

PROPOSITIONAL DYNAMIC LOGIC FOR SEARCHING GAMES WITH ERRORS

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ABSTRACT. We investigate some finitely-valued generalizations of propositional dynamic logic with tests. We start by introducing the $n + 1$ -valued KRIPKE models and a corresponding language based on a modal extension of ŁUKASIEWICZ many-valued logic. We illustrate the definitions by providing a framework for an analysis of the RÉNYI - ULAM searching game with errors.

Our main result is the axiomatization of the theory of the $n + 1$ -valued KRIPKE models. This result is obtained through filtration of the canonical model of the smallest $n + 1$ -valued propositional dynamic logic.

1. INTRODUCTION

Propositional dynamic logic (PDL) is a multi-modal logic designed to reason about programs. The general idea behind the semantic of this system is the following. Program states are gathered in a set W . Any program α is encoded by its input/output relation on W . Programs are built from atomic ones and test operators using regular operations. Their associated relations are defined as to respect this algebra of programs. The goal is to provide a framework for formal verification through input/output specifications.

Since its introduction by FISCHER and LADNER in [9], the scope of dynamic logic has widened to many other areas such as game theory (see [12, 18, 27]), epistemic logic (see [32]) or natural language (see [31]). The subject is under constant and active development (see [1, 2, 8, 19] for example) and we refer to [17] for an introductory monograph.

Informally, PDL is a mixture of modal logic and algebra of regular programs. Recently, some authors have considered generalizations of modal logics to many-valued realms (see [3, 4, 10, 11, 13, 24]). These modal many-valued systems can naturally be considered as building blocks of many-valued generalizations of PDL. In other words, these developments raise the issue of describing the systems obtained by adding a many-valued flavor to the modal logic used to define PDL. Such many-valued propositional dynamic

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logics would provide a language to state correctness criteria in the form of input/output specifications that could be *partly* satisfied.

We address this problem for the modal extensions of the $n + 1$ -valued ŁUKASIEWICZ logics (see [20, 21, 22]) studied in [15, 16]. Hence, the truth values of the propositions range in a set of finite cardinality $n + 1$ where $n \geq 1$.

Our starting point is the definition of a language (with test operator) for such generalizations and their corresponding $n + 1$ -valued KRIPKE models. In these models, relations associated to programs are crisp and valuation maps are many-valued. As an illustration of the new possibilities allowed by this language, we explain how it can be used to construct a dynamic model for formal verification of strategies of the RÉNYI - ULAM searching game with errors.

The goal of this paper is the characterization of the theory of these $n + 1$ -valued KRIPKE models (*i.e.*, the set of formulas that are true in any model). In this view, Theorem 5.13 is our main result. It gives an axiomatization of this theory through an $n + 1$ -valued propositional dynamic deductive system that we denote by PDL_n .

This result is obtained by the way of the canonical model. This construction defects to be a ‘standard’ $n + 1$ -valued KRIPKE model and we need a filtration result to obtain Theorem 5.13.

The construction of the canonical model for PDL is algebraic in disguise. This model is built upon the set of the maximal filters of the LINDENBAUM - TARSKI algebra of PDL which is a multi-modal Boolean algebra. Naturally, the canonical model for PDL_n also has an algebraic flavor. The system PDL_n is based on modal extensions of ŁUKASIEWICZ $n + 1$ -valued logic. Hence, MV-algebras - which are the algebraic counterpart of ŁUKASIEWICZ logics - replace Boolean algebras in this setting.

The techniques used in the proofs in this paper are generalizations of the corresponding techniques for PDL . It is worth noting that by considering $n = 1$, our results boil down to the existing ones for PDL .

This paper is organized as follows. In the next section we introduce some many-valued generalizations of the language and models of PDL . Section 3 provides an example that illustrates the possibilities offered by these generalizations. Section 4 is devoted to the development of a sound deductive system PDL_n for the $n + 1$ -valued KRIPKE models. The many-valued forms of the intrinsic axioms of PDL , such as the induction axiom, are discussed when needed. Eventually, in section 5 we prove the deductive completeness of PDL_n with respect to the $n + 1$ -valued KRIPKE-models (proof of the filtration lemma is provided in Appendix). In order to keep the paper self-contained, we recall the necessary definitions and results about algebras of regular programs and MV-algebras.

2. MANY-VALUED KRIPKE MODELS FOR DYNAMIC LOGICS

The starting point of the developments of this paper is a generalization to an $n + 1$ -valued realm of the definitions of the propositional dynamic language and the KRIPKE models.

Let us denote by Π_0 a nonempty set of atomic programs (denoted by a, b, \dots) and by \mathbf{Prop} a countable set of propositional variables (denoted by p, q, \dots). The sets Π of programs and \mathbf{Form} of well formed formulas are given by the following BACKUS-NAUR forms (where ϕ are formulas and α are programs) :

$$(2.1) \quad \begin{aligned} \phi &::= p \mid 0 \mid \neg\phi \mid \phi \rightarrow \phi \mid [\alpha]\phi \\ \alpha &::= a \mid \phi? \mid \alpha; \alpha \mid \alpha \cup \alpha \mid \alpha^*. \end{aligned}$$

To extend the definition of a KRIPKE model to a $[0, 1]$ -valued realm, we use ŁUKASIEWICZ interpretation $\rightarrow^{[0,1]}$ and $\neg^{[0,1]}$ of the binary connector \rightarrow and the unary connector \neg respectively. These maps are defined on $[0, 1]$ by

$$(2.2) \quad \neg^{[0,1]}x = 1 - x \quad \text{and} \quad x \rightarrow^{[0,1]}y = \min(1 - x + y, 1).$$

Hence, (2.2) allows us to define in the obvious inductive way the $[0, 1]$ -interpretation $\tau^{[0,1]}$ of any well formed formula τ constructed only with propositional variables and connectives \neg and \rightarrow (if τ has k propositional variables then $\tau^{[0,1]} : [0, 1]^k \rightarrow [0, 1]$). To shorten notation, when no confusion is possible we usually denote by \neg , \rightarrow and τ the maps $\neg^{[0,1]}$, $\rightarrow^{[0,1]}$ and $\tau^{[0,1]}$ respectively.

The results we are interested in are related to finitely-valued ŁUKASIEWICZ logics. It means that we only allow valuations of propositional variables in the finite subsets of $[0, 1]$ that are closed for the connectors \neg and \rightarrow (and that contain 0 and 1). It is not difficult to realize that these are exactly the subsets $\mathbb{L}_n = \{\frac{i}{n} \mid 0 \leq i \leq n\}$ where n is an integer greater than 1 (see [7] for details). If τ is a formula constructed from k propositional variables by using only connectives \neg and \rightarrow , we denote by $\tau^{\mathbb{L}_n}$ the restriction of $\tau^{[0,1]}$ to \mathbb{L}_n^k .

Recall that if R and R' are unary relations on W then the *composition* $R \circ R'$ is defined by $R \circ R' = \{(u, w) \in W \times W \mid \exists v \in W (uRv \ \& \ vR'w)\}$. Moreover, the k -th power R^k of R is inductively defined by $R^0 = \{(u, u) \mid u \in W\}$ and $R^{k+1} = R \circ R^k$ for $k \in \omega$.

Definition 2.1. An $n + 1$ -valued KRIPKE model $\mathcal{M} = \langle W, R, \text{Val} \rangle$ is given by a nonempty set W , a map $R : \Pi_0 \rightarrow 2^{W \times W}$ that assigns a binary relation R_a to any $a \in \Pi_0$ and a map $\text{Val} : W \times \mathbf{Prop} \rightarrow \mathbb{L}_n$ that assigns a truth value to any propositional variable p of \mathbf{Prop} in any world w of W .

The maps R and Val are extended by mutual induction to formulas and programs by the following rules:

- (1) $R_{\alpha;\beta} = R_\alpha \circ R_\beta$;
- (2) $R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$;
- (3) $R_{\phi?} = \{(u, u) \mid \text{Val}(u, \phi) = 1\}$;

- (4) $R_{\alpha^*} = \bigcup_{k \in \omega} (R_{\alpha})^k$;
- (5) $\text{Val}(w, 0) = 0$;
- (6) $\text{Val}(w, \phi \rightarrow \psi) = \text{Val}(w, \phi) \rightarrow^{[0,1]} \text{Val}(w, \psi)$;
- (7) $\text{Val}(w, \neg \psi) = \neg^{[0,1]} \text{Val}(w, \psi)$;
- (8) $\text{Val}(w, [\alpha]\psi) = \bigwedge \{ \text{Val}(v, \psi) \mid (w, v) \in R_{\alpha} \}$.

Throughout the paper, n stands for a fixed integer greater or equal to 1. We sometimes call *KRIPKE model* an $n + 1$ -valued *KRIPKE model*.

Clearly, we intend to interpret the operator ‘;’ as the concatenation program operator, ‘ \cup ’ as the alternative program operator and the operator ‘ $*$ ’ as the *KLEENE* program operator. Hence, if α and β are programs, the connective $[\alpha]$ is read ‘after any execution of α ’, the connective $[\alpha \cup \beta]$ is read ‘after any execution of α or β ’, the connective $[\alpha; \beta]$ is read ‘after any execution of α followed by an execution of β ’ and $[\alpha^*]$ is read ‘after an undetermined number of executions of α ’ (rule (4) means that R_{α^*} is defined as the transitive and reflexive closure of R_{α}).

Definition 2.2. If w is a world of a *KRIPKE model* \mathcal{M} and if ϕ is a formula such that $\text{Val}(w, \phi) = 1$, we write $\mathcal{M}, w \models \phi$ and say that ϕ is *true in w*. If ϕ is a formula that is true in each world of a model \mathcal{M} then ϕ is *true in M*. A formula that is true in every *KRIPKE model* is called a *tautology*.

We use of the well established following abbreviations for any $\phi, \psi \in \text{Form}$: the formula $\phi \vee \psi$ stands for $(\phi \rightarrow \psi) \rightarrow \psi$, the formula $\phi \wedge \psi$ for $\neg(\neg\phi \vee \neg\psi)$, the formula $\phi \oplus \psi$ for $\neg\phi \rightarrow \psi$, the formula $\phi \odot \psi$ for $\neg(\neg\phi \oplus \neg\psi)$, the formula $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \odot (\psi \rightarrow \phi)$. Moreover, we assume associativity of \oplus and \odot (this is justified by associativity of $\oplus^{[0,1]}$ and $\odot^{[0,1]}$). Hence, the formula $k.\psi$ and ψ^k ($k \in \omega$) stands respectively for $\psi \oplus \dots \oplus \psi$ and $\psi \odot \dots \odot \psi$ where the factor ψ is repeated k times. We adopt the convention that $\psi^0 = 1$ and $0.\psi = 0$. It is easily checked that the resulting $[0, 1]$ -interpretations of these abbreviations are the following:

- (1) $x \oplus^{[0,1]} y = \min\{x + y, 1\}$,
- (2) $x \odot^{[0,1]} y = \max\{x + y - 1, 0\}$,
- (3) $x \leftrightarrow^{[0,1]} y = 1 - |x - y|$,
- (4) $x \vee^{[0,1]} y = \max\{x, y\}$,
- (5) $x \wedge^{[0,1]} y = \min\{x, y\}$.

In ŁUKASIEWICZ logic, connectors \oplus and \odot are respectively called *strong disjunction* and *strong conjunction* because the equations $(p \odot q)^{[0,1]} \leq (p \wedge q)^{[0,1]}$ and $(p \oplus q)^{[0,1]} \geq (p \vee q)^{[0,1]}$ are satisfied. Recall that $\odot^{[0,1]}$ is a left-continuous t-norm with residuum $\neg^{[0,1]}$. This means that equation

$$(2.3) \quad ((p \odot (p \rightarrow q)) \rightarrow q)^{[0,1]} = 1,$$

which can be considered as the fuzzy version of *modus ponens*, is satisfied. It should be noted that $(p^{k+1})^{[0,1]} \neq (p^k)^{[0,1]}$ for any $k \in \omega$ but $(p^{k+1})^{\mathbb{L}_n} = (p^k)^{\mathbb{L}_n}$ for every $k \geq n$. Finally, the formula $\langle \alpha \rangle \phi$ stands for $\neg[\alpha]\neg\phi$.

Moreover, we write $\mathcal{M}, w \models \Gamma$ (respectively $\mathcal{M} \models \Gamma$) if Γ is a set of formulas that are true in w (respectively in \mathcal{M}).

Proposition 2.3. *The following formulas are tautologies for any programs α and β (where n is the integer that we have fixed to define E_n).*

- | | |
|--|---|
| (1) $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p.$ | (8) $[\alpha^*]p \rightarrow [\alpha]p.$ |
| (2) $[\alpha; \beta]p \leftrightarrow [\alpha][\beta]p.$ | (9) $\langle \alpha \rangle p \rightarrow \langle \alpha^* \rangle p.$ |
| (3) $\langle \alpha \cup \beta \rangle p \leftrightarrow \langle \alpha \rangle p \vee \langle \beta \rangle p.$ | (10) $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p).$ |
| (4) $\langle \alpha; \beta \rangle p \leftrightarrow \langle \alpha \rangle \langle \beta \rangle p.$ | (11) $\langle \alpha^* \rangle p \leftrightarrow (p \vee \langle \alpha \rangle \langle \alpha^* \rangle p).$ |
| (5) $[q?]p \leftrightarrow (\neg q^n \vee p).$ | (12) $(p \wedge [\alpha^*](p \rightarrow [\alpha]p)^n) \rightarrow [\alpha^*]p.$ |
| (6) $[\alpha^*]p \rightarrow p.$ | (13) $[\alpha^*]p \rightarrow [\alpha^*][\alpha^*]p.$ |
| (7) $p \rightarrow \langle \alpha^* \rangle p.$ | |

Moreover, the following formulas are tautologies for any program α , because they are tautologies of the modal $n + 1$ -valued ŁUKASIEWICZ logic.

- (14) $[\alpha](p \rightarrow q) \rightarrow ([\alpha]p \rightarrow [\alpha]q).$
 (15) $[\alpha](p \wedge q) \leftrightarrow [\alpha]p \wedge [\alpha]q$ and $\langle \alpha \rangle(p \vee q) \leftrightarrow \langle \alpha \rangle p \vee \langle \alpha \rangle q.$
 (16) $([\alpha]p \vee [\alpha]q) \rightarrow [\alpha](p \vee q).$
 (17) $([\alpha]\phi \odot \langle \alpha \rangle \psi) \rightarrow \langle \alpha \rangle(\phi \odot \psi).$
 (18) $\langle \alpha \rangle(\phi \odot \psi) \rightarrow (\langle \alpha \rangle \phi \odot \langle \alpha \rangle \psi)$
 (19) If $\tau(q)$ is a formula with a single variable q which is constructed only with the connectors \neg and \rightarrow and whose $[0, 1]$ -interpretation is increasing then $\tau([\alpha]p) \leftrightarrow [\alpha]\tau(p)$ and $\tau(\langle \alpha \rangle p) \leftrightarrow \langle \alpha \rangle \tau(p).$

Example 2.4. It is worth noting that the formula $(p \wedge [\alpha^*](p \rightarrow [\alpha]p)) \rightarrow [\alpha^*]p$ is not a tautology. It would have been the most natural many-valued generalization of the Induction Axiom of PDL. As a counterexample, consider the model $\mathcal{M} = \langle \{u, v\}, R, \text{Val} \rangle$ where $R_a = \{(u, v)\}$, $\text{Val}(u, p) = 3/4$ and $\text{Val}(v, p) = 1/4$. It follows that on the one hand $\text{Val}(u, [\alpha^*]p) = \text{Val}(u, p) \wedge \text{Val}(v, p) = 1/4$. On the other hand, we obtain successively

$$\begin{aligned}
 (2.4) \quad \text{Val}(u, [\alpha^*](p \rightarrow [\alpha]p)) &= \text{Val}(u, p \rightarrow [a]p) \wedge \text{Val}(v, p \rightarrow [a]p) \\
 (2.5) &= 1/2 \wedge 1 \\
 (2.6) &= 1/2.
 \end{aligned}$$

It follows that $\text{Val}(u, p \wedge [\alpha^*](p \rightarrow [\alpha]p)) = 3/4 \wedge 1/2 = 1/2 \neq 1/4 = \text{Val}(u, [\alpha^*]p).$

3. AN ILLUSTRATION, THE RÉNYI - ULAM GAME

We can use the previously defined models to provide a framework for an analysis of the famous RÉNYI - ULAM game. ULAM's formulation of the game in [29], which was previously and independently introduced by RÉNYI, is the following:

Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number

can be guessed by asking first: is the number in the first half-million? and again reduce the reservoir of numbers in the next question by one-half, and so-on. Finally, the number is obtained in less than $\log_2 1000000$. Now, suppose that one were allowed to lie once or twice, then how many questions would one need to get the right answer?

Many researchers (mainly computer scientists) have focused their attention on that game since the publication of ULAM's book [29]. The success of the game is due to its connections with the theory of error-correcting codes with feedbacks in a noisy channel and the complexity of the problem of defining optimal strategies for the game. We refer to [28] for an overview of the literature about the RÉNYI - ULAM game.

The game has also been considered by many-valued logicians as a way to give a concrete interpretation of ŁUKASIEWICZ finitely-valued calculi and their associated algebras (see [23]). Mathematicians have modeled the game by coding algebraically *questions* and *answers*. We recall this model, which is due to MUNDICI, and then build a dynamic layer upon it in order to model the interactions between the two gamers.

3.1. Algebraic approach of the states of knowledge. We call the first gamer (the one who chooses a number and can lie) Pinocchio, and the second gamer Geppetto. Let us denote by M the *search space*, *i.e.*, the finite set of integers (or whatever) in which Pinocchio can pick up his number. Let us also assume that Pinocchio can lie $n - 1$ times.

We set up a way to algebraically encode the information defined by Pinocchio's answers, *i.e.*, to model Geppetto's state of knowledge of the game after each of Pinocchio's answers. This can be done by considering at step i of the game (after i answers) the map $r_i : M \rightarrow \{0, 1, \dots, n\}$ where $r_i(m)$ is the number of the i previous answers that refute the element m of M as Pinocchio's number. Indeed, once $r(m) = n$, since Pinocchio is allowed to lie $n - 1$ times, Geppetto can safely conclude that m is not the 'right' number. Hence, the game ends once Geppetto encodes its knowledge by a map r which is equal to n in any element m of M but in the searched number.

In order to introduce ŁUKASIEWICZ language in the interpretation of the game, we consider an equivalent representation of Geppetto's states of knowledge. This approach was introduced in [23].

Definition 3.1. A *state of knowledge* is a map $f : M \rightarrow \mathbb{L}_n$. The *state of knowledge* f at some step of the game is defined by $f(m) = 1 - \frac{r(m)}{n}$ where $r(m)$ denotes for any m in M the number of Pinocchio's answers that refute m as the searched number.

Hence, informally speaking, if f is a state of knowledge at some step of the game, the number $f(m)$ can be viewed for any m in M as the relative distance between m and the set of the elements of M that can be safely discarded as inappropriate.

3.2. Questions and answers. Note that during the game any question is equivalent to a question of the form ‘Does the searched number belong to Q ?’ for a subset Q of the search space M . Hence, for the remainder of this section, we denote any question by its associated subset Q of M .

Let us assume that Geppetto has reached the state of knowledge f and that he asks question Q . What is the state of knowledge f' of the game after Pinocchio’s answer? If Pinocchio answers positively (‘Yes, the number belongs to Q ’) then Gepetto increments $r(m)$ by one (if necessary) for any m in $M \setminus Q$ since a positive answer to Q is equivalent to a negative answer to $M \setminus Q$, *i.e.*,

$$(3.1) \quad f' : M \rightarrow \mathbb{L}_n = m \mapsto \begin{cases} f(m) & \text{if } m \in Q \\ \max\{f(m) - \frac{1}{m}, 0\} & \text{if } m \in M \setminus Q. \end{cases}$$

On the contrary, if Pinocchio answers negatively to Q , then Gepetto increments $r(m)$ by one (if necessary) for any m in Q , *i.e.*,

$$(3.2) \quad f' : M \rightarrow \mathbb{L}_n = m \mapsto \begin{cases} f(m) & \text{if } m \in M \setminus Q \\ \max\{f(m) - \frac{1}{m}, 0\} & \text{if } m \in Q. \end{cases}$$

This line of argument justifies the following definition.

Definition 3.2. If Q is a subset of M , the *positive answer* to Q is the map

$$f_Q : M \rightarrow \{\frac{n-1}{n}, 1\} : m \mapsto \begin{cases} 1 & \text{if } m \in Q \\ \frac{n-1}{n} & \text{if } m \in M \setminus Q. \end{cases}$$

The *negative answer* to Q is the positive answer $f_{M \setminus Q}$ to $M \setminus Q$.

We can thus encode algebraically any of Pinocchio’s answers. Recall that the interpretation of the binary connector \odot on $[0, 1]$ is defined by $x \odot^{[0,1]} y = \max(x + y - 1, 0)$.

Fact 3.3. Assume that Geppetto has reached the state of knowledge f and that he asks question Q . After Pinocchio’s answer to Q , the stage of knowledge f' of the game is $f \odot f_Q$ if Pinocchio’s answer is positive and $f \odot f_{M \setminus Q}$ if it is negative.

3.3. A dynamic layer. Roughly speaking, we have modeled the game in a static way. There is no structure to model the possible *sequences* of states of games. We provide such a structure through the Question/Answer relations on the set of the states of knowledge. The atomic programs are the possible questions, *i.e.* $\Pi_0 = 2^M$. The set of propositional variables $\{p_m \mid m \in M\}$ that are relevant to the problem is made of a variable p_m for any m in M that can be read as ‘ m is far from the set of rejected element’ or ‘the relative distance between m and the set of rejected elements is’.

Definition 3.4. The model of the RÉNYI - ULAM game with search space M and $n - 1$ lies is the $n + 1$ -valued KRIPKE model $\mathcal{M} = \langle \mathbb{L}_n^M, R, \text{Val} \rangle$ where

- (1) for any Q in 2^M , the relation R_Q contains (f, f') if $f' = f \odot f_Q$ or if $f' = f \odot f_{M \setminus Q}$,
- (2) for any m in M and any f in \mathbb{L}_n^M , we set $\text{Val}(f, p_m) = f(m)$.

This model provides a way to interpret any run of the game as a path from the initial state $f : m \mapsto 1$ to any final winning state.

Example 3.5. Examples of formulas that state correctness specifications for ‘honest’ sequences of states of knowledge include the following. We denote by $\tau_{i/n}(p)$ a formula whose interpretation on \mathbb{L}_n is valued in $\{0, 1\}$ and satisfies $\tau_{i/n}^{\mathbb{L}_n}(x) = 1 \iff x \geq i/n$. See Definition 4.3 for a formal definition.

- (1) $[Q]p_m \rightarrow p_m$
- (2) $\tau_{\frac{i}{n}}(p_m) \rightarrow [Q; M \setminus Q]\tau_{\frac{i-2}{n}}(p_m)$ (if we agree that $\tau_{\frac{i-2}{n}}(p_m) = 1$ if $i - 2 \leq 0$).

As mentioned in the introduction, it is not the purpose of this paper to push further the investigation of the new possibilities allowed by the $n + 1$ -valued KRIPKE models. Nevertheless, we give some ideas of possible applications in section 6.

4. $n + 1$ -VALUED PROPOSITIONAL DYNAMIC LOGICS

We aim to provide a set of rules that allow to syntactically generate the theory of the $n + 1$ -valued KRIPKE models defined in section 2. The underlying modal system on which we base the following definition is the modal \mathbb{L}_n -valued logic introduced in [15, 16].

Definition 4.1. An $n + 1$ -valued propositional dynamic logic (or simply a logic) is a subset \mathbf{L} of \mathbf{Form} that is closed under the rules of *modus ponens*, uniform substitution and necessitation (generalization) and that contains the following axioms:

- (1) tautologies of the $n + 1$ -valued ŁUKASIEWICZ logic;
- (2) for any program α , axioms defining modality $[\alpha]$:
 - (a) $[\alpha](p \rightarrow q) \rightarrow ([\alpha]p \rightarrow [\alpha]q)$,
 - (b) $[\alpha](p \oplus p) \leftrightarrow [\alpha]p \oplus [\alpha]p$,
 - (c) $[\alpha](p \odot p) \leftrightarrow [\alpha]p \odot [\alpha]p$,
- (3) the axioms that define the program operations: for any programs α and β ;
 - (a) $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$,
 - (b) $[\alpha; \beta]p \leftrightarrow [\alpha][\beta]p$,
 - (c) $[q?]p \leftrightarrow (\neg q^n \vee p)$,
 - (d) $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p)$,
 - (e) $[\alpha^*]p \rightarrow [\alpha^*][\alpha^*]p$,
- (4) the induction axiom $(p \wedge [\alpha^*](p \rightarrow [\alpha]p)^n) \rightarrow [\alpha^*]p$ for any program α .

We denote by \mathbf{PDL}_n the smallest $n + 1$ -valued propositional dynamic logic.

As usual, a formula ϕ that belongs to a logic \mathbf{L} is called a *theorem* of \mathbf{L} and we often write $\vdash \phi$ instead of $\phi \in \mathbf{PDL}_n$.

Note that formulas of item (2) of Definition 4.1 are tautologies according to items (14) and (19) of Proposition 2.3. Similarly, formulas in (3) and (4) of Definition 4.1 are formulas (1), (2), (5), (10), (12), (13) of Proposition 2.3.

Remark 4.2. Note that conditions (1) and (2) and the deduction rules of Definition 4.1 together with deductive completeness for the modal \mathbb{L}_n -valued logic (see Theorem 6.2 in [16]) ensure that if ψ is a tautology of the modal \mathbb{L}_n -valued ŁUKASIEWICZ logic and if $\alpha \in \Pi$ then the formula obtained from ψ by substitution of any occurrence of \Box by $[\alpha]$ is a theorem of PDL_n .

Informally, the induction axiom (4) means ‘if after an undetermined number of executions of α the truth value of p cannot decrease after a new execution of α , then the truth value of p cannot decrease after any undetermined number of executions of α ’. Hence, it is a natural generalization of the induction axiom of PDL (which could not have been adopted without modification according to Example 2.4).

Let us introduce some notations in order to comment the axioms $[\alpha](p \oplus p) \leftrightarrow ([\alpha]p \oplus [\alpha]p)$ and $[\alpha](p \odot p) \leftrightarrow ([\alpha]p \odot [\alpha]p)$.

Definition 4.3. Let i be an element of $\{1, \dots, n\}$. We denote by $\tau_{i/n}$ a composition (fixed throughout the paper) of the formulas $p \oplus p$ and $p \odot p$ whose interpretation on \mathbb{L}_n is defined by $\tau_{i/n}^{\mathbb{L}_n}(x) = 0$ if $x < \frac{i}{n}$ and $\tau_{i/n}^{\mathbb{L}_n}(x) = 1$ if $x \geq \frac{i}{n}$ (see [24] for the existence and the construction of such formulas).

For any $i \in \{0, \dots, n\}$, we denote by $I_{i/n}$ the formula $\tau_{i/n} \wedge \neg \tau_{(i+1)/n}$ (where we set $\tau_{(n+1)/n} = \tau_{0/n} = p \oplus \neg p$).

Hence, the interpretation on \mathbb{L}_n of $I_{i/n}$ is the characteristic function of $\{\frac{i}{n}\}$. The following result is a consequence of deductive completeness for modal \mathbb{L}_n -valued ŁUKASIEWICZ logic (see [16]).

Fact 4.4. *In the definition of PDL_n , for any $\alpha \in \Pi$, the pair of axioms*

$$(4.1) \quad \{[\alpha](p \star p) \leftrightarrow ([\alpha]p \star [\alpha]p) \mid \star \in \{\odot, \oplus\}\}$$

can be equivalently replaced by the axioms

$$(4.2) \quad \{[\alpha]\tau_{i/n}(p) \leftrightarrow \tau_{i/n}([\alpha]p) \mid i \in \{1, \dots, n\}\}.$$

Hence, informally speaking, the content of the pair of axioms $\{[\alpha](p \star p) \leftrightarrow ([\alpha]p \star [\alpha]p) \mid \star \in \{\odot, \oplus\}\}$ is essentially the following.

For any $i \leq n$, the truth value of the statement ‘after any execution of α , formula ϕ holds’ is at least $\frac{i}{n}$ if and only if it holds that ‘after any execution of α the truth value of ϕ is at least $\frac{i}{n}$ ’.

Proposition 2.3 states that the axioms of PDL_n are tautologies. Tautologies are preserved by application of the deduction rules. It follows that any theorem of PDL_n is a tautology.

As an illustration of Definition 4.1, we prove that PDL_n is closed under a loop invariance rule. We say that a rule of inference is *derivable* in PDL_n if its consequence can be obtained from its premises by application of rules and axiom schemes of PDL_n .

Lemma 4.5. *For any $\alpha \in \Pi$, the rule*

$$(LI) \quad \frac{(\phi \rightarrow [\alpha]\phi)^n}{(\phi \rightarrow [\alpha^*]\phi)}$$

is derivable in PDL_n .

Proof. Assume that $\vdash (\phi \rightarrow [\alpha]\phi)^n$. Then

$$(4.3) \quad \vdash [\alpha^*](\phi \rightarrow [\alpha]\phi)^n$$

$$(4.4) \quad \vdash [\alpha^*](\phi \rightarrow [\alpha]\phi)^n \rightarrow (\phi \rightarrow (\phi \wedge [\alpha^*](\phi \rightarrow [\alpha]\phi)^n))$$

where (4.3) is obtained by generalization and 4.4 by the fact that $p \rightarrow (t \rightarrow (p \wedge t))$ is a tautology of the $n + 1$ -valued ŁUKASIEWICZ logic (and we apply substitution $p := [\alpha^*](\phi \rightarrow [\alpha]\phi)^n$ and $t := \phi$). It follows that

$$(4.5) \quad \vdash \phi \rightarrow (\phi \wedge [\alpha^*](\phi \rightarrow [\alpha]\phi)^n)$$

$$(4.6) \quad \vdash \phi \rightarrow [\alpha^*]\phi$$

where (4.5) is obtained by *modus ponens* and (4.6) by double *modus ponens* and induction axiom applied to the tautology of the $n + 1$ -valued ŁUKASIEWICZ logic $(p \rightarrow q) \rightarrow ((q \rightarrow t) \rightarrow (p \rightarrow t))$ with substitution $p := \phi$, $q := \phi \wedge [\alpha^*](\phi \rightarrow [\alpha]\phi)^n$ and $t := [\alpha^*]\phi$. \square

Remark 4.6. We say that a rule of inference RI is *admissible* in PDL_n if the system formed by PDL_n and RI has the same theorems as PDL_n . Since for any $k \in \omega$ the rule ϕ/ϕ^k is admissible in ŁUKASIEWICZ $n + 1$ -valued logic, we can deduce from Lemma 4.5 that the rule

$$(LI^\sharp) \quad \frac{(\phi \rightarrow [\alpha]\phi)}{(\phi \rightarrow [\alpha^*]\phi)}$$

is admissible in PDL_n .

5. DEDUCTIVE COMPLETENESS FOR PDL_n

The main result of the paper is Theorem 5.13 that states that PDL_n is complete with respect to the $n + 1$ -valued KRIPKE models. To obtain this result, we use the technique of the canonical model. We follow Part II of [17] to guide us in our constructions and developments.

As in the case of propositional dynamic logic, in the construction of the canonical model for PDL_n , the relation associated to a program is not built inductively from the relations associated to its atomic programs. Instead, we directly associate to each α of Π a relation R_α defined in a canonical way. In fact, the inductive rules involving the operators ‘;’, ‘ \cup ’ and ‘?’ are satisfied in the canonical model, but R_{α^*} may strictly contain the transitive and reflexive closure of R_α . We use the technique of filtration to construct L_n -valued KRIPKE models from this canonical model.

5.1. Filtration lemma. The canonical model of PDL_n will turn out to be non standard in the following sense.

Definition 5.1. A *weak non standard $n + 1$ -valued KRIPKE model* $\mathcal{M} = \langle W, R, \text{Val} \rangle$ is given by a nonempty set W a map $R : \Pi \rightarrow 2^{W \times W}$ and a valuation map $\text{Val} : W \times \text{Prop} \rightarrow \mathbb{L}_n$. The valuation map is extended to formulas by way of the rules (6), (7) and (8) of Definition 2.1. If $w \in W$ and $\phi \in \text{Form}$, we write $\mathcal{M}, w \models \phi$ if $\text{Val}(w, \phi) = 1$. We write $\mathcal{M} \models \phi$ if $\mathcal{M}, w \models \phi$ for any w in W .

A *non standard $n + 1$ -valued KRIPKE model* is a weak non standard $n + 1$ -valued KRIPKE model $\mathcal{M} = \langle W, R, \text{Val} \rangle$ such that for any programs α and β and any formula ψ ,

- (1) the following identities are satisfied in \mathcal{M} :
 - (a) $R_{\alpha;\beta} = R_\alpha \circ R_\beta$,
 - (b) $R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$,
 - (c) $R_{\psi?} = \{(u, u) \mid \text{Val}(u, \psi) = 1\}$;
- (2) the relation R_α^* is a transitive and reflexive extension of R_α ;
- (3) For any $\phi \in \text{Form}$, $\mathcal{M} \models \{[\alpha^*]\phi \rightarrow (\phi \wedge [\alpha][\alpha^*]\phi), [\alpha^*]\phi \rightarrow [\alpha^*][\alpha^*]\phi, (\phi \wedge [\alpha^*](\phi \rightarrow [\alpha]\phi^n) \rightarrow [\alpha^*]\phi\}$.

Note that in condition (2) of the previous definition we allow R_α^* to be *any* reflexive and transitive extension of R_α .

Remark 5.2. Conditions (1) and (3) ensure that if ϕ is a theorem of PDL_n then $\mathcal{M} \models \phi$ for any $n + 1$ -valued non standard KRIPKE model \mathcal{M} (because axioms and rules of PDL_n are sound for non standard KRIPKE frames).

Filtration lemmas are usually proved by induction on the subformula relation. In $(n + 1)$ -valued propositional dynamic logic, the use of induction is somehow cumbersome because of the interdependence of the definitions of formulas and programs. We use the FISCHER - LADNER closure $\text{FL}(\phi)$ of a formula ϕ to prove a filtration lemma for $n + 1$ -valued non standard models.

To ease readability, proof of the Filtration Lemma is moved in Appendix in which we also recall the definition (Definition A.1) of the FISHER - LADNER closure of a formula (see also [17]) .

Definition 5.3. If $\mathcal{M} = \langle W, R, \text{Val} \rangle$ is a weak $n + 1$ -valued non standard KRIPKE model and if ϕ is a formula then we define the equivalence relation \equiv_ϕ on W by

$$(5.1) \quad u \equiv_\phi v \quad \text{if} \quad \forall \psi \in \text{FL}(\phi) \quad \text{Val}(u, \psi) = \text{Val}(v, \psi).$$

We denote by $[W]_\phi$ (or simply by $[W]$) the quotient of W by \equiv_ϕ and by $[u]_\phi$ (or simply $[u]$) the class of an element u of W for \equiv_ϕ .

Then, for any atomic program a of Π_0 we define the relation $R_a^{[\mathcal{M}]_\phi}$ by

$$(5.2) \quad R_a^{[\mathcal{M}]_\phi} = \{([u], [v]) \mid (u, v) \in R_a\}$$

and the valuation map $\text{Val}^{[W]}$ on $[W] \times \text{Prop}$ by

$$(5.3) \quad \text{Val}^{[\mathcal{M}]_\phi}([u], p) = \bigvee \text{Val}([u], p).$$

The $n + 1$ -valued KRIPKE model $[\mathcal{M}]_\phi = \langle [W]_\phi, R^{[\mathcal{M}]_\phi}, \text{Val}^{[\mathcal{M}]_\phi} \rangle$ is called *the filtration of \mathcal{M} through ϕ* . If no confusion is possible we prefer to denote this model by $[\mathcal{M}] = \langle [W], R^{[\mathcal{M}]}, \text{Val}^{[\mathcal{M}]} \rangle$.

Note that the number of worlds in $[\mathcal{M}]_\phi$ is finite and bounded by $(n + 1)^{|\text{FL}(\phi)|}$.

The proof of the following result is provided in Appendix A.

Lemma 5.4 (Filtration). *Assume that $\mathcal{M} = \langle W, R, \text{Val} \rangle$ is an $n + 1$ -valued non standard KRIPKE model and that ϕ is a formula.*

- (1) *If ψ is in $\text{FL}(\phi)$ then $\text{Val}(u, \psi) = \text{Val}^{[\mathcal{M}]}([u], \psi)$.*
- (2) *For every $[\alpha]\psi$ in $\text{FL}(\phi)$,*
 - (a) *if $(u, v) \in R_\alpha$ then $([u], [v]) \in R_\alpha^{[\mathcal{M}]}$;*
 - (b) *if $([u], [v]) \in R_\alpha^{[\mathcal{M}]}$ then $\text{Val}(u, [\alpha]\psi) \leq \text{Val}(v, \psi)$.*

We obtain the decidability of the satisfiability problem for PDL_n as an immediate consequence of Lemma 5.4.

Definition 5.5. A formula ϕ of **Form** is *satisfiable* if there is an $n + 1$ -valued KRIPKE model and a world in this model in which ϕ is true.

Corollary 5.6. *The problem of deciding if a formula of **Form** is satisfiable is decidable.*

Proof. If ϕ is satisfiable in an $n + 1$ -valued KRIPKE model, Lemma 5.4 ensures that it is satisfiable in a model with at most $(n + 1)^{|\text{FL}(\phi)|}$ worlds. \square

5.2. The canonical model. We construct the canonical $n + 1$ -valued KRIPKE model of PDL_n on the set of homomorphisms from the LINDENBAUM - TARSKI algebra of PDL_n to \mathbb{L}_n . We assume that the reader has some acquaintance with the theory of MV-algebras which are the algebras of the many-valued ŁUKASIEWICZ logics. We only recall the necessary definitions. See [14] for an introduction or [7] for a monograph on the subject.

Recall that the variety \mathcal{MV} of MV-algebras is generated by the algebra $\langle [0, 1], \rightarrow, \neg, 1 \rangle$ where \neg and \rightarrow are defined on $[0, 1]$ as their ŁUKASIEWICZ interpretation (see section 2). \mathcal{MV} can be described as the class of algebras $A = \langle A, \rightarrow, \neg, 1 \rangle$ of type $(2, 1, 0)$ that satisfy the following equations¹:

$$(5.4) \quad \begin{aligned} x \rightarrow 1 &= x, & (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) &= 1, \\ (x \rightarrow) \rightarrow y &= (y \rightarrow x) \rightarrow x, & (\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x) &= 1. \end{aligned}$$

The variety \mathcal{MV}_n is the subvariety of \mathcal{MV} generated by the subalgebra \mathbb{L}_n of $[0, 1]$. We denote by $\mathcal{MV}(A, \mathbb{L}_n)$ the set of the MV-algebra homomorphisms from A to \mathbb{L}_n for any $A \in \mathcal{MV}_n$.

In any MV-algebra A , the relation \leq defined by $a \leq b$ if $a \rightarrow b = 1$ is a bounded distributive lattice order on A . The variety \mathcal{MV} was introduced by CHANG (see [5, 6]) in order to obtain an algebraic completeness result for ŁUKASIEWICZ infinite-valued logic.

¹This axiomatization is not the most commonly used axiomatization of \mathcal{MV} , but it is the most efficient for our purpose. See [7] for details.

Definition 5.7. We denote by \mathcal{F}_n the LINDENBAUM - TARSKI algebra of PDL_n , that is, the quotient of **Form** by the syntactic equivalence relation \equiv defined by $\phi \equiv \psi$ if $\text{PDL}_n \vdash \phi \leftrightarrow \psi$. This quotient is equipped with the operations \rightarrow , \neg and $[\alpha]$ ($\alpha \in \Pi$) defined in the obvious way: $(\phi / \equiv) \rightarrow (\psi / \equiv) = (\phi \rightarrow \psi) / \equiv$, $\neg(\psi / \equiv) = (\neg\psi) / \equiv$ and $[\alpha](\psi / \equiv) = ([\alpha]\psi) / \equiv$.

For the sake of readability, we prefer to denote by ϕ the class ϕ / \equiv .

Lemma 5.8. *The reduct of \mathcal{F}_n to the language $\{\rightarrow, \neg, 1\}$ belongs to \mathcal{MV}_n .*

Proof. We have included the tautologies of ŁUKASIEWICZ $n+1$ -valued logic in our axiomatization of PDL_n . \square

The preceding lemma leads to the definition of the canonical model for PDL_n . The classical construction of the canonical model for PDL is based on the set of the maximal (Boolean) filters of the LINDENBAUM - TARSKI algebra \mathcal{F} of PDL. One of the key element of this construction is a separation result (a consequence of the Ultrafilter Theorem) that states that for any ϕ, ψ such that $\phi, \psi, \phi \leftrightarrow \psi \notin \text{PDL}$, there is a maximal filter of \mathcal{F} that contains ϕ / \equiv but not ψ / \equiv . We can state this result using homomorphisms. Indeed, a subset F of a Boolean algebra A is a (proper) maximal filter if and only if the map $\pi_F : A \rightarrow \mathbf{2}$ (where $\mathbf{2}$ denotes the two element Boolean algebra) defined by $\pi_F^{-1}(1) = F$ is an homomorphism. Hence, the separation result can be stated in this way: for any ϕ, ψ such that $\phi, \psi, \phi \leftrightarrow \psi \notin \text{PDL}$, there is an homomorphism $v : \mathcal{F} \rightarrow \mathbf{2}$ such that $v(\phi) = 1$ and $v(\psi) = 0$.

There is an analogous separation result for the variety \mathcal{MV}_n : if $A \in \mathcal{MV}_n$ and $a \neq b \in A$, there is an homomorphism $v : A \rightarrow \mathbb{L}_n$ such that² $v(a) \neq v(b)$. This result, together with Lemma 5.8, indicates that the set $\mathcal{MV}(\mathcal{F}_n, \mathbb{L}_n)$ is a good candidate for the universe of the canonical model of PDL_n . Before proceeding with the construction of this model, let us recall how $\mathcal{MV}(A, \mathbb{L}_n)$ is linked with the set of maximal filters of $A \in \mathcal{MV}_n$.

A *filter* of an MV-algebra A is a subset F of A that contains 1 and that contains y whenever it contains x and $x \rightarrow y$. Equivalently, a filter of A is a nonempty increasing subset of A closed under \odot . If X is a nonempty subset of an MV-algebra A , the filter generated by X is the filter

$$(5.5) \quad \langle X \rangle = \{b \in A \mid \exists k \in \omega, \epsilon \in \omega^k, x \in X^k (b \geq x_1^{\epsilon_1} \odot \dots \odot x_k^{\epsilon_k})\}.$$

Filters are ordered by set inclusion and the proper maximal elements are called *maximal filters* and correspond to homomorphisms from A to \mathbb{L}_n in the following way. For any maximal filter F of $A \in \mathcal{MV}_n$, there is only one homomorphism $v_F : A \rightarrow \mathbb{L}_n$ that satisfies $v_F^{-1}(1) = F$. The map $v : F \mapsto v_F$ has converse $\cdot^{-1}(1) : v \mapsto v^{-1}(1)$ that associates a maximal filter for any $v \in \mathcal{A}(A, \mathbb{L}_n)$.

Definition 5.9. The *canonical model* of PDL_n is defined as the model $\mathcal{M}^c = \langle W^c, R^c, \text{Val}^c \rangle$ where

²Without going into details, note that this is a consequence of the characterization of subdirectly irreducible elements of \mathcal{MV}_n . See [7].

- (1) $W^c = \mathcal{MV}(\mathcal{F}_n, \mathbb{L}_n)$;
- (2) if $\alpha \in \Pi$, the relation R_α^c is defined as

$$R_\alpha^c = \{(u, v) \mid \forall \phi \in \mathcal{F}_n \ (u([\alpha]\phi) = 1 \Rightarrow v(\phi) = 1)\};$$

- (3) the map Val^c is defined as

$$\text{Val}^c : W^c \times \text{Form} : (u, \phi) \mapsto u(\phi).$$

When no confusion arises, we prefer to write W , R and Val instead of W^c , R^c and Val^c respectively.

Lemma 5.10. *If $\alpha \in \Pi$, then*

$$(5.6) \quad R_\alpha^c = \{(u, v) \mid \forall \phi \in \mathcal{F}_n (v(\phi) = 1 \Rightarrow u(\langle \alpha \rangle \phi) = 1)\}.$$

Proof. Assume that $(u, v) \in R_\alpha^c$ and that ϕ is an element of \mathcal{F}_n such that $v(\phi) = 1$. If $u(\langle \alpha \rangle \phi) < 1$ then $u([\alpha]\neg\phi) = 1 - u(\langle \alpha \rangle \phi) > 0$. Let i be the element of $\{0, \dots, n-1\}$ such that $u([\alpha]\neg\phi) = \frac{i}{n}$. It follows that $\tau_{i/n}(u([\alpha]\neg\phi)) = u([\alpha]\tau_{i/n}(\neg\phi)) = 1$ and so, that $v(\tau_{i/n}(\neg\phi)) = 1$. It means that $v(\neg\phi) \geq \frac{i}{n}$ or equivalently that $v(\phi) \leq 1 - \frac{i}{n} < 1$, a contradiction.

Proceed in a similar way to prove that the condition is sufficient. \square

Note that since we have defined an accessibility relation for *every* program α and the image of the valuation maps on *every* formula ϕ , it is not clear that the canonical model is an $n+1$ -valued (non-standard) model. Indeed, in (non-standard) models, valuations are defined on atomic objects and inductively extended to all formulas.

The canonical model will actually turn out to be an $n+1$ -valued non standard model. The following lemma is a major step in the proof of this result. The proof of this lemma is given in more general settings in [16]. We include a stand alone proof for the sake of readability.

Note that for any MV-homomorphism $u : \mathcal{F}_n \rightarrow \mathbb{L}_n$, the set $[\alpha]^{-1}u^{-1}(1)$ is a filter of \mathcal{F}_n since the formula $[\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$ belongs to PDL_n for any program α and any formulas ϕ and ψ .

Lemma 5.11. *If $\phi \in \text{Form}$, if $\alpha \in \Pi$ and if $u \in W^c$ then*

$$(5.7) \quad \text{Val}^c(u, [\alpha]\phi) = \bigwedge \{\text{Val}^c(v, \phi) \mid v \in R_\alpha^c u\}.$$

Proof. We have to prove that

$$(5.8) \quad u([\alpha]\phi) = \bigwedge \{v(\phi) \mid v \in R_\alpha u\}.$$

First, assume that $u([\alpha]\phi) = \frac{i}{n}$ for some $i \in \{1, \dots, n\}$. It follows that

$$(5.9) \quad 1 = \tau_{i/n}(u([\alpha]\phi)) = u(\tau_{i/n}([\alpha]\phi)) = u([\alpha]\tau_{i/n}(\phi)),$$

where the first equality is obtained by definition of $\tau_{i/n}$, the second one holds because u is an MV-homomorphism and the last one from item (19) of Proposition 2.3. Hence, for any $v \in R_\alpha u$, we get $v(\tau_{i/n}(\phi)) = \tau_{i/n}(v(\phi)) = 1$, which means that $v(\phi) \geq \frac{i}{n}$. We have proved that

$$(5.10) \quad u([\alpha]\phi) \leq \bigwedge \{v(\phi) \mid v \in R_\alpha u\}.$$

For the other inequality, assume *ad absurdum* that there is an $i \leq n$ such that

$$(5.11) \quad u([\alpha]\phi) < \frac{i}{n} \leq \bigwedge \{v(\phi) \mid v \in R_\alpha u\},$$

i.e., such that $u([\alpha]\tau_{i/n}(\phi)) \neq 1$ and $v(\tau_{i/n}(\phi)) = 1$ for any $v \in R_\alpha u$. Note that the definition of R_α means that the maximal filters above $[\alpha]^{-1}u^{-1}(1)$ are exactly the $v^{-1}(1)$ where v belongs to $R_\alpha u$. Hence, the element $\tau_{i/n}(\phi)$ belongs to any maximal filter that contains $[\alpha]^{-1}u^{-1}(1)$ but is not an element of $[\alpha]^{-1}u^{-1}(1)$, a contradiction. \square

Theorem 5.12. *The canonical model of PDL_n is an $n+1$ -valued non standard KRIPKE model.*

Proof. We prove the following properties of the canonical model.

- (1) $R_{[\alpha \cup \beta]} = R_\alpha \cup R_\beta$,
- (2) $R_{\alpha;\beta} = R_\alpha \circ R_\beta$,
- (3) $R_{\psi?} = \{(u, u) \mid \text{Val}(u, \psi) = 1\}$,
- (4) R_{α^*} is reflexive, transitive and contains R_α ,
- (5) $\mathcal{M} \models \{[\alpha^*]\phi \leftrightarrow (\phi \wedge [\alpha][\alpha^*]\phi), (\phi \wedge [\alpha^*](\phi \rightarrow [\alpha]\phi)^n) \rightarrow [\alpha^*]\phi, [\alpha]^*\phi \rightarrow [\alpha^*][\alpha^*]\phi\}$.

For (1), we note that the inequality $R_\alpha \cup R_\beta \subseteq R_{\alpha \cup \beta}$ is trivial. For the other inequality, let us assume that (u, v) belongs to $R_{\alpha \cup \beta}$ but not to $R_\alpha \cup R_\beta$. There are formulas ϕ and ψ such that $\text{Val}(u, [\alpha]\phi) = 1$, $\text{Val}(u, [\beta]\psi) = 1$ and $\text{Val}(v, \phi \vee \psi) < 1$. Then, thanks to Lemma 5.11,

$$(5.12) \quad \text{Val}(u, [\gamma](\phi \vee \psi)) \geq \text{Val}(u, [\gamma]\phi \vee [\gamma]\psi) = 1,$$

for $\gamma \in \{\alpha, \beta\}$. Hence,

$$(5.13) \quad \text{Val}(u, [\alpha \cup \beta](\phi \vee \psi)) = \text{Val}(u, [\alpha](\phi \vee \psi) \wedge [\beta](\phi \vee \psi)) = 1$$

while $\text{Val}(v, \phi \vee \psi) < 1$. We conclude that (u, v) does not belong to $R_{\alpha \cup \beta}$, a contradiction.

The inequality $R_\alpha \circ R_\beta \subseteq R_{\alpha;\beta}$ of (2) is clear. Let us prove the other inequality. Assume that $(u, v) \in R_{\alpha;\beta}$. We prove that the filter generated by $[\alpha]^{-1}u^{-1}(1) \cup \langle \beta \rangle v^{-1}(1)$ is a proper filter of \mathcal{F}_n . Assume that ϕ_1, \dots, ϕ_k belong to $[\alpha]^{-1}u^{-1}(1)$, that ψ_1, \dots, ψ_l belong to $v^{-1}(1)$ and that $\epsilon_1, \dots, \epsilon_k, \eta_1, \dots, \eta_l$ are nonnegative integers. We prove that

$$(5.14) \quad \Phi \odot \Psi \neq 0$$

where Φ denotes the formula $\phi_1^{\epsilon_1} \odot \dots \odot \phi_k^{\epsilon_k}$ and Ψ the formula $(\langle \beta \rangle \psi_1)^{\eta_1} \odot \dots \odot (\langle \beta \rangle \psi_l)^{\eta_l}$.

Let us denote by Ψ' the formula $\psi_1^{\eta_1} \odot \dots \odot \psi_l^{\eta_l}$. Since (u, v) belongs to $R_{\alpha;\beta}$ and $v(\Psi') = 1$ we obtain thanks to Lemma 5.10 that $u(\langle \alpha; \beta \rangle \Psi') = 1$. It follows that $u([\alpha]\Phi \odot \langle \alpha \rangle \langle \beta \rangle \Psi') = 1$ and hence, according to Lemma 2.3 (16) that $u(\langle \alpha \rangle (\Phi \odot \langle \beta \rangle \Psi')) = 1$. Then, according to Lemma 5.11 and Remark 4.2,

$$(5.15) \quad u(\langle \alpha \rangle (\Phi \odot \langle \beta \rangle \Psi')) = \bigvee \{w(\Phi \odot \langle \beta \rangle \Psi') \mid w \in R_\alpha u\}.$$

Hence, there is a w in $R_\alpha u$ such that $w(\Phi \odot \langle \beta \rangle \Psi') = 1$ which proves that $\Phi \odot \langle \beta \rangle \Psi' \neq 0$ in \mathcal{F}_n . It follows from Lemma 2.3 (18) that $(\Phi \odot \langle \beta \rangle \Psi') \rightarrow (\Phi \odot \Psi)$ is a theorem of PDL_n which implies that

$$(5.16) \quad \Phi \odot \Psi \geq \Phi \odot \langle \beta \rangle \Psi' > 0$$

in \mathcal{F}_n which is the desired conclusion.

(3) Thanks to axiom $[q?]p \leftrightarrow \neg q^n \vee p$ and the rule of uniform substitution, we obtain that $(u, v) \in R_{\psi?}$ if either $u(\psi) < 1$ and $v(\phi) = 1$ for any $\phi \in \mathcal{F}_n$ (which is impossible since $v^{-1}(1)$ is a proper filter of \mathcal{F}_n) or $u(\psi) = 1$ and $u^{-1}(1) \subseteq v^{-1}(1)$, which means that $v = u$ by maximality.

(4) Let us prove that $R_\alpha \subseteq R_{\alpha^*}$. Assume that $(u, v) \in R_\alpha$ and that $u([\alpha^*]\phi) = 1$ for some $\phi \in \mathcal{F}_n$. Thanks to the axioms that define the operator $*$, it means that $u(\phi \wedge [\alpha][\alpha^*]\phi) = 1$. It follows that $u([\alpha][\alpha^*]\phi) = 1$. Since $(u, v) \in R_\alpha$ we deduce that $v([\alpha^*]\phi) = 1$, hence that $v(\phi \wedge [\alpha][\alpha^*]\phi) = 1$ and finally that $v(\phi) = 1$.

Eventually, reflexivity and transitivity of R_{α^*} are easily obtained.

(5) is obtained by construction. \square

Theorem 5.13. *The logic PDL_n is complete with respect to the $n+1$ -valued KRIPKE models.*

Proof. If ϕ is a tautology, then ϕ is valid in $[\mathcal{M}^c]_\phi$ which is an $n+1$ -valued KRIPKE model. It follows from Lemma 5.4 that ϕ is true in \mathcal{M}^c . We thus conclude that ϕ is in any maximal filter of \mathcal{F}_n , i.e. that $\phi \equiv 1$ and thus that ϕ is a theorem of PDL_n . \square

6. CONCLUDING REMARKS

This paper deals with some theoretical issues of a many-valued generalization of PDL. We believe that this generalization could reveal to be a valuable tool for analysis of problems arising from various fields such as computer science, epistemic logic or game theory. We present a few ideas about possible areas in which this new language could be applied or generalized.

6.1. Distributed algorithms. Some of the problems that can be solved by a distributed or parallel algorithm could be modeled with the language of PDL_n .

Consider as a toy example the problem of encoding a string w of length $2n$ over an alphabet Σ into an alphabet Σ' using a coding function $c : \Sigma^2 \rightarrow \Sigma'^*$. Assume that this task is distributed over two processes P_1 and P_2 and that P_1 starts encoding from the head of the string and P_2 starts from its tail.

Let us consider two propositional variables p_1 and p_2 that are evaluated at each step u of the algorithm as $\text{Val}(u, p_i) = k_i/n$ where for any $i \in \{1, 2\}$, k_i denotes the number of substrings of length 2 that process P_i has already encoded in step u . The algorithm can be modeled by a many-valued KRIPKE model. It terminates in step u if $\text{Val}(u, p_1 \oplus p_2) = 1$.

6.2. Dynamic epistemic logic. In [30], the authors show how to use the language of PDL to design a dynamic epistemic logic LLC for multi-agent systems that allows to deal with different kinds of information changes (public announcements, subgroup announcements, partial observations...) or factual changes in the state of the world. In their settings, agents are represented by elements of Π_0 and the program operators $';$, $'\cup'$ and $'*'$ have epistemic interpretations (for example $[a;b]\phi$ is read ‘agent a knows that agent b knows ϕ ’). Their semantic uses two kind of models: *epistemic models* (which are standard KRIPKE models for PDL) and *update models* used to capture information changes. It also provides rules to update the former with the latter. The generalization of these constructions to a many-valued realm, using the language and the KRIPKE models introduced in this paper, could help to model situations involving partial or shared knowledge.

6.3. Modal logic for games. Modal logic turned out to be a valuable tool to study several kinds of game forms ([26]). For example, PAULY introduced in [25] a logic, called CL_N to reason about effective power in coalitional games (with set of players N). A set of outcome states X is effective for a coalition $C \subseteq N$ if the players in C can choose a joint strategy that leads to a state in X no matter which strategies are adopted by the players not belonging to C . CL_N is a multi-modal logic that is complete for a class of neighborhood models. Some of the tools introduced in this paper could be used to set up a generalization of CL_N designed to capture the degree with which a coalition C can encompass a fuzzy set of outcome states.

APPENDIX A. PROOF OF THE FILTRATION LEMMA

Definition A.1 ([9]). Assume that X is a set of formulas. The FISHER-LADNER *closure* $\text{FL}(X)$ of X is the smallest subset Y of **Form** such that

- | | |
|---|--|
| (1) $X \subseteq Y$, | (6) $\{[\alpha]\phi, [\beta]\phi\} \subseteq Y$ if $[\alpha \cup \beta]\phi \in Y$, |
| (2) $\phi \in Y$ if $\neg\phi \in Y$, | |
| (3) $\{\phi, \psi\} \subseteq Y$ if $\phi \rightarrow \psi \in Y$, | (7) $[\alpha][\alpha^*]\phi \in Y$ if $[\alpha^*]\phi \in Y$, |
| (4) $\phi \in Y$ if $[\alpha]\phi \in Y$, | (8) $\{\psi, \phi\} \subseteq Y$ if $[\psi?]\phi \in Y$. |
| (5) $[\alpha][\beta]\phi \in Y$ if $[\alpha;\beta]\phi \in Y$, | |

Lemma A.2. *If $\langle W, R, \text{Val} \rangle$ is a weak non-standard $n + 1$ -valued model, if $\phi \in \text{Form}$ and if E is a subset of W which is \equiv_ϕ -saturated (i.e., E contains $[u]_\phi$ whenever it contains u), then there is a formula Ψ_E such that $E = \text{Val}^{-1}(\cdot, \Psi_E)(1)$.*

Proof. For any $[t] \in [W]$, any $\rho \in \text{FL}(\phi)$ and any $i \in \{0, \dots, n\}$ such that $\text{Val}([t], \rho) = \frac{i}{n}$, let us denote by $I_{\rho, [t]}$ the formula $I_{\frac{i}{n}}(\rho)$. Then, set

$$(A.1) \quad \psi_{[t]} = \bigwedge_{\rho \in \text{FL}(\phi)} I_{\rho, [t]}.$$

Then $u \in [t]$ if and only if $\text{Val}(u, \psi_{[t]}) = 1$. The formula

$$(A.2) \quad \psi_E = \bigvee_{[t] \subseteq E} \psi_{[t]}.$$

has the desired property. \square

Lemma A.3. *Assume that \mathcal{M} is a non-standard KRIPKE model and that $\phi \in \text{Form}$ and $\alpha \in \Pi$. If $\mathcal{M} \models (\phi \rightarrow [\alpha]\phi)^n$ then $\mathcal{M} \models \phi \rightarrow [\alpha^*]\phi$.*

Proof. Remark (5.1) states that axioms and rules of PDL_n are sound in non-standard KRIPKE frames. Lemma 4.5 states that $\phi \rightarrow [\alpha^*]\phi$ can be obtained from $(\phi \rightarrow [\alpha]\phi)^n$ by applications of axioms and rules of PDL_n . It follows that if $\mathcal{M} \models (\phi \rightarrow [\alpha]\phi)^n$ then $\mathcal{M} \models \phi \rightarrow [\alpha^*]\phi$. \square

We now provide the proof of the Filtration Lemma.

Proof of Lemma 5.4. The proofs of (1) and (2) are done by mutual induction.

(1) If $\psi \in \text{Prop}$, the result follows directly from the definition of $[\mathcal{M}]$. If $\psi = \rho \rightarrow \mu$ or $\psi = \neg\rho$, the result follows by applying induction hypothesis to ρ and μ .

If $\psi = [\alpha]\rho \in \text{FL}(\phi)$ then $\rho \in \text{FL}(\phi)$. We have to prove that

$$(A.3) \quad \text{Val}(u, [\alpha]\rho) = \text{Val}([u], [\alpha]\rho).$$

First, we prove inequality \leq . We obtain successively

$$(A.4) \quad \text{Val}(u, [\alpha]\rho) \leq \bigwedge \{ \text{Val}(v, \rho) \mid ([u], [v]) \in R_\alpha \}$$

$$(A.5) \quad = \bigwedge \{ \text{Val}([v], \rho) \mid ([u], [v]) \in R_\alpha \}$$

$$(A.6) \quad = \text{Val}([u], [\alpha]\rho),$$

where (A.4) and (A.5) are obtained by (2) (b) and by induction hypothesis for ρ .

Now, we prove inequality \geq in (A.3). From (2)(a) we obtain that $R_\alpha^{[\mathcal{M}]}$ contains $([u], [v])$ whenever $(u, v) \in R_\alpha$. It follows that

$$(A.7) \quad \text{Val}([u], [\alpha]\rho) = \bigwedge \{ \text{Val}([v], \rho) \mid ([u], [v]) \in R_\alpha \}$$

$$(A.8) \quad = \bigwedge \{ \text{Val}(v, \rho) \mid ([u], [v]) \in R_\alpha \}$$

$$(A.9) \quad \leq \bigwedge \{ \text{Val}(v, \rho) \mid (u, v) \in R_\alpha \}$$

$$(A.10) \quad = \text{Val}(u, [\alpha]\rho),$$

where (A.8) is obtained by induction hypothesis. Hence, we have proved (A.3).

(2) There are five cases to consider according to the form of α .

If $\alpha \in \Pi_0$ then, knowing that $[\alpha]\psi$ and ψ are in $\text{FL}(\phi)$, the result follows easily from the definition of $[\mathcal{M}]$.

If $\alpha = \beta \cup \gamma$ then (a) is easily obtained by application of induction hypothesis to $[\alpha]\psi$ and $[\beta]\psi$ and the fact that $R_{[\beta \cup \gamma]} = R_\beta \cup R_\gamma$ in any (non standard) $n + 1$ -valued KRIPKE model. For (b), assume that $([u], [v]) \in R_{[\beta \cup \gamma]} = R_{[\beta]} \cup R_{[\gamma]}$. We apply induction hypothesis to $[\beta]\psi$ and $[\gamma]\psi$ and we

obtain that either $\text{Val}(u, [\beta]\psi) \leq \text{Val}(v, \psi)$ or $\text{Val}(u, [\gamma]\psi) \leq \text{Val}(v, \psi)$. The result is then obtained thanks to item (1) of Proposition 2.3.

If $\alpha = \beta; \gamma$, we can proceed in a similar way by application of induction hypothesis to $[\beta][\gamma]\psi, [\gamma]\psi \in \text{FL}(\phi)$.

If $\alpha = \rho?$ then $\rho \in \text{FL}(\phi)$ and we obtain by (1) that $\text{Val}(u, \rho) = \text{Val}([u], \rho)$, which gives a proof of (a). For (b), we note that if $([u], [u]) \in R_\rho?$ then $1 = \text{Val}([u], \rho) = \text{Val}(u, \rho)$ thanks to (1) applied to $\rho \in \text{FL}(\phi)$. It follows that

$$(A.11) \quad \text{Val}(u, [\rho?]\psi) = \text{Val}(u, \neg\rho^n \vee \psi) = \text{Val}(u, \psi).$$

If $\alpha = \beta^*$ then $[\beta][\beta^*]\psi \in \text{FL}(\phi)$ and we can apply the induction hypothesis to R_β . To prove (a), assume that $(u, v) \in R_{\beta^*}$. Let us consider

$$(A.12) \quad E = \{t \in W \mid ([u], [t]) \in R_{\beta^*}\}.$$

The set E is clearly \equiv_ϕ -saturated. By Lemma A.2, there exists a formula Ψ_E such that $E = \text{Val}(\cdot, \Psi_E)^{-1}(1)$.

Since $R_{\beta^*}^{[\mathcal{M}]}$ is a reflexive extension of $R_\beta^{[\mathcal{M}]}$, it follows that u belongs to E .

Now, assume that $s \in E$ and that $sR_\beta t$. By induction hypothesis, we obtain that $([s], [t])$ is in $R_\beta^{[\mathcal{M}]}$. Then $([u], [t])$ is in $R_{\beta^*}^{[\mathcal{M}]}$ since this relation is a transitive extension of $R_\beta^{[\mathcal{M}]}$. Hence $\mathcal{M} \models (\Psi_E^n \rightarrow [\beta]\Psi_E^n)^n$. By Lemma A.3 it follows that $\mathcal{M} \models \Psi_E^n \rightarrow [\beta^*]\Psi_E^n$. As $u \in E$, we conclude that $\text{Val}(u, \Psi_E^n) = 1$ so that $\text{Val}(u, [\beta^*]\Psi_E^n) = 1$ and $\text{Val}(v, \Psi_E^n) = 1$ since $(u, v) \in R_{\beta^*}$. Thus, we have proved that $([u], [v]) \in R_{\beta^*}^{[\mathcal{M}]}$.

To prove (b), assume that $([u], [v]) \in R_{\beta^*}$. Then, since $R_{\beta^*}^{[\mathcal{M}]}$ is the reflexive and transitive closure of R_β , there are some $[w_i]$ ($i \in \{0, \dots, m+1\}$) in $[\mathcal{M}]$ such that $([w_i], [w_{i+1}]) \in R_\beta$ for any $i \leq m$ and such that $[u] = [w_0]$ and $[v] = [w_{m+1}]$. We obtain by induction hypothesis that for any $i \leq m$,

$$(A.13) \quad \text{Val}(w_i, [\beta^*]\psi) \leq \text{Val}(w_i, \psi \wedge [\beta][\beta^*]\psi) \leq \text{Val}(w_{i+1}, [\beta^*]\psi).$$

Hence, we eventually obtain that $\text{Val}(u, [\beta^*]\psi) \leq \text{Val}(v, \psi)$. \square

REFERENCES

- [1] Mario R.F. Benevides, Renata de Freitas, and Petrucio Viana. Propositional dynamic logic with storing, recovering and parallel composition. *Electronic Notes in Theoretical Computer Science*, 269(0):95 – 107, 2011.
- [2] Mario R.F. Benevides and L. Menasché Schechter. A propositional dynamic logic for concurrent programs based on the π -calculus. *Electronic Notes in Theoretical Computer Science*, 262(0):49 – 64, 2010.
- [3] Félix Bou, Francesc Esteva, Lluís Godo, and Ricardo Oscar Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.
- [4] Xavier Caicedo and Ricardo O. Rodríguez. Standard Gödel modal logics. *Studia Logica*, 94(2):189–214, 2010.
- [5] C. C. Chang. Algebraic analysis of many valued logics. *Transactions of the American Mathematical Society*, 88:467–490, 1958.
- [6] C. C. Chang. A new proof of the completeness of the lukasiewicz axioms. *Transactions of the American Mathematical Society*, 93:74–80, 1959.

- [7] Roberto L. O. Cignoli, Itala M. L. D'Ottaviano, and Daniele Mundici. *Algebraic foundations of many-valued reasoning*, volume 7 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2000.
- [8] Ernst-Erich Doberkat. A stochastic interpretation of propositional dynamic logic: expressivity. *Journal of Symbolic Logic*, 77(2):687–716, 2012.
- [9] Michael J. Fischer and Richard E. Ladner. Propositional dynamic logic of regular programs. *Journal of Computer and System Sciences*, 18(2):194–211, 1979.
- [10] Melvin Fitting. Many-valued modal logics. II. *Fundamenta Informaticae*, 17(1-2):55–73, 1992.
- [11] Melvin Fitting. Tableaus for many-valued modal logic. *Studia Logica*, 55(1):63–87, 1995.
- [12] Melvin Fitting. Reasoning about games. *Studia Logica*, 99(1-3):143–169, 2011.
- [13] Melvin C. Fitting. Many-valued modal logics. *Fundamenta Informaticae*, 15(3-4):235–254, 1991.
- [14] Joan Gispert and Daniele Mundici. MV-algebras: a variety for magnitudes with Archimedean units. *Algebra Universalis*, 53(1):7–43, 2005.
- [15] Georges Hansoul and Bruno Teheux. Completeness results for many-valued\Łukasiewicz modal systems and relational semantics. *arXiv preprint math/0612542*, 2006.
- [16] Georges Hansoul and Bruno Teheux. Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics. *Studia Logica*, 101(3):505–545, 2013.
- [17] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic logic*. Foundations of Computing Series. MIT Press, Cambridge, MA, 2000.
- [18] Paul Harrenstein, Wiebe van der Hoek, John-Jules Meyer, and Cees Witteveen. A modal characterization of Nash equilibrium. *Fundamenta Informaticae*, 57(2-4):281–321, 2003. 1st International Workshop on Knowledge Representation and Approximate Reasoning (KR&AR) (Olsztyn, 2003).
- [19] Daniel Leivant. Propositional dynamic logic for recursive procedures. In Natarajan Shankar and Jim Woodcock, editors, *Verified Software: Theories, Tools, Experiments*, volume 5295 of *Lecture Notes in Computer Science*, pages 6–14. Springer Berlin Heidelberg, 2008.
- [20] Jan Łukasiewicz. O logice trójwartościowej. *Ruch Filozoficzny*, 5:170–171, 1920.
- [21] Jan Łukasiewicz. *Selected works*. North-Holland Publishing Co., Amsterdam, 1970. Edited by L. Borkowski, Studies in Logic and the Foundations of Mathematics.
- [22] Jan Łukasiewicz and Tarski Alfred. Untersuchungen über den ausagenkalkül. *Compte Rendus Séances Société des Sciences et Lettres Varsovie*, 23:30–50, 1930.
- [23] Daniele Mundici. The logic of Ulam’s game with lies. In *Knowledge, belief, and strategic interaction (Castiglione, 1989)*, Cambridge Stud. Probab. Induc. Decis. Theory, pages 275–284. Cambridge Univ. Press, Cambridge, 1992.
- [24] Pascal Ostermann. Many-valued modal propositional calculi. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 34(4):343–354, 1988.
- [25] M. Pauly. A Modal Logic for Coalitional Power in Games. *Journal of Logic and Computation*, 12(1):149–166, 2002.
- [26] Marc Pauly. *Logic for social software*. PhD thesis, Universiteit van Amsterdam, 2001.
- [27] Marc Pauly and Wiebe van der Hoek. Modal logics for games and informations. In van Benthem J. van Blackburn P. and Wolter F., editors, *Handbook of modal logic*, 3, Studies in logic and practical reasoning. Elsevier, 2006.
- [28] Andrzej Pelc. Searching games with errors—fifty years of coping with liars. *Theoretical Computer Science*, 270(1-2):71–109, 2002.
- [29] Stanisław Marcin Ulam. *Adventures of a mathematician*. Charles Scribner’s Sons, New York, 1976.
- [30] Johan van Benthem, Jan van Eijck, and Barteld Kooi. Logics of communication and change. *Information and Computation*, 204(11):1620–1662, 2006.

- [31] Johan F. A. K. van Benthem and Alice ter Meulen. *Handbook of Logic and Language*. Elsevier Science Publishers B. V., Amsterdam, The Netherlands, The Netherlands, 2nd edition, 2010.
- [32] Hans van Ditmarsch, Wiebe van der Hoek, and Barteld Kooi. *Dynamic Epistemic Logic*, volume 337 of *Synthese Library*. Springer, 2007.

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