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Nonlinear consensus protocols for multi-agent systems based on centre manifold reduction*

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Nonlinear consensus protocols for dynamic directed networks of multi-agent systems with fixed and switching topologies are investigated separately in this paper. Based on the centre manifold reduction technique, nonlinear consensus protocols are presented. We prove that a group of agents can reach a β-consensus, the value of which is the group decision value varying from the minimum and the maximum values of the initial states of the agents. Moreover, we derive the conditions to guarantee that all the agents reach a β–consensus on a desired group decision value. Finally, a simulation study concerning the vertical alignment manoeuvere of a team of unmanned air vehicles is performed. Simulation results show that the nonlinear consensus protocols proposed are more effective than the linear protocols for the formation control of the agents and they are an improvement over existing protocols.

Keywords: nonlinear consensus protocol, centre manifold reduction, multi-agent systems, switching topology

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1. Introduction

In recent years, the distributed coordination of multiple agents has attracted considerable attention. This is mainly due to its important applications, including the cooperative control of unmanned air vehicles (UAVs), autonomous underwater vehicles, congestion control in communication networks, swarms of autonomous vehicles or robots, autonomous formation flight, etc. In all cases the aim is to control a group of agents connected through a communication network to reach an agreement on certain quantities of interest. This problem is usually called the consensus problem. Many results have been obtained on this problem.[1–13] For example, Vicsek *et al*[1] proposed a simple model for the phase transition of a group of self-driven particles and numerically demonstrated the complex dynamics of the model. Jadbabaie *et al*[2] demonstrated that a simple neighbour rule made all agents eventually move in the same direction despite the absence of centralized coordination and each agent's set of neighbours changed with time as the system evolved under a joint connection condition. Lin *et al*[6] studied three formation strategies for groups of mobile autonomous agents. Fax and Murray[7] gave the stability analysis of multi-vehicle formations with a Nyquist-type criterion. Moreau[8] used a set-valued Lyapunov approach to study consensus problems with unidirectional time-dependent communication links. Moreover, by a Lyapunov-based approach, Olfati-Saber and Murray[9] solved the average-consensus problem for a network of agents with switching topology and time-delays. Cao *et al*[11,12] investigated a consensus in a dynamically changing environment. The studies mentioned above are all concerned with linear protocols design rule allowing consensus on certain quantities of interest. However, the problem of attitude alignment for robots and spacecraft is a special type of consensus problem. For these physical systems, it is not reasonable to assume that their attitudes can be changed by an unbounded value, i.e. the input torque is bounded. This suggests developing consensus protocols that guarantee that the overall input of each node stays bounded. This naturally leads to the design and analysis of nonlinear consensus protocols. Bauso *et al*[13] and Olfati-Saber and Murray[14] separately, considered nonlinear consensus protocols for an undirected network of agents with fixed topologies. The stability analysis in the existing literature is based

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on the traditional Lyapunov theory.[3−10] However, there still exists considerable difficulty in constructing a concrete Lyapunov functional for a nonlinear system, which motivates us to find a more suitable tool to judge the stability of nonlinear systems.

In the present paper, a β–consensus problem for directed networks of nonlinear multi-agent systems with fixed and switching topologies is discussed, separately. Here, each agent has only local information from its neighbours. What we are interested in is to design a nonlinear protocol to make the agents reach consensus on group decision value $\beta(x(0))$, the value which varies from the minimum value to the maximum value of the initial states of the agents. The convergence analysis is carried out based on centre manifold reduction theory. It is different from the results presented in the existing literature, where the convergence analysis is based on the Lyapunov theory.[3−10] The idea of centre manifold reduction is to reduce an infinite-dimensional network dynamic system into a one-dimensional system by projecting the original dynamics onto the eigenvectors corresponding to zero real-part eigenvalue. Therefore, the stability of the original system is completely dependent on the stability of the reduced system. Recently, centre manifold reduction has been introduced as a tool for the design of stabilizing control laws for nonlinear systems in critical cases. Critical cases occur when the linearized system at an equilibrium point has at least one zero real-part eigenvalue, while the remaining eigenvalues all have negative real-parts. So, if the communication links between agents are kept strongly connected, the nonlinear dynamic multi-agent systems with protocol (9) are said to be in the critical cases. We prove that the asymptotic consensus is reachable and we also derive conditions to guarantee that all agents reach a β-consensus on a desired group decision value. Finally, we perform a simulation study concerning the vertical alignment manoeuvere of a team of UAVs. Simulation results show that the nonlinear consensus protocols proposed are more effective than the linear protocols for the formation control of the agents and they are an improvement over existing protocols.

The remainder of the present paper is organized as follows. In Section 2, some fundamental conceptions on graph theory and centre manifold theory are introduced. In Section 3, the nonlinear consensus problem is described. In Section 4 are presented the main results, including our designed distributed nonlinear consensus protocol for a network of multi-agents with fixed and switching topologies according to the centre manifold theory, the analysis of the convergence, and the derived conditions to guarantee that all agents reach consensus on a group decision value of interest. In Section 5, the vertical alignment manoeuvere of a team of UAVs is simulated. Finally, some conclusions drawn from the present study are presented in Section 6.

2. Preliminaries

2.1. Graph theory

Let graph $G = (\mathit{\Gamma}, E, A)$ be a directed graph denoting the dynamic network with a set of nodes $\mathit{\Gamma}$, where $\mathit{\Gamma} = \{1, 2, \cdots, n\}$ is composed of all agents, and a set of edges $E = \{(i, j)\}$, where $(i, j) \in E$ which means that i and j are adjacent or that j is one of the neighbours of i. We refer to i and j as the tail and head of the edge (i, j), respectively. $A = [a_{ij}]$ is a weighed adjacency matrix, here we define $a_{ij} > 0$, $i, j = 1, \ldots, n$, if $(i, j) \in E$, while $a_{ij} = 0$ if $(i, j) \notin E$. Moreover, we assume that $a_{ii} = 0$ for all $i \in \mathit{\Gamma}$. The neighbours of agent i are denoted by $N_i = \{j \in \mathit{\Gamma} : (i, j) \in E\}$ and $|N_i|$ denotes the number of N_i. A directed path that connects i and j in the directed graph G is a sequence of distinct nodes $i_1, i_2, \ldots, i_m$, where $i_1 = i$, $i_m = j$ and $(i_l, i_{l+1}) \in E$, $0 \le l \le m-1$. The directed graph turns into an undirected graph if $a_{ij} = a_{ji}$ for any $i, j \in \mathit{\Gamma}$. If there is a directed path from a node to any other node, the graph is said to be strongly connected while the undirected graph is said to be a connected graph.[15] The in-degree and out-degree of node i are defined, respectively, as

$$d_{\text{in}}(i) = \sum_{j=1}^{n} a_{ji}, \quad \text{and} \quad d_{\text{out}}(i) = \sum_{j=1}^{n} a_{ij}.$$

A directed graph $G = (\mathit{\Gamma}, E, A)$ is said to be balanced if and only if all of its nodes are balanced, i.e. $d_{\text{in}}(i) = d_{\text{out}}(i)$, $i \in \mathit{\Gamma}$.[9]

2.2. Centre manifold theory

Consider the following system:

$$\begin{aligned} \dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y) \end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R^m$, and A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. The functions f and g are C^2 with $f(0, 0) = 0$, $Df(0, 0) = 0$, $g(0, 0) = 0$, and $Dg(0, 0) = 0$ (Here, Df denotes the Jacobian matrix of f). In general, if $y = h(x)$ is an invariant manifold for Eq.(1) and h is smooth, then it is called a centre manifold if $h(0) = 0$, and $Dh(0) = 0$.

Theorem 1[16] There exists a centre manifold for Eq.(1), $y = h(x)$, $|x| < \delta$ where h is C^2. The flow on the centre manifold is governed by the n-dimensional system

$$\dot{u} = Au + f(u, h(u)). \tag{2}$$

The next theorem shows that equation (2) contains all the necessary information that is needed to determine the asymptotic behaviour of small solutions of Eq.(1).

Theorem 2[16] (a) Suppose that the zero solution of Eq.(2) is stable (asymptotically stable) (unstable), then the zero solution of Eq.(1) will be stable (asymptotically stable) (unstable). (b) Suppose that the zero solution of Eq.(2) is stable and let $(x(t), y(t))$ be a solution of Eq.(1) with $(x(0), y(0))$ sufficiently small, then there will exist a solution $u(t)$ of Eq.(2) such that as $t \to \infty$

$$\begin{aligned} x(t) &= u(t) + o(\mathrm{e}^{-\gamma t}) \\ y(t) &= h(u(t)) + o(\mathrm{e}^{-\gamma t}) \end{aligned} \tag{3}$$

where $\gamma > 0$ is a constant.

Substituting $y(t) = h(x(t))$ into the second equation in Eq.(1) yields

$$Dh(x)[Ax + f(x, h(x))] = Bh(x) + g(x, h(x)). \tag{4}$$

Equation (4) together with the conditions $h(0) = 0$ and $Dh(0) = 0$ is the system to be solved for the centre manifold. It is impossible to accurately solve the equation, in general, since it is equivalent to solving Eq.(1). The next result, however, shows that, in principle, the centre manifold can be approximated to any degree of accuracy.

Function $\phi : R^n \to R^m$ which is C^1 in a neighbourhood of the origin is defined as

$$\begin{aligned} (M\phi)(x) = {} & D\phi(x)[Ax + f(x, \phi(x))] \\ & - B\phi(x) - g(x, \phi(x)). \end{aligned} \tag{5}$$

Note that by Eq.(4), $(Mh)(x) = 0$.

Theorem 3[16] Let ϕ be a C^1 mapping of a neighbourhood of the origin in R^n into R^m with $\phi(0) = 0$ and $D\phi(0) = 0$. Suppose that as $x \to 0, (M\phi)(x) = 0(|x|^q)$ where $q > 1$, then as $x \to 0$, $|h(x) - \phi(x)| = 0(|x|^q)$ will hold.

3. Problem description

Suppose that the network system under consideration is composed of n agents. Each agent is regarded as a node in a directed graph G. Each edge $(j, i) \in E(G)$ corresponds to an available information link from agent i to agent j. Moreover, each agent updates its current state based on the information received from its neighbours. Let x_i be the state of the i-th agent. Suppose the i-th agent $(i \in \Gamma)$ has the dynamics as follows:

$$\dot{x}_i(t) = u_i(t), \forall i \in \Gamma, j \in N_i \tag{6}$$

with initial condition $x_i(s) = x_i(0)$, $s \in (-\infty, 0]$ where u_i is the control protocol.

Our objective is to find an appropriate nonlinear protocol to suppress disturbances of agents and make all agents reach agreement.

Consider the following nonlinear protocol:

$$u_i(t) = \sum_{j \in N_i} \varphi(x_i, x_j), \tag{7}$$

where $i, j = 1, \ldots, n$; $\varphi(\cdot)$ $R \to R$ satisfies the following properties: 1) $\varphi(\cdot)$ is continuous and locally Lipschitz, and 2) $\varphi(x_i, x_j) = 0 \Leftrightarrow x_i = x_j$. $D(D \subseteq R)$ denotes the domains of definition of function $\varphi(\cdot)$, i.e. $x_i \in D, i = 1, \ldots, n$. Given protocol (7), the network dynamics of these n agents may be written in the vector form as

$$\dot{x}(t) = q(x), \tag{8}$$

where $x(t) := (x_1(t), \ldots, x_n(t))^T$, $t \geq 0$ and $q = (q_1, \ldots, q_n)^T$ is such that $q_i(x) = \sum_{j \in N_i} \varphi(x_i, x_j)$ and $q(\cdot)$ is continuous on R^n.

First, we define the invariant subspace Ω as $\Omega = \{x \in D^n \cap R^n : x_1 = x_2 = \ldots = x_n\}$. This implies that $q(x^*) = 0$ for all $x^* \in \Omega$. Hence, the subspace Ω is an equilibrium set for the system (8).

For the convenience of discussion, we assume the nonlinear protocol (7) is in the following form:

$$u_i(t) = \sum_{j \in N_i} a_{ij}(x_j - x_i) + \sum_{j \in N_i} \phi_{ij}(x_i, x_j), \tag{9}$$

where $\phi_{ij}(x_i, x_j)$ is a purely nonlinear function and for all $(i, j) \in \varepsilon$ satisfies the following properties: $\phi_{ij}(\cdot)$ is C^2 on R^{mn}, with $\phi_{ij}(x_i, x_j) = 0$ and $D\phi_{ij}(x_i, x_j) = 0$, if and only if $x_i = x_j$ (Here, $D\phi_{ij}$ is the Jacobian matrix of ϕ_{ij}).

Remark 1 Note that based on the Taylor series expansion, nonlinear protocol (7) can always be represented as the sum of a linear part and a purely nonlinear part. Therefore, the discussion of protocol (9) is without loss of generality.

Given protocol (9), the network dynamics can be summarized as

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$$\dot{x}(t) = -Lx(t) + f(x), \tag{10}$$

where L is called the graph Laplacian induced by the information flow G and defined as

3104
$$l_{ij} = \begin{cases} \sum\limits_{k=1,\, k\neq i}^{n} a_{ik}, & j = i, \\ -a_{ij}, & j \neq i, \end{cases}$$

3115 where $f = (f_1, \ldots, f_n)^T$ is such that $f_i(x) = \sum\limits_{j\in N_i} \phi_{ij}(x_i, x_j)$ and $f(x)$ satisfies the following properties:

3122 1) $f(\cdot)$ is C^2 on R^{mn}, with $f(x^*) = 0$ and $Df(x^*) = 0$, (Here, x^* is the equilibrium point of
3131 system (10), Df is the Jacobian matrix of f.

$$\dot{x}(t) = -Lx(t) \tag{11}$$

is a corresponding linearized system of nonlinear sys-
3135 tem (10). Apparently, L has a zero eigenvalue and $\mathbf{1} = (1, \ldots, 1)^T \in R^n$ is the corresponding eigenvec-
3139 tor with the eigenvalue $\lambda = 0$.

If for any initial state $x(0) \in D^n$, $x(t)$ converges to asymptotically stable equilibrium point $x^* \in \Omega$ of sys-
3145 tem (8) as $t \to \infty$ we say that all agents have asymptotically reached consensus in infinite time $t > 0$; let
3150 $\beta : R^n \to R$ be a continuous and differentiable function on $x = (x_1, \ldots, x_n)^T$, if $\|x_i - \beta(x(0))\| \to 0$ for all $i, j \in \Gamma$ as $t \to \infty$ we say that protocol $u_i(t)$ makes the agents asymptotically reach the β-consensus on a
3155 group decision value $\beta(x(0))\mathbf{1}$ function of their initial states or we say that the system asymptotically
3158 converges to $\beta(x(0))\mathbf{1}$; especially for the cases with $\beta(x) = Ave(x) = (1/n)\mathbf{1}^T x(0)\mathbf{1}$ we say that the system asymptotically reaches an average-consensus.

In this paper, we are interested in discussing the
3163 β-consensus problem in networks of agents with both fixed and switching topologies, separately. In particu-
3169 lar, we are interested in agreement functions varying in the following range:

$$\min_{i\in\Gamma}\{x_i(0)\} \le \beta(x) \le \max_{i\in\Gamma}\{x_i(0)\}. \tag{12}$$

The above condition means that the group decision value must be restricted between the minimum and the maximum values of the initial states of agents.

Remark 2 The β-consensus mentioned in this paper is significant in some areas, especially, in the biological and the chemical areas where only certain life-form groups or chemical reactors are required to reach an agreement but not to be maintained at a fixed value.

Remark 3 In a network of continuous-time integrator agents, convergence analysis of protocol (9) is equivalent to stability analysis for system (10) at equilibrium point $x^* = \beta(x(0))\mathbf{1}$.

The following lemmas are needed for the main result of next section.

Lemma 1[9] If the graph G is strongly connected, then its Laplacian L satisfies

(i) $\mathrm{rank}(L) = n - 1$;

(ii) $\lambda = 0$ is one eigenvalue of L, and $\mathbf{1}$ is the corresponding eigenvector;

(iii) The remaining $n-1$ eigenvalues all have positive real-parts, in particular, for an undirected graph, they are all positive and real.

3.1. Nonlinear consensus protocols based on centre manifold reduction

Recently, centre manifold reduction has been employed in nonlinear stabilization to stabilize the control law designs for varieties of nonlinear systems in the so-called "critical cases". Critical cases occur when the linearized system at an equilibrium point has at least one zero real-part eigenvalue, while the remaining eigenvalues all have negative real-parts. So, according to Lemma 1, nonlinear system (10) is said to be in the critical case, if the graph G is strongly connected.

To investigate the stability of nonlinear system (10) at equilibrium point $\beta(x(0))\mathbf{1}$ $(\beta(x(0))\mathbf{1} \in \Omega)$, we use a coordinate transformation $y_i(t) = x_i(t) - \beta(x(0))$ then system (10) becomes $\dot{y}(t) = -Ly(t) - L\beta(x(0)\mathbf{1} + f(y + \beta(x(0)))$ due to $L\beta(x(0)\mathbf{1} = \beta(x(0)L\mathbf{1} = 0$ and denote $\bar{f}(y) = f(y + \beta(x(0)))$ then

$$\dot{y}(t) = -Ly(t) + \bar{f}(y). \tag{13}$$

Thus, discussing the stability of system (10) at the equilibrium point, $\beta(x(0))\mathbf{1}$, is equivalent to discussing the stability of the origin of system (13).

Lemma 2 For system (13), there exists a unitary matrix $S, (S \in C^{n\times n})$ such that system (13) can be normalized to the form of system (1), if the communication links between agents are kept strongly connected.

Proof From Lemma 1, if the communication links between agents are kept strongly connected, then Laplacian matrix L has one 0 eigenvalue and $n-1$ positive real-part eigenvalues. Let $0<\lambda_2\le\lambda_3\le\ldots\le\lambda_n$ denote the $n-1$ nonzero eigenvalues of Laplacian matrix L and let $A=-L$, according to Schur's unitary triangularisation theorem, there will exist unitary matrix $S\in C^{n\times n}$ such that

$$S^H AS=\begin{bmatrix}0 & A_{12}\\ 0 & J_{n-1}\end{bmatrix},$$

and

$$J_{n-1}=\begin{bmatrix}-\lambda_2 & * & *\\ 0 & \ddots & *\\ 0 & 0 & -\lambda_n\end{bmatrix}.$$

We denote $S=\begin{bmatrix}\bar{S}_1 & S_1\end{bmatrix}$, $\xi(t)=\bar{S}_1^H y(t)$, and $\eta(t)=S_1^H y(t)$, where S^H is the conjugate transpose of S, $\xi(t)$ is the one-dimensional vector, and $\eta(t)$ is the $(n-1)$-dimensional vector. Noting that S is a unitary matrix, we have $S_1^H\bar{S}_1=0$. Further, let $\bar{S}_1=1/\sqrt{n}$, then we will have

$$\begin{aligned}S^H\dot{y}(t)&=\begin{bmatrix}\dot{\xi}(t)\\ \dot{\eta}(t)\end{bmatrix}\\ &=S^H AS\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix}+S^H f(\bar{S}_1\xi+S_1\eta)\\ &=\begin{bmatrix}0 & A_{12}\\ 0 & J_{n-1}\end{bmatrix}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix}+\begin{bmatrix}\bar{S}_1^H f(\bar{S}_1\xi+S_1\eta)\\ S_1^H f(\bar{S}_1\xi+S_1\eta)\end{bmatrix},\end{aligned}$$

i.e.

$$\begin{bmatrix}\dot{\xi}(t)\\ \dot{\eta}(t)\end{bmatrix}=\begin{bmatrix}0 & A_{12}\\ 0 & J_{n-1}\end{bmatrix}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix}+\begin{bmatrix}\tilde{f}(\xi,\eta)\\ \tilde{G}(\xi,\eta)\end{bmatrix}\quad(14)$$

where $\tilde{f}(\xi,\eta)$ and $\tilde{G}(\xi,\eta)$ are smooth functions and satisfy $\tilde{f}(0,0)=0$, $\tilde{G}(0,0)=0$, $D\tilde{f}(0,0)=0$, and $D\tilde{G}(0,0)=0$. The proof of Lemma 2 is completed.

Remark 4 For undirected connected graphs, since A is symmetric, there exists an orthogonal matrix $U,(U\in R^{n\times n})$ such that $U^TAU=\begin{bmatrix}0 & 0\\ 0 & J_{n-1}\end{bmatrix}$. Then system (13) can be normalized into the form of system (1).

From the centre manifold theorem it is easy to obtain the following lemma.

Lemma 3 If the directed graph G is strongly connected, then there exists a centre manifold $\eta(t)=h(\xi)$ such that the origin of system (14) is asymptotically stable if the origin is asymptotically stable for the reduced model

$$\dot{\xi}(t)=A_{12}h(\xi)+\tilde{f}(\xi,h(\xi)),\quad(15)$$

where h satisfies the partial differential equation

$$\begin{aligned}Dh(\xi)\{A_{12}h(\xi)+\tilde{f}(\xi,h(\xi))\}-J_{n-1}h(\xi)\\ -\tilde{G}(\xi,h(\xi))=0\end{aligned}\quad(16)$$

with boundary conditions: $h(0)=0$ and $Dh(0)=0$.

3.2. Analysis of stabilization

Consider a scalar real nonlinear system

$$\dot{y}=dy^2+ey^3+\ldots.\quad(17)$$

The stability condition for the system is given by the following lemma.

Lemma 4[17] The origin is asymptotically stable for system (17) if $d=0$ and $e<0$, but it is unstable if $d\neq0$.

We employ Taylor series expansions below, using multilinear function notation for the terms in these expansions. The definition of the multilinear function is recalled as follows.

Definition 1[18] Let $V_1, V_2,\ldots, V_k$ and W be vector spaces over the same field. A map Ψ : $V_1\times V_2\times\ldots\times V_k\to W$ is multilinear (or k-linear) if it is linear in each of its arguments, that is, for any scalar, we have $\Psi(v_1,\ldots,av_i+\tilde{a}\tilde{v}_i,\ldots,v_k)=a\Psi(v_1,\ldots,v_i,\ldots,v_k)+\tilde{a}\Psi(v_1,\ldots,\tilde{v}_i,\ldots,v_k)$.

The integer k is the degree of the multilinear function Ψ.

Next, we give a simple description for Taylor series expansions and the notations which are used in the following formula. Using the Taylor series expansion, a real-valued function $f(x_1,x_2,\ldots,x_n)$ at origin $M_0(0,0,\ldots,0)$ can be expressed as

$$f(x_1, x_2, \ldots, x_n) = f(0, 0, \ldots, 0) + \left(\frac{\partial}{\partial x_1}x_1 + \frac{\partial}{\partial x_2}x_2 + \ldots + \frac{\partial}{\partial x_n}x_n\right) f|_{M_0} + \frac{1}{2!}\left(\frac{\partial}{\partial x_1}x_1 + \frac{\partial}{\partial x_2}x_2 + \ldots + \frac{\partial}{\partial x_n}x_n\right)^2 f|_{M_0} + \ldots + \frac{1}{n!}\left(\frac{\partial}{\partial x_1}x_1 + \frac{\partial}{\partial x_2}x_2 + \ldots + \frac{\partial}{\partial x_n}x_n\right)^n f|_{M_0} + R_n,$$

where

$$R_n = \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x_1}x_1 + \frac{\partial}{\partial x_2}x_2 + \ldots + \frac{\partial}{\partial x_n}x_n\right)^{n+1} f(\theta x_1, \theta x_2, \ldots, \theta x_n), \quad (0 < \theta < 1).$$

If f is vector-valued instead, i.e. $f = (f_1, f_2, \ldots, f_n)$, then the array of m-order partial derivatives is an n-dimensional tensor of order m and has n^m components. The concept of tensor can be found in Ref.[19].

Now consider system (14), applying the concept of tensor and the multilinear function, and due to $\tilde{f}(0, 0) = 0$, $\tilde{G}(0, 0) = 0$, $D\tilde{f}(0, 0) = 0$, $D\tilde{G}(0, 0) = 0$, the Taylor series expansion of $\tilde{f}(\xi, \eta)$ and $\tilde{G}(\xi, \eta)$ can be described as

$$\tilde{f}(\xi, \eta) := \tilde{f}_{\xi\xi}\xi^2 + \xi\tilde{f}_{\xi\eta}\eta + \eta^T\tilde{f}_{\eta\eta}\eta + \tilde{f}_{\xi\xi\xi}\xi^3 + \xi^2\tilde{f}_{\xi\xi\eta}\eta + \xi\eta^T\tilde{f}_{\xi\eta\eta}\eta + \tilde{f}_{\eta\eta\eta}(\eta, \eta, \eta) + o(\|(\xi, \eta)\|^4), \tag{18}$$

$$\tilde{G}(\xi, \eta) := \xi^2\tilde{G}_{\xi\xi} + \xi\tilde{G}_{\xi\eta}\eta + \tilde{G}_{\eta\eta}(\eta, \eta) + \xi^3\tilde{G}_{\xi\xi\xi} + \xi^2\tilde{G}_{\xi\xi\eta}\eta + \xi\tilde{G}_{\xi\eta\eta}(\eta, \eta) + \tilde{G}_{\eta\eta\eta}(\eta, \eta, \eta) + o(\|(\xi, \eta)\|^4). \tag{19}$$

The coefficients in the Taylor series expansions (18) and (19) are either constants or symmetric multilinear functions of their arguments. For instance, $\tilde{f}_{\eta\eta\eta}(\eta, \eta, \eta)$ and $\tilde{G}_{\eta\eta}(\eta, \eta)$ denote a symmetric trilinear scalar function and a bilinear vector function of η, respectively.

From Lemma 2 and Lemma 4 we have the following theorem.

Theorem 4 Consider a directed network of multi-agents with fixed topology $G(\Gamma, E)$ that is strongly connected. Given nonlinear protocol (9), there exists a centre manifold $\eta(t) = h(\xi)$ such that the stability of system (6) is completely dependent on the stability of the reduced system $\dot{\xi}(t) = A_{12}h(\xi) + \tilde{f}(\xi, h(\xi))$ and if $\tilde{f}_{\xi\xi} - A_{12}J_{n-1}^{-1}\tilde{G}_{\xi\xi} = 0$ and $\tilde{f}_{\xi\xi\xi} - \tilde{f}_{\xi\eta}J_{n-1}^{-1}\tilde{G}_{\xi\xi} < 0$, the agents asymptotically reach β-consensus at $\beta(x(0))\mathbf{1}$ for any initial state $x(0)$.

Proof First, observe system (9) where the consensus that is reached at the equilibrium point $\beta(x(0))\mathbf{1}$ corresponds to asymptotic stability of variable $y = \{y_i, i \in \Gamma\}$ where $y(t) = x(t) - \beta(x(0))\mathbf{1}$ and $y = 0$ corresponds to $x(t) = \beta(x(0))\mathbf{1}$. Substituting $y(t) = x(t) - \beta(x(0))\mathbf{1}$ into system (10), then system (10) is transformed into system (13). It follows that if $\lim\limits_{t\to\infty}\xi(t) = 0$, then $\lim\limits_{t\to\infty}y(t) = 0$. From Lemma 2, if the dynamic graph is strongly connected, there exists a unitary matrix such that system (13) turns into system (14). Thus from Lemma 3, we can obtain a centre manifold $\eta = h(\xi)$ such that the stability of the origin of the reduced model (15) determines the stability of the origin of system (14). Solving the partial differential equation (16), we have

$$h(\xi) = \xi^2h_{\xi\xi} + o(|\xi|^3), \tag{20}$$

where

$$h_{\xi\xi} = -J_{n-1}^{-1}\tilde{G}_{\xi\xi}.$$

Substituting expressions (18), (19) and (20) into Eq.(15) yields

$$\dot{\xi}(t) = (\tilde{f}_{\xi\xi} - A_{12}J_{n-1}^{-1}\tilde{G}_{\xi\xi})\xi^2 + (\tilde{f}_{\xi\xi\xi} - \tilde{f}_{\xi\eta}J_{n-1}^{-1}\tilde{G}_{\xi\xi})\xi^3 + o(\|\xi\|^3)$$

from Lemma 4. If $\tilde{f}_{\xi\xi} - A_{12}J_{n-1}^{-1}\tilde{G}_{\xi\xi} = 0$ and $\tilde{f}_{\xi\xi\xi} - \tilde{f}_{\xi\eta}J_{n-1}^{-1}\tilde{G}_{\xi\xi} < 0$, then $\xi \to 0$ as $t \to \infty$, i.e. $y \to 0 \Leftrightarrow x(t) \to \beta(x(0))\mathbf{1}$, as $t \to \infty$, that is, consensus is reachable.

Corollary 1 Assume that all the conditions in Theorem 4 will hold and if $\tilde{f}_{\xi\xi} - A_{12}J_{n-1}^{-1}\tilde{G}_{\xi\xi} = 0$ and $\tilde{f}_{\xi\xi\xi} - \tilde{f}_{\xi\eta}J_{n-1}^{-1}\tilde{G}_{\xi\xi} = 0$, then every agreement state will be stable for the nonlinear system (14).

Remark 5 Theorem 4 provides a rule to find nonlinear protocols for directed networks of agents and it sheds new light on consensus behaviour: the original infinite-dimensional system is reduced into a one-dimensional system via centre manifold reduction and the agreement of the original system is completely dependent on the stability of the reduced system, if and only if the dynamic graph G is kept strongly connected.

In the case where there is no nonlinear term in system (10), that is $f(x)$ is assumed to be zero, nonlinear system (10) degenerates into linear system (11). Theorem 4 is specialized into the following corollary.

Corollary 2 Consider a directed network of multi-agents with fixed topology $G(\Gamma, E)$ that is strongly connected. Given a linear protocol $u_i(t) = \sum\limits_{j\in N_i} a_{ij}(x_j - x_i)$, then there exists a centre manifold $\eta(t) = 0$ such that system (10) is stable and the agents asymptotically reach an average-consensus at $(1/n)\sum\limits_{i=1}^{n} x_i(0)\mathbf{1}$ if and only if the topology $G = (\Gamma, E)$ is a balanced digraph.

Proof First, observe system (11) where the consensus that is reached at the equilibrium point $\beta(x(0))\mathbf{1}$ corresponds to asymptotic stability of variable $y = \{y_i, i \in \Gamma\}$, where $y(t) = x(t) - \beta(x(0))\mathbf{1}$ and $y = 0$ corresponds to $x(t) = \beta(x(0))\mathbf{1}$. Substituting $y(t) = x(t) - \beta(x(0))\mathbf{1}$ into system (11), then system (11) is transformed into system $\dot{y}(t) = -Ly(t)$. From Lemma 2, if the dynamic graph is strongly connected, there exists an unitary matrix S such that system $\dot{y}(t) = -Ly(t)$ becomes system

$$\begin{bmatrix} \dot{\xi}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} 0 & A_{12} \\ 0 & J_{n-1} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}. \tag{21}$$

Thus from Lemma 3, we can obtain a centre manifold $\eta = h(\xi)$ such that the stability of the origin of the reduced model $\dot{\xi}(t) = A_{12}h(\xi)$ determines the stability of the origin of system (21). Solving the partial differential equation $Dh(\xi)\{A_{12}h(\xi)\} - J_{n-1}h(\xi) = 0$ with boundary conditions: $h(0) = 0$ and $Dh(0) = 0$, we have $\eta = h(\xi) = 0$, and $\dot{\xi}(t) = 0$. So, linear system (16) is stable in the origin. Then $\xi \to 0$ as $t \to \infty$, i.e. $y \to 0 \Leftrightarrow x(t) \to \beta(x(0))\mathbf{1}$ as $t \to \infty$, that is, consensus is reachable. Moreover, $\beta(x(0))\mathbf{1} = (1/n)\sum\limits_{i=1}^{n} x_i(0)\mathbf{1}$ is an equilibrium point of system (11) if and only if the topology $G = (\Gamma, E)$ is a balanced digraph, thus by the above discussion, the agents can asymptotically reach an average-consensus on $(1/n)\sum\limits_{i=1}^{n} x_i(0)\mathbf{1}$.

Remark 6 The corollary is compatible with Theorem 5 in Ref.[9], therefore, the nonlinear protocol proposed in this paper includes the case investigated in Ref.[20].

3.3. Network consensus with switching topology

Communication links among multi-agent systems are often unreliable due to multipath effects and exogenous disturbances leading to dynamic information exchange topologies. In this section, we develop a static nonlinear consensus protocol to achieve agreement over a network with switching topology. Consider a hybrid system with continuous-state $x \in R^n$ and discrete-state G that belongs to a finite collection of digraphs $\Gamma_n = \{G\}$ such that G is a digraph of order n and strongly connected. This set can be analytically expressed as $F_n = \{G = (\Gamma, E, A) : \mathrm{rank}(L(G)) = n - 1\}$.

Given protocol (9), the network dynamics is summarized as

$$\dot{x}(t) = L_{\sigma(t)}x(t) + f_{\sigma(t)}(x), \tag{22}$$

where $L_{\sigma(t)} = L(G_{\sigma(t)})$ is the Laplacian of graph $G_{\sigma(t)}$, $G_{\sigma(t)} \in F$ is a random switching signal that determines the communication topology G, P is a finite set of indices corresponding to all graphs over n nodes, $f_{\sigma(t)}(x)$ is a purely nonlinear function vector and C^2 on R^{mn} and satisfies the following conditions: $f_{\sigma(t)}(x^*) = 0$ and $Df_{\sigma(t)}(x^*) = 0$.

Resembling the analysis in cases with fixed topology, there exists unitary matrix $S_{\sigma(t)}$ such that system (22) can be normalized to satisfy the form

$$\begin{bmatrix} \dot{\xi}(t) \\ \dot{\eta}(t) \end{bmatrix} = \begin{bmatrix} 0 & A_{\sigma(t)_{12}} \\ 0 & J_{\sigma(t)_{n-1}} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} \tilde{f}_{\sigma(t)}(\xi, \eta) \\ \tilde{G}_{\sigma(t)}(\xi, \eta) \end{bmatrix}. \tag{23}$$

According to the centre manifold theorem, if the switching topology $G_{\sigma(t)}(\Gamma, E)$ is kept strongly connected, then there exists centre manifold $\eta(t) = h_{\sigma(t)}(\xi)$ such that the stability of system (23) is completely determined by the stability of the reduced system $\dot{\xi}(t) = A_{\sigma(t)_{12}}h_{\sigma(t)}(\xi) + \tilde{f}_{\sigma(t)}(\xi, h_{\sigma(t)}(\xi))$. Therefore, from Lemma 2 and Lemma 4 we can obtain the following theorem.

Theorem 5 Consider a directed network of agents with switching topology $G_{\sigma(t)}(\Gamma, E)$ that is kept strongly connected. Given a nonlinear protocol (9), then there exists centre manifold $\eta(t) = h_{\sigma(t)}(\xi)$ such that the stability of system (22) is completely dependent on the stability of the reduced system $\dot{\xi}(t) = A_{\sigma(t)_{12}}h_{\sigma(t)}(\xi) + \tilde{f}_{\sigma(t)}(\xi, h_{\sigma(t)}(\xi))$ and, if $\tilde{f}_{\sigma(t)\xi\xi} - A_{\sigma(t)_{12}}J^{-1}_{\sigma(t)_{n-1}}\tilde{G}_{\sigma(t)\xi\xi} = 0$ and $\tilde{f}_{\sigma(t)\xi\xi\xi} - \tilde{f}_{\sigma(t)\xi\eta}J^{-1}_{\sigma(t)_{n-1}}\tilde{G}_{\sigma(t)\xi\xi} < 0$, the agents asymptotically reach β-consensus at $\beta(x(0))\mathbf{1}$ for any initial state $x(0)$.

Proof The proof of Theorem 5 follows straightforwardly the way of proving Theorem 4, hence, it is omitted here.

Remark 7 It should be pointed out that we only require all the dynamic directed graphs in this paper to be kept strongly connected, but not all are required to be balanced graphs. It is different from the situations in Refs.[9] and [10], where all digraphs are required to be balanced. Therefore, the method presented in this paper may relax the restriction on mulit-agents reaching an agreement state to some extent.

Remark 8 With switching signals varying, the group decision value is not a fixed-value any more, but a variable of a bounded-value between the minimum and the maximum values of the initial states of agents.

Remark 9 In this paper are investigated the nonlinear consensus protocols under the condition of strongly connected between agents. However, the strongly connected condition may be relaxed into a more general one. For example, the topology graph $G(\Gamma, E)$ is quasi-strongly connected or uniformly quasi-strongly connected,[20] even if there exist communication delays between agents.[11] Considering that quasi-strongly connected graph and communication delays between agents might be useful for further improving our results, it deserves to be further studied.

4. Simulation results

We consider a team of tenUAVs in longitudinal flight and initially at different heights. Each UAV controls the vertical rate with knowing only the relative positions of its neighbours, but without knowing the relative positions of all the other UAVs according to the communication network topology depicted in Fig.1.

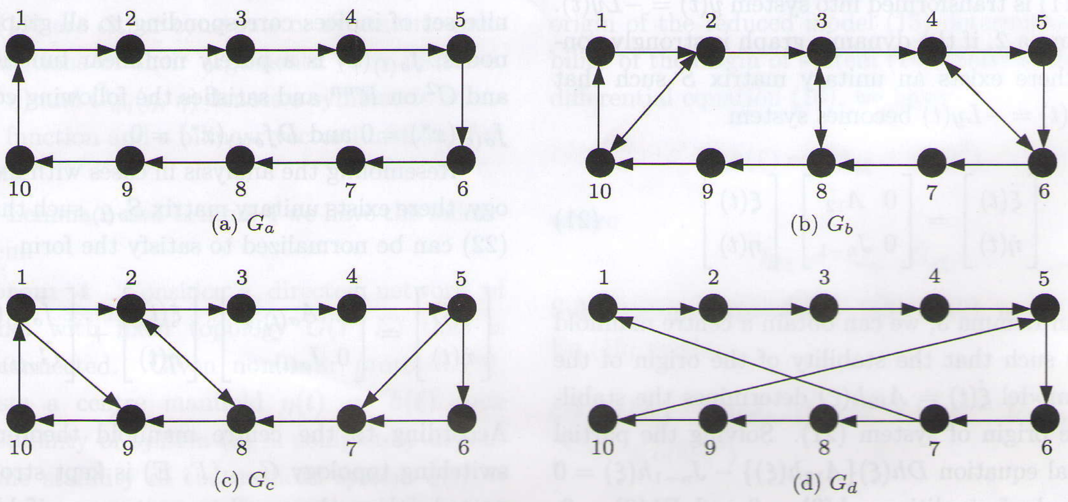


Fig.1. Four strongly connected digraphs, where G_a and G_b are balanced digraphs and G_c and G_d are unbalanced digraphs.

All digraphs in this figure have 0–1 weights. Moreover, they are all strongly connected, but not all graphs are required to be balanced, for instance, where G_c and G_d, are not balanced graphs. It is different from the situation in Refs.[9] and [10], where all digraphs are required to be balanced. Shown in Fig.2 is a finite state machine with four states $\{G_a, G_b, G_c, G_d\}$ representing the discrete-states of a network with switching topology as a hybrid system. The hybrid system starts at the discrete-state G_a and switches to the next state every other one second, i.e. $T = 1$ s according to the state machine in Fig.2.

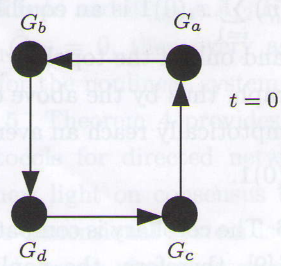


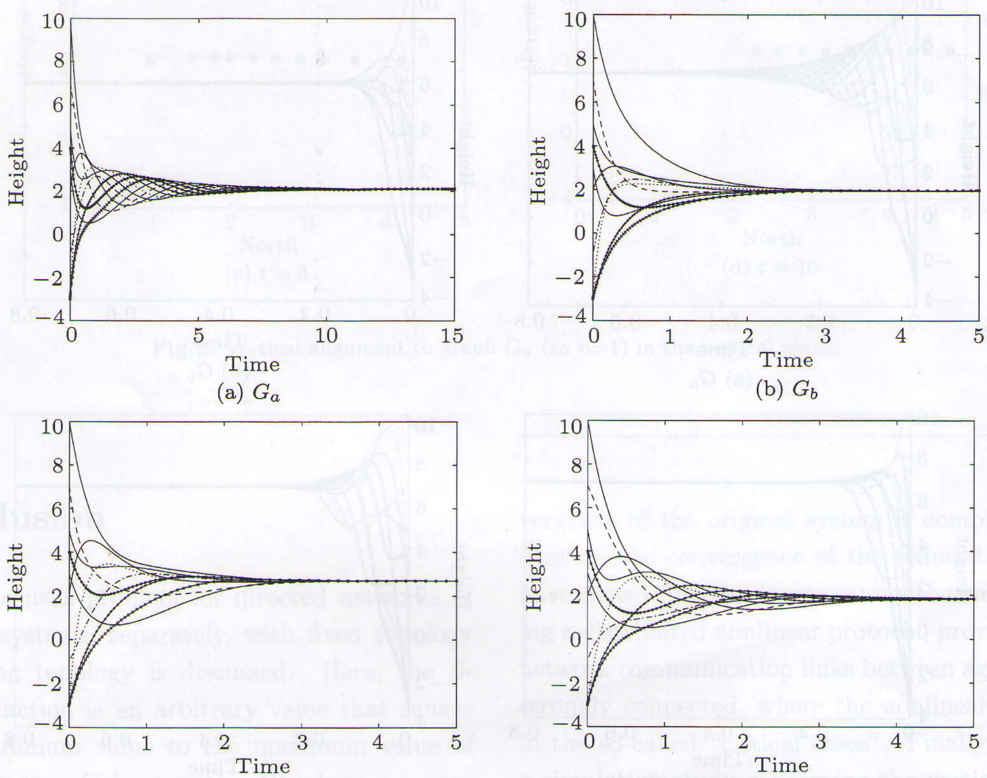
Fig.2. Finite machine with four states representing the discrete states of a network with switching topology.

As mentioned above, we are interested in determining a suitable distributed vertical rate control strategy that allows the UAVs to align their paths according to the formation centre at time 0 which we assume to be expressed as the β agreement function of the initial heights of UAVs and to satisfy $\min_{i\in\Gamma}\{x_i(0)\} \leq \beta\{(x(0)) \leq \max_{i\in\Gamma}\{x_i(0)\}$. The initial height is $x(0) = [2; 7; 3; -3; 4; -3; 5; -3; 10; -1]^T$. The challenging point is that the UAV knows the heights of only its neighbours (but it does not know the heights of all the other UAVs) and is required to align its path according to the path of the formation centre $\beta(x(0))$, which in turn depends on the positions of all the UAVs.

We give the UAVs a nonlinear protocol as follows:

$$u_i = \sum_{j\in N_i}(x_j - x_i) + \sum_{j\in N_i} x_j^n(x_j - x_i)). \qquad (24)$$

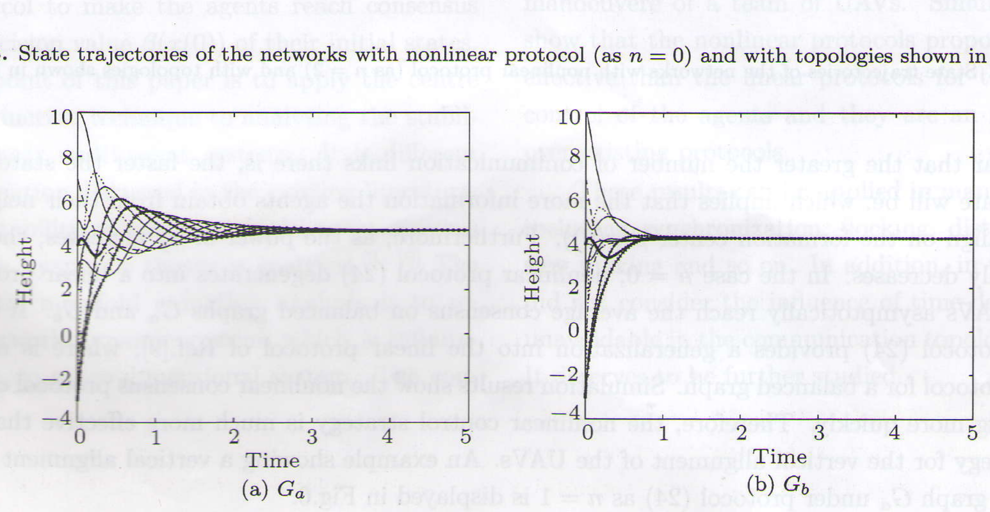
Applying Theorem 5, the UAVs asymptotically align on $\beta(x(0))$. Figures 3, 4 and 5 give the state trajectories of the UAVs as $n = 0$, $n = 1$, and $n = 2$, respectively.



(a) G_a (b) G_b

(c) G_c (d) G_d

Fig.3. State trajectories of the networks with nonlinear protocol (as $n = 0$) and with topologies shown in Fig.1.



(a) G_a (b) G_b

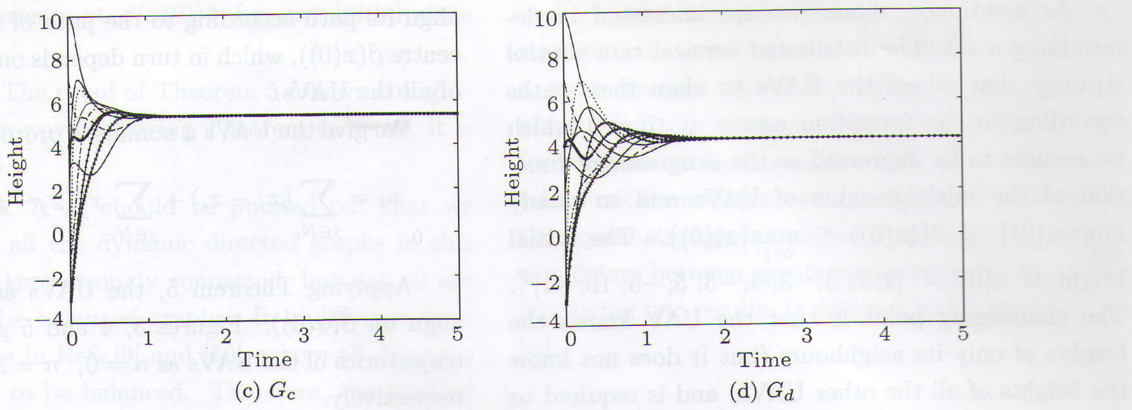


Fig.4. State trajectories of the networks with nonlinear protocol (as $n = 1$) and with topologies shown in Fig.1.

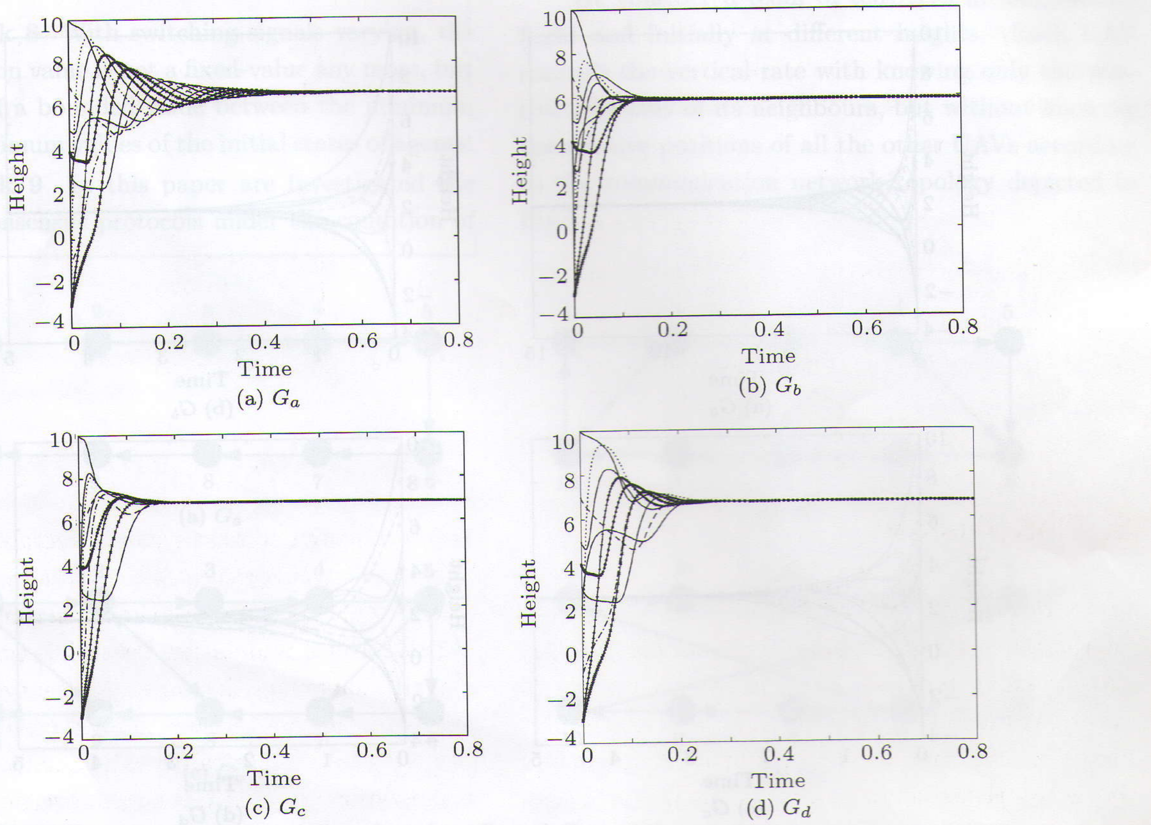


Fig.5. State trajectories of the networks with nonlinear protocol (as $n = 2$) and with topologies shown in Fig.1.

It is clear that the greater the number of communication links there is, the faster the state trajectories converging rate will be, which implies that the more information the agents obtain from their neighbours, the faster they align on the formation center $\beta(x(0))$. Furthermore, as the power of x_j increases, the converging time gradually decreases. In the case $n = 0$, nonlinear protocol (24) degenerates into a linear protocol. From Fig.3, the UAVs asymptotically reach the average consensus on balanced graphs G_a and G_b. It implies that nonlinear protocol (24) provides a generalization into the linear protocol of Ref.[9], where is established a consensus protocol for a balanced graph. Simulation results show the nonlinear consensus protocol characterized by converging more quickly. Therefore, the nonlinear control strategy is much more effective than the linear control strategy for the vertical alignment of the UAVs. An example showing a vertical alignment manoeuvre for topology graph G_a under protocol (24) as $n = 1$ is displayed in Fig.6.

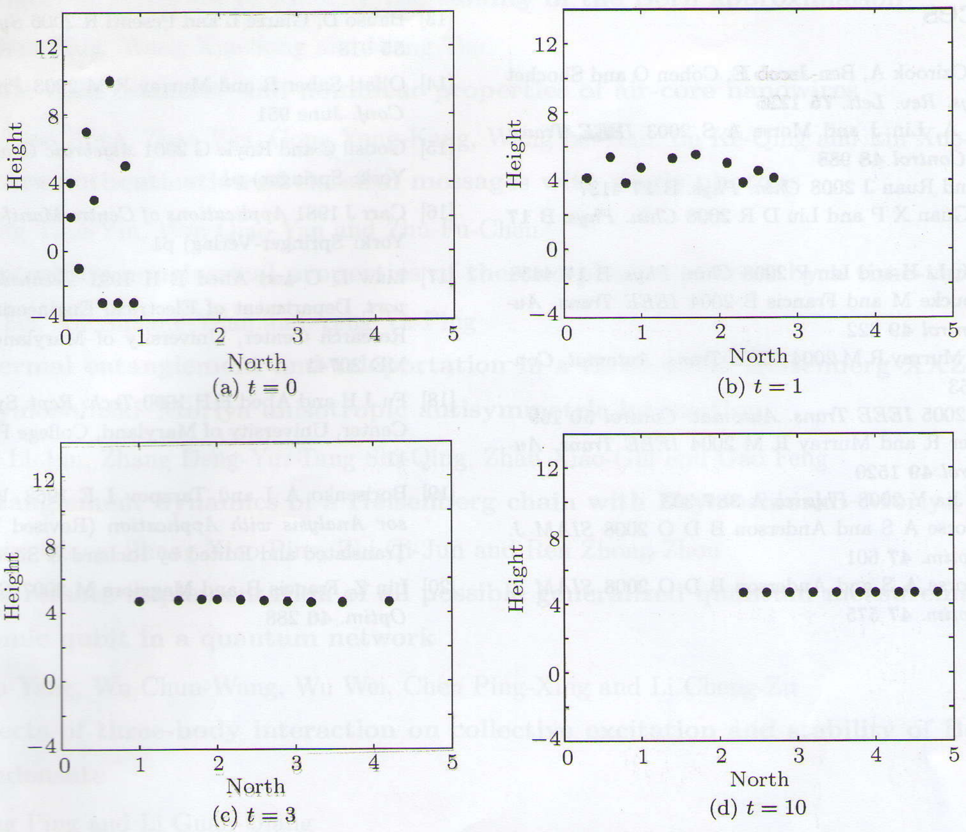


Fig.6. Vertical alignment to graph G_a (as n=1) in the vertical plane.

5. Conclusion

A β-consensus problem for directed networks of multi-agent systems, separately, with fixed topology and switching topology is discussed. Here, the β-consensus function is an arbitrary value that ranges from the minimum value to the maximum value of the initial states of the agents. We design a nonlinear protocol to make the agents reach consensus on group decision value $\beta(x(0))$ of their initial states. This main point of this paper is to apply the centre manifold reduction technique to analysing the stability of nonlinear multi-agent systems. It is different from the situation discussed in the existing literature, where the stability of nonlinear multi-agent systems based on the Lyapunov theory is analysed.[3–10] The idea of centre manifold reduction analysis is to reduce the network dynamic system, which is infinite-dimensional, to a one-dimensional system. The convergence of the original system is completely dependent on the convergence of the reduced system. We have shown that the agents can reach consensus by using a distributed nonlinear protocol, provided that the network communication links between agents are kept strongly connected, where the nonlinear systems are in the so-called "critical cases". Finally, we perform a simulation study concerning the vertical alignment manoeuvere of a team of UAVs. Simulation results show that the nonlinear protocols proposed are more effective than the linear protocols for the formation control of the agents and they are an improvement over existing protocols.

These results can be applied in many other fields including synchronization, flocking, distributed decision making and so on. In addition, in the paper we did not consider the influence of time-delay, which is unavoidable in the communication topology of agents. It deserves to be further studied.

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