^t b(X_s) ds$.

Theorem 2.18 Let $\sigma \in C_b^2$, b be a Lipschitz function and x_0 be a real. Then the equation (2) admits a solution X in the sense of Definition 2.17. Moreover, if σ is analytic, then X is the unique solution of the form $f(B, A)$ with f analytic (resp. of class C^1) in the first (resp. second) variable and A a process having C^1 -trajectories and verifying $A_0 = 0$.

Remark 2.19 • If $H > 1/3$, one can improve Theorem 2.18. Indeed, as shown in [26], uniqueness holds without any supplementary condition on σ . Moreover, in that reference, another meaning to (2) than Definition 2.17 is given, using the concept of Lévy area.

- In [25], one studies the problem of absolute continuity in equation (2), where the solution is in the sense of Definition 2.17. It is proved that, if $\sigma(x_0) \neq 0$, then $\mathcal{L}(X_t)$ is absolutely continuous with respect to the Lebesgue measure for all $t \in]0, T]$. More precisely, the Bouleau-Hirsch criterion is shown to hold: if $x_t = x_0 + \int_0^t b(x_s) ds$ and $t_x = \sup\{t \in [0, T] : x_t \notin \text{Int}J\}$ where $J = \sigma^{-1}(\{0\})$ then $\mathcal{L}(X_t)$ is absolutely continuous if and only if $t > t_x$.
- We already said that, among the m -order Newton-Cotes functionals, only the first one (that is, the symmetric integral, defined by (3)) is a "true" integral. For this integral, the main results contained in this paper are summarized in the following table (where f denotes a regular enough function and A a process having C^1 -trajectories):

If we use Definition	we have to choose $H \in$	X is then of the form	we have existence if	and uniqueness if moreover	See Theorem
2.8	(1/6, 1)	$f(B)$	$\sigma \in C^5 \cap \text{Lip}$, $b \in C^0$ and $b _{\mathcal{S}(\mathbb{R})} \equiv 0$	-	2.10
2.13	(1/6, 1)	$f(B + A)$	$\sigma \in C^5 \cap \text{Lip}$, $b \in C^0$ + i) $\sigma(x_0) = 0$ $b(x_0) = 0$ or ii) $\sigma(x_0) \neq 0$	i) - ii) $\inf_{\mathbb{R}} \sigma > 0$ and $b \in \text{Lip}$	2.14
2.17	(1/3, 1)	$f(B, A)$	$\sigma \in C_b^2$ and $b \in \text{Lip}$	-	2.18 and [25]

Table 1. *Existence and uniqueness in SDE $X_t = x_0 + \int_0^t \sigma(X_s) d^\circ B_s + \int_0^t b(X_s) ds$*

Proof of Theorem 2.18. Let us remark that the classical Doss-Sussmann [9, 33] method gives a natural solution X of the form $f(B, A)$. Then, in the remainder of the proof, we will concentrate on the uniqueness. Assume that $X = f(B, A)$ is a solution to (2) in the sense of Definition 2.17. On the one hand, we have

$$\begin{aligned} X_t &= x_0 + \int_0^t \sigma(X_s) d^{\text{NC}} B_s + \int_0^t b(X_s) ds \\ &= x_0 + \int_0^t \sigma \circ f(B_s, A_s) d^{\text{NC}} B_s + \int_0^t b \circ f(B_s, A_s) ds. \end{aligned} \quad (9)$$

On the other hand, using the change of variables formula, we can write

$$X_t = x_0 + \int_0^t f'_b(B_s, A_s) d^{\text{NC}} B_s + \int_0^t f'_a(B_s, A_s) A'_s ds. \quad (10)$$

Using (9) and (10), we deduce that $t \mapsto \int_0^t \varphi(B_s, A_s) d^{\text{NC}} B_s$ has C^1 -trajectories where $\varphi := f'_b - \sigma \circ f$. As in the proof of Theorem 2.14, we show that, a.s.,

$$\forall t \in]0, T[, \varphi(B_t, A_t) = 0. \quad (11)$$

Similarly, we can obtain that, a.s.,

$$\forall k \in \mathbb{N}, \forall t \in]0, T[, \frac{\partial^k \varphi}{\partial b^k}(B_t, A_t) = 0.$$

If σ and $f(\cdot, y)$ are analytic, then $\varphi(\cdot, y)$ is analytic and

$$\forall t \in]0, T[, \forall x \in \mathbb{R}, \varphi(x, A_t) = f'_b(x, A_t) - \sigma \circ f(x, A_t) = 0. \quad (12)$$

By uniqueness, we deduce

$$\forall t \in [0, T], \forall x \in \mathbb{R}, f(x, A_t) = u(x, A_t),$$

where u is the unique solution to $u'_b = \sigma(u)$ with initial value $u(0, y) = y$ for any $y \in \mathbb{R}$. In particular, we obtain a.s.

$$\forall t \in [0, T], X_t = f(B_t, A_t) = u(B_t, A_t). \quad (13)$$

Identity (9) can then be rewritten as:

with G a centered Gaussian random variable independent of X_1 , whose variance depends only on H . Note also that, in [14], the exact rate of convergence associated to the schemes introduced in [24] are computed and results of the type (17)-(19) are obtained.

In this section, we are interested in whether scheme (16) converges, according to the value of H and the expression of σ . First of all, this problem looks easier than computing the exact rate of convergence, as in [14, 22]. But, in these two papers, no optimality is sought in the domain of validity of H . For instance, in (17), we impose that $H > 1/3$ although it seems more natural to only assume that $H > 1/6$.

Unfortunately, we were able to find the exact barrier of convergence for (16) only for particular σ , namely those which verify (18). In this case, we prove in Theorem 3.1 below that the barrier of convergence is $H = 1/6$. In the other cases, it is nevertheless possible to prove that the scheme (16) converge when $H > 1/3$ (see the proof of Theorem 3.1). But the exact barrier remains an open question.

The class (18) is quite restricted. In particular, I must acknowledge that Theorem 3.1 has a limited interest. However, its proof is instructive. Moreover it contains a useful formula for $\widehat{X}_{k/n}^{(n)}$ (see Lemma 3.4), which is the core of all the results concerning the Crank-Nicholson scheme proved in [22] (see also [14]).

Now, we state the main result of this section:

Theorem 3.1 *Assume that $\sigma \in C^1(\mathbb{R})$ verifies (18). Then the sequence $\{\widehat{X}_1^{(n)}\}$ defined by (16) converges in L^2 if and only if $H > 1/6$. In this case, the limit is the unique solution at time 1 to the SDE $X_t = x_0 + \int_0^t \sigma(X_s) d^\circ B_s$, in the sense of Definition 2.8 and given by Theorem 2.10.*

Remark 3.2 When $\sigma(x) = x$ it is easy to understand why $\widehat{X}_1^{(n)}$ converges in L^2 if and only if $H > 1/6$. Indeed, setting $\Delta_k^n = B_{(k+1)/n} - B_{k/n}$, we have

$$\widehat{X}_1^{(n)} = x_0 \prod_{k=0}^{n-1} \frac{1 + \frac{1}{2}\Delta_k^n}{1 - \frac{1}{2}\Delta_k^n} = x_0 \exp\left\{\sum_{k=0}^{n-1} \ln \frac{1 + \frac{1}{2}\Delta_k^n}{1 - \frac{1}{2}\Delta_k^n}\right\};$$

but

$$\ln \frac{1 + \frac{1}{2}\Delta_k^n}{1 - \frac{1}{2}\Delta_k^n} = \Delta_k^n + \frac{1}{12}(\Delta_k^n)^3 + \frac{1}{80}(\Delta_k^n)^5 + O((\Delta_k^n)^6),$$

and, because $\sum_{k=0}^{n-1} \Delta_k^n = B_1$ and by using Lemma 3.3 below, one has that $\widehat{X}_1^{(n)}$ converges if and only if $H > 1/6$ and that, in this case, the limit is $x_0 \exp(B_1)$.

As a preliminary of the proof of Theorem 3.1, we need two lemmas:

Lemma 3.3 *Let $m \geq 1$ be an integer.*

- *We have*

$$\sum_{k=0}^{n-1} (B_{(k+1)/n} - B_{k/n})^{2m} \text{ converges in } L^2 \text{ as } n \rightarrow \infty \text{ if and only if } H \geq \frac{1}{2m}.$$

In this case, the limit is zero if $H > 1/2m$ and is $(2m)!/(2^m m!)$ if $H = 1/2m$.

- *We have*

$$\sum_{k=0}^{n-1} (B_{(k+1)/n} - B_{k/n})^{2m+1} \text{ converges in } L^2 \text{ as } n \rightarrow \infty \text{ if and only if } H > \frac{1}{4m+2}.$$

In this case, the limit is zero.

Proof of Lemma 3.3. The first point is an obvious consequence of the well-known convergence

$$n^{2mH-1} \sum_{k=0}^{n-1} (B_{(k+1)/n} - B_{k/n})^{2m} \xrightarrow{L^2} (2m)!/(2^m m!), \text{ as } n \rightarrow \infty.$$

Let us then prove the second point. On the one hand, for $H > 1/(4m + 2)$, we can prove directly that

$$\sum_{k,\ell=0}^{n-1} E[(B_{(k+1)/n} - B_{k/n})^{2m+1} (B_{(\ell+1)/n} - B_{\ell/n})^{2m+1}] \longrightarrow 0, \text{ as } n \rightarrow \infty,$$

by using a Gaussian linear regression, see for instance [16], Proposition 3.8. On the other hand, it is well known that, when $H < 1/2$,

$$n^{(2m+1)H-1/2} \sum_{k=0}^{n-1} (B_{(k+1)/n} - B_{k/n})^{2m+1} \xrightarrow{\mathcal{L}} \mathbb{N}(0, \sigma_{m,H}^2), \text{ as } n \rightarrow \infty, \text{ for some } \sigma_{m,H} > 0$$

(use, for instance, the main result by Nualart and Peccati [29]). We can then deduce the non-convergence when $H \leq 1/(4m + 2)$ as in [15], Proof of 2(c), page 796. \square

Lemma 3.4 *Assume that $\sigma \in C^5(\mathbb{R})$ is bounded together with its derivatives. Consider ϕ the flow associated to σ , that is, $\phi(x, \cdot)$ is the unique solution to $y' = \sigma(y)$ with initial value $y(0) = x$. Then we have, for any $\ell \in \{0, 1, \dots, n\}$:*

$$\widehat{X}_{\ell/n}^{(n)} = \phi\left(x_0, B_{\ell/n} + \sum_{k=0}^{\ell-1} f_3(\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^3 + \sum_{k=0}^{\ell-1} f_4(\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^4 + \sum_{k=0}^{\ell-1} f_5(\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^5 + O(n\Delta^6(B))\right). \quad (20)$$

Here we set

$$f_3 = \frac{(\sigma^2)''}{24}, f_4 = \frac{\sigma(\sigma^2)'''}{48} \text{ and } f_5 = \frac{\sigma'^4}{80} + \frac{\sigma^2 \sigma' \sigma'''}{15} + \frac{3\sigma \sigma'^2 \sigma''}{40} + \frac{\sigma^2 \sigma''^2}{20} + \frac{\sigma^3 \sigma^{(4)}}{80},$$

$$\Delta_k^n = B_{(k+1)/n} - B_{k/n}, \text{ when } n \in \mathbb{N} \text{ and } k \in \{0, 1, \dots, n-1\}$$

and

$$\Delta^p(B) = \max_{k=0, \dots, n-1} |(\Delta_k^n)^p|, \text{ when } p \in \mathbb{N}^*.$$

Proof of Lemma 3.4. Assume, for an instant, that σ does not vanish. In this case, $\phi(x, \cdot)$ is a bijection from \mathbb{R} to himself for any x and we can consider $\varphi(x, \cdot)$ such that

$$\forall x, t \in \mathbb{R} : \varphi(x, \phi(x, t)) = t \text{ and } \phi(x, \varphi(x, t)) = t. \quad (21)$$

On the one hand, thanks to (21), it is a little long but easy to compute that

$$\begin{aligned} \varphi(x, x) &= 0, \\ \varphi'_t(x, x) &= 1/\sigma(x), \\ \varphi''_{tt}(x, x) &= [-\sigma'/\sigma^2](x), \\ \varphi^{(3)}_{ttt}(x, x) &= [(2\sigma'^2 - \sigma\sigma'')/\sigma^3](x), \\ \varphi^{(4)}_{tttt}(x, x) &= [(-6\sigma'^3 + 6\sigma\sigma'\sigma'' - \sigma^2\sigma''')/\sigma^4](x) \\ \varphi^{(5)}_{ttttt}(x, x) &= [(24\sigma'^4 - 36\sigma\sigma'^2\sigma'' + 8\sigma^2\sigma'\sigma''' + 6\sigma^2\sigma''^2 - \sigma^3\sigma^{(4)})/\sigma^5](x). \end{aligned}$$

Then, for u sufficiently small, we have

$$\varphi(x, x+u) = \frac{1}{\sigma}(x)u - \frac{\sigma'}{2\sigma^2}(x)u^2 + \frac{2\sigma'^2 - \sigma\sigma''}{6\sigma^3}(x)u^3 + \frac{-6\sigma'^3 + 6\sigma\sigma'\sigma'' - \sigma^2\sigma'''}{24\sigma^4}(x)u^4 + \frac{24\sigma'^4 - 36\sigma\sigma'^2\sigma'' + 8\sigma^2\sigma'\sigma''' + 6\sigma^2\sigma''^2 - \sigma^3\sigma^{(4)}}{\sigma^5}(x)u^5 + O(u^6).$$

On the other hand, using (16) and some basic Taylor expansions, one has for $k \in \{0, 1, \dots, n-1\}$:

$$\begin{aligned} \widehat{X}_{(k+1)/n}^{(n)} &= \widehat{X}_{k/n}^{(n)} + \sigma(\widehat{X}_{k/n}^{(n)})\Delta_k^n + \frac{\sigma\sigma'}{2}(\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^2 + \frac{\sigma\sigma'^2 + \sigma^2\sigma''}{4}(\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^3 \\ &\quad + \left(\frac{\sigma\sigma'^3}{8} + \frac{3\sigma^2\sigma'\sigma''}{8} + \frac{\sigma^3\sigma'''}{12}\right)(\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^4 \\ &\quad + \left(\frac{\sigma\sigma'^4}{16} + \frac{3\sigma^2\sigma'^2\sigma''}{8} + \frac{\sigma^3\sigma'\sigma'''}{6} + \frac{\sigma^3\sigma''^2}{8} + \frac{\sigma^4\sigma^{(4)}}{48}\right)(\widehat{X}_{k/n}^{(n)}) (\Delta_k^n)^5 \\ &\quad + O(\Delta^6(B)). \end{aligned}$$

Then, we have

$$\begin{aligned} \varphi(\widehat{X}_{k/n}^{(n)}, \widehat{X}_{(k+1)/n}^{(n)}) &= \varphi(\widehat{X}_{k/n}^{(n)}, \widehat{X}_{k/n}^{(n)} + [\widehat{X}_{(k+1)/n}^{(n)} - \widehat{X}_{k/n}^{(n)}]) \\ &= \Delta_k^n + \frac{\sigma'^2 + \sigma\sigma''}{12}(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^3 + \left(\frac{\sigma\sigma'\sigma''}{8} + \frac{\sigma^2\sigma'''}{24}\right)(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^4 \\ &\quad + \left(\frac{\sigma'^4}{80} + \frac{\sigma^2\sigma'\sigma'''}{15} + \frac{3\sigma\sigma'^2\sigma''}{40} + \frac{\sigma^2\sigma''^2}{20} + \frac{\sigma^3\sigma^{(4)}}{80}\right)(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^5 + O(\Delta^6(B)) \\ &= \Delta_k^n + f_3(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^3 + f_4(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^4 + f_5(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^5 + O(\Delta^6(B)). \end{aligned}$$

We deduce, using (21):

$$\widehat{X}_{(k+1)/n}^{(n)} = \phi(\widehat{X}_{k/n}^{(n)}, \Delta_k^n + f_3(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^3 + f_4(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^4 + f_5(\widehat{X}_{k/n}^{(n)})(\Delta_k^n)^5 + O(\Delta^6(B))).$$

Finally, by using the semi-group property verified by ϕ , namely

$$\forall x, s, t \in \mathbb{R} : \phi(\phi(x, t), s) = \phi(x, t + s).$$

we easily deduce (20).

In fact, we assumed that σ does not vanish only for having the possibility to introduce φ . But (20) is an algebraic formula then it is also valid for general σ , as soon as it is bounded together with its derivatives. \square

Proof of Theorem 3.1. Assume that σ verifies (18). Although σ is not bounded in general, it is easy to verify that we still have $O(n\Delta^6(B))$ as remainder in (20). Moreover, simple but tedious computations show that we can simplify in (20) to obtain

$$\widehat{X}_1^{(n)} = \phi(x_0, B_1 + \frac{\alpha}{12} \sum_{k=0}^{n-1} (\Delta_k^n)^3 + \frac{\alpha^2}{80} \sum_{k=0}^{n-1} (\Delta_k^n)^5 + O(n\Delta^6(B))).$$

Thus, as a conclusion of Lemma 3.3, we obtain easily that $\widehat{X}_1^{(n)}$ converges to $\phi(x_0, B_1)$ if and only if $H > 1/6$. \square

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References

1. Alos, E., Mazet, O., Nualart, D. *Stochastic calculus with respect to fractional Brownian motion with Hurst parameter less than $\frac{1}{2}$* . Stochastic Process. Appl. **86** (2000), 121-139.
2. Alos, E., Leon, J.A., Nualart, D. *Stratonovich calculus for fractional Brownian motion with Hurst parameter less than $\frac{1}{2}$* . Taiwanese J. Math. **5** (2001), 609-632.
3. Bender, C. *An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter*. Stochastic Process. Appl. **104** (2003), 81-106.
4. Baudoin, F., Coutin, L. *Etude en temps petit du flot d'équations conduites par des mouvements browniens fractionnaires*. C.R. Math. Acad. Sci. Paris I **341** (2005), 39-42.
5. Boufoussi, B., Tudor, C.A. *Kramers-Smoluchowski approximation for stochastic equations with fBm*. Rev. Roumaine Math. Pures Appl. **50** (2005), 125-136.
6. Carmona, P., Coutin, L. *Intégrales stochastiques pour le mouvement brownien fractionnaire*. C.R. Math. Acad. Sci. Paris I **330** (2000), 213-236.
7. Coutin, L., Qian, Z. *Stochastic analysis, rough path analysis and fractional Brownian motions*. Probab. Theory Related Fields **122** (2002), no. 1, 108-140.
8. Decreusefond, L., Ustunel, A.S. *Stochastic analysis of the fractional Brownian motion*. Potential Anal. **10** (1998), 177-214.
9. Doss, H. *Liens entre équations différentielles stochastiques et ordinaires*. Ann. Inst. H. Poincaré Sect. B **13** (1977), 99-125.
10. Duncan, T.E., Hu, Y., Pasik-Duncan, B. *Stochastic calculus for fractional Brownian motion I. Theory*. SIAM J. Control Optim. **38** (2000), 582-612.
11. Errami, M., Russo, F. *n-covariation and symmetric SDEs driven by finite cubic variation process*. Stochastic Process. Appl. **104** (2003), 259-299.

12. Feyel, D., De La Pradelle, A. *On fractional Brownian processes*. Potential Anal. **10** (1999), no. 3, 273-288.
13. Feyel, D., De La Pradelle, A. *Curvilinear integrals along enriched paths*. Electron. J. Probab. **11** (2006), 860-892.
14. Gradinaru, M., Nourdin, I. *Weighted power variations of fractional Brownian motion and application to approximating schemes*. Preprint Paris VI.
15. Gradinaru, M., Nourdin, I., Russo, F., Vallois, P. *m-order integrals and Itô's formula for non-semimartingale processes; the case of a fractional Brownian motion with any Hurst index*. Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), 781-806.
16. Gradinaru, M., Russo, F., Vallois, P. *Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$* . Ann. Probab. **31** (2001), 1772-1820.
17. Klingenhöfer, F., Zähle, M. *Ordinary differential equations with fractal noise*. Proc. AMS **127** (1999), 1021-1028.
18. Lejay, A. *An Introduction to Rough Paths*. Séminaire de probabilités XXXVII, vol. **1832** of Lecture Notes in Mathematics (2003), 1-59.
19. Lin, S.J. *Stochastic analysis of fractional Brownian motion*. Stochastics Stochastics Rep. **55** (1995), 121-140.
20. Lyons, T.J. *Differential equations driven by rough signals*. Rev. Math. Iberoamer. **14** (1998), 215-310.
21. Neuenkirch, A. *Optimal approximation of SDEs with additive fractional noise*. J. Complexity **22** (4), 459-475, 2006.
22. Neuenkirch, A., Nourdin, I. *Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion*. J. Theor. Probab., to appear.
23. Nourdin, I. *Calcul stochastique généralisé et applications au mouvement brownien fractionnaire; Estimation non-paramétrique de la volatilité et test d'adéquation*. PhD thesis, University of Nancy (2004).
24. Nourdin, I. *Schémas d'approximation associés à une équation différentielle dirigée par une fonction höldérienne; cas du mouvement brownien fractionnaire*. C.R. Math. Acad. Sci. Paris, Ser. I **340** (2005), 611-614.
25. Nourdin, I., Simon, T. *On the absolute continuity of one-dimensional SDE's driven by a fractional Brownian motion*. Statist. Probab. Lett. **76** (2006), no. 9, 907-912.
26. Nourdin, I., Simon, T. *Correcting Newton-Côtes integrals corrected by Lévy areas*. Bernoulli **13** (2007), no. 3, 695-711.
27. Nualart, D. *Stochastic calculus with respect to the fractional Brownian motion and applications*. Contemp. Math. **336** (2003), 3-39.
28. Nualart, D., Ouknine, Y. *Stochastic differential equations with additive fractional noise and locally unbounded drift*. Progr. Probab. **56** (2003), 353-365.
29. Nualart, D., Peccati, G. *Central limit theorems for sequences of multiple stochastic integrals*. Ann. Probab. **33** (1) (2005), 177-193.
30. Nualart, D., Răşcanu, A. *Differential equations driven by fractional Brownian motion*. Collect. Math. **53** (2002), no. 1, 55-81.
31. Russo, F., Vallois, P. *Forward, backward and symmetric stochastic integration*. Probab. Theory Related Fields **97** (1993), 403-421.
32. Russo, F., Vallois, P. *Stochastic calculus with respect to a finite quadratic variation process*. Stochastics Stochastics Rep. **70** (2000), 1-40.
33. Sussmann, H.J. *An interpretation of stochastic differential equations as ordinary differential equations which depend on a sample point*. Bull. Amer. Math. Soc. **83** (1977), 296-298.
34. Talay, D. *Résolution trajectorielle et analyse numérique des équations différentielles stochastiques*. Stochastics **9** (1983), 275-306.
35. Young, L. C. *An inequality of the Hölder type connected with Stieltjes integration*. Acta Math. **67** (1936), 251-282.
36. Zähle, M. *Integration with respect to fractal functions and stochastic calculus I*. Probab Theory Related Fields **111** (1998), 333-374.