Relaxing the compatibility condition in (extended) finite element methods: applications to fracture and nano-mechanics

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Recently, novel finite element methods were proposed from the coupling of stabilized conforming nodal integration with the standard finite element method [1]. An overarching theory has been developed in the recent paper [2]. The main premise of this theory is the wish to achieve reliable results using lower order elements, i.e. simple meshes (triangles, tetrahedra). SFEM retains the accuracy and inherit the advantages of triangular and tetrahedral meshes to represent complex geometries and can benefit directly from any advance in automatic remeshing.

Furthermore, smoothed FEMs are a lot less sensitive to locking (volumetric and shear) as well as mesh distortion (because Jacobians are not required since no isoparametric mapping is used. In this sense, SFEMs are a way to improve the quality of the results obtained by simplex elements, thereby significantly reducing the need for human-intervention in the generation of hexahedral meshes.

A few salient features of the smoothed finite element method can be summarised as follows:

- The formulation can be written within a mixed Hellinger Reissner formalism.
- Reduced sensitivity to mesh distortion.
- Softer response compared to linear simplex elements (triangular and tetrahedral) thereby allowing the use of unstructured meshes, much easier to generate automatically than structured hexahedral meshes.
- A range of methods producing results comprised between quasi-equilibrium methods and displacement-based finite elements.
- Possibility to obtain "ultra accurate solutions" with the "α" FEM (see e.g. [3] and [4]).

The basic ideas of strain smoothing in FEM [**?**] and XFEM have been reviewed recently in [5] and [6].

1 Methodology and results

We first show how various variants of strain smoothing techniques can be applied to fracture mechanics (e.g. by coupling to the extended finite element method [7]). It is shown that strain smoothing leads to efficient numerical schemes for 2D linear elastic fracture mechanics, where the computational time versus accuracy ratio can be reduced compared to competing methods.

We then address nano-composites and tackle in particular the case of nanosized microstructures, where interfacial energy plays an important role in determining equilibrium morphologies, due to the high interface-to-volume ratio. We propose an hybrid smoothed extended finite element/level set method to model nanoscale inhomogeneities including the interfacial energy effect (similar in principle to the work of [8], in which the finite element mesh can be completely independent of the interface geometry. The Gurtin-Murdoch surface elasticity model is used to account for the interface stress effect and the Wachspress interpolants are used for the first time to construct the shape functions in the smoothed extended finite element method. Selected numerical results are presented to study the accuracy and efficiency of the proposed method as well as the equilibrium shapes of misfit particles in elastic solids. The results compare very well with those obtained from theoretical solutions and experimental observations, and the computational efficiency of the method is shown to be superior to that of its most advanced competitors.

2 Future work and open questions

The fact that the smoothed FEM can be written in a mixed finite element formalism indicates interesting research directions for non-polynomial enrichment, where satisfying the "inf-sup" condition for stability of the approximation is expected to be non obvious. Indeed, the displacement and strain fields are approximated using different spaces. To this point, the space used for the (smoothed) strain approximation has been constant and the displacement approximation generally linear. This is in accordance with the stability condition for mixed methods. For non-polynomial enrichment of the displacement field, however, it is not obvious what space should be use to approximate the strain field to retain stability. This remark may help explain the results of [6].

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