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MINIMAL LAGRANGIAN DIFFEOMORPHISMS BETWEEN HYPERBOLIC CONE SURFACES AND ANTI-DE SITTER GEOMETRY

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“Смешно, н правда ли, смешно?
А н спешил - н доспешил.
Осталось недореш н ,
Всё то, что н недорешил.”

Владимир Высоцкий

Abstract

We study minimal diffeomorphisms between hyperbolic cone-surfaces (that is diffeomorphisms whose graph are minimal submanifolds). We prove that, given two hyperbolic metrics with the same number of conical singularities of angles less than π, there always exists a minimal diffeomorphism isotopic to the identity.

When the cone-angles of one metric are strictly smaller than the ones of the other, we prove that this diffeomorphism is unique.

When the angles are the same, we prove that this diffeomorphism is unique and area-preserving (so is minimal Lagrangian). The last result is equivalent to the existence of a unique maximal space-like surface in some Globally Hyperbolic Maximal (GHM) anti-de Sitter (AdS) 3-manifold with particles.

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Introduction

Finding a preferred diffeomorphism between closed Riemann surfaces (Σ, J_1) and (Σ, J_2) in a given isotopy class (for example the one of the identity) is an old problem first solved by O. Teichmüller [Tei40] with the extremal map. The extremal map f is by definition the map minimizing the dilatation coefficient $\sup_{x\in\Sigma}\left|\frac{\overline{\partial}_z f}{\partial_z f}\right|(x)$. This map is defined using the complex structures J_1 and J_2.

On the other hand, one can use the unique hyperbolic metrics g_1 and g_2 associated to J_1 and J_2 respectively in order to define, in a more Riemannian geometric way, a canonical diffeomorphism. One possibility, as introduced by J.J. Eells and J.H. Sampson [ES64], is to find a global minimizer of the L^2-norm of the differential (the so-called harmonic maps). It follows from the global theory that there exists a unique harmonic diffeomorphism isotopic to the identity between (Σ, g_1) and (Σ, g_2). These harmonic maps carry very nice geometric properties and were used by M. Wolf, A.J. Tromba and others to study the geometry of the Teichmüller space (see [Wol89, Tro92]). However, a problem of harmonic maps is the lack of symmetry. Namely, if $u : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ is harmonic, then $u^{-1} : (\Sigma, g_2) \longrightarrow (\Sigma, g_1)$ is not, in general, harmonic.

A diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ is called **minimal** if its graph $\Gamma \subset (\Sigma\times\Sigma, g_1+g_2)$ is a minimal surface (that is, if Γ is area-minimizing). One immediately notes that if Ψ is minimal, then Ψ^{-1} also is. Minimal diffeomorphisms between hyperbolic surfaces have been studied first by R. Schoen [Sch93] (see also [Lab92]). They proved that, given two hyperbolic surfaces (Σ, g_1) and (Σ, g_2), there exists a unique minimal diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ isotopic to the identity and that this Ψ is area-preserving (so its graph is a Lagrangian submanifold of $(\Sigma \times \Sigma, \omega_1 + (-\omega_2))$, where ω_i is the area-form associated to g_i). Such a diffeomorphism is called **minimal Lagrangian**. Later, S. Trapani and G. Valli [TV95] generalized this result by proving that, whenever (Σ, g_1) and (Σ, g_2) are negatively curved surfaces, there exists a unique minimal diffeomorphism Ψ isotopic to the identity so that Ψ preserves the curvature form (that is $\Psi^* K_2\omega_2 = K_1\omega_1$ where K_i is the Gauss curvature of (Σ, g_i)).

Minimal Lagrangian diffeomorphisms between hyperbolic surfaces have also deep connections with anti-de Sitter (AdS) geometry (that is with the geometry of constant curvature -1 Lorentz manifolds), as discovered by K. Krasnov and J.-M. Schlenker [KS07]. In its ground breaking work, G. Mess [Mes07] proved that the moduli space of Globally Hyperbolic Maximal (GHM) AdS structures on $M := \Sigma \times \mathbb{R}$ (see Chapter 1 for precise definitions and statements) is parametrized by two copies of the Fricke space $\mathscr{F}(\Sigma)$ of Σ (that is the space of marked hyperbolic structure on Σ). In [BBZ07] (see also [KS07] for the link with minimal Lagrangian diffeomorphisms), the authors proved that every AdS GHM manifold (M, g) contains a unique space-like area-maximizing surface (a so-called **maximal surface**). This result is actually equivalent to the result of Schoen of the existence of a unique minimal Lagrangian diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ isotopic to the identity where g_1 and g_2 parametrize the AdS GHM metric g (see Proposition 1.4.1 and [AAW00] for case of the hyperbolic disk).

A natural question is whether these results generalize to the case of surfaces with cone singularities. For example, the theory of harmonic maps between cone surfaces has been studied by many people: E. Kuwert [Kuw96] for flat surfaces, E Lamb [Lam12] for hyperbolic surfaces with branched points and by J. Gell-Redman [Gel10] for negatively curved surfaces with cone singularities of angles smaller than 2π.

The goal of this thesis is to extend this picture to the case of manifolds with cone singularities of angles less than π. For $\alpha \in \left(0, \frac{1}{2}\right)$, consider the singular metric obtained by gluing by a rotation an angular sector of angle $2\pi\alpha$ between two half-lines in the hyperbolic disk. This space $(\mathbb{H}^2_\alpha, g_\alpha)$ is called **local model for hyperbolic metric with cone singularity of angle** $2\pi\alpha$.

Let $\Sigma_\mathfrak{p}$ be the surface obtained by removing a finite number of points $\mathfrak{p} := (p_1, ..., p_n)$ on a closed oriented surface Σ. For $\alpha := (\alpha_1, ..., \alpha_n) \in \left(0, \frac{1}{2}\right)^n$, a metric g on $\Sigma_\mathfrak{p}$ is **hyperbolic with cone singularities of angle** $2\pi\alpha$ is g is a smooth metric of constant curvature -1 outside $\mathfrak{p}$ and each $p_i \in \mathfrak{p}$ has a neighborhood isometric to the center of $(\mathbb{H}^2_{\alpha_i}, g_{\alpha_i})$. When $\chi(\Sigma_\mathfrak{p}) + \sum_{i=1}^n(\alpha_i - 1) < 0$, $\Sigma_\mathfrak{p}$ admits hyperbolic metric with cone singularities of angle $2\pi\alpha$ (see Troyanov and McOwen [Tro91, McO88]) and one can construct the Fricke space $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ as the moduli space of marked hyperbolic metrics with cone singularities of angle α (see Chapter 2 for the construction). We prove the following

Main Theorem 1. Given $g_1, g_2 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$, there exists a unique minimal Lagrangian diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ isotopic to the identity.

This theorem was proved in [Tou13]. The proof of this result uses the deep connections with AdS geometry. In [KS07], the authors constructed the so-called "AdS GHM manifolds with particles" which are globally hyperbolic AdS manifolds with conical singularities along time-like curves. The parametrization of Mess extends to the case of AdS manifolds with particles. Namely, in [BS09], the authors constructed a parametrization of the moduli space $\mathscr{A}_\alpha(\Sigma_\mathfrak{p})$ of AdS GHM structures with particles of angle α by $\mathscr{F}_\alpha(\Sigma_\mathfrak{p}) \times \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$. To prove Main Theorem 1, we first prove:

Main Theorem 2. Given an AdS GHM manifold (M, g) with particles of angle $\alpha \in \left(0, \frac{1}{2}\right)^n$, there exists a unique maximal space-like surface $S \hookrightarrow (M, g)$ which is orthogonal to the particles.

To prove the existence part, we consider a sequence of globally hyperbolic space-times $\left((M, g_n)\right)_{n\in\mathbb{N}}$ which converges in some sense to (M, g). Using the geometry of the convex core of (M, g) and general existence results for maximal surfaces in globally hyperbolic spacetimes (see [Ger83]), we prove that each (M, g_n) contains a maximal surface S_n. By elliptic regularity, we show that the sequence $(S_n)_{n\in\mathbb{N}}$ converges to a maximal surface $S \hookrightarrow (M, g)$ which is space-like and orthogonal to the particles. Uniqueness is obtained by a maximum principle.

Finally, we show that this result is equivalent to Main Theorem 1 where g_1 and g_2 parametrize (M, g).

After this, we address the question of existence of a minimal Lagrangian diffeomorphism between hyperbolic surfaces with cone singularities (Σ, g_1) and (Σ, g_2) when the cone-angles of (Σ, g_1) are different from the cone angles of (Σ, g_2). This question has been solved in [Tou14]. Namely, we proved:

Main Theorem 3. For $\alpha, \alpha' \in \left(0, \frac{1}{2}\right)^n$, $g_1 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_\mathfrak{p})$, there exists a minimal diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ isotopic to the identity. Moreover, if $\alpha_i < \alpha'_i$ for all $i = 1, ..., n$, then Ψ is unique.

In this case, there is no longer an interpretation in terms of AdS geometry. So we study the energy functional on $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$. It has been recently proved by J. Gell-Redman [Gel10] that, given a conformal class of metric $\mathfrak{c}$ and a negatively curved metric g with cone singularities of angles less than π on $\Sigma_\mathfrak{p}$, there exists a unique harmonic diffeomorphism $u_{\mathfrak{c},g} : (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, g)$ isotopic to the identity. It follows that, given a hyperbolic metric $g \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$, one can define the energy functional $\mathscr{E}_g : \mathscr{T}(\Sigma_\mathfrak{p}) \longrightarrow \mathbb{R}$ (where $\mathscr{T}(\Sigma_\mathfrak{p})$ is the Teichmüller space of $\Sigma_\mathfrak{p}$, that is the space of marked conformal structures on $\Sigma_\mathfrak{p}$) associating to a conformal structure $\mathfrak{c}$ the energy of the harmonic diffeomorphism $u_{\mathfrak{c},g}$ (in fact, we prove that the energy only depends on the isotopy class of $\mathfrak{c}$). We show that this functional is proper and so, given $g_1 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_\mathfrak{p})$, the functional $\mathscr{E}_{g_1} + \mathscr{E}_{g_2}$ admits a minimum. We prove that such a minimum corresponds to a minimal diffeomorphism and, proving the stability of minimal surface in $(\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 + g_2)$ when the angles of g_1 are strictly smaller that the angles of g_2, we prove the uniqueness part of Main Theorem 3.

Note that, in the case of different angles, the minimal diffeomorphism fails to be Lagrangian.

Outline of the thesis:

In Chapter 1, we recall the classical theory: we define harmonic and minimal maps, explain their relations and give a proof of Schoen's theorem. We also reinterpret this result in terms of Codazzi operators. Then we define AdS GHM manifolds, explain the Mess parametrization and state the result of Barbot, Béguin and Zeghib of existence of a unique maximal surface. Finally, we explain the equivalence between maximal surfaces and minimal Lagrangian diffeomorphisms.

In Chapter 2, we define metrics with conical singularities. We explicitly construct the Fricke space $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ of marked hyperbolic metrics on $\Sigma_\mathfrak{p}$ with cone singularities of angle α. We also define AdS GHM manifolds with particles and explicit the extension of Mess' parametrization.

In Chapter 3, we prove Main Theorem 2. The proof follows [Tou13]. We also show the equivalence between Main Theorem 2 and Main Theorem 1.

In Chapter 4, we define and study the energy functional on $\mathscr{T}(\Sigma_\mathfrak{p})$ and prove Main Theorem 3. The proof follows [Tou14].

Chapter 1

Classical theory

1.1 Harmonic maps and minimal surfaces

1.1.1 Harmonic maps

To a smooth map between compact Riemannian manifolds $u : (M, g) \longrightarrow (N, h)$, one can associate its energy

$$E(u) := \int_M e(u) dv_g,$$

where $e(u) = \frac{1}{2}\|du\|^2$ is the energy density. Here, $\|du\|$ is the norm of $du \in \Gamma(T^*M \otimes u^*TN)$ where the vector bundle $T^*M \otimes u^*TN$ is endowed with the product metric and dv_g is the volume form of (M, g).

Definition 1.1.1. A smooth map $u : (M, g) \longrightarrow (N, h)$ is **harmonic** if it is a critical point of the energy functional.

Remark 1.1.1. When $\dim M = 2$, the energy of u only depends on the conformal class of the metric g. In particular, we can define harmonic maps from a conformal surface to a Riemannian manifold.

The pull-back by u of the Levi-Civita connection ∇^h on (N, h) allows us to define the differential of vector-valued k-forms on M

$$d_\nabla : \Omega^k(M, u^*TN) \longrightarrow \Omega^{k+1}(M, u^*TN)$$

by

$$d_\nabla(\eta \otimes s) = d\eta \otimes s + (-1)^k \eta \wedge u^*\nabla^h s,$$

where $\eta \in \Omega^k(M)$ and $s \in \Omega^0(M, u^*TN)$.

The operator d_∇ admits an adjoint $d_\nabla^* : \Omega^{k+1}(M, u^*TN) \longrightarrow \Omega^k(M, u^*TN)$ defined by the equation

$$\langle d_\nabla \theta, \eta \rangle_{\Omega^{k+1}} = \langle \theta, d_\nabla^* \eta \rangle_{\Omega^k},$$

where $\theta \in \Omega^k(M, u^*TN)$, $\eta \in \Omega^{k+1}(M, u^*TN)$ and $\langle ., . \rangle_{\Omega^k}$ is the L^2-scalar product induced by g and u^*h on $\Omega^k(M, u^*TN)$. In other words,

$$\langle \alpha, \beta \rangle_{\Omega^k} = \int_M (\alpha, \beta)(x) dv_g(x).$$

Proposition 1.1.2. *Let $(u_t)_{t \in I}$ be a family of smooth maps so that $u_0 = u$. Denote by*

$\psi := \frac{d}{dt}_{|t=0} u_t \in \Gamma(u^*TN)$. *We have:*

$$\frac{d}{dt}_{|t=0} E(u_t) = \langle \psi, d_\nabla^* du \rangle_{\Omega^0}.$$

Proof. We have $\frac{d}{dt}_{|t=0} du_t = d_\nabla \psi$. So

$$\begin{aligned} \frac{d}{dt}_{|t=0} E(u_t) &= \frac{d}{dt}_{|t=0} \left(\frac{1}{2} \int_M (du_t, du_t) dv_g \right) \\ &= \langle d_\nabla \psi, du \rangle_{\Omega^1} \\ &= \langle \psi, d_\nabla^* du \rangle_{\Omega^0}. \end{aligned}$$

□

Definition 1.1.3. Given a smooth map $u : (M, g) \longrightarrow (N, h)$, one defines its **tension field** by

$$\tau(u) := d_\nabla^* du \in \Gamma(u^*TN).$$

It follows that the tension field can be thought as the gradient of the energy functional. We have

Proposition 1.1.4. *A smooth map* $u : (M, g) \longrightarrow (N, h)$ *is harmonic if and only if* $\tau(u) = 0$.

We recall the following theorem whose existence is due to J.J. Eells and J.H. Sampson [ES64, Theorem 11.A] and uniqueness to S.I. Al'ber [Al′68] and P. Hartman [Har67]:

Theorem 1.1.5. *(Eells-Sampson, Al'ber, Hartman) If* (N, h) *has non-positive sectional curvature, each isotopy class of map from* (M, g) *to* (N, h) *contains a harmonic map* u *which is unique if* u *does not send* (M, g) *onto a geodesic or a totally geodesic flat subspace.*

1.1.2 Minimal surfaces

The theory of harmonic maps when (M, g) is an oriented surface Σ endowed with a conformal metric $\mathfrak{c}$ has many nice properties. Here we explicit some of them.

Definition 1.1.6. Let $u : (\Sigma, \mathfrak{c}) \longrightarrow (N, h)$ be a smooth map. We define the **Hopf differential** of u by

$$\Phi(u) := u^* h^{(2,0)},$$

that is, the $(2, 0)$-part (with respect to the complex structure $J_\mathfrak{c}$ associated to $\mathfrak{c}$) of the pull-back metric.

We have a result of Eells and Sampson [ES64, Section 9.]:

Proposition 1.1.7. *(Eells-Sampson) If* u *is harmonic, then* $\Phi(u)$ *is a holomorphic quadratic differential. If* $\dim N = 2$ *and the Jacobian of* u *does not vanish, then the converse is also true.*

Write $g = \rho^2(z)|dz|^2$ where z are complex coordinates on (Σ, g). We have the following expression

$$u^* h = \Phi(u) + \rho^2(z) e(f) |dz|^2 + \overline{\Phi(u)}.$$

In particular, the area $A(\Gamma)$ of $u(\Sigma, g)$ is given by

$$\begin{aligned} A(\Gamma) &= \int_\Sigma \det(u^*h)^{1/2}|dz|^2 \\ &= \int_\Sigma \left(e(u)^2 - 4\|\Phi(u)\|^2\right)^{1/2} dv_g. \end{aligned}$$

We easily get

$$A(\Gamma) \leq E(u).$$

Moreover, equality holds if and only if $\Phi(u) = 0$, that is if and only if the conformal class of u^*h is the same as the one of g (that is, u is conformal).

Definition 1.1.8. A minimal surface is an area-minimizing immersion $f : (\Sigma, g) \hookrightarrow (N, h)$.

Minimal surfaces (and more generally minimal submanifolds) have been widely studied in differential geometry. For a surface $\Sigma \hookrightarrow (M, g)$ embedded in a Riemannian manifold (M, g), we denote by $H \in \Gamma((T\Sigma)^\perp)$ its mean curvature field (where $(T\Sigma)^\perp$ is the normal bundle). It is classical (see for example [Spi79]) that given a normal deformation $\Psi := \frac{d}{dt}_{|t=0}\Sigma_t \in \Gamma((T\Sigma)^\perp)$ where $(\Sigma_t)_{t\in I}$ is a family of surfaces so that $\Sigma_0 = \Sigma$, the variation of the area is given by:

$$\frac{d}{dt}_{|t=0} A(\Sigma_t) = \langle H, \Psi \rangle_g.$$

It follows that being a minimal surface is a local property characterized by the vanishing of the mean curvature field. Minimal surfaces are related to harmonic maps, we have (see [ES64, Proposition 4.B]):

Proposition 1.1.9. *(Eells-Sampson) f is a minimal surface if and only if f is harmonic and conformal.*

1.1.3 Energy functional on $\mathscr{T}(\Sigma)$

The general existence Theorem of J.J. Eells and J.H. Sampson have a very nice Corollary which is due to R. Schoen and S.T. Yau [SY78] and independently to J.H. Sampson [Sam78]:

Corollary 1.1.10. *If Σ is a closed oriented surface of genus $g(\Sigma) > 1$, then for each conformal class $\mathfrak{c}$ and hyperbolic metric g on Σ, there exists a unique harmonic diffeomorphism isotopic to the identity*

$$u : (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, g).$$

In particular, given a hyperbolic metric g on Σ, one can define the energy functional $\widetilde{\mathscr{E}}_g$ on the space of conformal structure on Σ by

$$\widetilde{\mathscr{E}}_g(\mathfrak{c}) := E(u_{\mathfrak{c},g}),$$

where $u_{\mathfrak{c},g} : (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, g)$ is the unique harmonic diffeomorphism isotopic to the identity provided by Corollary 1.1.10 and E is the energy. Note that, (see for example [Tro92, Chapter 3]) $\widetilde{\mathscr{E}}_g(\mathfrak{c})$ only depends on the isotopy class of $\mathfrak{c}$ (and g) so descends to a functional

$$\mathscr{E}_g : \mathscr{T}(\Sigma) \longrightarrow \mathbb{R}.$$

We have very important result [Tro92, Theorem 3.1.3 and 3.2.4]:

Theorem 1.1.11. *(Tromba) $\mathscr{E}_g$ is proper and its Weil-Petersson gradient at a point $\mathfrak{c} \in \mathscr{T}(\Sigma)$ is given by $-2\Phi(u_{\mathfrak{c},g})$.*

1.1.4 Some formulae

Here we recall some important formulae for harmonic maps between surfaces. Let $U \subset (\Sigma, g)$ and $V \subset (\Sigma, h)$ be open sets, $(x^1, x^2) : U \longrightarrow \mathbb{R}^2$ and $(v^1, v^2) : V \longrightarrow \mathbb{R}^2$ be local coordinates. For a harmonic diffeomorphism $u : (\Sigma, g) \longrightarrow (\Sigma, h)$ sending U on V, we consider the differential du as a section of the bundle $T^*\Sigma \otimes u^*T\Sigma$. Note that locally, this bundle is generated by the sections $\{dx^i \otimes u^* \frac{\partial}{\partial v^j},\ i, j = 1, 2\}$. Denote by $u^j := v^j(u)$, and let ∂_{u^j} be the vector field dual to du^j (in particular $\partial_{u^j} = u^* \frac{\partial}{\partial v^j}$ is a section of $u^*T\Sigma$). Finally, set $\partial_i = \frac{\partial}{\partial x^i}$. In the (local) framing $\{dx^i \otimes \partial_{u^j},\ i, j = 1, 2\}$, the differential du is given by:

$$du = \sum_{i,j=1}^{2} \partial_i u^j dx^i \otimes \partial_{u^j}.$$

Now, consider $du \in \Gamma(T^*\Sigma \otimes u^*T\Sigma \otimes \mathbb{C})$ and denote the functions $u, \overline{u} : U \longrightarrow \mathbb{C}$ by $u := u^1 + iu^2$ and $\overline{u} := u^1 - iu^2$ (note that these notations are misleading because we identify the diffeomorphism u with its expression in a coordinate system). As usually, set

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), & \overline{\partial}_z = \frac{1}{2}(\partial_1 + i\partial_2) \\ dz = dx^1 + idx^2, & d\overline{z} = dx^1 - idx^2 \\ \partial_u = \frac{1}{2}(\partial_{u^1} - i\partial_{u^2}), & \overline{\partial}_u = \frac{1}{2}(\partial_{u^1} + i\partial_{u^2}). \end{cases}$$

It follows that the bundle $T^*\Sigma \otimes u^*T\Sigma \otimes \mathbb{C}$ is (locally) generated by the sections $\{dz\partial_u,\ d\overline{z}\partial_u,\ dz\overline{\partial}_u,\ d\overline{z}\overline{\partial}_u\}$ (note that we omitted the tensor products). In this framing, the differential du is given by:

$$du = \partial_z u dz \partial_u + \overline{\partial}_z u d\overline{z} \partial_u + \partial_z \overline{u} dz \overline{\partial}_u + \overline{\partial}_z \overline{u} d\overline{z} \overline{\partial}_u.$$

According to the complex structures associated to g and h, the space $\Omega^1(\Sigma, u^*T\Sigma \otimes \mathbb{C})$ of 1-forms on Σ with value in $u^*T\Sigma \otimes \mathbb{C}$ splits into $\mathbb{C}$-linear and $\overline{\mathbb{C}}$-linear ones that we denote by $\Omega^{1,0}(\Sigma, u^*T^{\mathbb{C}}\Sigma)$ and $\Omega^{0,1}(\Sigma, u^*T^{\mathbb{C}}\Sigma)$ respectively. Under this decomposition, set

$$du = \sqrt{2}(\partial u + \overline{\partial} u),$$

where $\partial u \in \Omega^{1,0}(\Sigma, u^*T^{\mathbb{C}}\Sigma)$ and $\overline{\partial} u \in \Omega^{0,1}(\Sigma, u^*T^{\mathbb{C}}\Sigma)$ (we define ∂u and $\overline{\partial} u$ with a coefficient $\sqrt{2}$ to get the well-known formula $\frac{1}{2}\|du\|^2 = e(u) = \|\partial u\|^2 + \|\overline{\partial} u\|^2$). In coordinates, we get the following expression:

$$\begin{cases} \partial u = \frac{1}{\sqrt{2}}(\partial_z u dz \partial_u + \overline{\partial}_z \overline{u} d\overline{z} \overline{\partial}_u) \\ \overline{\partial} u = \frac{1}{\sqrt{2}}(\overline{\partial}_z u d\overline{z} \partial_u + \partial_z \overline{u} dz \overline{\partial}_u). \end{cases}$$

Now, assume that z and u are complex coordinates for g and h respectively, so that we have

$$g = \rho^2(z)|dz|^2,\ h = \sigma^2(u)|du|^2.$$

We have the following expression:

$$\begin{aligned} \Phi(u) &= u^*h(\partial_z, \partial_z)dz^2 \\ &= h(du(\partial_z), du(\partial_z))dz^2 \\ &= \sigma^2(u)\partial_z u \partial_z \overline{u} dz^2. \end{aligned}$$

Moreover, for g^{ij} the coefficient of the metric dual to g,

$$\begin{aligned} e(u) &= \frac{1}{2}\sum_{\alpha,\beta,i,j=1}^{2} g^{ij}h_{\alpha\beta}\partial_i u^\alpha \partial_j u^\beta \\ &= \rho^{-2}(z)\sigma^2(u)\left(|\partial_z u|^2 + |\overline{\partial}_z u|^2\right). \end{aligned}$$

we have the following expressions:

$$\begin{cases} \Phi(u) &= \sigma^2(u)\partial_z u \partial_z \overline{u} dz^2 \\ e(u) &= \rho^{-2}(z)\sigma^2(u)(|\partial_z u|^2 + |\overline{\partial}_z u|^2)|dz|^2 \\ \|\partial u\|^2 &= \rho^{-2}(z)\sigma^2(u)|\partial_z u|^2 \\ \|\overline{\partial} u\|^2 &= \rho^{-2}(z)\sigma^2(u)|\overline{\partial}_z u|^2. \end{cases}$$

In particular, writing $J(u)$ the Jacobian of u, we get the relations:

$$\begin{cases} \|\Phi(u)\| = \|\partial u\|\|\overline{\partial} u\| \\ e(u) = \|\partial u\|^2 + \|\overline{\partial} u\|^2 \\ J(u) = \|\partial u\|^2 - \|\overline{\partial} u\|^2. \end{cases}$$

These functions satisfy a Bochner type identities everywhere it is defined (see [SY78])

$$\begin{cases} \Delta \ln \|\partial u\|^2 = -2K_h J(u) + 2K_g \\ \Delta \ln \|\overline{\partial} u\|^2 = 2K_h J(u) + 2K_g, \end{cases}$$

where Δ is the Laplace-Beltrami operator (with negative spectrum) with respect to g and K_g (respectively K_h) is the scalar curvature of (Σ, g) (respectively (Σ, h)). When g and h are hyperbolic, it gives

$$\begin{cases} \Delta \ln \|\partial u\| = \|\partial u\|^2 - \|\overline{\partial} u\|^2 - 1 \\ \Delta \ln \|\overline{\partial} u\| = -\|\partial u\|^2 + \|\overline{\partial} u\|^2 - 1. \end{cases} \tag{1.1}$$

1.2 Minimal Lagrangian diffeomorphisms

Definition 1.2.1. A map $f : (M, g) \longrightarrow (N, h)$ is called **minimal** if its graph is a minimal submanifold of $(M \times N, g + h)$ (that is if its mean curvature field vanishes everywhere). If moreover M and N are endowed with symplectic forms ω_M and ω_N (respectively) and f is a symplectomorphism (or equivalently if graph(f) is a Lagrangian submanifold of $(M \times N, \omega_M + (-\omega_N))$), then f is called **minimal Lagrangian**.

In the case of surfaces $\Sigma = M = N$, the area form associated to a metric is symplectic. Minimal Lagrangian diffeomorphisms associated to hyperbolic metrics on closed surface have been studied by R. Schoen [Sch93] (see also F. Labourie [Lab92]) and latter and in a more general setting by S. Trapani and G. Valli [TV95]. In this section, we expose their results, explain the proof of R. Schoen and re-interpret this result in terms of Codazzi operators.

1.2.1 Main result

Theorem 1.2.2. *(Schoen) Let $g_1,\ g_2 \in \mathscr{F}(\Sigma)$. There exists a unique minimal diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ which is isotopic to the identity. Moreover, Ψ is area-preserving, hence is minimal Lagrangian.*

Proof of R. Schoen. Let $g_1, g_2 \in \mathscr{F}(\Sigma)$, for $\mathfrak{c} \in \mathscr{T}(\Sigma)$, consider the map

$$\begin{array}{rccl} f_{\mathfrak{c}}: & (\Sigma, \mathfrak{c}) & \longrightarrow & (\Sigma \times \Sigma, g_1 + g_2) \\ & x & \longmapsto & (u_{\mathfrak{c},g_1}(x), u_{\mathfrak{c},g_2}(x)), \end{array}$$

(recall that $u_{\mathfrak{c},g_i}$ is the unique harmonic map isotopic to the identity). We have $E(f_{\mathfrak{c}}) = E(u_{\mathfrak{c},g_1}) + E(u_{\mathfrak{c},g_2})$. As the functional $\mathscr{E}_{g_1} + \mathscr{E}_{g_2}$ is proper, it admits a minimum $\mathfrak{c}_o$. As $\mathfrak{c}_o$ is a critical point, the gradient of the energy of $f_{\mathfrak{c}}$ vanishes, and so $\Phi(f_{\mathfrak{c}_o}) = \Phi(u_1) + \Phi(u_2) = 0$ (where $u_i(\Sigma, \mathfrak{c}_o) \longrightarrow (\Sigma, g_i)$ is harmonic). It follows that $f_{\mathfrak{c}_o}$ is a harmonic conformal immersion, hence $f_{\mathfrak{c}_o}(\Sigma, \mathfrak{c}_o)$ is a minimal surface in $(\Sigma \times \Sigma, g_1 + g_2)$.

Denoting $\pi_i : (\Sigma \times \Sigma, g_1 + g_2) \longrightarrow (\Sigma, g_i)$ the projection on the i-th factor, we get that $u_i = \pi_i \circ f_{\mathfrak{c}_o}$ and $\Psi := u_2 \circ u_1^{-1} : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ is such that graph$(\Psi) = f_{\mathfrak{c}_o}(\Sigma, \mathfrak{c}_o)$. It follows that Ψ is a minimal diffeomorphism isotopic to the identity.

Now, applying Equation (1.1) to $w_i := \ln\left(\frac{\|\partial u_i\|}{\|\overline{\partial} u_i\|}\right)$, we get:

$$\Delta w_i = 2\|\partial u_i\|\|\overline{\partial} u_i\|(e^{w_i} - e^{-w_i}) = 4\|\Phi(u_i)\| \sinh w_i.$$

Note that, as u_i is a diffeomorphism, $J(u_i) > 0$ so $\|\partial u_i\| > \|\overline{\partial} u_i\|$ and so the singularities of w_i corresponds to zeros of $\|\overline{\partial} u_i\|$ (that is to zeros of $\|\Phi(u_i)\|$).

As $\Phi(u_1) + \Phi(u_2) = 0$, $\|\Phi(u_1)\| = \|\Phi(u_2)\| =: \|\Phi\|$ and so w_1 and w_2 have the same singularities. It follows that $w_1 - w_2$ is a regular function satisfying

$$\Delta(w_1 - w_2) = 4\|\Phi\|(\sinh w_1 - \sinh w_2).$$

Applying the maximum principle, we obtain that $w_1 = w_2$. In particular, we get that $\|\partial u_1\| = \|\partial u_2\|$ and $\|\overline{\partial} u_1\| = \|\overline{\partial} u_2\|$. It follows that $J(u_2) = J(u_2)$ and $J(\Psi) = 1$. So Ψ is area-preserving.

Now, we have that $\Gamma :=$ graph(Ψ) is a minimal Lagrangian surface in $(\Sigma \times \Sigma, g_1 + g_2, \omega_1 - \omega_2)$ which is Kähler-Einstein. By a result of M. Micallef and J. Wolfson [MW93], the area of graph(Ψ) is a strict minimum. Since $A(\Gamma) \leq E(f_{\mathfrak{c}})$, the critical points of $\mathscr{E}_{g_1} + \mathscr{E}_{g_2}$ can only be minima, so it is unique. ☐

1.2.2 One-harmonic maps

Given a diffeomorphism $u : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$, one can define another energy:

Definition 1.2.3. Given a diffeomorphism $u : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$, one defines its **one-energy** by

$$E_\partial(u) := \int_\Sigma \|\partial u\| dv_{g_1},$$

where $\frac{1}{\sqrt{2}} du = \partial u + \overline{\partial} u$. Such a diffeomorphism u is called **one-harmonic** if it is a critical point of the one-energy.

This functional has been studied by S. Trapani and G. Valli in [TV95]. In particular, they proved the following:

Theorem 1.2.4. *(Trapani, Valli) Given (Σ, g_1) and (Σ, g_2) two surfaces with negatively curved metrics, there exists a unique one-harmonic diffeomorphism $\phi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ isotopic to the identity.*

They also proved (see [TV95, Lemma 3.3]) that this diffeomorphism has very nice geometric properties: its graph is a minimal surface in $\left(\Sigma \times \Sigma, \sqrt{\frac{K_{g_1}}{K_{g_2}}} g_1 + \sqrt{\frac{K_{g_2}}{K_{g_1}}} g_2\right)$ (where

K_{g_i} is the scalar curvature of g_i), and ϕ preserves the curvature form, that is

$$K_{g_1}\omega_1 = \phi^*(K_{g_2}\omega_2),$$

where ω_i is the area-form of g_i. It means that ϕ is a minimal Lagrangian map

$$\phi : \left(\Sigma, \sqrt{\frac{K_{g_1}}{K_{g_2}}} g_1, K_{g_1}\omega_1\right) \longrightarrow \left(\Sigma, \sqrt{\frac{K_{g_2}}{K_{g_1}}} g_2, K_{g_2}\omega_2\right).$$

Note that in particular, when g_1 and g_2 are hyperbolic, ϕ corresponds to the unique minimal Lagrangian diffeomorphism isotopic to the identity of Theorem 1.2.2.

1.2.3 Interpretation in terms of Codazzi operators

Given a diffeomorphism $f : (M, g) \longrightarrow (N, h)$, there exists a unique self-adjoint operator $b \in \Gamma(\mathrm{End}(TM))$ with positive eigenvalues so that $f^*h(.,.) = g(b.,b.)$.

Definition 1.2.5. Let (M, g) be a Riemannian manifold. A bundle morphism $b \in \Gamma(\mathrm{End}(TM))$ is **Codazzi** if $d_\nabla b = 0$. That is if for each X, Y vector fields on M we have

$$d_\nabla b(X,Y) = (\nabla_X b)(Y) - (\nabla_Y b)(X) = \nabla_X(b(Y)) - \nabla_Y(b(X)) - b([X,Y]) = 0.$$

Codazzi operators provide a way to characterize minimal Lagrangian diffeomorphisms:

Proposition 1.2.6. *Let $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ be a diffeomorphism, then Ψ is minimal Lagrangian (with respect to the area-form) if and only if its associated operator $b \in \Gamma(End(T\Sigma))$ is Codazzi and has determinant one (with respect to g_1).*

Proof. Let (dx^1, dx^2) be an orthonormal framing of $(T^*\Sigma, g_1)$ so that in this framing $b = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$. The area form associated to $\Psi^* g_2 = g_1(b.,b.)$ is given by:

$$\Psi^*\omega_2 = k_1 k_2 dx^1 \wedge dx^2 = (\det_{g_1} b)\omega_1.$$

So Ψ is Lagrangian if and only if $\det_{g_1} b = 1$.

Now, we have

$$\begin{aligned}
\Psi \text{ is minimal} \quad &\Longleftrightarrow \quad \Gamma := \mathrm{graph}(\Psi) \subset (\Sigma \times \Sigma, g_1 + g_2) \text{ is a minimal surface} \\
&\Longleftrightarrow \quad p_j : (\Gamma, i^*(g_1 + g_2)) \longrightarrow (\Sigma, g_j) \text{ is harmonic for } j = 1, 2 \\
&\Longleftrightarrow \quad \Phi(p_j) \text{ is holomorphic for } j = 1, 2 \\
&\Longleftrightarrow \quad \Phi(p_1) \text{ is holomorphic (as } \Phi(p_1) = -\Phi(p_2)) \\
&\Longleftrightarrow \quad \varphi := 2\Re(\Phi(p_1)) \text{ is divergence-free.}
\end{aligned}$$

We have

$$p_1^* g_1 = \lambda(i^*(g_1 + g_2)) + \Phi(p_1) + \overline{\Phi}(p_2)$$

for some $\lambda > 0$. Using an orthonormal framing so that $g_1 = Id$ (the identity) and

$$b = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix},$$

we get

$$Id = \lambda(Id + b) + \varphi.$$

Writing

$$\varphi := \begin{pmatrix} a & c \\ c & -a \end{pmatrix},$$

we get

$$\begin{cases} 1 &=& \lambda(1+k) + a \\ 1 &=& \lambda(1+k^{-1}) - a \\ c &=& 0. \end{cases}$$

That is

$$\begin{cases} c &=& 0 \\ \lambda &=& \dfrac{2k}{(1+k)^2} \\ a &=& \dfrac{1-k}{1+k}, \end{cases}$$

so $\varphi = (E+b)^{-1}(E-b)$ (where we have identified symmetric 2-forms on Γ and sections of $\mathrm{End}(T\Sigma)$). Writing (X_1, X_2) an orthonormal framing of $T\Gamma$ with $[X_1, X_2] = 0$, we get that

$$\begin{aligned} \mathrm{div}_{g_\Gamma}\varphi = 0 &\iff \mathrm{div}_{g_\Gamma}\varphi(X_1) = \mathrm{div}_{g_\Gamma}\varphi(X_2) = 0 \\ &\iff X_1\big((1+k)^{-1}(1-k)\big) = X_2\big((1+k)^{-1}(1-k)\big) = 0 \\ &\iff X_1(k) = X_2(k) = 0 \\ &\iff \nabla_{X_1}(k^{-1}X_2) - \nabla_{X_2}(kX_1) = 0 \\ &\iff d_\nabla(X_1, X_2) = 0. \end{aligned}$$

□

1.3 Anti-de Sitter geometry

In this section we introduce the anti-de Sitter (AdS) geometry, explain the parametrization of Mess and give the main result of T. Barbot, F. Béguin and A. Zeghib of existence of a maximal surface. We also introduce the moduli space of maximal AdS germs and parametrize it. Classical references for this material are [Mes07, BBZ07, KS07, BS09].

1.3.1 The AdS 3-space

Let $\mathbb{R}^{2,2}$ be the usual real 4-space with the quadratic form:

$$q(x) := x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

The anti-de Sitter (AdS) 3-space is defined by:

$$\mathrm{AdS}^3 = \{x \in \mathbb{R}^{2,2} \text{ such that } q(x) = -1\}.$$

With the induced metric, AdS^3 is a Lorentzian symmetric space of dimension 3 of constant curvature -1 diffeomorphic to $\mathbb{D} \times S^1$ (where $\mathbb{D}$ is a disk of dimension 2). In particular, AdS^3 is not simply connected. We will consider two models for the AdS 3-space:

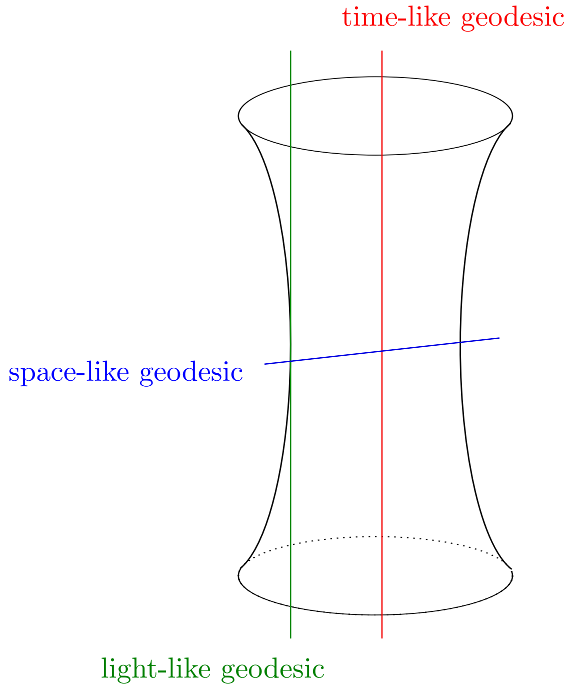

Figure 1.1: Klein model of AdS^3 in an affine chart

The Klein model of AdS^3. Consider the projection

$$\pi : \mathbb{R}^{2,2} \setminus \{0\} \longrightarrow \mathbb{RP}^3.$$

The image of AdS^3 under this projection is called the Klein model of the AdS 3-space. Note that in this model, AdS^3 is not proper (it is not contained in an affine chart). In the affine chart $x_4 \neq 0$ of $\mathbb{RP}^3$, AdS^3 is the interior of the hyperboloid of one sheet given by the equation $\{x_1^2 + x_2^2 - x_3^2 = 1\}$, and this hyperboloid identifies with the boundary $\partial\mathrm{AdS}^3$ of AdS^3 in this chart.

This model is called the Klein model by analogy with the Klein model of the hyperbolic space. In fact, in this model, the geodesics of AdS^3 are given by straight lines: space-like geodesics are the ones which intersect the boundary $\partial\mathrm{AdS}^3$ in two points, time-like geodesics are the ones which do not have any intersection and light-like geodesics are tangent to $\partial\mathrm{AdS}^3$ (see Picture 1.1).

The Lie group model. Consider the group $PSL_2(\mathbb{R})$ endowed with its Killing form. As $PSL_2(\mathbb{R})$ is not compact, its Killing form is not positive definite positive, but has signature $(2, 1)$. With this metric, $PSL_2(\mathbb{R})$ is isometric to AdS^3. Note that in this model, the 1-parameter subgroup associated to rotations correspond to time-like geodesics, the ones associated to hyperbolic transformations correspond to space-like geodesics and the ones associated to parabolic transformations correspond light-like geodesics.

The isometry group. It follows from the definition of AdS^3 that the group $\mathrm{Isom}_0(\mathrm{AdS}^3)$ of space and time-orientation preserving isometries of AdS^3 is the connected component of the identity of Lie group $SO(2, 2)$ of linear transformations of $\mathbb{R}^4$ preserving the signature $(2, 2)$ quadratic form q.

Consider now the Klein model of AdS^3. In an affine chart, $\partial\mathrm{AdS}^3$ is identified with a hyperboloid of one sheet. It is well-known that such a hyperboloid is foliated by two families of straight lines. We call one family the right one and the other, the left one. The group $\mathrm{Isom}_0(\mathrm{AdS}^3)$ preserves each family of the foliation. Fix a space-like plane P_0 in AdS^3, its boundary is a space-like circle in $\partial\mathrm{AdS}^3$ which intersects each line of the right (respectively the left) family exactly once. Then P_0 provides an identification of each family with $\mathbb{RP}^1$ (when changing P_0 to another space-like plane, the identification changes by a conjugation by an element of $PSL_2(\mathbb{R})$). It is proved in [Mes07, Section 7] that each element of $\mathrm{Isom}_0(\mathrm{AdS}^3)$ acts by projective transformations on each $\mathbb{RP}^1$ and so extend to a pair of elements in $PSL_2(\mathbb{R})$. So $\mathrm{Isom}_0(\mathrm{AdS}^3) \cong PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.

Remark 1.3.1. Fixing a space-like plane P_0 also provides an identification between $\partial\mathrm{AdS}^3$ and $\mathbb{RP}^1 \times \mathbb{RP}^1$. Each point $x \in \partial\mathrm{AdS}^3$ is the intersection of two lines: one in the right family, one in the left one. It follows that $x \in \partial\mathrm{AdS}^3$ gives a point in $\mathbb{RP}^1 \times \mathbb{RP}^1$. This application is bijective.

Remark 1.3.2. We have a kind of projective duality in the Klein model of AdS^3: given a point $x \in \mathrm{AdS}^3$, there exists a unique space-like plane $P_x \subset \mathrm{AdS}^3$ so that the intersection of P_x with $\partial\mathrm{AdS}^3$ coincides with the intersection of null-cone at x with $\partial\mathrm{AdS}^3$. It follows that the stabilizer of x is also the stabilizer of P_x. From the description of the isometry group given above, on easily checks that if we denote by x_0 the point dual to P_0, then

$$\mathrm{Stab}(x_0) = \{(g, g) \in \mathrm{Isom}_0(\mathrm{AdS}^3),\ g \in PSL_2(\mathbb{R})\}.$$

As $\mathrm{Isom}_0(\mathrm{AdS}^3)$ acts transitively on AdS^3, we have the following homogeneous space description:

$$\mathrm{AdS}^3 = \big(PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})\big)/PSL_2(\mathbb{R}).$$

In the Lie group model, $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ acts by left and right multiplication on $PSL_2(\mathbb{R})$.

1.3.2 Globally Hyperbolic Maximal AdS 3-manifolds

Definition 1.3.1. An AdS 3-manifold is a manifold M endowed with a (G, X)-structure, where $G = \mathrm{Isom}_0(\mathrm{AdS}^3)$, $X = \mathrm{AdS}^3$. That is, M is endowed with an atlas of charts taking values in AdS^3 so that the transition functions are restriction of elements in $\mathrm{Isom}_0(\mathrm{AdS}^3)$.

In this thesis, we are going to consider a special class of AdS manifolds, namely the Globally Hyperbolic Maximal ones:

Definition 1.3.2. An AdS 3-manifold M is Globally Hyperbolic Maximal (GHM) if it satisfies the following two conditions:

1. **Global Hyperbolicity:** M contains a space-like Cauchy surface, that is a surface which intersects every inextensible time-like curve exactly once.

2. **Maximality:** M cannot be strictly embedded in an AdS manifold satisfying the same properties.

Note that the Global Hyperbolicity condition implies strong restrictions on the topology of M. In particular, M has to be homeomorphic to $\Sigma \times \mathbb{R}$ where $\dim(\Sigma) = 2$ and Σ is homeomorphic to the Cauchy surface. Note that, when Σ is closed, oriented and connected, its genus has to be strictly positive.

Remark 1.3.3. In the original paper [Mes07], Mess claimed that the genus of a closed Cauchy surface in an AdS GHM space-time had to be strictly bigger than one. It is explained in [ABB$^+$07] that this statement was false: there exists AdS GHM space-times whose Cauchy surface is a torus, the so-called Torus Universe, see [BBZ07, Car98].

1.3.3 Mess' parametrization

Let Σ be a closed oriented surface of genus $g > 1$. We denote by $\mathscr{A}(\Sigma)$ the space of AdS GHM structures on $\Sigma \times \mathbb{R}$ considered up to isotopy.

We have a fundamental result due to G. Mess [Mes07, Proposition 20]:

Theorem 1.3.3 (Mess)**.** *There is a parametrization* $\mathfrak{M} : \mathscr{A}(\Sigma) \longrightarrow \mathscr{F}(\Sigma) \times \mathscr{F}(\Sigma)$.

Construction of the parametrization. To an AdS GHM structure on M is associated its holonomy representation $\rho : \pi_1(M) \to \mathrm{Isom}_0(\mathrm{AdS}^3)$ (well defined up to conjugation). As $\mathrm{Isom}_0(\mathrm{AdS}^3) \cong PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ and as $\pi_1(M) = \pi_1(\Sigma)$, one can split the representation ρ into two morphisms

$$\rho_1, \rho_2 : \pi_1(\Sigma) \to PSL_2(\mathbb{R}).$$

G. Mess proved [Mes07, Proposition 19] that these holonomies have maximal Euler class e (that is $|e(\rho_l)| = |e(\rho_r)| = 2g - 2$). Using Goldman's criterion [Gol88], he proved that these morphisms are Fuchsian holonomies and so define a pair of points in $\mathscr{F}(\Sigma)$.

Reciprocally, as two Fuchsian holonomies ρ_1, ρ_2 are conjugated by an orientation preserving homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ and as $\partial\mathrm{AdS}^3$ identifies with $\mathbb{RP}^1 \times \mathbb{RP}^1$ (fixing a totally geodesic space-like plane P_0, see Remark 1.3.1), one can see the graph of ϕ as a closed curve in $\partial\mathrm{AdS}^3$. G. Mess proved that this curve is nowhere time-like and is contained in an affine chart. In particular, one can construct the convex hull $K(\phi)$ of the graph of ϕ. The holonomy $(\rho_1, \rho_2) : \pi_1(\Sigma) \to \mathrm{Isom}_0(\mathrm{AdS}^3)$ acts properly discontinuously on $K(\phi)$ and the quotient is a piece of globally hyperbolic AdS manifold (see Figure 1.2). It follows from a Theorem of Y. Choquet-Bruhat and R. Geroch [CBG69] that this piece of AdS globally hyperbolic manifold uniquely embeds in a maximal one. So the map $\mathfrak{M}$ is one-to-one.

□

K. Krasnov and J.-M. Schlenker [KS07, Section 3] reinterpreted this parametrization in terms of space-like surfaces embedded in an AdS GHM manifold.

We can associate to a space-like surface $S \hookrightarrow (M, g)$ embedded in an AdS GHM manifold (M, g) some natural objects:

- Its first fundamental form $\mathrm{I} \in \Gamma(S^2T^*S)$ corresponding to the induced metric on S.
- Its shape operator $B : TS \longrightarrow TS$ defined by $B(u) = -\nabla_u N$ where ∇ is the Levi-Civita connection on S and N is the unit future pointing normal vector field along S. Note that B is a self-adjoint operator satisfying the Codazzi equation: for all u, v vector fields on S, we have

  $$d_\nabla B(u, v) = R(v, u)N,$$

  where $R(v, u)N = \nabla_v \nabla_u N - \nabla_u \nabla_v N + \nabla_{[v,u]} N$ is the Riemann curvature tensor.
- A complex structure $J \in \Gamma(\mathrm{End}(TS))$ associated to the induced metric.

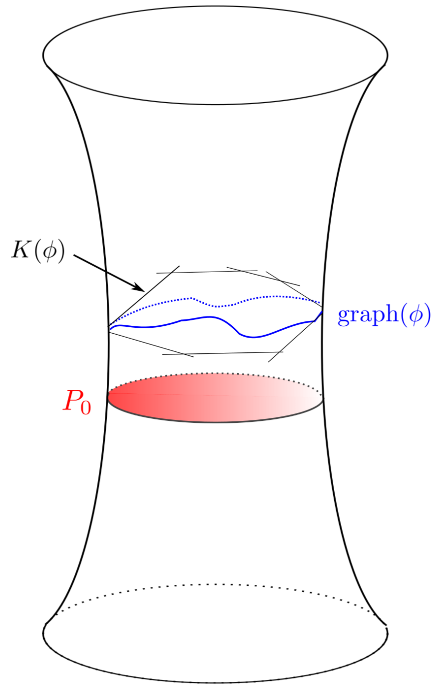

Figure 1.2: Convex hull of graph(ϕ)

- The second fundamental form $\mathrm{II} \in \Gamma(S^2T^*S)$ defined by $\mathrm{II}(.,.) = \mathrm{I}(B.,.)$.
- The third fundamental form $\mathrm{III} \in \Gamma(S^2T^*S)$ defined by $\mathrm{III}(.,.) = \mathrm{I}(B.,B.)$.

We have the following (cf. [KS07, Lemma 3.16]):

Proposition 1.3.4 (K. Krasnov, J.-M. Schlenker)**.** *Let $S \hookrightarrow (M, g)$ be an embedded space-like surface whose principal curvatures are in $(-1, 1)$. If $E \in \Gamma(End(TS))$ is the identity morphism, then we have the following expression for the Mess parametrization:*

$$\mathfrak{M}(M) = (g_1, g_2),$$

where $g_{1,2}(x, y) = I((E \pm JB)x, (E \pm JB)y)$.

Remark 1.3.4. In particular, they proved that the metrics g_1 and g_2 are hyperbolic and hat their isotopy class do not depend on the choice of S.

1.3.4 Maximal surfaces and maximal AdS germs

In Lorentzian geometry, there is no minimal space-like surface. Nevertheless, it makes sense to maximize the area, and maximal surfaces are characterized (as minimal surface in Riemannian geometry) by the vanishing of their mean curvature field. We have a fundamental result [KS07, BBZ07, Theorem 3.17]

Theorem 1.3.5 (Barbot, Béguin, Zeghib and Krasnov, Schlenker)**.** *Every AdS GHM 3-manifold contains a unique maximal space-like surface. Moreover, its principal curvatures are in $(-1, 1)$.*

In the spirit of C.H. Taubes [Tau04], (see also [KS07]) one can define an interesting moduli space related to maximal surfaces in AdS manifolds:

Definition 1.3.6. The moduli space $\mathscr{H}(\Sigma)$ of **maximal AdS germs** is the space of pairs (h, m) where h is a metric on Σ and m is a symmetric 2-tensor on Σ so that:

i. $\mathrm{tr}_h m = 0$ (traceless condition),

ii. $d_\nabla m = 0$ (Codazzi's equation),

iii. $K_h = -1 - \det_h(m)$ where K_h is the Gauss curvature of (Σ, h) (modified Gauss' equation).

Recall that (see for instance [Spi79]), given a metric h on Σ and a symmetric 2-tensor m satisfying Gauss-Codazzi equation, one can find a unique (up to isometry) surface embedded in $\mathbb{R}^3$ whose first and second fundamental form correspond to h and m respectively. This is the so-called "Fundamental Theorem of surfaces in $\mathbb{R}^3$".

The same is true for space-like surfaces embedded in globally hyperbolic AdS manifolds: given a pair (h, m) on Σ where h is a metric and m a symmetric 2-tensor satisfying the Codazzi equation and the modified Gauss equation, there exists a unique space-like surface $S \hookrightarrow (M, g)$ embedded in a AdS GHM manifold (M, g) so that the first and second fundamental form on S corresponds to h and m respectively.

Note that, the condition $\mathrm{tr}_h m = 0$ implies that the corresponding surface $S \hookrightarrow (M, g)$ is maximal (so it justifies the name for $\mathscr{H}(\Sigma)$). It follows that we get a natural map

$$\mathscr{H}(\Sigma) \longrightarrow \mathscr{A}(\Sigma),$$

associating to a maximal AdS germ the AdS GHM structure g so that $S \hookrightarrow (M, g)$. From Theorem 1.3.5, this map is one-to-one.

We also have a natural parametrization of $\mathscr{H}(\Sigma)$.

Proposition 1.3.7. *(Krasnov, Schlenker) The moduli space $\mathscr{H}(\Sigma)$ of maximal AdS germs on Σ is naturally parametrized by $T^*\mathscr{T}(\Sigma)$.*

Proof. Let $(h, m) \in \mathscr{H}(\Sigma)$. It is classical (see for instance [Hop51] or [Tro92, Section 2.4]) that a symmetric 2-tensor m on (Σ, h) is traceless if and only if it is the real part of a quadratic differential on (Σ, J_h) where J_h is the complex structure naturally associated to h. Moreover, m is Codazzi if and only the quadratic differential is holomorphic.

It is proved in [KS07, Lemma 3.6] that given a symmetric traceless Codazzi tensor η on a surface (Σ, g) (for some metric g), there exists a unique metric g' in the conformal class of g so that (g', η) satisfies the modified Gauss equation. It follows that $(g', \eta) \in \mathscr{H}(\Sigma)$ and $\mathscr{H}(\Sigma)$ canonically identifies with the space of pairs $(\mathfrak{c}, q)$ where $\mathfrak{c}$ is a conformal structure on Σ and q is a holomorphic quadratic differential (with respect to the complex structure associated to $\mathfrak{c}$). In other words, $(\mathfrak{c}, q) \in T^*\mathscr{T}(\Sigma)$. □

Remark 1.3.5. Mess' parametrization and the natural parametrization of $\mathscr{H}(\Sigma)$ provides a bijection

$$\varphi : \mathscr{F}(\Sigma) \times \mathscr{F}(\Sigma) \longrightarrow T^*\mathscr{T}(\Sigma),$$

associating to a pair of hyperbolic metrics (g_1, g_2) the first and second fundamental form of the unique maximal surface $S \hookrightarrow (M, g)$ where $g \in \mathscr{A}(\Sigma)$ is parametrized by (g_1, g_2). We will see in Proposition 1.4.2 that this map has a very nice geometric interpretation.

1.4 Relations between minimal Lagrangian and AdS geometry

We have the fundamental result which provides deep connections between AdS geometry and hyperbolic surfaces:

Proposition 1.4.1. *The following are equivalent:*

1 *There exists a unique minimal Lagrangian diffeomorphism isotopic to the identity* $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$

2 *There exists a unique maximal surface in the AdS GHM space-time* (M, g) *where* $g \in \mathscr{A}(\Sigma)$ *is parametrized by* (g_1, g_2).

Proof.

1 ⇒ 2 Let $b : TS \longrightarrow TS$ be the self-adjoint Codazzi operator with respect to g_1 associated to the minimal Lagrangian diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ by Proposition 1.2.6. Define a metric h on Σ by

$$h := \frac{1}{4} g_1((E + b).,(E + b).).$$

For J the complex structure associated to h, define the bundle morphism $B \in \Gamma(\mathrm{End}(TS))$ by

$$B := -J(E + b)^{-1}(E - b).$$

A computation shows (see [KS07, Theorem 3.17]) that B is a traceless self-adjoint operator (with respect to h) satisfying the Codazzi and modified Gauss equations. It follows that the pair $(h, h(B.,.))$ defines a maximal AdS germ corresponding to a maximal surface embedded in (M, g), where (M, g) is parametrized by $h((E \pm JB).,(E \pm JB).)$.

Moreover, the metrics $h((E \pm JB).,(E \pm JB).)$ are equal to g_1 and g_2, so S is a maximal surface in (M, g). Uniqueness comes from uniqueness of b.

2 ⇒ 1 Let $S \hookrightarrow (M, g)$ be a maximal space-like surface. As proved in [KS07, Lemma 3.11], S has principal curvatures in $(-1, 1)$. So

$$\begin{cases} g_1 &= \mathrm{I}((E + JB).,(E + JB).) \\ g_2 &= \mathrm{I}((E - JB).,(E - JB).). \end{cases}$$

Define $b := (E + JB)^{-1}(E - JB) \in \Gamma(\mathrm{End}(TS))$. One gets that $g_2 = g_1(b.,b.)$ and that b is a self-adjoint Codazzi tensor with positive eigenvalues and determinant one. So b provides a minimal Lagrangian map $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$. Uniqueness comes from uniqueness of the maximal surface. See Section 3.3 for more details.

□

Remark 1.4.1. This theorem gives a very nice picture in terms of equivariant conformal harmonic maps into symmetric spaces. Given a pair of Fuchsian representations $\rho_1, \rho_2 : \pi_1(\Sigma) \longrightarrow PSL_2(\mathbb{R})$, one can construct

$$\rho_1 \times \rho_2 : \pi_1(\Sigma) \times \pi_1(\Sigma) \longrightarrow PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}).$$

As such a representation is reductive, the Theorem of Corlette-Donaldson [Cor88, Don87] implies that, for each conformal structure $\mathfrak{c}$ on Σ, there exists a unique $(\rho_1\times\rho_2)$-equivariant harmonic map

$$u : (\widetilde{\Sigma}, \mathfrak{c}) \longrightarrow (PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}))/(SO(2) \times SO(2)) = \mathbb{H}^2 \times \mathbb{H}^2.$$

The Theorem of Schoen implies that there exists a unique conformal structure $\mathfrak{c}_o$ on Σ so that u is harmonic and conformal.

The equivalence between minimal Lagrangian diffeomorphism and maximal surface in AdS GHM implies that for the same conformal structure $\mathfrak{c}_o$, there is a unique conformal harmonic ρ-equivariant map

$$v : (\widetilde{\Sigma}, \mathfrak{c}_o) \longrightarrow (PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) = \mathrm{AdS}^3,$$

where $\rho = (\rho_1, \rho_2)$ (in fact, the notion of harmonic maps extends to Lorentz manifolds and immersion of maximal space-like surfaces are harmonic and conformal).

One can ask whether its picture holds in general. For example, let G be a real split semi-simple Lie group. In [Hit92], N. Hitchin described a connected component of the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ which is homeomorphic to a ball of dimension $\dim G(2g-2)$. These representations, now called Hitchin representations, have been widely studied by F. Labourie, O. Guichard, A. Wienhard... In particular, one can associate to such a representation ρ an energy functional on $\mathscr{T}(\Sigma)$ by associating to a conformal structure the energy of the unique ρ-equivariant harmonic map from $(\widetilde{\Sigma}, \mathfrak{c}) \longrightarrow G/K$ where $K \subset G$ is a maximal compact subgroup. It is proved in [Lab08] that this energy functional is proper. It implies in particular that, given two Hitchin representations ρ_1 and ρ_2, there exists a conformal structure $\mathfrak{c}_o$ (not necessarily unique) so that the $(\rho_1 \times \rho_2)$-equivariant harmonic map

$$u : (\widetilde{\Sigma}, \mathfrak{c}_o) \longrightarrow (G \times G)/K \times K$$

corresponds to a immersed minimal surface in $\rho_1(\pi_1\Sigma)\backslash G/K \times \rho_2(\pi_1\Sigma)\backslash G/K$. It is thus natural to wonder if one can associate to this a ρ-equivariant conformal harmonic map

$$v : (\widetilde{\Sigma}, \mathfrak{c}_o) \longrightarrow (G \times G)/G,$$

where $(G \times G)/G$ is a semi-Riemannian symmetric space isometric to G with its Killing form.

The existence of a unique maximal surface in AdS GHM manifolds also provides a very nice geometric interpretation of minimal Lagrangian diffeomorphisms.

Proposition 1.4.2. *Let $\varphi : \mathscr{F}(\Sigma) \times \mathscr{F}(\Sigma) \longrightarrow T^*\mathscr{T}(\Sigma)$ be the homeomorphism described in Remark 1.3.5. For each $g_1, g_2 \in \mathscr{F}(\Sigma)$, $\varphi(g_1, g_2) = (\mathfrak{c}, q)$ where $\mathfrak{c}$ is the conformal class of the induced metric on Γ, the graph of the unique minimal Lagrangian diffeomorphism $\Psi : (\Sigma, g_1) \longrightarrow (\Sigma, g_2)$ and $q = i\Phi(u_1)$ where $\Phi(u_1)$ is the Hopf differential of the unique harmonic map $u_1 : (\Gamma, \mathfrak{c}) \longrightarrow (\Sigma, g_1)$.*

Proof. See Proposition 3.3.4 for the precise proof in a more general context. □

We can summarize theses connections in the following picture:
where

(1) $\Psi^* g_2 = g_1(b., b.)$

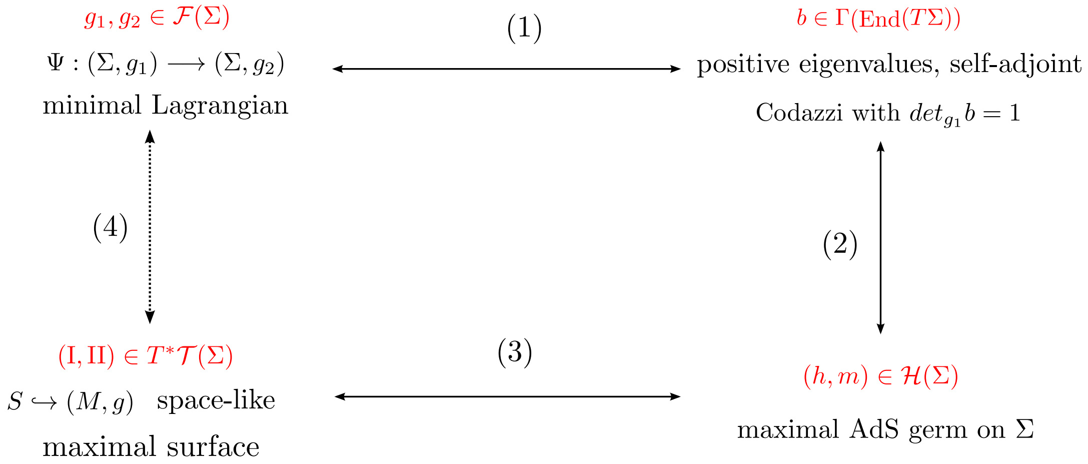

Figure 1.3: Global picture

(2) $\begin{cases} h = \frac{1}{4} g_1((E+b).,(E+b).), & m = h(B.,.) \text{ with } B = -J_h(E+b)^{-1}(E-b) \\ b = (E + J_h B)^{-1}(E - J_h B), & \text{where } m = h(B.,.). \end{cases}$

(3) Fundamental Theorem of surface theory in AdS GHM manifolds.

(4) Proposition 1.4.1.

Chapter 2

Manifolds with cone singularities

2.1 Fricke space with cone singularities

2.1.1 Hyperbolic disk with cone singularity

Let $\alpha \in (0,1)$ and $\mathbb{H}^2 := (\mathbb{D}^2, g_p)$ be the unit disk equipped with the Poincaré metric. Cut $\mathbb{D}^2$ along two half-lines making an angle $2\pi\alpha$ intersecting at the center 0 of $\mathbb{D}^2$ and define $\mathbb{H}^2_\alpha$ as the space obtained by gluing the boundary of the angular sector of angle $2\pi\alpha$ by a rotation fixing 0. Topologically, $\mathbb{H}^2_\alpha = \mathbb{D}^2 \setminus \{0\}$ and the induced metric g_α (which is not complete) is hyperbolic outside 0 and carries a conical singularity of angle $2\pi\alpha$ at 0. We call $\mathbb{H}^2_\alpha = (\mathbb{D}^*, g_\alpha)$ the **hyperbolic disk with cone singularity of angle** $2\pi\alpha$.

In conformal coordinates, we have the well-known expression:

$$g_p = \frac{4}{(1-|\widetilde{z}|^2)^2}|d\widetilde{z}|^2.$$

Using the coordinates $\widetilde{z} = \frac{1}{\alpha}z^\alpha$, we obtain:

$$g_\alpha = \frac{4|z|^{2(\alpha-1)}}{(1-\alpha^{-2}|z|^{2\alpha})^2}|dz|^2.$$

In cylindrical coordinates $(\rho, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}/2\pi\alpha\mathbb{Z}$, we have:

$$g_\alpha = d\rho^2 + \sinh^2 \rho d\theta^2.$$

2.1.2 Hyperbolic surfaces with cone singularities

Here we define the moduli space of hyperbolic metrics with cone singularities. Before that, we need to introduce weighted Hölder spaces adapted to the study of metrics with conical singularities and to the existence of harmonic maps (see [Gel10, Section 2.2]). The regularity of the metric that we impose here is exactly the one we need to use the existence result for harmonic maps (see [Gel10]):

Definition 2.1.1. For $R > 0$, let $D(R) := \{z \in \mathbb{C},\ |z| \in (0,R)\}$. We say that a function $f : D(R) \longrightarrow \mathbb{C}$ is in $\chi_b^{0,\gamma}(D(R))$ with $\gamma \in (0,1)$ if, writing $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$,

$$\|f\|_{\chi_b^{0,\gamma}} := \sup_{D(R)} |f| + \sup_{z,z' \in D(R)} \frac{|f(z) - f(z')|}{|\theta - \theta'|^\gamma + \frac{|r-r'|^\gamma}{|r+r'|^\gamma}} < +\infty.$$

We say that $f \in \chi_b^{k,\gamma}(D(R))$ if, for all linear differential operator L of order k, $L(f) \in \chi_b^{0,\gamma}(D(R))$ (note that in particular, $f \in \mathscr{C}^k(D(R))$).

From now and so on, all the cone angles will be considered strictly smaller than π.

Let Σ be a closed oriented surface, $\mathfrak{p} = (p_1, ..., p_n) \subset \Sigma$ be a set of points. Denote by $\Sigma_\mathfrak{p} := \Sigma \setminus \mathfrak{p}$ and let $\alpha := (\alpha_1, ..., \alpha_n) \in \left(0, \frac{1}{2}\right)^n$ be such that $\chi(\Sigma_\mathfrak{p}) - \sum_{i=1}^n (\alpha_i - 1) < 0$ (this condition implies the existence of hyperbolic metric with cone singularities).

Definition 2.1.2. A hyperbolic metric on $\Sigma_\mathfrak{p}$ with cone singularities of angle $2\pi\alpha$ is a metric g so that

- For each compact $K \subset \Sigma_\mathfrak{p}$, $g_{|K}$ is $\mathscr{C}^2$ and has constant curvature -1,
- for each puncture $p_i \in \mathfrak{p}$, there exists a neighborhood U with local conformal coordinates z centered at p_i together with a local diffeomorphism $\psi \in \chi_b^{2,\gamma}(U)$ (see Definition 2.1.1) so that
$$g_{|U} = \psi^* g_{\alpha_i}.$$

We denote by $\mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})$ the space of such metrics.

Remark 2.1.1. In the general case, one says that a metric g on $\Sigma_\mathfrak{p}$ has a conical singularity of angle $2\pi\alpha$ at $p \in \mathfrak{p}$ if in a neighborhood of p, g has the form

$$g = e^{2u}|z|^{2(\alpha-1)}|dz|^2,$$

where u a bounded $\mathscr{C}^2$ function which extends to a $\mathscr{C}^0$ function at p (see [Tro91]).

Definition 2.1.3. Let $\mathscr{D}_0(\Sigma_\mathfrak{p})$ be the space of diffeomorphisms ψ of $\Sigma_\mathfrak{p}$ isotopic to the identity (in the isotopy class fixing each $p_i \in \mathfrak{p}$) so that, for each compact $K \subset \Sigma_\mathfrak{p}$, $\psi_{|K}$ is of class $\mathscr{C}^3$ and, for each marked point $p_i \in \mathfrak{p}$, there exists an open neighborhood U so that $\psi \in \chi_b^{2,\gamma}(U)$ in some complex coordinates system.

Note that, $\mathscr{D}_0(\Sigma_\mathfrak{p})$ acts by pull-back on $\mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})$ and the quotient space $\mathscr{F}_\alpha(\Sigma_\mathfrak{p}) := \mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})/\mathscr{D}_0(\Sigma_\mathfrak{p})$ is a smooth manifold called the **Fricke space with cone singularities of angles** $2\pi\alpha$.

Proposition 2.1.4. *For a fixed $\alpha \in \left(0, \frac{1}{2}\right)^n$ and all $i \in \{1, ..., n\}$, there exists $r_i > 0$ such that for each hyperbolic metric with cone singularities $g \in \mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})$ the open set $V_i := \{x \in \Sigma_\mathfrak{p},\ d_g(x, p_i) < r_i\}$ is isometric to a neighborhood of 0 in $\mathbb{H}^2_{\alpha_i}$ (here $d_g(.,.)$ is the distance w.r.t. g).*

Proof. The result follows from the fact that the distance between two conical singularities of angles less than π on a hyperbolic surface is bounded from below.

Let p_1 and p_2 two conical singularities of angles $2\pi\alpha_1 < \pi$ and $2\pi\alpha_2 < \pi$ respectively on a hyperbolic cone surface. Let β be an embedded geodesic segment joining p_1 and p_2, and denote by γ the unique geodesic in a regular neighborhood of β homotopic to a simple closed curve around p_1 and p_2. Finally, denote by δ_i the geodesic arc from p_i making an angle $\pi\alpha_i$ with β $(i = 1, 2)$.

We claim that, as $2\pi\alpha_1$ and $2\pi\alpha_2$ are (strictly) smaller than π, the distance between β and γ is uniformly bounded from below by a strictly positive constant. In fact, take a regular neighborhood U of β, and cut it along β, δ_1 and δ_2. We get two connected components V and W, each containing β, δ_1 and δ_2 in their boundary. By a hyperbolic isometry, send V to the upper half-plane model of $\mathbb{H}^2$, sending β on the imaginary axis. Denote by N the unit (for the Euclidian metric) vector field orthogonal to β pointing to the interior of V. Note that N is a Jacobi field. For $\epsilon > 0$ small enough, the length of the geodesic arc $\beta_\epsilon := \exp(\epsilon N) \cap V$ is strictly smaller than the length of β (see Figure 2.1). It implies that if γ is too close to β (or even coincide), then a local deformation of γ along the vector field N would strictly decreases its length. So the distance between γ and β is strictly positive.

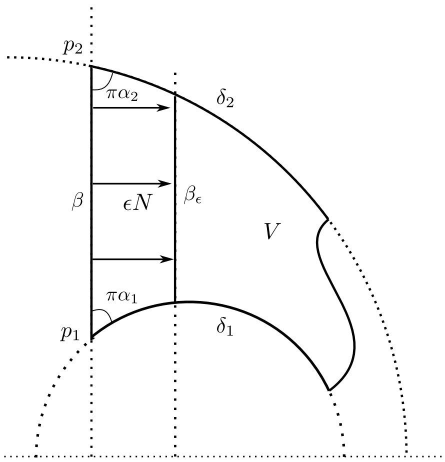

Figure 2.1: The geodesic β_ϵ

Now, consider the connected component S of $\Sigma \setminus \gamma$ containing p_1 and p_2, and cut it along β, δ_1 and δ_2. The remaining surfaces are two isometric hyperbolic quadrilaterals (see Figure 2.2). When the length of γ tends to zero, each quadrilateral tends to a hyperbolic triangle of angles $\pi\alpha_1$, $\pi\alpha_2$ and 0. In such a triangle, the length on β satisfies

$$\cosh(l(\beta)) = \frac{1 + \cos(\pi\alpha_1)\cos(\pi\alpha_2)}{\sin(\pi\alpha_1)\sin(\pi\alpha_2)}.$$

It corresponds to the lower bound for the distance between two hyperbolic cone singularities of angles $2\pi\alpha_1$ and $2\pi\alpha_2$.

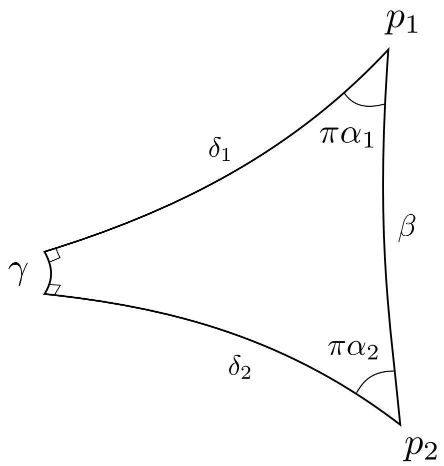

Figure 2.2: Hyperbolic quadrilateral

Applying this result to the universal covering of $\Sigma_{\mathfrak{p}}$, we get a lower bound for the injectivity radius of the singular points on a hyperbolic cone surface. $\square$

From now and so on, we fix a cylindrical coordinates system $(\rho_i, \theta_i) : V_i \to \mathbb{H}^2_{\alpha_i}$ centered at p_i for each $i \in \{1, ..., n\}$ (where the V_i are as in Proposition 2.1.4). Note that Proposition 2.1.4 implies that, up to a gauge, we can always assume that for each $i \in \{1, ..., n\}$, every metric $g \in \mathscr{M}^{\alpha}_{-1}(\Sigma_{\mathfrak{p}})$ has the following expression:

$$g_{|U_i} = d\rho_i^2 + \sinh^2 \rho_i d\theta_i^2.$$

We get the following Corollary:

Corollary 2.1.5. *Let $g_0 \in \mathscr{M}^{\alpha}_{-1}(\Sigma_{\mathfrak{p}})$ and let $\widetilde{h} := \frac{d}{dt}_{|t=0} g_t$ be an infinitesimal deformation of g_0. There exists a vector field $v \in Lie(\mathscr{D}_0(\Sigma_{\mathfrak{p}}))$ (the Lie algebra of $\mathscr{D}_0(\Sigma_{\mathfrak{p}})$) and a $\mathscr{C}^2$ symmetric 2-tensor h so that*

$$\widetilde{h} = h + \mathscr{L}_v g_0, \text{ and } h_{|V_i} = 0 \quad \forall i \in \{1, ..., n\}.$$

Here $\mathscr{L}_v g_0$ is the Lie derivative of g in the direction v and the V_i are defined as in Proposition 2.1.4. We call such a h a ***normalized deformation****.*

Analysis on hyperbolic cone manifolds. Let $(\Sigma_{\mathfrak{p}}, g)$ be a hyperbolic surface with cone singularities of angle $2\pi\alpha$ for $\alpha \in \left(0, \frac{1}{2}\right)^n$. It is not obvious that classical results of geometric analysis on Riemannian manifolds (as integration by parts) extend to hyperbolic cone surfaces. In this section, we study differential operators on vector bundles over $(\Sigma_{\mathfrak{p}}, g)$ in the framework of unbounded operators. For the convenience of the reader, we recall here basic facts about unbounded operators between Hilbert spaces. A good reference for the subject is [Sch12].

Unbounded operators. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces with scalar product $\langle ., . \rangle_1$ and $\langle ., . \rangle_2$ respectively.

Definition 2.1.6. An *unbounded operator* is a linear map

$$T : \mathscr{D}(T) \subset \mathscr{H}_1 \longrightarrow \mathscr{H}_2$$

where $\mathscr{D}(T)$ is a linear subset of $\mathscr{H}_1$ called the domain of T.

Example. Let $I \subset \mathbb{R}$ be an interval and D an order $n \in \mathbb{N}$ linear differential operator. We see $D : \mathscr{C}^\infty_0(I) \subset L^2(I) \longrightarrow L^2(I)$ as an unbounded operator (here $\mathscr{C}^\infty_0(I)$ is the space of $\mathscr{C}^\infty$ real valued functions over I with compact support).

Of course, one notes that in this example, $\mathscr{C}^\infty_0$ is probably not the biggest set (with respect to the inclusion) where D can be defined. This motivates the following definitions:

Definition 2.1.7. Let T_1 and T_2 two unbounded operators from $\mathscr{H}_1$ to $\mathscr{H}_2$. We say that T_1 **extends** T_2 (and we denote by $T_2 \subset T_1$) if $\mathscr{D}(T_2) \subset \mathscr{D}(T_1)$ and $T_{1|_{\mathscr{D}(T_2)}} = T_2$.

We have the important notion of closed and closable operators:

Definition 2.1.8. An unbounded operator T is **closed** if its graph $\mathscr{G}(T)$ is closed in $\mathscr{H}_1 \oplus \mathscr{H}_2$. T is called **closable** if the closure of $\mathscr{G}(T)$ in $\mathscr{H}_1 \oplus \mathscr{H}_2$ is the graph of an unbounded operator $\overline{T}$. In this case, $\overline{T}$ is called the closure of T.

We have the following characterization (cf. [Sch12, Proposition 1.5]):

Proposition 2.1.9. *T is closable if and only if, for each sequence $(x_n)_{n\in\mathbb{N}} \subset \mathscr{D}(T)$ such that* $\lim_{n\to\infty} x_n = 0$ *and $(Tx_n)_{n\in\mathbb{N}}$ converges to $y \in \mathscr{H}_2$ we have $y = 0$.*

Remark 2.1.2. If T is continuous, $\lim_{n\to\infty} x_n = 0$ implies $\lim_{n\to\infty} Tx_n = 0 \in \mathscr{H}_2$, and so T is closable by Proposition 2.1.9. For T being closable, we just require that if $(Tx_n)_{n\in\mathbb{N}}$ converges in $\mathscr{H}_2$, then it converges to the "good" limit. Hence the closability condition can be thought as a weakening of continuity.

Using the scalar products of $\mathscr{H}_1$ and $\mathscr{H}_2$, we can define the adjoint of an unbounded operator with dense domain:

Definition 2.1.10. Let $T : \mathscr{D}(T) \subset \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be an unbounded operator such that $\mathscr{D}(T)$ is dense in $\mathscr{H}_1$. We define the **adjoint** of T as the unbounded operator $T^* : \mathscr{D}(T^*) \subset \mathscr{H}_2 \longrightarrow \mathscr{H}_1$ where:

$$\mathscr{D}(T^*) := \{y \in \mathscr{H}_2,\ \text{there exists } u \in \mathscr{H}_1 \text{ such that } \langle Tx, y\rangle_2 = \langle x, u\rangle_1,\ \forall x \in \mathscr{D}(T)\}.$$

As $\mathscr{D}(T)$ is dense, u is uniquely defined and we set $T^*y := u$.

Determining the domain of an adjoint operator is generally difficult. Hence we have the notion of a formal adjoint:

Definition 2.1.11. Let T be an unbounded operator with dense domain. We say that an operator $T^t : \mathscr{D}(T^t) \subset \mathscr{H}_2 \longrightarrow \mathscr{H}_1$ is a **formal adjoint** of T is for all $x \in \mathscr{D}(T),\ y \in \mathscr{D}(T^t)$ we have $\langle Tx, y\rangle_2 = \langle x, T^t y\rangle_1$.

Remark 2.1.3. Note that, by Riesz' theorem, $y \in \mathscr{D}(T^*)$ if and only if the application $x \longmapsto \langle Tx, y\rangle$ is continuous on $\mathscr{D}(T)$. In particular, for every formal adjoint T^t of T, we have $\mathscr{D}(T^t) \subset \mathscr{D}(T^*)$ and by density $T^*_{|\mathscr{D}(T^t)} = T^t$. So T^* extends every formal adjoint of T.

We have the following classical properties (see e.g. [Sch12, Chapter 1]):

Proposition 2.1.12. *Let S and T be two unbounded operators from $\mathscr{H}_1$ to $\mathscr{H}_2$ with dense domain. Then:*

i. T^* is closed.

ii. If $T \subset S$ then $S^* \subset T^*$.

iii. $\mathscr{D}(T^*)$ is dense if and only if T is closable. In this case, $\overline{T} = T^{**}$.

iv. $\Im(T) = Ker(T^*)^\perp$ (where $\Im$ and Ker design the image and the kernel respectively).

Application to geometric analysis on cone surfaces. Let E, F be two vector bundles over a hyperbolic cone surface $(\Sigma_{\mathfrak{p}}, g)$ (recall that the cone angles are supposed strictly smaller than π), and equip E and F with Riemannian metrics $(.,.)_E$ and $(.,.)_F$ respectively. For $k \in \mathbb{N}$, denote by $\mathscr{C}^k_0(E)$ (respectively $\mathscr{C}^k(E)$ and $L^2(E)$) the space of sections of E which are $\mathscr{C}^k$ with compact support (respectively $\mathscr{C}^k$ and L^2). The Riemannian metric on E turns $L^2(E)$ into a Hilbert space with respect to the following scalar product:

$$\langle f, g\rangle_E := \int_{\Sigma_{\mathfrak{p}}} (f, g)_E vol_g.$$

Note that $\mathscr{C}^\infty_0(E) \subset L^2(E)$ is a dense subset.

Notations. *Denote by* $T^{(r,s)}\Sigma_{\mathfrak{p}}$ *the bundle of* (r,s)*-tensors (that is* r*-covariant and* s*-contravariant) over* $\Sigma_{\mathfrak{p}}$ *and by* $S^k\Sigma_{\mathfrak{p}} \subset T^{(k,0)}\Sigma_{\mathfrak{p}}$ *the bundle of* k*-symmetric tensors. The metric* g *on* $\Sigma_{\mathfrak{p}}$ *induces a metric on these bundles, also denoted by* g*.*

We need some results of integration by parts in cone manifolds. Some good references for this theory are [Che80],[Mon05b, Part 3] and [Mon05a].

Operators on covariant tensors. We denote by $\mathring{\nabla}$ the covariant derivative associated to g. We see $\mathring{\nabla}$ as an unbounded operator:

$$\mathring{\nabla} : \mathscr{D}(\nabla) := \mathscr{C}_0^1\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right) \subset L^2\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right) \longrightarrow L^2\left(T^{(r+1,0)}\Sigma_{\mathfrak{p}}\right).$$

Stokes formula for compactly supported tensors implies that $\mathring{\nabla}$ admits a formal adjoint

$$\nabla^t : \mathscr{D}(\nabla^t) = \mathscr{C}_0^1\left(T^{(r+1,0)}\Sigma_{\mathfrak{p}}\right) \subset L^2\left(T^{(r+1,0)}\Sigma_{\mathfrak{p}}\right) \longrightarrow L^2\left(T^{(r,0)}\Sigma_{\mathfrak{p}}\right),$$

where

$$\nabla^t\eta(X_1,...,X_r) = -\sum_{i=1}^{2}(\nabla_{e_i}\eta)(e_i,X_1,...,X_r),$$

for (e_1,e_2) an orthonormal framing of $T\Sigma_{\mathfrak{p}}$.

As $\mathscr{C}_0^\infty\left(T^{(r+1,0)}\Sigma_{\mathfrak{p}}\right) \subset \mathscr{D}(\nabla^t)$ and $\nabla^t \subset \nabla^*$ (here ∇^* is the adjoint of $\mathring{\nabla}$), then $\mathring{\nabla}$ is closable (by Proposition 2.1.12). Denote by ∇ its closure (so $\nabla = \nabla^{**}$). The restrictions of the operators ∇ and ∇^* to smooth sections are described above.

Operators on symmetric tensors. For $k > 0$, we define the divergence operator $\mathring{\delta}$ by

$$\mathring{\delta} := \nabla^*_{|\mathscr{C}_0^1(S^k\Sigma_{\mathfrak{p}})}.$$

Again, Stokes formula for compactly supported symmetric tensors implies that $\mathring{\delta}$ admits a formal adjoint,

$$\delta^t : \mathscr{C}_0^1(S^{k-1}\Sigma_{\mathfrak{p}}) \subset L^2(S^{k-1}\Sigma_{\mathfrak{p}}) \longrightarrow L^2(S^k\Sigma_{\mathfrak{p}})$$

which is the composition of the covariant derivative with the symmetrization.

It follows that δ^* (the adjoint of $\mathring{\delta}$) has dense domain, and so $\mathring{\delta}$ is closable. We denote by δ its closure.

Notations. *By analogy with classical Sobolev spaces, we introduce the following notations:*

- $H^1(S^1\Sigma_{\mathfrak{p}}) := \mathscr{D}(\delta^*) \subset L^2(S^1\Sigma_{\mathfrak{p}})$,
- $H^1(S^2\Sigma_{\mathfrak{p}}) := \mathscr{D}(\delta) \subset L^2(S^2\Sigma_{\mathfrak{p}})$,
- $H^2(S^1\Sigma_{\mathfrak{p}}) := \mathscr{D}(\delta \circ \delta^*) \subset L^2(S^1\Sigma_{\mathfrak{p}})$,
- $H^1(\Sigma_{\mathfrak{p}}) = \mathscr{D}(\delta^*) \subset L^2(\Sigma_{\mathfrak{p}})$ *(the space of* L^2 *functions over* $\Sigma_{\mathfrak{p}}$*)*,
- $H^2(\Sigma_{\mathfrak{p}}) = \mathscr{D}(\delta \circ \delta^*) \subset L^2(\Sigma_{\mathfrak{p}})$.

We have a result of integration by parts for symmetric tensors on $(\Sigma_{\mathfrak{p}}, g)$. The proof is analogous to the proof of [Mon05b, Theorem 1.4.3], however, as it is a central result in what follows, we include it.

Theorem 2.1.13. *For all* $u \in H^1(S^1\Sigma_{\mathfrak{p}}) \cap \mathscr{C}^1(S^1\Sigma_{\mathfrak{p}})$ *and* $h \in H^1(S^2\Sigma_{\mathfrak{p}}) \cap \mathscr{C}^1(S^2\Sigma_{\mathfrak{p}})$*, we have:*

$$\langle \delta^* u, h\rangle_{S^2} = \langle u, \delta h\rangle_{S^1}.$$

For all $f \in \mathscr{C}^1(\Sigma_{\mathfrak{p}}) \cap H^1(\Sigma_{\mathfrak{p}})$ *and* $\alpha \in \mathscr{C}^1(S^1\Sigma_{\mathfrak{p}}) \cap H^1(S^1\Sigma_{\mathfrak{p}})$*,*

$$\langle \delta^* f, \alpha\rangle_{S^1} = \langle f, \delta\alpha\rangle_{L^2(\Sigma_{\mathfrak{p}})}.$$

Proof. The proof of the two statements are analogous, so we just prove the first one (which is a little bit more technical).

Let's prove the result when $(\Sigma_{\mathfrak{p}}, g)$ contains a unique cone singularity p of angle $2\pi\alpha$. To prove the result in the general case, we just apply the following computation to each puncture.

Fix cylindrical coordinates $(\rho, \theta) \in (0, r) \times \mathbb{R}/2\pi\alpha\mathbb{Z}$ in a neighborhood of p so that

$$g_{|V} = d\rho^2 + \sinh^2 \rho d\theta^2.$$

For $t \in (0, r)$, denote by $U_t := \{(\rho, \theta) \in V, \rho < t\}$.

For $u \in H^1(S^1\Sigma_{\mathfrak{p}}) \cap \mathscr{C}^1(S^1\Sigma_{\mathfrak{p}})$ and $h \in H^1(S^2\Sigma_{\mathfrak{p}}) \cap \mathscr{C}^1(S^2\Sigma_{\mathfrak{p}})$, we have:

$$\int_{\Sigma\setminus U_t} (g(u, \delta h) - g(\delta^* u, h))dv_g = \int_{\Sigma\setminus U_t} \left(g(u, \nabla^* h) - \frac{1}{2}\left(g(\nabla u, h) + g(F \circ \nabla u, h)\right)\right) dv_g,$$

where $F : T^{(2,0)}\Sigma_{\mathfrak{p}} \longrightarrow T^{(2,0)}\Sigma_{\mathfrak{p}}$ is defined by $F\eta(x, y) := \eta(y, x)$. Note that, for $\theta, \eta \in L^2\left(T^{(2,0)}\Sigma_{\mathfrak{p}}\right)$,

$$\langle F\theta, \eta\rangle_{T^{(2,0)}} = \langle \theta, F\eta\rangle_{T^{(2,0)}}.$$

As h is symmetric, and applying Stokes formula, we get:

$$\int_{\Sigma\setminus U_t} (g(u, \delta h) - g(\delta^* u, h))dv_g = \int_{\Sigma\setminus U_t} (g(u, \nabla^* h) - g(\nabla u, h))dv_g = \int_{\partial U_t} g_{|\partial U_t}(u, i_{e_\rho} h)dv_g,$$

where $i_{e_\rho} h = h(e_\rho, .)$ and $e_\rho = \partial_\rho$ is the unit vector field normal to ∂U_t.

As t tends to 0, the left hand side tends to $\langle u, \delta h\rangle_{S^1} - \langle \delta^* u, h\rangle_{S^2}$. Denote by I_t the right hand side. By the Cauchy-Schwarz inequality,

$$|I_t| \leq \int_{\partial U_t} |u||i_{e_\rho} h| dv_g \leq \left(\int_{\partial U_t} |u|^2 dv_g\right)^{1/2} \left(\int_{\partial U_t} |i_{e_\rho} h|^2 dv_g\right)^{1/2}.$$

When $u \neq 0$, $|u|$ is differentiable and $d|u|(x) = g\left(\nabla_x u, \frac{u}{|u|}\right)$, so we set

$$\partial_\rho |u| = g\left(\nabla_{e_\rho} u, \frac{u}{|u|}\right);$$

and if $u = 0$, set $\partial_\rho |u| = 0$. Note that $\partial_\rho |u|$ is the partial derivative of $|u|$ is the sense of distributions. In fact, for all $t, a \in (0, r)$ and θ fixed, we have

$$|u(t, \theta)| - |u(a, \theta)| = \int_a^t \partial_\rho |u(\rho, \theta)| d\rho.$$

In particular, as $|\partial_\rho|u|| \leq |\nabla_{e_\rho} u|$,

$$|u(t,\theta)| \leq |u(a,\theta)| + \int_t^a |\nabla_{e_\rho} u| d\rho.$$

So

$$|u(t,\theta)|^2 \leq 2|u(a,\theta)|^2 + 2\left(\int_t^a |\nabla_{e_\rho} u| d\rho\right)^2.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left(\int_t^a |\nabla_{e_\rho} u| d\rho\right)^2 &\leq \int_t^a \frac{d\rho}{\rho} \int_t^a \rho |\nabla_{e_\rho} u|^2 d\rho \\ &\leq \left|\ln\left(\frac{t}{a}\right)\right| \int_t^a \rho |\nabla_{e_\rho} u|^2 d\rho.\end{aligned}$$

Finally, we get

$$\begin{aligned}\int_{\partial U_t} |u|^2 dv_g &\leq 2\int_{\partial U_t} |u(a)|^2 dv_g + \int_{\partial U_t} \left(2\left|\ln(\frac{t}{a})\right| \int_t^a \rho |\nabla_{e_\rho} u|^2 d\rho\right) dv_g \\ &\leq 2t\int_{\theta=0}^{2\pi\alpha} |u(a,\theta)|^2 d\theta + 2\left|\ln(\frac{t}{a})\right| \int_{\partial U_t} \left(\int_t^a \rho |\nabla_{e_\rho} u|^2 d\rho\right) dv_g \\ &\leq 2t\int_{\theta=0}^{2\pi\alpha} |u(a,\theta)|^2 d\theta + 2t\left|\ln(\frac{t}{a})\right| \int_{\theta=0}^{2\pi\alpha} \int_t^a |\nabla_{e_\rho} u|^2 \rho d\rho d\theta \\ &\leq 2t\int_{\theta=0}^{2\pi\alpha} |u(a,\theta)|^2 \, d\theta + 2t\left|\ln(\frac{t}{a})\right| \int_{U_a} |\nabla_{e_\rho} u|^2 dv_g \\ &= O(t\ln t).\end{aligned}$$

Now, as $h \in L^2(S^2\Sigma_{\mathfrak{p}})$,

$$\int_0^a \left(\int_{\partial U_t} |i_{e_\rho} h|^2 dv_g\right) \leq \int_0^a \left(\int_{\partial U_t} |h|^2 dv_g\right) = \int_{U_a} |h|^2 dv_g < +\infty,$$

that is, the function $t \longmapsto \int_{\partial U_t} |h|^2$ is integrable on $(0,a)$. As the function $(t\ln t)^{-1}$ is not integrable in 0, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ such that

$$\int_{\partial U_{t_n}} |h|^2 dv_g = o((t_n \ln t_n)^{-1}).$$

It follows that $\lim_{n\to\infty} I_{t_n} = 0$. □

We have a very useful corollary:

Corollary 2.1.14. *For $i = 1,2$, the operator $\delta\delta^* : H^2(S^i\Sigma_{\mathfrak{p}}) \longrightarrow L^2(S^i\Sigma_{\mathfrak{p}})$ is self-adjoint with strictly positive spectrum.*

Proof. The fact that $\delta\delta^*$ are self-adjoint follows directly from Theorem 2.1.13. Let $\lambda \geq 0$ such that, for $f \in H^2(S^i\Sigma_{\mathfrak{p}})$ $(i = 1,2)$,

$$\delta\delta^* f + \lambda f = 0.$$

Taking the scalar product with f, and using Proposition 2.1.13, we get:

$$\langle \delta\delta^* f + \lambda f, f\rangle_{S^i} = \|\delta^* f\|^2_{S^{i+1}} + \lambda \|f\|^2_{S^i} = 0,$$

and so $f = 0$. □

2.1.3 Tangent space to $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$

Here we prove the following result:

Proposition 2.1.15. *For $[g_0] \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$, there is a natural identification of $T_{[g_0]}\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ with the space of meromorphic quadratic differentials on $\Sigma = \Sigma_\mathfrak{p} \cup \mathfrak{p}$ with at most simple poles at the $p_i \in \mathfrak{p}$ (where the complex structure on $(\Sigma_\mathfrak{p}, g_0)$ is the one associated to g_0).*

Proof. Fix $g_0 \in \mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})$ and let

$$\widetilde{h} = \frac{d}{dt}_{|t=0} g_t \in T_{g_0}\mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p}),$$

where $(g_t)_{t\in I}$ is a smooth path in $\mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})$ with $g_{t=0} = g_0$ (and $0 \in I \subset \mathbb{R}$ is an interval). Note that, by Corollary 2.1.5, there exists a vector field $v \in \mathrm{Lie}(\mathscr{D}_0(\Sigma_\mathfrak{p}))$ (the Lie algebra of $\mathscr{D}_0(\Sigma_\mathfrak{p})$), such that

$$\widetilde{h} = h + \mathscr{L}_v g,\ h_{|V_i} = 0\ \forall i \in \{1, ..., n\},$$

where $\mathscr{L}_v g$ is the Lie derivative of g in the direction v and the V_i are defined as in Proposition 2.1.4. We call such a h a **normalized deformation** (note that in particular, $h \in \mathscr{C}_0^2(S^2\Sigma_\mathfrak{p})$).

Such a symmetric 2-tensor h on $\Sigma_\mathfrak{p}$ is tangent to the space $\mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})$ of hyperbolic metrics with cone singularities if and only if the differential of the sectional curvature dK_{g_0} in the direction h is equal to 0.

First, we have a canonical orthogonal splitting:

Lemma 2.1.16. *For all normalized deformation $h \in T_{g_0}\mathscr{M}_{-1}^\alpha(\Sigma_\mathfrak{p})$, there exists $u \in H^2(S^1\Sigma_\mathfrak{p})$ and $h_0 \in H^1(S^2\Sigma_\mathfrak{p})$ with $\delta h_0 = 0$ such that:*

$$h = h_0 + \mathscr{L}_{u^\sharp} g_0,$$

where $u^\sharp$ is the vector field dual to u. Moreover, this splitting is orthogonal with respect to the scalar product of $L^2(S^2\Sigma_\mathfrak{p})$.

Proof. As $h \in \mathscr{C}_0^2(S^2\Sigma_\mathfrak{p})$, $\delta h \in \mathscr{C}_0^1(S^1\Sigma_\mathfrak{p}) \subset L^2(S^1\Sigma_\mathfrak{p})$. So we want to find $u \in H^2(S^1\Sigma_\mathfrak{p})$ so that

$$2\delta\delta^* u = \delta h. \tag{2.1}$$

It is possible to solve (2.1) if and only if $\delta h \in \Im(\delta\delta^*)$ (where $\Im$ stands for the image).

By Corollary 2.1.14, $\delta\delta^*$ is self-adjoint, so $\Im(\delta\delta^*) = \mathrm{Ker}(\delta\delta^*)^\perp$ (cf. Proposition 2.1.12). Hence we can solve (2.1) if and only if δh is orthogonal to the kernel of $\delta\delta^*$.

Take $w \in \mathrm{Ker}(\delta\delta^*) \subset H^2(S^1\Sigma_\mathfrak{p})$. By elliptic regularity, such a w is smooth. So, by Theorem 2.1.13, we get:

$$\langle \delta\delta^* w, w\rangle_{S^1} = 0 = \langle \delta^* w, \delta^* w\rangle_{S^2}.$$

In particular, $\delta^* w = 0$, and we obtain:

$$\langle \delta h, w\rangle_{S^1} = \langle h, \delta^* w\rangle_{S^2} = 0.$$

So $\delta h \in \Im(\delta\delta^*)$ and we can solve (2.1).

Now, such a solution u is smooth (at least $\mathscr{C}^4$), so we know the expression of $\delta^* u$. We have:

$$\delta^* u(x,y) = \frac{1}{2}((\nabla_x u)(y) + (\nabla_y u)(x)) = \frac{1}{2}\left(g_0(\nabla_x u^\sharp, y) + g_0(x, \nabla_y u^\sharp)\right),$$

which is the expression of $\frac{1}{2}\mathscr{L}_{u^\sharp} g_0$. In particular, setting $h_0 := h - \frac{1}{2}\delta^* u$, we get the decomposition.

Note that, if u_1 and u_2 are two solutions of (2.1), they satisfy

$$\delta\delta^*(u_1 - u_2) = 0.$$

By integration by parts, we get that $\delta^* u_1 = \delta^* u_2$. In particular, $\mathscr{L}_{u_1^\sharp} g_0 = \mathscr{L}_{u_2^\sharp} g_0$, so the decomposition is independant on the choice of the solution of (2.1).

Now we prove the orthogonal splitting. Let u and h_0 as above. As such sections are smooths, we have:

$$\langle \mathscr{L}_{u^\sharp} g_0, h\rangle_{S^2} = 2\langle \delta^* u, h_0\rangle_{S^2} = \langle u, \delta h_0\rangle = 0.$$

□

We explicit now the condition $dK_{g_0}(\widetilde{h}) = 0$. We have the well-know formula (e.g. [Tro92, Formula 1.5 p.33]):

$$dK_{g_0}(\widetilde{h}) = \delta\delta^*_{g_0}(\mathrm{tr}_{g_0}\widetilde{h}) + \delta\delta\widetilde{h} + \frac{1}{2}\mathrm{tr}_{g_0}\widetilde{h},$$

where tr_{g_0} is the trace with respect to the metric g_0.

Applying this formula to the divergence-free part h_0 (which is transverse to the fiber of the projection), we get

$$\delta\delta^*_{g_0}(\mathrm{tr}_{g_0} h_0) + \frac{1}{2}\mathrm{tr}_{g_0} h_0 = 0.$$

By Corollary 2.1.14, we get $tr_{g_0} h_0 = 0$. Moreover, one easily checks that each $h \in H^2(S^2\Sigma_\mathfrak{p}) \cap \mathscr{C}^2(S^2\Sigma_\mathfrak{p})$ such that $\delta h = 0$ and $tr_{g_0} h = 0$ defines a tangent vector to $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ at $[g_0]$. So, we get the following identification

$$T_{[g_0]}\mathscr{F}_\alpha(\Sigma_\mathfrak{p}) = \left\{h \in H^2(S^2\Sigma_\mathfrak{p}) \cap \mathscr{C}^2(S^2\Sigma_\mathfrak{p}),\ \delta h = 0 \text{ and } \mathrm{tr}_{g_0} h = 0\right\}.$$

But we can go further. For (dx, dy) an orthonormal framing of $T^*\Sigma_\mathfrak{p}$, write

$$h_0 = u(x,y)dx^2 - v(x,y)(dxdy + dydx) + w(x,y)dy^2.$$

The condition $tr_{g_0} h = 0$ implies $w(x,y) = -u(x,y)$. Write (∂_x, ∂_y) the framing dual to (dx, dy). Let us explicit the divergence-free condition:

$$\begin{aligned} 0 &= \delta h(\partial_x) \\ &= -(\nabla_{\partial_x} h)(\partial_x, \partial_x) - (\nabla_{\partial_y} h)(\partial_y, \partial_y) \\ &= -\partial_x u + \partial_y w. \end{aligned}$$

In the same way, we get:

$$0 = \delta h(\partial_y) = \partial_x v + \partial_y u.$$

These are the Cauchy-Riemann equations. It implies in particular that $f = u + iv$ is holomorphic on $\Sigma_\mathfrak{p}$.

Now, for $z = x + iy$, $dz = dx + idy$, set $\psi = f(z)dz^2$. It is a holomorphic quadratic differential on $\Sigma_\mathfrak{p}$ such that $h = \Re(\psi)$. It follows that ψ is meromorphic on Σ with possible poles at the $p_i \in \mathfrak{p}$.

We claim that, as $h = \Re(\psi) \in L^2\left(S^2\Sigma_\mathfrak{p}\right)$, the poles of ψ at the p_i are at most simples. In fact, let $p \in \mathfrak{p}$ be a cone singularity of angle $2\pi\alpha$, z be a local holomorphic coordinates around p and

$$\psi(z) = \left(\frac{a}{z^n} + g(z)\right) dz^2$$

for $a \in \mathbb{C}^*$, $n \geq 0$ and g meromorphic so that $z^n g(z) \underset{z\to 0}{\longrightarrow} 0$.

It follows from Proposition 2.1.4 that around p, each lifting $g_0 \in \mathscr{M}^\alpha_{-1}(\Sigma_\mathfrak{p})$ of $[g_0] \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ is isometric to the expression g_α given in section 2.1.1. In particular,

$$\psi\overline{\psi} = \left(O(|z|^{-2n}\right)|dz|^4,$$

so

$$g_0(\psi, \overline{\psi})(z) = O\left(|z|^{2(2-2\alpha-n)}\right).$$

It follows,

$$g_0(\psi, \overline{\psi})dv_{g_0} = O\left(|z|^{2(1-\alpha-n)}\right)|dz|^2.$$

As $\alpha \in \left(0, \frac{1}{2}\right)$, $g_0(\psi, \overline{\psi})dv_{g_0}$ is integrable in 0 is and only if $n \leq 1$, and the same is true for h.

On the other hand, given a meromorphic quadratic differential ψ with at most simple poles at the p_i, its real part $h = \Re(\psi)$ is a zero trace divergence-free symmetric $(2, 0)$ tensor in $L^2(S^2\Sigma_\mathfrak{p})$. Hence, as it is smooth on $\Sigma_\mathfrak{p}$, $h \in T_{[g_0]}\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$. □

A Weil-Petersson metric on $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$. Let $h, k \in T_{[g_0]}\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$. Fix a lifting $g_0 \in \mathscr{M}^\alpha_{-1}(\Sigma_\mathfrak{p})$ of $[g_0]$. It follows from the above construction that there exists a unique lifting $\widetilde{h}, \widetilde{k} \in T_{g_0}\mathscr{M}^\alpha_{-1}(\Sigma_\mathfrak{p})$ of h and k respectively which are divergence-free symmetric tensors of zero trace. We call such a lifting a **horizontal lifting**. Define:

$$\frac{1}{8}\langle h, k\rangle_{WP_\alpha} := \langle \widetilde{h}, \widetilde{k}\rangle_{S^2}.$$

Obviously, $\langle ., .\rangle_{WP_\alpha}$ is a metric on $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$. This metric is analogous to the metric defined in the non-singular case by A.E. Fischer and A.G. Tromba (see [FT84]). They proved in [FT84, Theorem (0.8)] that, up to a constant, this metric coincides with the Weil-Petersson metric, so we call it **Weil-Petersson metric with cone singularities of angles $2\pi\alpha$**.

Remark 2.1.4. In [ST11], the authors proved that $\langle ., .\rangle_{WP_\alpha}$ is a Kähler metric. It seems possible, by using the renormalized volume of quasi-Fuchsian manifolds with particles to prove that $\langle ., .\rangle_{WP_\alpha}$ admits a Kähler potential (see [KS08, KS12]).

Uniformization. Here, we recall a fundamental result proved by R.C. McOwen [McO88] and independently M. Troyanov [Tro91]. Let $\mathscr{T}(\Sigma)$ be the Teichmüller space of $\Sigma_\mathfrak{p}$, that is the moduli space of marked conformal structures on $\Sigma_\mathfrak{p}$. We have

Theorem 2.1.17. *(McOwen, Troyanov) Given $\mathfrak{c} \in \mathscr{T}(\Sigma_\mathfrak{p})$, there exists a unique $h \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ in the conformal class $\mathfrak{c}$ as long as $\chi(\Sigma) + \sum_{i=1}^n(\alpha_i - 1) < 0$ (where $\Sigma = \Sigma_\mathfrak{p} \cup \mathfrak{p}$).*

This theorem provides a family of identification $\Theta_\alpha : \mathscr{T}(\Sigma_\mathfrak{p}) \longrightarrow \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ for each $\alpha \in \mathbb{R}^n_{>0}$ such that $\chi(\Sigma_\mathfrak{p}) + \sum_{i=1}^n(\alpha_i - 1) < 0$. In particular, one can define a family

$(\Theta_\alpha^*\langle.,.\rangle_{WP_\alpha})_{\alpha\in(0,\frac{1}{2})^n}$ of Weil-Petersson metric on $\mathscr{T}(\Sigma)$.

2.2 AdS convex GHM 3-manifolds with particles

In this section, we introduce and study AdS convex GHM manifolds with particles. These manifolds have been defined and studied in [KS07] and [BS09]. We recall their results.

2.2.1 The moduli space $\mathscr{A}_\alpha(\Sigma_\mathfrak{p})$

First, we define the singular AdS space of dimension 3:

Definition 2.2.1. Let $\alpha > 0$, we define AdS^3_α as the space $\mathbb{R}\times\mathbb{R}_{>0}\times(\mathbb{R}/2\pi\alpha\mathbb{Z})$ with the metric:

$$g_\alpha = -\cosh^2 t dt^2 + d\rho^2 + \sinh^2(\rho)d\theta^2$$

where $t\in\mathbb{R}, \rho\in\mathbb{R}_{>0}$ and $\theta\in(\mathbb{R}/2\pi\alpha\mathbb{Z})$.

Remark 2.2.1.

- The totally geodesic plane $\mathscr{P}_0 := \{(\rho,\theta,t)\in\mathrm{AdS}^3_\alpha,\ t=0\}$ is canonically isometric to the hyperbolic space with cone singularity $\mathbb{H}^2_\alpha$.
- AdS^3_α can be obtained by cutting the universal cover of AdS^3 along two time-like planes intersecting along the line $l := \{\rho = 0\}$, making an angle $2\pi\alpha$, and gluing the two sides of the angular sector of angle $2\pi\alpha$ by the rotation of angle $2\pi(1-\alpha)$ fixing l. A simple computation shows that, outside of the singular line, AdS^3_α is a Lorentz manifold of constant curvature -1, and AdS^3_α carries a conical singularity of angle $2\pi\alpha$ along l.
- In the neighborhood of the totally geodesic plane $\mathscr{P}_0$ given by the points at a (causal) distance less than $\pi/2$ from $\mathscr{P}_0$, the metric of AdS^3_α expresses:

$$g_\alpha = -dt^2 + \cos^2 t(d\rho^2 + \sinh^2\rho d\theta^2).$$

Definition 2.2.2. An AdS cone-manifold is a (singular) Lorentzian 3-manifold (M,g) in which any point x has a neighborhood isometric to an open subset of AdS^3_α for some $\alpha > 0$. If α can be taken equal to 1, x is a smooth point, otherwise α is uniquely determined.

Remark 2.2.2. When dealing with AdS cone-manifolds, we assume that we use the same kind of weighted Hölder spaces as in Subsection 2.1.2 to define regularity of the metric tensor. However, in general, we will not remove the singular locus: we will just assume our manifold is closed and that the metric tensor is singular at the cone singularities.

To define the global hyperbolicity in the singular case, we need to define the orthogonality to the singular locus:

Definition 2.2.3. Let $S\subset\mathrm{AdS}^3_\alpha$ be a space-like surface which intersect the singular line l at a point x. S is said to be orthogonal to l at x if the causal distance (that is the "distance" along a time-like line) to the totally geodesic plane P orthogonal to the singular line at x is such that:

$$\lim_{y\to x, y\in S}\frac{d(y,P)}{d_S(x,y)} = 0$$

where $d_S(x,y)$ is the distance between x and y along S.

Now, a space-like surface S in an AdS cone-manifold (M, g) which intersects a singular line d at a point y is said to be orthogonal to d if there exists a neighborhood $U \subset M$ of y isometric to a neighborhood of a singular point in AdS^3_α such that the isometry sends $S \cap U$ to a surface orthogonal to l in AdS^3_α.

Now we are able to define the AdS convex GHM manifolds with particles.

Definition 2.2.4. An AdS convex GHM manifold with particles is an AdS cone-manifold (M, g) which is homeomorphic to $\Sigma_{\mathfrak{p}} \times \mathbb{R}$ (where $\Sigma_{\mathfrak{p}}$ is a closed oriented surface with n marked points), such that the singularities are along time-like lines $d_1, ..., d_n$ and have fixed cone angles $2\pi\alpha_1, .., 2\pi\alpha_n$ with $\alpha_i < \frac{1}{2}$. Moreover, we impose two conditions:

1. **Convex Global Hyperbolicity** M contains a space-like future-convex Cauchy surface orthogonal to the singular locus.
2. **Maximality** M cannot be strictly embedded in another manifold satisfying the same conditions.

Remark 2.2.3. - The condition of convexity in the definition will allow us to use a convex core. As pointed out by the authors in [BS09], we do not know if every AdS GHM manifold with particles is convex GHM.

- The name "particle" comes from the physic litterature. In fact, such a conical singularity is often used to modelise a massive point particle. The defect of angle being related to the mass of the particle (see for instance [tH96, tH93, BG00]).

Definition 2.2.5. For $\alpha := (\alpha_1, ..., \alpha_n) \in \left(0, \frac{1}{2}\right)^n$, let $\mathscr{A}_\alpha(\Sigma_{\mathfrak{p}})$ be the space of isotopy classes of AdS convex GHM metrics on $M = \Sigma_{\mathfrak{p}} \times \mathbb{R}$ with particles of cone angles $2\pi\alpha_i$ along d_i.

2.2.2 Parametrization of $\mathscr{A}_\alpha(\Sigma_{\mathfrak{p}})$

The Mess parametrization naturally extends to the moduli space $\mathscr{A}_\alpha(\Sigma_{\mathfrak{p}})$ of AdS convex GHM manifolds with particles. We have ([BS09]):

Theorem 2.2.6. *(Bonsante, Schlenker) Let $g \in \mathscr{A}_\alpha(\Sigma_{\mathfrak{p}})$, the map associating to a space-like surface $S \hookrightarrow (M, g)$ the following metrics*

$$\begin{cases} g_1 &= I((E + JB).,(E + JB).) \\ g_2 &= I((E - JB).,(E - JB).) \end{cases}$$

gives a homeomorphism

$$\mathfrak{M}_\alpha : \mathscr{A}_\alpha(\Sigma_{\mathfrak{p}}) \longrightarrow \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}}) \times \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}}).$$

Here I is the first fundamental form of S, E is the identity, J is the complex structure associated to S and B is its shape operator.

Remark 2.2.4. It follows that the metrics g_1 and g_2 are hyperbolic with cone singularities of angles $2\pi\alpha$ and are independant on the choice of the space-like surface S.

Each AdS convex GHM 3-manifold with particles (M, g) contains a minimal non-empty convex subset called its "convex core" whose boundary is a disjoint union of two pleated space-like surfaces orthogonal to the singular locus (except in the Fuchsian case which corresponds to the case where the two metrics of the parametrization are equal. In this case, the convex core is a totally geodesic space-like surface).

2.2.3 Maximal surfaces and germs

Let $g \in \mathscr{A}_\alpha(\Sigma_\mathfrak{p})$ be an AdS convex GHM metric with particles on $M = \Sigma_\mathfrak{p} \times \mathbb{R}$.

Definition 2.2.7. A maximal surface in (M, g) is a locally area-maximizing space-like Cauchy surface $S \hookrightarrow (M, g)$ which is orthogonal to the singular lines.

In particular, such a maximal surface $S \hookrightarrow (M, g)$ has everywhere vanishing mean curvature. Note that our definition differs from [KS07, Definition 5.6] where the authors impose the boundedness of the principal curvatures of S. The following Proposition shows that a maximal surface in our sense has bounded principal curvatures:

Proposition 2.2.8. *For a maximal surface $S \hookrightarrow (M, g)$ with shape operator B and induced metric g_S, $\det_{g_S}(B)$ tends to zero at the intersections with the particles. In particular, B is the real part of a meromorphic quadratic differential with at most simple poles at the singularities.*

Proof. Let d be a particle of angle $2\pi\alpha$ and set $0 := d \cap S$. We see locally S as the graph of a function $u : P_0 \longrightarrow \mathbb{R}$ where P_0 is the (piece of) totally geodesic plane orthogonal to d at 0. We will show that, the induced metric g_S on S carries a conical singularity of angle $2\pi\alpha$. We need the following lemma:

Lemma 2.2.9. *The gradient of u tends to zero at the intersections with the particles.*

Proof. To prove this lemma, we will use Schauder estimates for solutions of uniformly elliptic PDE's. For the convenience of the reader, we recall these estimates. The main reference for the theory is [GT01].

A second order linear operator L on a domain $\Omega \subset \mathbb{R}^n$ is a differential operator of the form

$$Lu = a^{ij}(x)D_{ij}u + b^k(x)D_k u + c(x)u, \ u \in \mathscr{C}^2(\Omega), \ x \in \Omega,$$

where we sum over all repeated indices. We say that L is uniformly elliptic if the smallest eigenvalue of the matrix $\big(a_{ij}(x)\big)$ is bounded from below by a strictly positive constant.

We finally define the following norms for a function u on Ω:

- $|u|_k := \|u\|_{\mathscr{C}^k(\Omega)}$.
- $|u|_0^{(i)} := \sup\limits_{x\in\Omega} d_x^i |u(x)|, \text{ where } d_x = \mathrm{dist}(x, \partial\Omega).$
- $|u|_k^* = \sum_{i=0}^k \sup\limits_{x\in\Omega,\ |\alpha|=i} d_x^i |D^\alpha u|$.

The following theorem can be found in [GT01, Theorem 6.2]

Theorem 2.2.10. *(Schauder interior estimates) Let $\Omega \subset \mathbb{R}^n$ be a domain with $\mathscr{C}^2$ boundary and $u \in \mathscr{C}^2(\Omega)$ be solution of the equation*

$$Lu = 0$$

where L is uniformly elliptic so that

$$\left|a^{ij}\right|_0^{(0)}, \ \left|b^k\right|_0^{(1)}, \ |c|_0^{(2)} < \Lambda.$$

Then there exists a positive constant C depending only on Ω and L so that

$$|u|_2^* \le C|u|_0.$$

For every domain $\Omega \subset P_0$ which does not contain the singular point, u satisfies the maximal surface equation (see for example [Ger83]) which is given by:

$$\mathscr{L}(u) := \operatorname{div}_{g_S}(v(-1, \pi^* \nabla u)) = 0.$$

Here, $\pi : S \longrightarrow P_0$ is the orthogonal projection, $v = \left(1 - \|\pi^* \nabla u\|^2\right)^{-1/2}$ and so $v(-1, \pi^* \nabla u)$ is the unit future pointing normal vector field to S. Also, one easily checks that this equation can be written

$$\operatorname{div}_{g_S}(v\pi^* \nabla u) + a(x, u, \nabla u) = 0, \text{ for some function } a. \tag{2.2}$$

The proof of Proposition 3.1.12 applies in this case and implies the S is uniformly space-like. It follows that π is uniformly bi-Lipschitz and so v is uniformly bounded.

It follows that Equation (2.2) is a quasi-linear elliptic equation in the divergence form. Moreover, if we write it in the following way:

$$a^{ij}(x, u, Du)D_{ij}u + b^k(x, u, Du)D_k u + c(x, Du, u)u = 0,$$

it is easy to see that the equation is uniformly elliptic (in fact $a^{ij}(x, u, Du) \geq 1$) and the the coefficients satisfy conditions of Theorem 2.2.10 (as they are uniformly bounded on Ω). Hence, we are in the good framework to apply the Schauder estimates.

Let $x_0 \in P_0 \setminus \{0\}$ and let $2r := \operatorname{dist}_S(x_0, 0)$. Consider the disk D_r of radius r centered at x_0. It follows from the previous discussion that $u : D_r \longrightarrow \mathbb{R}$ satisfies $\mathscr{L}u = 0$. By a homothety of ratio $1/r$, send the disk D_r to the unit disk (D, h_r) where h_r is the metric of constant curvature $-r^2$. The function u is sent to a new function

$$u_r : (D, h_r) \longrightarrow \mathbb{R},$$

and satisfies the equation

$$\mathscr{L}_r u_r = 0.$$

Here, the operator $\mathscr{L}_r$ is the maximal surface operator for the rescaled metric $g_r := -dt^2 + \cos^2 t.h_r$. In particular, $\mathscr{L}_r$ is a quasi-linear uniformly elliptic operator whose coefficients applied to u_r satisfy the condition of Theorem 2.2.10.

In a polar coordinates system (ρ, θ), the metric h_r expresses

$$h_r = d\rho^2 + r^{-2} \sinh^2(r.\rho) d\theta^2.$$

As r tends to zero, the metric h_r converges $\mathscr{C}^\infty$ on D to the flat metric $h_0 = d\rho^2 + \rho^2 d\theta^2$. It follows that the coefficients of the family of operators $(\mathscr{L}_r)_{r\in(0,1)}$ applied to u_r converge to the ones of the operator $\mathscr{L}_0$ applied to $u_0 = \lim\limits_{r\to 0} u_r$ where $\mathscr{L}_0$ is the maximal surface operator associated to the metric $g_0 = -dt^2 + \cos^2 t h_0$.

As a consequence, the family of constants $\{C_r\}$ associated to the Schauder interior estimates applied to $\mathscr{L}_r(u_r)$ are uniformly bounded by some $C > 0$.

Now, to obtain a bound on the norm of the gradient $\|\nabla u\|$ at a point x_0 at a distance $2r$ from the singularity, we apply the Schauder interior estimates to $\mathscr{L}_r(u_r)$, where $u_r : (D, h_r) \longrightarrow \mathbb{R}$. We get

$$|u_r|_2^* \leq C_r |u_r|_0 \leq C|u_r|_0.$$

As $\|\nabla u_r\|(x_0) \leq |u|_2^*$, and as $u_r(x_0) = o(2r)$ (because S is orthogonal to d), we obtain

$$\|\nabla u_r\|(x_0) \leq C.o(r).$$

But as u_r is obtained by rescaling u with a factor r, so $\|\nabla u\| = r^{-1}\|\nabla u_r\|$ and we finally get:

$$\|\nabla u\| = o(1).$$

□

Lemma 2.2.11. *The induced metric g_S on S carries a conical singularity of angle $2\pi\alpha$ at its intersection with the particle d.*

Proof. Recall that (see [MRS13, Section 2.2] and [JMR11, Section 2.1]) a metric h carries a conical singularity of angle $2\pi\alpha$ if and only if there exists normal polar coordinates $(\rho, \theta) \in \mathbb{R}_{>0} \times [0, 2\pi)$ around the singularity so that

$$g = d\rho^2 + f^2(\rho,\theta)d\theta^2, \ \frac{f(\rho,\theta)}{\rho} \underset{\rho\to 0}{\longrightarrow} \alpha.$$

That is, if g can be written by the matrix

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2\rho^2 + o(\rho^2) \end{pmatrix}.$$

The metric of (M, g) can be locally written around the intersection of S and the particle d by

$$g = -dt^2 + \cos^2 t h_\alpha,$$

where $h_\alpha = d\rho^2 + \alpha^2 \sinh^2 \rho d\theta^2$ is the metric of $\mathbb{H}^2_\alpha$.

Setting $t = u(\rho, \theta)$, with $u(\rho, \theta) = o(\rho)$ and $\|\nabla u\| = o(1)$, we get

$$dt^2 = (\partial_\rho u)^2 d\rho^2 + 2\partial_\rho u \partial_\theta u d\rho d\theta + (\partial_\theta u)^2 d\theta^2.$$

Note that, as $\|\nabla u\| = o(1)$, $\partial_\rho u = o(1)$ and $\partial_\theta u = o(\rho)$.

Finally, using $\cos^2(u) = 1 + o(\rho^2)$, we get the following expression for the induced metric on S:

$$g_S = \begin{pmatrix} 1 + o(1) & o(\rho) \\ o(\rho) & \alpha^2\rho^2 + o(\rho^2) \end{pmatrix}.$$

One easily checks that, with a change of variable, the induced metric carries a conical singularity of angle $2\pi\alpha$ at the intersection with d. □

Now the proof of Proposition 2.2.8 follows: suppose the second fundamental form $\mathrm{II} = g_S(B., .)$ is the real part of a meromorphic quadratic differential q with a pole of order n. In complex coordinates, write $q = f(z)dz^2$ and $g_S = e^{2u}|z|^{2(\alpha-1)}|dz|^2$ where u is bounded. Then B is the real part of the harmonic Beltrami differential

$$\mu := \frac{\overline{q}}{g_S} = e^{-2u}|z|^{-2(\alpha-1)}\overline{f}(z)d\overline{z}\partial_z.$$

Using the real coordinates $z = x + iy,\ dz = dx + idy,\ \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ we get

$$\begin{aligned} B &= \Re\left(\frac{1}{2}e^{-2u}|z|^{-2(\alpha-1)}(\Re(f)-i\Im(f))(dx-idy)(\partial_x-i\partial_y)\right) \\ &= \frac{1}{2}e^{-2u}|z|^{-2(\alpha-1)}(\Re(f)(dx\partial_x-dy\partial_y)-\Im(f)(dx\partial_y-dy\partial_x)) \\ &= \frac{1}{2}e^{-2u}|z|^{-2(\alpha-1)}\begin{pmatrix} \Re(f) & -\Im(f) \\ -\Im(f) & -\Re(f)\end{pmatrix}. \end{aligned}$$

It follows that

$$\det_{g_S}(B) = -\frac{1}{4}e^{-4u}|z|^{-4(\alpha-1)}|f|^2 = -e^v|z|^{-2(2\alpha-2+n)},$$

for some bounded v. By (modified) Gauss equation, the curvature K_S of S is given by

$$K_S = -1-\det_{g_S}(B).$$

By Gauss-Bonnet formula for surface with cone singularities (see for example [Tro91]), K_S has to be locally integrable. But we have:

$$K_s dvol_S = \big(-1+e^w|z|^{-2(\alpha-1+n)}\big)d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^2$. It follows that $K_s dvol_S$ is integrable if and only if $1-\alpha+n<1$, that is $n\leq 1$. Note also that, for $n\leq 1$, $\det_{g_S}(B) = O\left(|z|^{2(1-2\alpha)}\right)$ and so tends to zero at the singularity. □

As in the non-singular case (see Section 1.4), one can construct the moduli space $\mathscr{H}_\alpha(\Sigma_\mathfrak{p})$ of **maximal AdS germs with particles on $\Sigma_\mathfrak{p}$**. By definition, this space is the space of pairs (h,m) where $h\in\Gamma(S^2T\Sigma_\mathfrak{p})$ is a metric with cone singularities of angles $2\pi\alpha$ and $m\in\Gamma(S^2T\Sigma_\mathfrak{p})$ is a trace-less Codazzi tensor satisfying the modified Gauss equation. Note that, given a maximal surface $S\hookrightarrow(M,g)$ in an AdS convex GHM space-time with particles, the pair (I,II) of first and second fundamental form of S gives a point in $\mathscr{H}(\Sigma)$.

Conversely, given (h,m) a maximal AdS germ with particles on $\Sigma_\mathfrak{p}$, one can uniquely reconstruct an AdS convex GHM space-time with particles (M,g) together with an embedded maximal surface $S\hookrightarrow(M,g)$ so that h is the first fundamental form on S and m its second fundamental form. It gives a canonical map

$$\mathscr{H}(\Sigma)\longrightarrow\mathscr{A}_\alpha(\Sigma_\mathfrak{p}).$$

This map is bijective if and only if each AdS convex GHM space-time with particles contains a unique maximal surface.

As in the non-singular case, this space is canonically parametrized by $T^*\mathscr{T}(\Sigma_\mathfrak{p})$. Recall that the cotangent space $T^*_\mathfrak{c}\mathscr{T}(\Sigma_\mathfrak{p})$ to $\mathscr{T}(\Sigma_\mathfrak{p})$ at a conformal class $\mathfrak{c}\in\mathscr{T}(\Sigma_\mathfrak{p})$ is canonically identified with the space of meromorphic quadratic differentials on $(\Sigma_\mathfrak{p},J_\mathfrak{c})$ (where $J_\mathfrak{c}$ is the complex structure associated to $\mathfrak{c}$) with at most simple poles at the marked points. It is proved in [KS07] that the map associating to a maximal AdS germs with particles $(h,m)\in\mathscr{H}(\Sigma)$ the pair $(\mathfrak{c},q)$ where $\mathfrak{c}$ is the conformal class of h and q is the unique meromorphic quadratic differential so that $m=\Re(q)$ is bijective.

Moreover, given $(g,h)\in\mathscr{H}_\alpha(\Sigma_\mathfrak{p})$, using the Fundamental Theorem of surfaces in AdS convex GHM manifolds with particles, one can locally reconstruct a piece of AdS globally hyperbolic manifold with particles which uniquely embeds in a maximal one. It provides a map from $\mathscr{H}_\alpha(\Sigma_\mathfrak{p})$ to $\mathscr{A}_\alpha(\Sigma_\mathfrak{p})$. This map is bijective if and only if each AdS convex GHM manifold (M, g) contains a unique maximal surface.

Chapter 3

Case of same cone-angles

In this Chapter, we prove Main Theorem 2 and Main Theorem 1. In Section 3.1, we prove the existence part of Main Theorem 2 by convergence of maximal surfaces in some regularized manifold. In Section 3.2 we prove the uniqueness part by a maximum principle argument. Finally, in Section 3.3, we show the equivalence between Main Theorem 2 and 1 and generalize the global picture given in Chapter 1.

3.1 Existence of a maximal surface

Let $g \in \mathscr{A}_\alpha(\Sigma_{\mathfrak{p}})$ be an AdS convex GHM manifold with particles. We prove the following:

Proposition 3.1.1. *The AdS convex GHM manifold with particles (M, g) contains a maximal surface $S \hookrightarrow (M, g)$.*

First note that in the "Fuchsian" space-times (that is when the two metrics of the parametrization $\mathfrak{M}_\alpha(g)$ are equal), the convex core of (M, g) is reduced to a totally geodesic plane orthogonal to the singular locus. Such a surface is clearly a space-like maximal surface (its second fundamental form vanishes).

Hence, from now on, suppose that (M, g) is not Fuchsian (that is $\mathfrak{M}_\alpha(g) \in \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}}) \times \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}})$ are two distinct points). It follows from [BS09, Section 5] that (M, g) contains a convex core with non-empty interior whose boundary consist of two pleated surfaces: a future-convex one and a past-convex one.

The proof of Proposition 3.1.1 is done in four steps:

Step 1 Approximate the singular metric g by a sequence of smooth metrics $(g_n)_{n\in\mathbb{N}}$ which converges to the metric g, and prove the existence for each $n \in \mathbb{N}$ of a maximal surface $S_n \hookrightarrow (M, g_n)$.

Step 2 Prove that the sequence $(S_n)_{n\in\mathbb{N}}$ converges outside the singular lines to a smooth nowhere time-like surface S with vanishing mean curvature.

Step 3 Prove that the limit surface S is space-like.

Step 4 Prove that the limit surface S is orthogonal to the singular lines.

3.1.1 First step

Approximation of singular metrics. Take $\alpha \in (0,1)$ and let $\mathscr{C}_\alpha \subset \mathbb{R}^3$ be the cone given by the parametrization:

$$\mathscr{C}_\alpha := \{(u.\cos(v), u.\sin(v), u.\text{cotan}(\pi\alpha)),\ (u,v) \in \mathbb{R}_+ \times [0,2\pi)\}.$$

Now, consider the intersection of this cone with the Klein model of the hyperbolic 3-space, and denote by h_α the induced metric on $\mathscr{C}_\alpha$. Outside the apex, $\mathscr{C}_\alpha$ is a convex ruled surface in $\mathbb{H}^3$, and so has constant curvature -1. Moreover, one easily checks that h_α carries a conical singularity of angle $2\pi\alpha$ at the apex of $\mathscr{C}_\alpha$. Consider the orthogonal projection p from $\mathscr{C}_\alpha$ to the disk of equation $\mathbb{D} := \{z = 0\} \subset \mathbb{H}^3$. We have that $(\mathbb{D}, (p^{-1})^* h_\alpha)$ is isometric to the local model of hyperbolic metric with cone singularity $\mathbb{H}^2_\alpha$ as defined in Chapter 2.

Remark 3.1.1. The angle of the singularity is given by $\lim\limits_{\rho \to 0} \dfrac{l(C_\rho)}{\rho}$ where $l(C_\rho)$ is the length of the circle of radius ρ centred at the singularity.

Now, to approximate this metric, take $(\epsilon_n)_{n\in\mathbb{N}} \subset (0,1)$, a sequence decreasing to zero and define a sequence of even functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ so that for each $n \in \mathbb{N}$,

$$\begin{cases} f_n(0) = -\epsilon_n^2.\text{cotan}(\pi\alpha) \\ f_n''(x) < 0 \quad \forall x \in (-\epsilon_n, \epsilon_n) \\ f_n(x) = -\text{cotan}(\pi\alpha).x \text{ if } x \geqslant \epsilon_n. \end{cases}$$

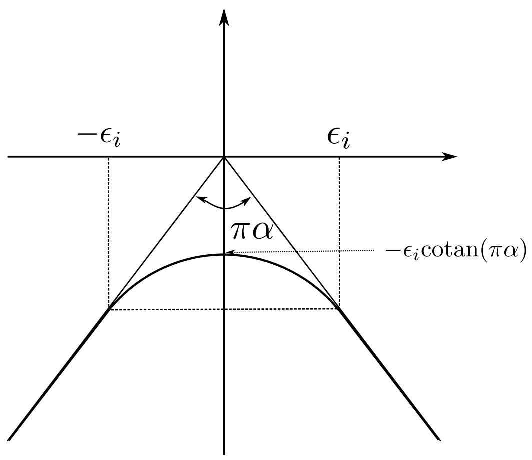

Figure 3.1: Graph of f_n

Consider the surface $\mathscr{C}_{\alpha,n}$ obtained by making a rotation of the graph of f_n around the axis $(0z)$ and consider its intersection with the Klein model of hyperbolic 3-space. Denote by $h_{\alpha,n}$ the induced metric on $\mathscr{C}_{\alpha,n}$, and define $\mathbb{H}^2_{\alpha,n} := (\mathbb{D}, (p^{-1})^* h_{\alpha,n})$ (where p is still the orthogonal projection to the disk $\mathbb{D} = \{z = 0\} \subset \mathbb{H}^3$). By an abuse of notations, we write $\mathbb{H}^2_{\alpha,n} = (\mathbb{D}, h_{\alpha,n})$. Denote by $B_i \subset \mathbb{D}$ the smallest set where the metric $h_{\alpha,n}$ does not have of constant curvature -1, by construction, $B_n \underset{n\to\infty}{\longrightarrow} \{0\}$, where $\{0\}$ is the center of $\mathbb{D}$. We have

Proposition 3.1.2. *For all compact $K \subset \mathbb{D} \setminus \{0\}$, there exists $i_K \in \mathbb{N}$ such that for all*

$n > n_K$, $h_{\alpha_{|K}} = h_{\alpha,n_{|K}}$.

We define the AdS 3-space with regularized singularity:

Definition 3.1.3. Let $\alpha > 0$, $n \in \mathbb{N}$, we define $\mathrm{AdS}^3_{\alpha,n}$ as the completion of $\mathbb{R} \times \mathbb{D}$ with the metric:

$$g_{\alpha,n} = -dt^2 + \cos^2(t) h_{\alpha,n}, \text{ where } t \in \mathbb{R}.$$

By construction, there exists a smallest tubular neighborhood V^n_α of $l = \{0\} \times \mathbb{R}$ such that $\mathrm{AdS}^3_{\alpha,n} \setminus V^n_\alpha$ is a Lorentzian manifold of constant curvature -1.

In this way, we are going to define the regularized AdS convex GHM manifold with particles.

For all $j \in \{1, ..., n\}$ and $x \in d_j$ where d_j is a singular line in (M, g), there exists a neighborhood of x in (M, g) isometric to a neighborhood of a point on the singular line in $\mathrm{AdS}^3_{\alpha_j}$. For $n \in \mathbb{N}$, we define the regularized metric g_n on M so that the neighborhoods of points of d_j are isometric to neighborhoods of points on the central axis in $\mathrm{AdS}^3_{\alpha_j,n}$. Clearly, the metric g_n is obtained taking the metric of $V^n_{\alpha_j}$ in a tubular neighborhood U^n_j of the singular lines d_j for all $j \in \{1, ..., n\}$. In particular, outside these U^n_j, (M, g_n) is a regular AdS manifold.

Proposition 3.1.4. *Let $K \subset M$ be a compact set which does not intersect the singular lines. There exists $n_K \in \mathbb{N}$ such that, for all $n > n_K$, $g_{n_{|K}} = g_{|K}$.*

Existence of a maximal surface in each (M, g_n)

We are going to prove Proposition 3.1.1 by convergence of maximal surfaces in each (M, g_n). A result of Gerhardt [Ger83, Theorem 6.2] provides the existence of a maximal surface in (M, g_n) given the existence of two smooth barriers, that is, a strictly future-convex smooth (at least $\mathscr{C}^2$) space-like surface and a strictly past-convex one. This result has been improved in [ABBZ12, Theorem 4.3] reducing the regularity conditions to $\mathscr{C}^0$ barriers.

The natural candidates for these barriers are equidistant surfaces from the boundary of the convex core of (M, g). It is proved in [BS09, Section 5] that the future (respectively past) boundary component ∂_+ (respectively ∂_-) of the convex core is a future-convex (respectively past-convex) space-like pleated surface orthogonal to the particles. Moreover, each point of the boundary components is either contained in the interior of a geodesic segment (a pleating locus) or of a totally geodesic disk contained in the boundary components.

For $\epsilon > 0$ fixed, consider the 2ϵ-surface in the future of ∂_+ and denote by $\partial_{+,\epsilon}$ the ϵ-surface in the past of the previous one. As pointed out in [BS09, Proof of Lemma 4.2], this surface differs from the ϵ-surface in the future of ∂_+ (at the pleating locus).

Proposition 3.1.5. *For n big enough, $\partial_{+,\epsilon} \hookrightarrow (M, g_n)$ is a strictly future-convex space-like $\mathscr{C}^{1,1}$ surface.*

Proof. Outside the open set $U^n := \bigcup_{j=1}^{n} U^n_j$ (where the U^n_j are tubular neighborhoods of d_j so that the curvature is different from -1), (M, g_n) is isometric to (M, g), and moreover, $U^n_j \underset{n\to\infty}{\longrightarrow} d_j$ for each j. As proved in [BS09, Lemma 5.2], each intersection of ∂_+ with a particle lies in the interior of a totally geodesic disk contained in ∂_+. So, there exists $n_0 \in \mathbb{N}$ such that, for $n > n_0$, $U^j_i \cap \partial_+$ is totally geodesic.

The fact that $\partial_{+,\epsilon}$ is $\mathscr{C}^{1,1}$ is proved in [BS09, Proof of Lemma 4.2].

For the strict convexity outside U^n, the result is proved in [BBZ07, Proposition 6.28]. So it remains to prove that $\partial_{+,\epsilon} \cap U_n$ is strictly future-convex.

Let $d = d_j$ be a singular line which intersects ∂_+ at a point x. As $U := U_n^j \cap \partial_+$ is totally geodesic, we claim that $U_\epsilon := U_n^j \cap \partial_{+,\epsilon}$ is the ϵ-surface of U with respect to the metric g_n. In fact, the space-like surface $\mathscr{P}_0 \subset \mathrm{AdS}^3_{\alpha,i}$ given by the equation $\{t = 0\}$ is totally geodesic and the one given by $\mathscr{P}_\epsilon := \{t = \epsilon\}$ is the ϵ-surface of $\mathscr{P}_0$ and corresponds to the ϵ-surface in the past of $\mathscr{P}_{2\epsilon}$. It follows that U_ϵ is obtained by taking the ϵ-time flow of U along the unit future-pointing vector field N normal to ∂_+ (extended to an open neighborhood of U by the condition $\nabla^n_N N = 0$, where ∇^n is the Levi-Civita connection of g_n). We are going to prove that the second fundamental form on U_ϵ is positive definite.

Note that in $\mathrm{AdS}^3_{\alpha_j,n}$, the surfaces $\mathscr{P}_{t_0} := \{t = t_0\}$ are equidistant from the totally geodesic space-like surface $\mathscr{P}_0$. Moreover, the induced metric on $\mathscr{P}_{t_0}$ is $\mathrm{I}_{t_0} = \cos^2(t_0) h_{\alpha,n}$ and so, the variation of I_{t_0} along the flow of N is given by

$$\frac{d}{dt}_{|t=t_0} \mathrm{I}_t(u_t, u_t) = -2\cos(t_0)\sin(t_0),$$

for u_t a unit vector field tangent to $\mathscr{P}_t$. On the other hand, this variation is given by

$$\frac{d}{dt}_{|t=t_0} \mathrm{I}_t(u_t, u_t) = \mathscr{L}_N \mathrm{I}_{t_0}(u_{t_0}, u_{t_0}) = 2\mathrm{I}_{t_0}(\nabla^i_{u_{t_0}} N, u_{t_0}) = -2\mathrm{II}_{t_0}(u_{t_0}, u_{t_0}),$$

where $\mathscr{L}$ is the Lie derivative and $Bu := -\nabla_u N$ is the shape operator.

It follows that II_{t_0} is positive-definite for $t_0 > 0$ small enough. So $\partial_{+,\epsilon} \hookrightarrow (M, g_n)$ is strictly past-convex (that is for each point $p \in \partial_{+,\epsilon}$, $\partial_{+,\epsilon}$ remains locally in the past of the totally geodesic space-like plane tangent to $\partial_{+,\epsilon}$ at p). □

So we get a $\mathscr{C}^{1,1}$ barrier. The existence of a $\mathscr{C}^{1,1}$ strictly future-convex surface is analogous. So, by [ABBZ12, Theorem 4.3], we get that for all $n > n_0$, there exists a maximal space-like Cauchy surface S_n in (M, g_n). By re-indexing, we finally have proved

Proposition 3.1.6. *There exists a sequence $(S_n)_{n\in\mathbb{N}}$ of space-like surfaces where each $S_n \hookrightarrow (M, g_n)$ is a maximal space-like surface.*

3.1.2 Second step

Proposition 3.1.7. *There exists a subsequence of $(S_n)_{n\in\mathbb{N}}$ converging uniformly on each compact which does not intersect the singular lines to a surface $S \hookrightarrow (M, g)$.*

Proof. For some fixed $n_0 \in \mathbb{N}$, (M, g_{n_0}) is a smooth globally hyperbolic manifold and so admits some smooth time function $f : (M, g_{n_0}) \longrightarrow \mathbb{R}$. This time function allows us to see the sequence of maximal surfaces $(S_n)_{n\in\mathbb{N}}$ as a sequence of graphs on functions over $f^{-1}(\{0\})$ (where we suppose $0 \in f(M)$). Let $K \subset f^{-1}(\{0\})$ be a compact set which does not intersect the singular lines and see locally the surfaces S_n as graphs of functions $u_n : K \longrightarrow \mathbb{R}$.

For n big enough , the graphs of u_n are pieces of space-like surfaces contained in the convex core of (M, g), so the sequence $(u_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded Lipschitz functions with uniformly bounded Lipschitz constant. By Arzelà-Ascoli's Theorem, this sequence admits a subsequence (still denoted by $(u_n)_{n\in\mathbb{N}}$) converging uniformly to a function $u : K \longrightarrow \mathbb{R}$. Applying this to each compact set of $f^{-1}(\{0\})$ which does not intersect the singular line, we get that the sequence $(S_n)_{n\in\mathbb{N}}$ converges uniformly outside the singular lines to a surface S. □

Note that, as the surface S is a limit of space-like surfaces, it is nowhere time-like. However, S may contains some light-like locus.

Proposition 3.1.8. *The light-like locus of the surface $S \hookrightarrow (M, g)$ lies in the set of light-like ray between two singular lines.*

Proof. Let $p \in S$ be a light-like point. Then, either p lies on a light-like segment contained in S, or p is isolated (that is, there exists a neighborhood $U \subset S$ of p so $U \setminus \{p\}$ is space-like).

The second case is impossible since from [Ger83, Theorem 4.1], if $v = \|N\|^{-1/2}$ (where $\|N\|$ is the norm of the normal to S) is bounded on ∂U, the v is bounded on U.

So the light-like points of S are contained in light-like segment. We recall a theorem of C. Gerhardt [Ger83, Theorem 3.1]:

Theorem 3.1.9. *(C. Gerhardt) Let S be a limit on compact subsets of a sequence of space-like surfaces in a globally hyperbolic space-time. Then if S contains a segment of a null geodesic, this segment has to be maximal, that is it extends to the boundary of M.*

So, if S contains a light-like segment, either this segment extends to the boundary of M, either it intersects two singular lines. The first is impossible as it would imply that S is not contained in the convex core. The result follows.

□

We now prove the following:

Proposition 3.1.10. *The sequence of space-like surfaces $(S_n)_{n\in\mathbb{N}}$ of Proposition 3.1.6 converges $\mathscr{C}^{1,1}$ on each compact which does not intersect the singular lines and light-like locus. Moreover, outside these loci, the surface S has everywhere vanishing mean curvature.*

Proof. For a point $x \in S$ which neither lies on a singular line nor on a light-like locus, see a neighborhood $K \subset S$ of x as the graph of a function u over a piece of totally geodesic space-like plane Ω. With an isometry Ψ, send Ω to the totally geodesic plane $P_0 \subset \mathrm{AdS}^3$ given by the equation $P_0 := \{(\rho, \theta, t) \in \mathrm{AdS}^3,\ t = 0\}$. We still denote by S_n (respectively S, u and Ω) the image by Ψ of S_n (respectively S, u and Ω). Note that, for $n \in \mathbb{N}$ big enough, the metric g_n coincides with the metric g in a neighborhood of K in M. So locally around x, the surfaces S_n have vanishing mean curvature in (M, g), hence their images in AdS^3 have vanishing mean curvature.

Let $u_n : \Omega \longrightarrow \mathbb{R}$ be such that $S_n = \mathrm{graph}(u_n)$. The unit future pointing normal vector to S_n at $(x, u_n(x))$ is given by

$$N_n = v_n.\pi^*(\nabla u_n, 1),$$

where $(\nabla u_n, 1) \in T_x\mathrm{AdS}^3$ is the vector $(\nabla_\rho u_n, \nabla_\theta u_n, \partial_t)$, $\pi : S_n \longrightarrow \Omega$ is the orthogonal projection on P_0 and $v_n = (1 - \|\pi^*\nabla u_n\|^2)^{-1/2}$. The vanishing of the mean curvature of S_n is equivalent to

$$-\delta_g N_n = 0,$$

where δ_g is the divergence operator. In coordinates, this equation reads (see also [Ger83, Equation 1.14]):

$$\frac{1}{\sqrt{\det g}}\partial_i(\sqrt{\det g} v_n g^{ij}\nabla_j u_n) + \frac{1}{2}v_n\partial_t g^{ij}\nabla_i u_n\nabla_j u_n - \frac{1}{2}v_n^{-1}g^{ij}\partial_t g_{ij} = 0. \tag{3.1}$$

Here, we wrote the metric

$$g = -dt^2 + g_{ij}(x,t)dx^i dx^j,$$

applying the convention of Einstein for the summation (with indices $i, j = 1, 2$). The metric g is taken at the points $(x, u_n(x),)$ and $\det g$ is the determinant of the metric.

We have the following

Lemma 3.1.11. *The solutions u_n of equation (3.1) are in $\mathscr{C}^\infty(\Omega)$.*

Proof. This is a bootstrap argument. From [Ger83, Theorem 5.1], we have $u_n \in \mathscr{W}^{2,p}(\Omega)$ for all $p \in [1, +\infty)$ (where $\mathscr{W}^{k,p}(\Omega)$ is the Sobolev space of functions over Ω admitting weak L^p derivatives up to order k).

As v_n is uniformly bounded from above and from below (because the surface S_n is space-like), and as $u_n \in \mathscr{W}^{2,p}(\Omega)$, the third term of equation (3.1) is in $\mathscr{W}^{1,p}(\Omega)$.

For the second term, we recall the multiplication law for Sobolev space: if $\frac{k}{2} - \frac{1}{p} > 0$, then the product of functions in $\mathscr{W}^{k,p}(\Omega)$ is still in $\mathscr{W}^{k,p}(\Omega)$. So, as the second term of equation (3.1) is a product of three terms in $\mathscr{W}^{1,p}(\Omega)$, it is in $\mathscr{W}^{1,p}(\Omega)$ (by taking $p > 2$).

Hence the first term is in $\mathscr{W}^{1,p}(\Omega)$, and so $\sqrt{\det g} v_n g^{ij} \nabla_j u_n \in \mathscr{W}^{2,p}(\Omega)$. Moreover, as we can write the metric g to that $g_{ij} = 0$ whenever $i \neq j$ and as $\sqrt{\det g} g^{ii}$ are $\mathscr{W}^{2,p}(\Omega)$ and bounded from above and from below, $v_n \nabla_i u_n \in \mathscr{W}^{2,p}(\Omega)$. We claim that it implies $u_n \in \mathscr{W}^{3,p}$. It fact, for f a never vanishing smooth function, consider the map

$$\begin{array}{rlcl} \varphi : & D \subset \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & p & \longmapsto & (1 - f^2(p)|p|^2)^{-1/2} p, \end{array}$$

where D is a domain such that $f^2(p)|p|^2 < 1 - \epsilon$ and $p \neq 0$. The map φ is a $\mathscr{C}^\infty$ diffeomorphism on its image, and we have $\big(\varphi(\nabla u_n)\big)_i \in \mathscr{W}^{2,p}(\Omega)$ for $i = 1, 2$ (in fact, as it is a local argument, we can always perturb Ω so that $\nabla u_n \neq 0$). Applying φ^{-1}, we get $\nabla_i u_n \in \mathscr{W}^{2,p}(\Omega)$ and so $u \in \mathscr{W}^{3,p}(\Omega)$.

Iterating the process, we obtain that $u_n \in \mathscr{W}^{k,p}(\Omega)$ for all $k \in \mathbb{N}$ and $p > 1$ big enough. Using the Sobolev embedding Theorem

$$\mathscr{W}^{j+k,p}(\Omega) \subset \mathscr{C}^{j,\alpha}(\Omega) \text{ for } 0 < \alpha < k - \frac{2}{p},$$

we get the result. □

Now, from Proposition 3.1.7, $u_n \xrightarrow{\mathscr{C}^{0,1}} u$, that is $u_n \xrightarrow{\mathscr{W}^{1,p}} u$ for all $p \in [1, +\infty)$.

Moreover, as the sequence of graphs of u_n converges uniformly to a space-like graph, the sequence $(\nabla u_n)_{n \in \mathbb{N}}$ is uniformly bounded. From equation (3.1), we get that there exists a constant $C > 0$ such that for each $n \in \mathbb{N}$,

$$|\partial_i(\sqrt{\det h} v_n g^{ii} \nabla_i u_n)| < C.$$

As $(\nabla u_n)_{n \in \mathbb{N}}$ is uniformly bounded, the terms $\partial_i v_n$ are also uniformly bounded and we obtain

$$|\partial_i(\nabla_i u_n)| < C',$$

for some constant C'.

Thus $(\nabla_i u_n)_{n \in \mathbb{N}}$ is a sequence of bounded Lipschitz functions with uniformly bounded Lipschitz constant so admits a convergent subsequence by Arzelà-Ascoli. It follows that

$$u_n \xrightarrow{\mathscr{W}^{2,p}} u,$$

for all $p \in [1, +\infty)$. Thus u is a solution of equation (3.1), and so $u \in \mathscr{C}^\infty(\Omega)$. Moreover, as u satisfies equation (3.1), S has locally vanishing mean curvature. □

3.1.3 Third step

Proposition 3.1.12. *The surface S of Proposition 3.1.7 is space-like.*

We are going to prove that, at its intersections with the singular lines, S does not contain any light-like direction. To prove this, we are going to consider the link of S at its intersection p with a particle d. The link is essentially the set of rays from p that are tangent to the surface. Denote by α the cone angle of the singular line. We see locally the surface as the graph of a function u over a small disk

$$D_\alpha = \{(\rho, \theta),\ \rho \in [0, r),\ \theta \in [0, \alpha)\}$$

contained in the totally geodesic plane orthogonal to d passing through p (in particular, $u(0) = 0$).

First, we describe the link at a regular point of an AdS convex GHM manifold, then the link at a singular point. The link of a surface at a smooth point is a circle in a sphere with an angular metric (called **HS-surface** in [Sch98]). As the surface S is not necessary smooth, we will define the link of S as the domain contained between the two curves given by the limsup and liminf at zero of $\dfrac{u(\rho, \theta)}{\rho}$.

The link of a point. Consider $p \in (M, g)$ not lying on a singular line. The tangent space T_pM identifies with the Minkowski 3-space $\mathbb{R}^{2,1}$. We define the link of M at p, that we denote by $\mathscr{L}_p(M)$, as the set of rays from p, that is the set of half-lines from 0 in T_pM, so $\mathscr{L}_p(M) = T_pM \setminus \{0\}/\mathbb{R}_{>0}$. Topologically, $\mathscr{L}_p(M)$ is a 2-sphere, and the metric is given by the angle "distance". It follows that $\mathscr{L}_p(M)$ is divided into five subsets (depending on the type of the rays and on the causality):

- The set of future and past pointing time-like rays, that carries a hyperbolic metric.
- The set of light-like rays defines two circles called **past and future light-like circles**.
- The set of space-like rays, which carries a de Sitter metric.

To obtain the link of a point lying on a singular line of angle $\alpha \leq 2\pi$, we cut $\mathscr{L}_p(M)$ along two meridian separated by an angle α and glue by a rotation. We get a surface denoted $\mathscr{L}_{\alpha,p}(M)$ (see Figure 3.2).

The link of a surface. Let Σ be a smooth surface in (M, g) and $p \in \Sigma$ not lying on a singular line. The space of rays from p tangent to Σ is just the projection of the tangent plane to Σ on $\mathscr{L}_p(M)$ and so describe a circle in $\mathscr{L}_p(M)$. Denote this circle by $\mathscr{C}_{\Sigma,p}$. Obviously, if Σ is a space-like surface, $\mathscr{C}_{\Sigma,p}$ is a space-like circle in the de Sitter domain of $\mathscr{L}_p(M)$ and if Σ is time-like or light-like, $\mathscr{C}_{\Sigma,p}$ intersects one the time-like circles in $\mathscr{L}_p(M)$.

Now, if $p \in \Sigma$ belongs to a singular line of angle α and is not smooth, we define the link of Σ at p as the domain $\mathscr{C}_{\Sigma,p}$ delimited by the limsup and the liminf of $\dfrac{u(\rho, \theta)}{\rho}$.

We have the following:

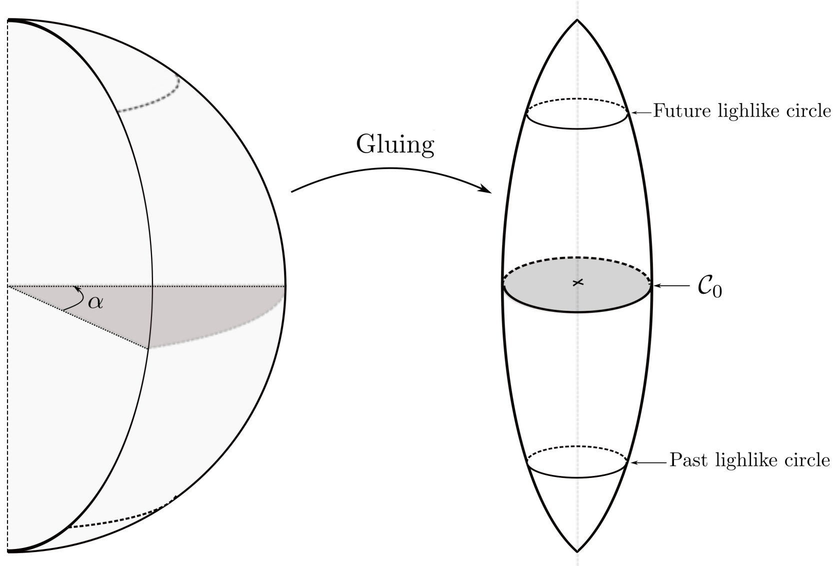

Figure 3.2: Link at a singular point

Lemma 3.1.13. *Let $\Sigma \hookrightarrow (M, g)$ be a nowhere time-like surface which intersects a singular line of angle $\alpha < \pi$ at a point p. If $\mathscr{C}_{\Sigma,p}$ intersects a light-like circle in $\mathscr{L}_{\alpha,p}(M)$, then $\mathscr{C}_{\Sigma,p}$ does not cross $\mathscr{C}_{0,p}$. That is, $\mathscr{C}_{\Sigma,p}$ remains strictly in one hemisphere (where a hemisphere is a connected component of $\mathscr{L}_{\alpha,p}(M) \setminus \mathscr{C}_{0,p}$).*

Proof. For a non-zero vector $v \in T_p(\Sigma)$ and $\theta \in [0, \alpha)$, denote by v_θ the unit vector making a positive angle θ with v. Suppose that v_{θ_0} corresponds to the direction where $\mathscr{C}_{\Sigma,p}$ intersects a light-like circle, for example, the future light-like circle. As the surface is nowhere time-like, Σ remains in the future of the light-like plane containing v_{θ_0}. But the link of a light-like plane at a non singular point p is a great circle in $\mathscr{L}_p(M)$ which intersects the two different light-like circles at the directions given by v_{θ_0} and $v_{\theta_0+\pi}$. So it intersects $\mathscr{C}_{0,p}$ at the directions $v_{\theta_0 \pm \pi/2}$.

Now, if p belongs to a singular line of angle $\alpha < \pi$, the link of the light-like plane which contains v_{θ_0} is obtained by cutting the link of p along the directions of $v_{\theta_0 \pm \alpha/2}$ and gluing the two wedges by a rotation (see the Figure 3.2). So, the link of our light-like plane remains in the upper hemisphere, which implies the result. ☐

Remark 3.1.2. In particular, if $\mathscr{C}_{\Sigma,p}$ intersects $\mathscr{C}_{0,p}$, it does not intersect a light-like circle.

It follows that if the link of Σ at p is continuous, there exists $\eta > 0$ (depending of α) so that:

- If $\mathscr{C}_{\Sigma,p}$ intersects the future light-like circle, then

$$u(\rho, \theta) \geq \eta.\rho \ \forall \theta \in [0, \alpha), \ \rho \ll 1. \tag{3.2}$$

- If $\mathscr{C}_{\Sigma,p}$ intersects $\mathscr{C}_{0,p}$, then

$$u(\rho, \theta) \leq (1 - \eta).\rho \ \forall \theta \in [0, \alpha) \ \rho \ll 1. \tag{3.3}$$

These two results will be used in the next part.

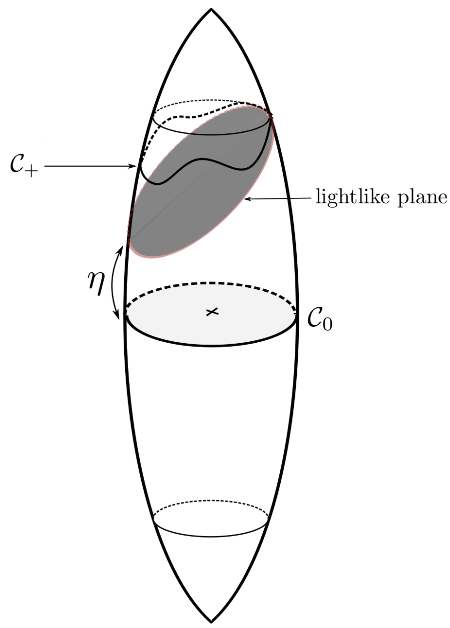

Figure 3.3: The link remains in the upper hemisphere

Link of S and orthogonality. Let S be the limit surface of Proposition 3.1.7 and let $p \in S$ be an intersection with a singular line d of angle $\alpha < \pi$. As previously, we consider locally S as the graph of a function

$$u : D_\alpha \to \mathbb{R}$$

in a neighborhood of p. Let $\mathscr{C}_{S,p} \subset \mathscr{L}_{\alpha,p}(M)$ be the "augmented" link of S at p, that is, the connected domain contained between the curves $\mathscr{C}_\pm$, where $\mathscr{C}_+$ is the curve corresponding to $\limsup_{\rho \to 0} \frac{u(\rho,\theta)}{\rho}$, and $\mathscr{C}_-$ corresponding to the liminf.

Lemma 3.1.14. *The curves $\mathscr{C}_+$ and $\mathscr{C}_-$ are $\mathscr{C}^{0,1}$.*

Proof. We give the proof for $\mathscr{C}_-$ (the one for $\mathscr{C}_+$ is analogue). For $\theta \in [0, \alpha)$, denote by

$$k(\theta) := \liminf_{\rho \to 0} \frac{u(\rho,\theta)}{\rho}.$$

Fix $\theta_0 \in [0, \alpha)$. By definition, there exists a decreasing sequence $(\rho_k)_{k \in \mathbb{N}} \subset \mathbb{R}_{>0}$ such that $\lim_{k \to \infty} \rho_k = 0$ and

$$\lim_{k \to \infty} \frac{u(\rho_k, \theta_0)}{\rho_k} = k(\theta_0).$$

As S is nowhere time-like, for each $k \in \mathbb{N}$, S remains in the cone of space-like and light-like geodesic from $((\rho_k, \theta_0), u(\rho_k, \theta_0)) \in S$. That is,

$$|u(\rho_k, \theta) - u(\rho_k, \theta_0)| \leq d_a(\theta, \theta_0)\rho_k,$$

where d_a is the angular distance between two directions. So we get

$$\lim_{k\to\infty} \frac{u(\rho_k,\theta)}{\rho_k} \leq k(\theta_0) + d_a(\theta,\theta_0),$$

and so

$$k(\theta) \leq k(\theta_0) + d_a(\theta,\theta_0).$$

On the other hand, for all $\epsilon > 0$ small enough, there exists $R > 0$ such that, for all $\rho \in (0,R)$ we have:

$$u(\rho,\theta_0) > (k(\theta_0)-\epsilon)\rho.$$

By the same argument as before, because S is nowhere time-like, we get

$$|u(\rho,\theta) - u(\rho,\theta_0)| \leq d_a(\theta,\theta_0)\rho,$$

that is

$$u(\rho,\theta) \geq u(\rho,\theta_0) - d_a(\theta,\theta_0)\rho.$$

So

$$u(\rho,\theta) > (k(\theta_0)-\epsilon)\rho - d_a(\theta,\theta_0)\rho,$$

taking $\epsilon \to 0$, we obtain

$$k(\theta) \geq k(\theta_0) - d_a(\theta,\theta_0).$$

So the function k is 1-Lipschitz □

Now we can prove Proposition 3.1.12. Suppose that S is not space-like, that is, S contains a light-like direction at an intersection with a singular line. For example, suppose that $\mathscr{C}_+$ intersects the upper light-like circle (the proof is analogue if $\mathscr{C}_-$ intersects the lower light-like circle). The proof will follow from the following Lemma:

Lemma 3.1.15. *If $\mathscr{C}_+$ intersects the future light-like circle, then $\liminf_{\rho\to 0} \frac{u(\rho,\theta)}{\rho} \geq \eta$ for all $\theta \in [0,\alpha)$.*

Proof. As $\mathscr{C}_+$ intersects the upper time-like circle, there exist $\theta_0 \in [0,\alpha)$, and $(\rho_k)_{k\in\mathbb{N}} \subset \mathbb{R}_{>0}$ a strictly decreasing sequence, converging to zero, such that

$$\lim_{k\to\infty} \frac{u(\rho_k,\theta_0)}{\rho_k} = 1.$$

From (3.2), for a fixed $\eta' < \eta$ and $k \in \mathbb{N}$ big enough,

$$u(\rho_k,\theta) \geq \eta'\rho_k \ \forall\theta \in [0,\alpha[.$$

As S has vanishing mean curvature outside its intersections with the singular locus, we can use a maximum principle. Namely if an open set U of S intersects a piece of totally geodesic plane, it has to intersect it at the boundary of U. It follows that on an open set $V \subset D_\alpha$,

$$\sup_{x\in V} u(x) = \sup_{x\in\partial V} u(x), \text{ and } \inf_{x\in V} u(x) = \inf_{x\in\partial V} u(x).$$

Now, applying the maximum principle to the open annulus $A_k := \{(\rho,\theta) \in D_\alpha,\ \rho \in (\rho_{k+1},\rho_k)\}$, we get:

$$\inf_{A_k} u = \min_{\partial A_k} u \geq \eta'\rho_{k+1}.$$

So, for all $\rho \in [0, r)$, there exists $k \in \mathbb{N}$ such that $\rho \in [\rho_{k+1}, \rho_k]$ and

$$u(\rho, \theta) \geq \eta' \rho_{k+1}. \tag{3.4}$$

We obtain that, $\forall \theta \in [0, \alpha)$, $u(\rho, \theta) > 0$ and so $\liminf_{\rho \to 0} \frac{u(\rho, \theta)}{\rho} \geq 0$.

Now, suppose that

$$\exists \theta_1 \in [0, \alpha) \text{ such that } \liminf_{\rho \to 0} \frac{u(\rho, \theta_1)}{\rho} = 0,$$

then there exists $(r_k)_{k \in \mathbb{N}} \subset \mathbb{R}_{>0}$ a strictly decreasing sequence converging to zero with

$$\lim_{k \to \infty} \frac{u(r_k, \theta_1)}{r_k} = 0.$$

Moreover, we can choose the sequences so that $r_k \in [\rho_{k+1}, \rho_k)$ $\forall k \in \mathbb{N}$.

This implies, by (3.3) that for k big enough,

$$u(r_k, \theta) \leq (1 - \eta') r_k \ \forall \theta \in [0, \alpha).$$

Now, applying the maximum principle to the open annulus $B_k := \{(\rho, \theta) \in D_\alpha, \ \rho \in (r_{k+1}, r_k)\}$, we get:

$$\sup_{B_k} u = \max_{\partial B_k} u \leq (1 - \eta') r_k.$$

And so we get that for all $\rho \in [0, r)$ there exists $k \in \mathbb{N}$ with $\rho \in [r_{k+1}, r_k]$ and we have:

$$u(\rho, \theta) \leq (1 - \eta') r_k \leq (1 - \eta') \rho_k. \tag{3.5}$$

Fix $\epsilon > 0$. As $\lim \frac{u(\rho_k, \theta_0)}{\rho_k} = 1$, for k big enough,

$$u(\rho_k, \theta_0) \geq (1 - \epsilon \eta') \rho_k.$$

By (3.5) we have

$$(1 - \epsilon.\eta').\rho_k \leq u(\rho_k, \theta_0) \leq (1 - \eta') \rho_{k+1}$$

and so:

$$\frac{\rho_{k+1}}{\rho_k} \leq \frac{1 - \epsilon.\eta'}{1 - \eta'}. \tag{3.6}$$

In the same way, using $\lim \frac{u(r_k, \theta_0)}{r_k} = 0$ and equation (3.4), we get (for k big enough):

$$\eta'.\rho_{k+1} \leq u(r_k, \theta_0) \leq \epsilon.\eta'.\rho_k$$

and so

$$\frac{\rho_{k+1}}{\rho_k} \leq \epsilon. \tag{3.7}$$

But, for $\epsilon < 1$, the conditions (3.6) and (3.7) are incompatible, so we get a contradiction □

Now, as the curve $\mathscr{C}_-$ does not cross $\mathscr{C}_{0,p}$ and is contained in the de Sitter domain, we obtain $l(\mathscr{C}_-) < l(\mathscr{C}_{0,p})$ (where l is the length). For $D_r \subset D_\alpha$ the disk of radius r and center 0 and $A_g(u(D_r))$ the area of the graph of $u_{|D_r}$, we get:

$$\begin{array}{rcl} A_g(u(D_r)) & \leq & \int_0^r l(\mathscr{C}_-)\rho d\rho \\ & < & \int_0^r l(\mathscr{C}_{0,p})\rho d\rho. \end{array}$$

The first inequality comes from the fact that $\int_0^r l(\mathscr{C}_-)\rho d\rho$ corresponds to the area of a flat piece of surface with link $\mathscr{C}_-$ which is bigger than the area of a curved surface (because we are in a Lorentzian manifold).

So, the local deformation of S sending a neighborhood of $S \cap d$ to a piece of totally geodesic disk orthogonal to the singular line would strictly increase the area of S. However, as S is a limit of a sequence of maximizing surfaces, such a deformation does not exist. So $\mathscr{C}_{S,p}$ cannot cross the light-like circles.

3.1.4 Fourth step

Here we prove the following:

Proposition 3.1.16. *The surface $S \hookrightarrow (M, g)$ of Proposition 3.1.7 is orthogonal to the singular lines.*

The proof uses a "zooming" argument: by a limit of a sequence of homotheties and rescaling, we send a neighborhood U of an intersection of the surface S with a singular line to a piece of surface U_∞ in the Minkowski space-time with cone singularity (that is in a flat singular space-time). Then we prove, using the Gauss map, that U_∞ is orthogonal to the singular line and we show that it implies the result.

Proof. For $\tau > 0$, define $\mathrm{AdS}^3_{\alpha,\tau}$ as the completion of $\mathbb{R}_{\geq 0} \times \mathbb{R}/\alpha\mathbb{Z} \times \mathbb{R}$ with the metric

$$g_{\alpha,\tau} = -dt^2 + \cos^2(t/\tau)(d\rho^2 + \tau^2 \sinh^2(\rho/\tau)d\theta^2),$$

where $(\rho, \theta, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}/\alpha\mathbb{Z} \times \mathbb{R}$. Given the coordinates (ρ, θ, t) on each $\mathrm{AdS}^3_{\alpha,\tau}$, one defines the "zoom" map

$$\begin{array}{rcccl} \mathscr{Z}_\tau & : & \mathrm{AdS}^3_\alpha & \longrightarrow & \mathrm{AdS}^3_{\alpha,\tau} \\ & & (\rho, \theta, t) & \longmapsto & (\tau\rho, \tau\theta, \tau t) \end{array}$$

and the set

$$K_\tau := (K, g_{\alpha,\tau}),$$

where $K := \{(\rho, \theta, t) \in [0, 1] \times \mathbb{R}/\alpha\mathbb{Z} \times \mathbb{R}\}$.

Let p be the intersection of the surface $S \hookrightarrow (M, g)$ of Proposition 3.1.7 with a singular line of angle α. By definition, there exists an isometry Ψ sending a neighborhood $V \subset M$ of p to a neighborhood of $0 := (0, 0, 0) \in \mathrm{AdS}^3_\alpha$. Set $U := \Psi(V \cap S)$ and $U_n := \mathscr{Z}_n(U) \cap K \subset \mathrm{AdS}^3_{\alpha,n}$ for all $n \in \mathbb{N}$. Note that the U_n are pieces of maximal space-like surface in $\mathrm{AdS}^3_{\alpha,n}$.

For all $n \in \mathbb{N}$, let $f_n : [0, 1] \times \mathbb{R}/\alpha\mathbb{Z} \longrightarrow [-1, 1]$ so that $U_n = \mathrm{graph}(f_n)$. With respect to the metric $d\rho^2 + \sinh^2(\rho)d\theta^2$ on $[0, 1] \times \mathbb{R}/\alpha\mathbb{Z}$, the sequence $(f_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded Lipschitz functions with uniformly bounded Lipschitz constant and so converges $\mathscr{C}^{0,1}$ to f_∞.

Lemma 3.1.17. *Outside its intersection with the singular line, the surface graph*$(f_\infty) \subset K$ *is space-like and has everywhere vanishing mean curvature with respect to the metric*

$$g_{\alpha,\infty} := -dt^2 + d\rho^2 + \rho^2 d\theta^2.$$

Proof. As the $U_n \subset (K, g_{\alpha,n})$ are space-like surfaces with everywhere vanishing mean curvature (outside the intersection with the singular line), they satisfy on $K \setminus \{0\}$ the following equation (see equation (3.1) using the fact that $g_{ij} = 0$ for $i \neq j$):

$$\frac{1}{\sqrt{\det g_n}}\partial_i(\sqrt{\det g_n} v_n g_n^{ii} \nabla_i f_n) + \frac{1}{2} v_n \partial_t g_n^{ii} |\nabla_i f_n|^2 - \frac{1}{2} v_n^{-1} g_n^{ii} \partial_t (g_n)_{ii} = 0.$$

Recall that here, $\det g_n$ is the determinant of the induced metric on $U_n \hookrightarrow (K, g_{\alpha,n})$, ∇f_n is the gradient of f_n and $v_n := (1 - \|\pi^* \nabla f_n\|^2)^{-1/2}$ for π the orthogonal projection of $\{(\rho, \theta, t) \in K,\ t = 0\}$.

As each f_n satisfies the vanishing mean curvature equation (equation (3.1), the same argument as in the proof of Proposition 3.1.10 implies a uniform bound on the norm of the covariant derivative of the gradient of f_n. It follows that

$$f_n \overset{\mathscr{C}^{1,1}}{\longrightarrow} f_\infty.$$

Moreover, one easily checks that on K, $g_{\alpha,n} \overset{\mathscr{C}^\infty}{\longrightarrow} g_{\alpha,\infty}$. In particular $\det h_n$ and v_n converge $\mathscr{C}^{1,1}$ to h_∞ and v_∞ (respectively). It follows that f_∞ is a weak solution of the vanishing mean curvature equation for the metric $g_{\alpha,\infty}$, and so, by a bootstrap argument, is a strong solution. In particular, graph(f_n) is a maximal surface in $(K, g_{\alpha,\infty})$. □

Remark 3.1.3. The metric $g_{\alpha,\infty}$ corresponds to the Minkowski metric with cone singularity, that is the metric obtained by cutting the classical Minkowski space $\mathbb{R}^{2,1}$ along two time-like half-planes making an angle α and intersecting along the time-like line $d := \{\rho = 0\}$, then gluing by a rotation. We denote by l the singular axis of $\mathbb{R}^{2,1}_\alpha$.

Denote by $\mathscr{N} : U_\infty \setminus \{0\} \longrightarrow \mathscr{U}\mathbb{R}^{2,1}_\alpha$ the Gauss map, that is the map send a point $p \in U_\infty \setminus \{0\}$ to the unit future pointing normal to U_∞ at x (here $\mathscr{U}\mathbb{R}^{2,1}_\alpha$ is the unit tangent bundle to $\mathbb{R}^{2,1}_\alpha$). We have the following

Lemma 3.1.18. *The Gauss map $\mathscr{N}$ takes value in the hyperbolic disk with cone singularity $\mathbb{H}^2_\alpha$ and is holomorphic with respect to the complex structure on $\mathbb{H}^2_\alpha$ associated to the reversed orientation.*

Proof. Fix a point $p \in U_\infty \setminus \{0\}$ and a simple closed loop $\gamma : [0,1] \longrightarrow U_\infty \setminus \{0\}$ based at p. By construction, $\mathbb{H}^2_\alpha$ is embedded in $\mathbb{R}^{2,1}_\alpha$ as a space-like surface orthogonal to the central axis. In fact, $\mathbb{H}^2_\alpha$ can be obtained by gluing the intersection of the angular sector of angle α in $\mathbb{R}^{2,1}$ with the future component of the hyperboloid by the rotation φ_α of angle $2\pi - \alpha$ preserving the central axis.

Fix $\hat{p} \in \widetilde{\mathbb{R}}^{2,1}_\alpha$ a lifting of p in the universal cover of $\mathbb{R}^{2,1}_\alpha \setminus d$ and denote by $\widetilde{\gamma} : [0,1] \longrightarrow \widetilde{U}_\infty \subset \widetilde{\mathbb{R}}^{2,1}_\alpha$ a piece of the lifting of $\gamma([0,1])$ with $\widetilde{\gamma}(0) = \hat{p}$. Note that $\mathbb{R}^{2,1}_\alpha \setminus d = \widetilde{\mathbb{R}}^{2,1}_\alpha / \rho([\gamma])$ where ρ is the holonomy representation of $\mathbb{R}^{2,1}_\alpha$ (so in particular, $\rho(\gamma) = \varphi_\alpha$, where now φ_α acts on $\widetilde{\mathbb{R}}^{2,1}_\alpha$).

To prove the result, it suffices to show that $\mathscr{N}(\widetilde{\gamma}([0,1])) \subset \widetilde{\mathbb{H}}^2_\alpha$. In fact, it will follow that $\mathscr{N}(\gamma([0,1])) \subset \widetilde{\mathbb{H}}^2_\alpha / \rho([\gamma]) = \mathbb{H}^2_\alpha \setminus \{0_\alpha\}$ (where $0_\alpha = l \cap \mathbb{H}^2_\alpha$).

As γ does not intersect 0, each point $m \in \widetilde{\gamma}([0,1])$ has a neighborhood U_m isometric to an open set in $\mathbb{R}^{2,1}$ by an isometry Ψ. The image V of $U_m \cap \widetilde{U}_\infty$ by Ψ is a piece of space-like surface in $\mathbb{R}^{2,1}$. Hence $\mathscr{N}(V) \subset \mathbb{H}^2$.

It follows that, if $\mathscr{N}(U_m)$ does not intersect 0_α, the set $\Psi^{-1}(\mathscr{N}(V)) \subset \widetilde{\mathbb{H}}^2_\alpha$. However, the condition $\mathscr{N}(U_m) \cap \{0_\alpha\} \neq \emptyset$ is not true in general. Denote by $\mathfrak{p} := \{p \in U_m,\ \mathscr{N}(x) = 0_\alpha\}$. We have

Lemma 3.1.19. $\mathfrak{p}$ *is discrete.*

Proof. Given $x \in \mathfrak{p}$, either the shape operator B of U_m at x is invertible or not. Let $\mathfrak{p} = \mathfrak{p}_1 \cup \mathfrak{p}_2$ where $\mathfrak{p}_1 := \{x \in \mathfrak{p},\ \det(B(x)) = 0\}$ and $\mathfrak{p}_2 := \{x \in \mathfrak{p},\ \det(B(x)) \neq 0\}$. As $\mathfrak{p}_2 = \det^{-1}(0)$ and $\det B$ is a regular map, $\mathfrak{p}_2$ is discrete. Now, if $x \in \mathfrak{p}_1$, then for each $y \in U_m$ in a neighborhood of x, $\mathscr{N}(y)$ is given by parallel transport of $\mathscr{N}(x)$ along the unique geodesic joining x to y. So $\mathscr{N}(y) \neq 0_\alpha$ and $\mathfrak{p}_1$ is discrete. □

It follows that $\mathscr{N}(U_m \setminus \mathfrak{p}) \subset \widetilde{\mathbb{H}}^2_\alpha$. Applying this construction to a finite open covering of $\widetilde{\gamma}([0,1])$, we get that, except on a discrete subset, $\mathscr{N}(\widetilde{\gamma}([0,1])) \subset \widetilde{\mathbb{H}}^2_\alpha$. In particular, there exists a discrete set $\mathscr{K} \subset U_\infty$ such that

$$\mathscr{N}(U_\infty \setminus \mathscr{K}) \subset \mathbb{H}^2_\alpha \setminus \{0_\alpha\}.$$

Now, as U_∞ is smooth at each $x \in \mathscr{K} \setminus \{0\}$, $\mathscr{N}(x)$ is well defined and by construction, $\mathscr{N}(x) = 0_\alpha \in \mathbb{H}^2_\alpha$ and so

$$\mathscr{N} : U_\infty \setminus \{0\} \longrightarrow \mathbb{H}^2_\alpha.$$

As $U_\infty \setminus \{0\}$ has everywhere vanishing mean curvature, we can choose an orthonormal framing on $U_\infty \setminus \{0\}$ such that the shape operator B of $U_\infty \setminus \{0\}$ as expression

$$B = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

Denoting h_α the metric of $\mathbb{H}^2_\alpha$, we obtain that

$$\mathscr{N}^* h_\alpha = \mathrm{I}(B.,B.) = k^2 \mathrm{I}(.,.),$$

where I is the first and third fundamental form of U_∞. That is $\mathscr{N}$ is conformal and reverses the orientation and so is holomorphic with respect to the holomorphic structure defined by the opposite orientation of $\mathbb{H}^2_\alpha$. □

Lemma 3.1.20. *The piece of surface* $U_\infty \hookrightarrow \mathbb{R}^{2,1}_\alpha$ *is orthogonal to the singular line.*

Proof. Fix complex coordinates $z : U_\infty \longrightarrow \mathbb{D}^*$ and $w : \mathbb{H}^2_\alpha \longrightarrow \mathbb{D}^*$. In these coordinates systems, the metric g_U and g_α of U_∞ and $\mathbb{H}^2_\alpha$ respectively express:

$$g_U = \rho^2(z)|dz|^2, \quad g_\alpha = \sigma^2(w)|dw|^2.$$

Note that, as $\mathbb{H}^2_\alpha$ carries a conical singularity of angle $2\pi\alpha$ at the center, $\sigma^2(w) = e^{2u}|w|^{2(\alpha-1)}$, where u is a bounded $\mathscr{C}^2$ function on $\mathbb{D}^*$ which extends to a $\mathscr{C}^0$ function on the whole disk (see Remark 2.1.1).

Denote by B the shape operator of U_∞. As U_∞ is maximal, the third fundamental form of U_∞ is given by:

$$\mathrm{III}(.,.) := g_U(B.,B.) = \mathscr{N}^* g_\alpha = k^2 g_U,$$

where $\pm k$ are the principal curvature of U_∞. In particular, $\mathscr{N} : U_\infty \longrightarrow \mathbb{H}^2_\alpha$ is conformal and so, choosing the orientation of $\mathbb{H}^2_\alpha$ so that $\mathscr{N}$ is orientation preserving, and assuming $\mathscr{N}$ does not have an essential singularity at 0, the expression of $\mathscr{N}$ in the complex charts has the form:

$$\mathscr{N}(z) = \frac{\lambda}{z^n} + f(z), \text{ where } z^n f(z) \underset{z\to 0}{\longrightarrow} 0$$

for some $n \in \mathbb{Z}$ and non-zero λ.

Denote by $e(\mathscr{N}) = \frac{1}{2}\|d\mathscr{N}\|^2$ the energy density of $\mathscr{N}$. The third fundamental form of U_∞ is thus given by

$$\mathscr{N}^* g_\alpha = e(\mathscr{N}) g_U.$$

Moreover, we have:

$$e(\mathscr{N}) = \rho^{-2}(z)\sigma^2(\mathscr{N}(z))|\partial_z \mathscr{N}|^2.$$

If $n \neq 0$, we have

$$|\partial_z \mathscr{N}|^2 = C|z|^{2(n-1)} + o\left(|z|^{2(n-1)}\right), \text{ for some } C > 0,$$

and

$$\sigma^2(\mathscr{N}(z)) = e^{2v}|z|^{2n(\alpha-1)}, \text{ for some bounded } v.$$

So we finally get,

$$\mathscr{N}^* g_\alpha = e^{2\varphi}|z|^{2(n\alpha-1)}|dz|^2, \text{ where } \varphi \text{ is bounded.}$$

For $n = 0$, the same computation gives

$$\mathscr{N}^* g_\alpha = e^{2\varphi}|dz|^2, \text{ where } \varphi \text{ is bounded from above.}$$

For $\mathscr{N}$ having an essential singularity, we get that for all $n < 0$, $|z|^n = o\left(\rho^2(z)e(\mathscr{N})\right)$ and so $\mathscr{N}^* g_\alpha$ cannot have a conical singularity.

It follows that the third fundamental form carries a conical singularity of angle $2\pi\alpha$ if and only if $n = 1$. In particular, we get that $\mathscr{N}(z) \underset{z\to 0}{\longrightarrow} 0$, which means that U_∞ is orthogonal to the singular line.

□

The proof of Proposition 3.1.16 follows:

For $\tau \in \mathbb{R}_{>0}$, let $u_\tau \in T_0\mathrm{AdS}^3_{\alpha,\tau}$ be the unit future pointing vector tangent to d at $0 = U_\tau \cap d$. For $x \in U_\tau$ close enough to 0, let $u_\tau(x)$ be the parallel transport of u_τ along the unique geodesic in U_τ joining 0 to x. Denoting by $\mathscr{N}_\tau$ the Gauss map of U_τ, we define a map:

$$\psi_\tau(x) := g_{\alpha,\tau}(u_\tau(x), \mathscr{N}_\tau(x)),$$

where $g_{\alpha,\tau}$ is the metric of $\mathrm{AdS}^3_{\alpha,\tau}$. Note that, by construction, the value of $\psi_\tau(0)$ is constant for all $\tau \in \mathbb{R}_{>0}$. As U_∞ is orthogonal to d, $\lim_{\tau\to\infty} \psi_\tau(0) = -1$ so in particular $\psi_1(0) = -1$, that is the surface S is orthogonal to the singular lines. □

3.2 Uniqueness

In this section, we show the uniqueness part of Main Theorem 2:

Proposition 3.2.1. *The maximal surface $S \hookrightarrow (M, g)$ of Proposition 3.1.1 is unique.*

Before, we give an explicit description of totally geodesic space-like plane and light-like geodesics in AdS^3_α. Let $(\rho, \theta, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}/\alpha\mathbb{Z} \times (-\pi/2, \pi/2)$ so that the metric g_α on AdS^3_α is (locally) given by

$$g_\alpha = -dt^2 + \cos^2 t(d\rho^2 + \sinh^2(\rho)d\theta^2).$$

Let $\mathscr{P}_0$ be the totally geodesic space-like plane given by the equation $\mathscr{P}_0 := \{(\rho, \theta, t) \in \mathrm{AdS}^3_\alpha,\ t = 0\}$.

Lemma 3.2.2. *In this coordinates system,*

1. *time-like geodesics orthogonal to $\mathscr{P}_0$ are given by the equations $\{\rho = cte.,\ \theta = cte.\}$.*
2. *The space-like surface at a distance $l \in (0, \pi/2)$ in the future of $\mathscr{P}_0$ is given by the equation $\{(\rho, \theta, t) \in AdS^3_\alpha,\ t = l\}$.*
3. *The totally geodesic space-like plane $\mathscr{P}_l$ orthogonal to the central axis and passing through the point $(l, 0, 0)$ is given by $\mathscr{P}_l = \{(\rho, \theta, t) \in AdS^3_\alpha,\ t = l\cosh\rho\}$.*

Proof. Let γ be a geodesic in AdS^3_α so that $|g_\alpha(\gamma', \gamma')| = 1$. The deformation of γ along the flow of a vector field J is a geodesic if and only if J satisfies the Jacobi equation

$$J'' + R(J, \gamma')\gamma' = 0, \tag{3.8}$$

where R is the Riemann curvature tensor of AdS^3_α. Note that, as AdS^3_α has constant sectional curvature -1, we have:

$$g_\alpha(R(J, \gamma')\gamma', J) = -\epsilon g_\alpha(J, J),$$

where $\epsilon = \mathrm{sign}(g_\alpha(\gamma', \gamma'))$. By taking the scalar product with J in equation (3.8), we get that J is a Jacobi field if and only if it satisfies

$$J'' - \epsilon J = 0.$$

1. If γ is a geodesic orthogonal to $\mathscr{P}_0$ passing through $(\rho_0, \theta_0, 0) \in \mathrm{AdS}^3_\alpha$, then it is a deformation of the central axis by the Jacobi field satisfying

$$\begin{cases} J'' + J = 0 \\ J(0) = u \\ J'(0) = 0, \end{cases}$$

where $u \in T_{(0,0,0)}\mathrm{AdS}^3_\alpha$ is such that $\exp(u) = (\rho_0, \theta_0, 0)$. So J is given by

$$J(t) = \cosh(t)u,$$

and $\gamma(t) = \exp(J(t))$. One easily checks that $\gamma(t) = (\rho_0, \theta_0, t)$.

2. It is a direct consequence of 1.

3. Such a $\mathscr{P}_0$ is obtained by a deformation along a Jacobi flow of every geodesic contained in $\mathscr{P}_0$ passing through $(0, 0, 0)$ satisfying

$$\begin{cases} J'' - J = 0 \\ J(0) = lN \\ J'(0) = 0, \end{cases}$$

where $N \in T_{(0,0,0)}\mathrm{AdS}^3_\alpha$ is the unit future pointing normal to $\mathscr{P}_0$. The equation of $\mathscr{P}_l$ follows.

□

Proof of Proposition 3.2.1. For a time-like curve $\gamma : [0,1] \to (M,g)$, we define its causal length by

$$l(\gamma) := \int_0^1 \big(-g(\gamma'(t),\gamma'(t))\big)^{1/2} dt.$$

Suppose that there exist two different maximal surfaces S_1 and S_2 in (M,g) where S_1 is the one of Proposition 3.1.1. Let

$$C := \sup_{\gamma\in\Gamma} l(\gamma) > 0,$$

where Γ is the set of time-like geodesic segments $\gamma : [0,1] \to M$ with $\gamma(0) \in S_1$ and $\gamma(1) \in S_2$.

Note that, from [BS09, Lemma 5.7], as $S_1 \hookrightarrow (M,g)$ is contained in the convex core, $C < \pi/2$. Consider $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ such that

$$\lim_{n\to\infty} l(\gamma_n) = C.$$

Lemma 3.2.3. *The sequence of geodesic segments $(\gamma_n)_{n\in\mathbb{N}}$ admits a subsequence which converges to $\gamma \in \Gamma$.*

Proof. Suppose for example that γ_n is future directed for n big enough, and denote by $(x_{1n})_{n\in\mathbb{N}} \subset S_1$ and $(x_{2n})_{n\in\mathbb{N}} \subset S_2$ where $x_{1n} = \gamma_n(0)$ and $x_{2n} = \gamma_n(1)$.

For $n \in \mathbb{N}$, choose a lifting $\widetilde{x_{1n}}$ of x_{1n} in the universal cover $\widetilde{M}$ of M. This choice fixes a lifting of the whole sequence $(x_{1n})_{n\in\mathbb{N}}$ and of $(\gamma_n)_{n\in\mathbb{N}}$, so of $(x_{2n})_{n\in\mathbb{N}}$ (by setting $\widetilde{x_{2n}} = \widetilde{\gamma}_n(1)$). Note that the sequence $(\widetilde{x_{1n}})_{n\in\mathbb{N}}$ converges to $\widetilde{x_1} \in \widetilde{S_1} \subset \widetilde{M}$ and, as the future of $\widetilde{x_1}$ intersects $\widetilde{S}_2$ in a compact set containing infinitely many $\widetilde{x_{2n}}$, the sequence $(\widetilde{x_{2n}})_{n\in\mathbb{N}}$ converges to $\widetilde{x}_2$ (up to a subsequence).

It follows that $\widetilde{x_2}$ projects to $x_2 \in S_2$ and C is equal to the length of the projection of the time-like geodesic segment joining $\widetilde{x}_1$ to $\widetilde{x}_2$. □

By an isometry Ψ, send the geodesic segment γ to the central axis in AdS^3_α (note that is γ is not contained in a singular line, $\alpha = 2\pi$), so that $\psi(0) = (0,0,0)$ (where we take the coordinates (ρ,θ,t) on AdS^3_α as in the beginning of this section). Note that, one easily checks that γ is orthogonal to S_1 and S_2, so Ψ sends the tangent plane to S_1 at x_1 to the plane $\mathscr{P}_0$ and the tangent plane to S_2 at x_2 to $\mathscr{P}_l$ (as defined in Lemma 3.2.2). We still denote by S_i and x_i their images by Ψ in AdS^3_α (for $i = 1,2$).

For $i = 1,2$, let $k_i \geq 0$ be the principal curvature of S_i at x_i. We can suppose, without loss of generality, that $k_1 \geq k_2$. Take $u_1 \in \mathscr{U}_{x_1}S_1$ (where $\mathscr{U}S_1$ is the unit tangent bundle of S_1) a principal direction corresponding to $-k$ and let $u_2 \in \mathscr{U}_{x_2}S_2$ be the image of u_1 by parallel transport along γ.

For $\epsilon > 0$, consider $\gamma_\epsilon \in \Gamma$ the ϵ-time deformation of γ along the Jacobi field given by $J(0) = u$ and $J'(0) = 0$. It follows from Lemma 3.2.2 that $\gamma_\epsilon \subset \{(\rho_0,\theta_0,t),\ \text{where } (\rho_0,\theta_0,0) = \exp(\epsilon u)\}$.

One easily check that the length of γ_ϵ has the following expansion:

$$l(\gamma_\epsilon) = l + \frac{1}{2}\epsilon^2 + (k_1 - \kappa_2)\epsilon^2 + o(\epsilon^2),$$

where κ_2 is the curvature of S_2 at x_2 in the direction u_2. Note that the first two terms correspond to the distance between the tangent planes $\mathscr{P}_0$ and $\mathscr{P}_l$.

It follows from our assumption $k_1 \geq k_2$ that $(k_1 - \kappa_2)\epsilon^2 \geq (k_1 - k_2)\epsilon^2 \geq 0$, so $l(\gamma_\epsilon) > l$ which is impossible. □

3.3 Consequences

3.3.1 Minimal Lagrangian diffeomorphisms

In this paragraph, we prove Main Theorem 1. Let Σ be a closed oriented surface endowed with a Riemannian metric g and let ∇ be the associated Levi-Civita connection.

Definition 3.3.1. A bundle morphism $b : T\Sigma \longrightarrow T\Sigma$ is **Codazzi** if $d^\nabla b = 0$, where d^∇ is the covariant derivative of vector valued form associated to the connection ∇.

We recall a result of [Lab92]:

Theorem 3.3.2 (Labourie)**.** *Let $b : T\Sigma \longrightarrow T\Sigma$ be a everywhere invertible Codazzi bundle morphism, and let h be the symmetric 2-tensor defined by $h = g(b.,b.)$. The Levi-Civita connection ∇^h of h satisfies*

$$\nabla^h_u v = b^{-1}\nabla_u(bv),$$

and its curvature is given by:

$$K_h = \frac{K_g}{\det(b)}.$$

Given $g_1, g_2 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and $\Psi : (\Sigma_\mathfrak{p}, g_1) \longrightarrow (\Sigma_\mathfrak{p}, g_2)$ a diffeomorphism isotopic to the identity, there exists a unique bundle morphism $b : T\Sigma_\mathfrak{p} \longrightarrow T\Sigma_\mathfrak{p}$ so that $g_2 = g_1(b.,b.)$. We have the following characterization (which proof is analogous to the one of Proposition 1.2.6):

Proposition 3.3.3. *The diffeomorphism Ψ is minimal Lagrangian if and only if*

1. *b is Codazzi with respect to g_1,*
2. *b is self-adjoint for g_1 with positive eigenvalues.*
3. $\det(b) = 1$.

We now prove Main Theorem 1:

Existence: Let $g_1, g_2 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$, by the extension of Mess' parametrization, there exists a unique AdS convex GHM metric g on $M = \Sigma_\mathfrak{p} \times \mathbb{R}$ parametrized by (g_1, g_2). From Section 2.2.2, for each space-like surface $S \hookrightarrow (M, g)$ with principal curvatures in $(-1, 1)$, first fundamental form I, shape operator B, complex structure J and identity map E, we have

$$\begin{cases} g_1(x,y) = & \mathrm{I}((E+JB)x, (E+JB)y) \\ g_2(x,y) = & \mathrm{I}((E-JB)x, (E-JB)y) \end{cases}$$

In particular, this equality holds if S is the unique maximal surface S provided by Main Theorem 2.

Define the bundle morphism $b : T\Sigma_\mathfrak{p} \longrightarrow T\Sigma_\mathfrak{p}$, by:

$$b = (E+JB)^{-1}(E-JB).$$

Note that, from the proof of Proposition 2.2.8, B extends continuously by 0 to the cone points, and so b is equal to the identity at the cone points.

Moreover, as the eigenvalues of B are in $(-1,1)$, (from [KS07, Lemma 5.15]) the morphism b is well defined. We easily check that $g_2 = g_1(b.,b.)$. We are going to prove that b satisfies the conditions of Proposition 3.3.3:

- *Codazzi:* Denote by D the Levi-Civita connection associated to I, and consider the bundle morphism $A = (E + JB)$. From Codazzi's equation for surfaces, $d^D A = 0$. From Proposition 3.3.2, the Levi-Civita connection ∇_1 of $\mathrm{I}(A.,A.)$ satisfies:

$$\nabla_{1u} v = A^{-1} D_u(Av).$$

  We get that $d^{\nabla_1} b = A^{-1} d^D (E - JB) = 0$.

- *Self-adjoint:*

$$\begin{aligned} g_1(bx,y) &= \mathrm{I}((E-JB)x,(E+JB)y) \\ &= \mathrm{I}((E+JB)(E-JB)x,y) \\ &= \mathrm{I}((E-JB)(E+JB)x,y) \\ &= \mathrm{I}((E+JB)x,(E-JB)y) \\ &= g_1(x,by). \end{aligned}$$

- *Positive eigenvalues:* From [KS07, Lemma 5.15], the eigenvalues of B are in $(-1,1)$. So $(E \pm JB)$ has strictly positives eigenvalues and the same hold for b.

- *Determinant 1:* $\det(b) = \dfrac{\det(E-JB)}{\det(E+JB)} = \dfrac{1+\det(JB)}{1+\det(JB)} = 1$, (as $tr(JB) = 0$).

Uniqueness: Suppose that there exist $\Psi_1, \Psi_2 : (\Sigma_{\mathfrak{p}}, g_1) \longrightarrow (\Sigma_{\mathfrak{p}}, g_2)$ two minimal Lagrangian diffeomorphisms. It follows from Proposition 3.3.3 that there exists $b_1, b_2 : T\Sigma_{\mathfrak{p}} \longrightarrow T\Sigma_{\mathfrak{p}}$ Codazzi self-adjoint with respect to g_1 with positive eigenvalues and determinant 1 so that $g_1(b_1.,b_1.)$ and $g_2(b_2.,b_2.)$ are in the same isotopy class.

For $i = 1,2$, define

$$\begin{cases} \mathrm{I}_i(.,.) &= \frac{1}{4} g_1((E+b_i).,(E+b_i).) \\ B_i &= -J_i(E+b_i)^{-1}(E-b_i), \end{cases}$$

where J_i is the complex structure associated to I_i.

One easily checks that B_i is well defined and self-adjoint with respect to I_i with eigenvalues in $(-1,1)$. Moreover, we have

$$b_i = (E + J_i B_i)^{-1}(E - J_i B_i).$$

Writing the Levi-Civita connection of g_1 by ∇ and the one of I_i by D^i, Proposition 3.3.2 implies

$$D^i_x y = (E + b_i)^{-1} \nabla_x ((E + b_i)y).$$

So we get:

$$\begin{aligned} D^i B_i(x,y) &= (E+b_i)^{-1}\nabla_y((E+b_i)By) - (E+b_i)^{-1}\nabla_y((E+b_i)x) - B_i[x,y] \\ &= (E+b_i)^{-1}(\nabla(E+b_i))(x,y) \\ &= 0. \end{aligned}$$

And the curvature of I_i satisfies

$$K_{\mathrm{I}_i} = -\det(E+JB_i) = -1-\det(B_i).$$

It follows that B_i is traceless, self-adjoint and satisfies the Codazzi and Gauss equation. Setting $\mathrm{II}_i := \mathrm{I}_i(B_i.,.)$, we get that I_i and II_i are respectively the first and second fundamental form of a maximal surface in an AdS convex GHM manifold with particles (that is, $(\mathrm{I}_i,\mathrm{II}_i) \in \mathscr{H}_\alpha(\Sigma_\mathfrak{p})$ where $\mathscr{H}_\alpha(\Sigma_\mathfrak{p})$ is defined in Section 2.2.3). Moreover, one easily checks that, for $i=1,2$

$$\begin{cases} g_1 &= \mathrm{I}_i\left((E+J_iB_i).,(E+J_iB_i).\right) \\ g_2 &= \mathrm{I}_i\left((E-J_iB_i).,(E-J_iB_i).\right) \end{cases}$$

It means that $(\mathrm{I}_i,\mathrm{II}_i)$ is the first and second fundamental form of a maximal surface in (M,g) (for $i=1,2$) and so, by uniqueness, $(\mathrm{I}_1,\mathrm{II}_1)=(\mathrm{I}_2,\mathrm{II}_2)$. In particular, $b_1=b_2$ and $\Psi_1=\Psi_2$.

3.3.2 Middle point in $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$

Main Theorem 1 provides a canonical identification between the moduli space $\mathscr{A}_\alpha(\Sigma_\mathfrak{p})$ of singular AdS convex GHM structure on $\Sigma_\mathfrak{p}\times\mathbb{R}$ with the space $\mathscr{H}_\alpha(\Sigma_\mathfrak{p})$ of maximal AdS germs with particles (as defined in Section 2.2.3). By the extension of Mess' parametrization, the moduli space $\mathscr{A}_\alpha(\Sigma_\mathfrak{p})$ is parametrized by $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})\times\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and by [KS07, Theorem 5.11], the space $\mathscr{H}_\alpha(\Sigma_\mathfrak{p})$ is parametrized by $T^*\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$.

It follows that we get a map

$$\varphi : \mathscr{F}_\alpha(\Sigma_\mathfrak{p})\times\mathscr{F}_\alpha(\Sigma_\mathfrak{p}) \longrightarrow T^*\mathscr{F}_\alpha(\Sigma_\mathfrak{p}).$$

We show that this map gives a "middle point" in $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$:

Proposition 3.3.4. *Let $g_1, g_2 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ be two hyperbolic metrics with cone singularities. There exists a unique conformal structure $\mathfrak{c}$ on $\Sigma_\mathfrak{p}$ so that*

$$\Phi(u_1) = -\Phi(u_2)$$

where $u_i : (\Sigma_\mathfrak{p},\mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, g_i)$ is the unique harmonic map isotopic to the identity provided by [Gel10] and $\Phi(u_i)$ is its Hopf differential. Moreover,

$$(g_1,g_2) = \varphi(\mathfrak{c}, i\Phi(u_1)).$$

Proof. Let $g_1, g_2 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and let I, B, E and J be respectively the first fundamental form, shape operator, identity and complex structure associated of the unique maximal surface S in the AdS convex GHM manifold with particles (M,g) where g is parametrized by (g_1,g_2). It follows from the definition of Mess' parametrization (see Section 2.2.2) that

$$\begin{cases} g_1(.,.) = \mathrm{I}((E+JB).,(E+JB).) \\ g_2(.,.) = \mathrm{I}((E-JB).,(E-JB).) \end{cases}$$

Let $\Psi : (\Sigma_\mathfrak{p}, g_1) \longrightarrow (\Sigma_\mathfrak{p}, g_2)$ be the unique minimal Lagrangian diffeomorphism isotopic to the identity given by Main Theorem 1.

Note that here the metrics g_1 and g_2 are normalized so that $\Psi = \mathrm{Id}$.

Denote by Γ the graph of Ψ in $(\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 \oplus g_2)$ and by h_Γ the induced metric on Γ. And easy computation shows that $h_\Gamma = 2(\mathrm{I} + \mathrm{III})$, where $\mathrm{III} = I(B., B.)$ is the third fundamental form of the maximal surface $S \hookrightarrow (M, g)$. In fact, for $u \in T\Sigma_\mathfrak{p}$, tangent vectors to Γ have the form $(u, d\Psi(u)) = (u, u)$ (and will be denoted by u when no confusion will be possible). It follows that

$$h_\Gamma(u, v) = h_l(u, v) + h_r(u, v) = 2\mathrm{I}(u, v) + 2\mathrm{I}(JBu, JBv) = 2(\mathrm{I} + \mathrm{III})(u, v).$$

Note that, as $S \hookrightarrow (M, g)$ is a maximal surface, $\mathrm{III} = k^2\mathrm{I}$. Thus the conformal class of h_Γ is equal to the conformal class of I (and will be denoted by $\mathfrak{c}$), and so J is also the complex structure of Γ.

Consider $\pi_1 : \Gamma \longrightarrow (\Sigma_\mathfrak{p}, g_1)$ and $\pi_2 : \Gamma \longrightarrow (\Sigma_\mathfrak{p}, g_2)$ the projections on the first and second factor respectively. As Γ is minimal in $(\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 \oplus g_2)$, these projections are harmonic.

The main Theorem of [Gel10] implies that these projections are the unique harmonic maps isotopic to the identity from $(\Sigma, \mathfrak{c})$ to (Σ, g_2) for $i = 1, 2$.

Now, we are going to compute $\Phi(\pi_i)$. By definition,

$$\Phi(\pi_i) = \pi_i^* h_l^{2,0},$$

that is, $\Phi(\pi_i)$ is the $(2, 0)$ part with respect to J of the pull-back of g_i.

Let (e_1, e_2) an orthonormal framing of principal directions of $S \hookrightarrow (M, g)$. So $Be_1 = ke_1$ and $Be_2 = -ke_2$.

Denote by $T^\mathbb{C}\Gamma = T\Gamma \underset{\mathbb{R}}{\otimes} \mathbb{C}$ the complexified tangent bundle of Γ, and set as usually:

$$\begin{cases} \partial_z = \frac{1}{2}(e_1 - iJe_1) = \frac{1}{2}(e_1 - ie_2) \\ \overline{\partial}_z = \frac{1}{2}(e_1 + iJe_1) = \frac{1}{2}(e_1 + ie_2) \end{cases}$$

and

$$\begin{cases} dz = dx + idy \\ d\overline{z} = dx - idy \end{cases}$$

(where dx and dy are the dual of e_1 and e_2 respectively).

Setting $\Phi(\pi_i) = \phi_i dz^2$, we get by definition

$$\phi_i = \pi_i^* g_i(\partial_z, \partial_z).$$

So

$$\begin{cases} \phi_1 = \frac{1}{4}\mathrm{I}((E + JB)(e_1 - ie_2), (E + JB)(e_1 - ie_2)) = -i\mathrm{I}(JBe_1, e_2) = -ik \\ \phi_2 = \frac{1}{4}\mathrm{I}((E - JB)(e_1 - ie_2), (E - JB)(e_1 - ie_2)) = i\mathrm{I}(JBe_1, e_2) = ik \end{cases}$$

Moreover,

$$\Re(i\Phi(\pi_1)) = \Re(kdz^2) = k(dx^2 - dy^2).$$

Uniqueness follows from the uniqueness of a minimal Lagrangian diffeomorphism isotopic to the identity. In fact, suppose that there exists $\mathfrak{c}_1$ and $\mathfrak{c}_2$ two conformal structures on $\Sigma_\mathfrak{p}$. Denoting by $u_{ij} : (\Sigma_\mathfrak{p}, \mathfrak{c}_i) \longrightarrow (\Sigma_\mathfrak{p}, g_j)$ the unique harmonic maps isotopic to the identity for $i, j = 1, 2$, then $\Psi_i := u_{i2} \circ u_{i1}^{-1} : (\Sigma_{\mathfrak{p}}, g_1) \longrightarrow (\Sigma_{\mathfrak{p}}, g_2)$ are minimal Lagrangian diffeomorphisms isotopic to the identity and so $\Psi_1 = \Psi_2 = \Psi$ by Main Theorem 1.

It follows that $\mathfrak{c}_1 = \mathfrak{c}_2$ corresponds to the conformal structure of the induced metric induced on the graph of Ψ. $\square$

Chapter 4

Case of different cone-angles

4.1 Energy functional on $\mathscr{T}(\Sigma_{\mathfrak{p}})$

Let $g_0 \in \mathscr{M}^{\alpha}_{-1}(\Sigma_{\mathfrak{p}})$ be a hyperbolic metric with cone singularities of angle $\alpha \in \left(0, \frac{1}{2}\right)^n$. We have the following result due to J. Gell-Redman [Gel10]:

Theorem 4.1.1 (J. Gell-Redman)**.** *For each $g \in \mathscr{M}^{\alpha}_{-1}(\Sigma_{\mathfrak{p}})$, there exists a unique harmonic diffeomorphism $u : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ in the isotopy class (fixing the each p_i) of the identity.*

Recall that (see Chapter 1) a harmonic map $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is a critical point of the energy, where the energy of f is defined as follow:

$$E(f) := \int_M e(f) vol_g,$$

and $e(f) = \frac{1}{2}\|df\|^2$ is called the **energy density of** f. Here, df is seen as a section of $T^*M \otimes f^*TN$ with the metric $g^* \otimes f^*h$ (g^* stands for the metric on T^*M dual to g).

Note that, when $\dim M = 2$, the energy functional only depends on the conformal class $\mathfrak{c}$ of the metric g. We denote by $u_{\mathfrak{c},g_0}$ the harmonic diffeomorphism isotopic to the identity from $(\Sigma_{\mathfrak{p}}, \mathfrak{c})$ to $(\Sigma_{\mathfrak{p}}, g_0)$.

Moreover, a complex structure $J_{\mathfrak{c}}$ on $\Sigma_{\mathfrak{p}}$ is canonically associated to $\mathfrak{c}$. It allows us to split each symmetric two forms on $\Sigma_{\mathfrak{p}}$ into its $(2,0)$, $(1,1)$ and $(0,2)$ part.

Definition 4.1.2. To a diffeomorphism $u : (\Sigma_{\mathfrak{p}}, \mathfrak{c}) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$, we associate its Hopf differential:

$$\Phi(u) := (u^* g_0)^{(2,0)},$$

that is the $(2,0)$ part of the pull-back by u of g_0.

Local expressions Let $u : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ be a diffeomorphism, z be local isothermal coordinates on (Σ, g). Set $g = \rho^2(z)|dz|^2$ and $g_0 = \sigma^2(u)|du|^2$. As usual, write $u = u^1 + iu^2$ and

$$\begin{cases} \partial_z &=& \frac{1}{2}(\partial_1 - i\partial_2), & \overline{\partial}_z &=& \frac{1}{2}(\partial_1 + i\partial_2) \\ dz &=& dx_1 + idx_2, & d\overline{z} &=& dx_1 - idx_2 \\ \partial_u &=& \frac{1}{2}(\partial_{u^1} - i\partial_{u^2}), & \overline{\partial}_u &=& \frac{1}{2}(\partial_{u^1} + i\partial_{u^2}) \end{cases}$$

We have the following expression:

$$\begin{aligned} du &= \sum_{i,j=0}^{2} \partial_i u^j dx_i \otimes \partial_{u^j} \\ &= \partial_z u dz \partial_u + \partial_z \overline{u} dz \overline{\partial}_u + \overline{\partial}_z u d\overline{z} \partial_u + \overline{\partial}_z \overline{u} d\overline{z} \overline{\partial}_u. \end{aligned}$$

It follows that

$$\begin{aligned} \Phi(u) &= u^* g_0(\partial_z, \partial_z) dz^2 \\ &= g_0\big(du(\partial_z), du(\partial_z)\big) dz^2 \\ &= \sigma^2(u) \partial_z u \partial_z \overline{u} dz^2. \end{aligned}$$

Moreover, for g^{ij} the coefficients of the metric dual to g,

$$\begin{aligned} e(u) &= \frac{1}{2} \sum_{\alpha,\beta,i,j=0}^{2} g^{ij} g_{0\alpha\beta} \partial_i u^\alpha \partial_j u^\beta \\ &= \rho^{-2}(z) \sigma^2(u) \left(|\partial_z u|^2 + |\overline{\partial}_z u|^2 \right). \end{aligned}$$

In particular, we have

$$\begin{aligned} (u^* g_0)^{(1,1)} &= \left(u^* g_0(\partial_z, \overline{\partial}_z) + u^* g_0(\overline{\partial}_z, \partial_z) \right) |dz|^2 \\ &= 2 g_0\big(du(\partial_z), du(\overline{\partial}_z)\big) |dz|^2 \\ &= \sigma^2(u)(|\partial_z u|^2 + |\overline{\partial}_z u|^2) |dz|^2 \\ &= \rho^2(z) e(u) |dz|^2. \end{aligned}$$

Note that we get the following equation for each section ξ of $T^*\Sigma_{\mathfrak{p}} \otimes u^* T\Sigma_{\mathfrak{p}}$ with the metric $g^* \otimes u^* g$:

$$\|\xi\|^2 = 4\rho^2 |\langle \xi(\partial_z), \xi(\overline{\partial}_z) \rangle|, \tag{4.1}$$

where $\langle ., . \rangle$ is the scalar product with respect to the metric g_0.

Finally, noting that the framing $(dz\partial_u, dz\overline{\partial}_u, d\overline{z}\partial_u, d\overline{z}\overline{\partial}_u)$ of $(T^*\Sigma_{\mathfrak{p}} \otimes u^* T\Sigma_{\mathfrak{p}}, g^* \otimes u^* g_0)$ is orthogonal and each vector has norm $\rho^{-1}(z)\sigma(u)$, we get the following expression for the Jacobian $J(u)$ of u:

$$\begin{aligned} J(u) &= \det{}_{g^* \otimes u^* g_0} \begin{pmatrix} \partial_z u & \partial_z \overline{u} \\ \overline{\partial}_z u & \overline{\partial}_z \overline{u} \end{pmatrix} \\ &= \rho^{-2}(z) \sigma^2(u) \left(|\partial_z u|^2 - |\overline{\partial}_z u|^2 \right). \end{aligned}$$

Remark 4.1.1.

- As in the classical case, $\Phi(u)$ is holomorphic on $(\Sigma_{\mathfrak{p}}, J_g)$ if and only if u is harmonic. So for u harmonic, $\Phi(u)$ is a meromorphic quadratic differential on (Σ, J_c) with at most simple poles at the p_i (cf. [Gel10, Section 5.1]).

- We have the following expression:

$$u^* g_0 = \Phi(u) + \rho^2(z) e(u) |dz|^2 + \overline{\Phi(u)}.$$

  Thus $\Phi(u)$ measures the difference of the conformal class of $u^* g_0$ with $\mathfrak{c}$.

Energy functional Fixing $g_0 \in \mathscr{M}^\alpha_{-1}(\Sigma_\mathfrak{p})$, we define the energy functional $\widetilde{\mathscr{E}}_{g_0}$ on the space of conformal structures of $\Sigma_\mathfrak{p}$ by:

$$\widetilde{\mathscr{E}}_{g_0}(\mathfrak{c}) := E(u_{\mathfrak{c},g_0}).$$

Proposition 4.1.3. *The energy functional $\widetilde{\mathscr{E}}_{g_0}$ descends to a functional $\mathscr{E}_{g_0}$ on $\mathscr{T}(\Sigma_\mathfrak{p})$.*

Proof. For each diffeomorphism isotopic to the identity $f \in \mathscr{D}_0(\Sigma)$, $f : (\Sigma_\mathfrak{p}, f^*\mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, \mathfrak{c})$ is holomorphic and E is invariant under holomorphic mapping (see [ES64, Proposition p.126]), that is $E(u_{\mathfrak{c},g_0}) = E(f^*u_{\mathfrak{c},g_0})$. Moreover, $f^*u_{\mathfrak{c},g_0} = u_{f^*\mathfrak{c},g_0}$. In fact,

$$f^*u_{\mathfrak{c},g_0} : (\Sigma_\mathfrak{p}, f^*\mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, g_0)$$

is harmonic. So, as $f \in \mathscr{D}_0(\Sigma)$ is isotopic to the identity, uniqueness of the harmonic diffeomorphism implies $f^*u_{\mathfrak{c},g_0} = u_{f^*\mathfrak{c},g_0}$. So $\widetilde{\mathscr{E}}_{g_0}$ is $\mathscr{D}_0(\Sigma)$-invariant and descends to a functional $\mathscr{E}_{g_0}$ on $\mathscr{T}(\Sigma_\mathfrak{p})$. $\square$

Remark 4.1.2. The same argument shows that $\mathscr{E}_{g_0}$ only depends on the class of g_0 in $\mathscr{F}_\alpha(\Sigma_\mathfrak{p})$.

Now, we are going to prove the following main result:

Theorem 4.1.4. *The energy functional $\mathscr{E}_{g_0}$ is a proper functional and its Weil-Petersson gradient at $[g] \in \mathscr{T}(\Sigma_\mathfrak{p})$ is given by $-2\Re(\Phi(u_{[g],g_0})) \in T_{[g]}\mathscr{T}(\Sigma_\mathfrak{p})$.*

4.1.1 Properness of $\mathscr{E}_{g_0}$

Recall that (Proposition 2.1.4), for each $g \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and $i \in \{1, ..., n\}$, there exists a neighborhood $V_i = \{x \in \Sigma_\mathfrak{p},\ d(x, p_i) < r_i\}$ of p_i such that

$$g_{|V_i} = d\rho_i^2 + \sinh^2 \rho_i d\theta_i^2$$

where (ρ_i, θ_i) are fixed cylindrical coordinates on V_i. We can choose the V_i such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. We denote $V := \bigcup_{i=1}^n V_i$. We need an important result, corresponding to Mumford's compactness theorem for the case of hyperbolic surfaces with cone singularities. The proof is an extension of Tromba's proof in the classical case [Tro92].

Proposition 4.1.5. *Let $(g_k)_{k\in\mathbb{N}} \subset \mathscr{M}^\alpha_{-1}(\Sigma_\mathfrak{p})$ be such that the length of every closed geodesic $\gamma^k \subset (\Sigma_\mathfrak{p} \setminus V, g_k)$ is uniformly bounded from below by $l > 0$. There exists $g \in \mathscr{M}^\alpha_{-1}(\Sigma_\mathfrak{p})$ and a sequence $(f_k)_{k\in\mathbb{N}} \subset$ Diff$(\Sigma_\mathfrak{p})$ such that*

$$f_k^* g_k \underset{\mathscr{C}^2}{\longrightarrow} g.$$

Proof. Let $(g_k)_{k\in\mathbb{N}}$ be as above. It follows that there exists $\rho > 0$ such that, for each $k \in \mathbb{N}$ and $x \in \Sigma_\mathfrak{p} \setminus V$, the injectivity radius of x is bigger than ρ (for example, take $\rho = \min\{l, r_1, ..., r_n\}$).

Fix $R > 0$ such that $R < \frac{1}{2}\rho$. As the area of $(\Sigma_\mathfrak{p} \setminus V, g_k)$ is independent of k, there exists $N > 0$ such that for each $k \in \mathbb{N}$, N is the maximum number of disjoint disks of radius $\frac{R}{2}$ in $\Sigma_\mathfrak{p}$.

That is, for each $k \in \mathbb{N}$, there exists $\left(x_1^k, ..., x_N^k\right) \subset \Sigma_\mathfrak{p} \backslash V$ such that $D_{\frac{R}{2}}\left(x_1^k\right), ..., D_{\frac{R}{2}}\left(x_N^k\right)$, $V_1, ..., V_n$ are disjoints (here $D_{\frac{R}{2}}(x_i^k) \subset \Sigma_\mathfrak{p}$ is the disk of center x_i^k and radius $\frac{R}{2}$) and $D_R(x_1^k), ..., D_R(x_N^k), V_1, ..., V_n$ is a covering of Σ_p.

For each $i, j \in \{1, ..., N\}$ with $D_R(x_i^k) \cap D_R(x_j^k) \neq \emptyset$, note that $x_i^k \in D_{2R}(x_j^k)$, $x_j^k \in D_{2R}(x_i^k)$ and, as $2R < \rho$, there exist isometries Ψ_i^k and Ψ_j^k sending $D_{2R}(x_i^k)$ (resp. $D_{2R}(x_j^k)$) to the disk B of radius $2R$ centered at 0 in $\mathbb{H}^2$.

It follows that the map $\tau_{ij}^k := \Psi_i^k \circ (\Psi_j^k)^{-1}$ is a positive local isometry of $\mathbb{H}^2$ which uniquely extend to $\tau_{ij}^k \in PSL(2, \mathbb{R})$. Moreover, for each k,

$$\tau_{ij}^k(\Psi_j^k(x_i^k)) = \Psi_i^k(x_j^k) \in B,$$

that is $(\tau_{ij}^k)_{k\in\mathbb{N}}$ is compact. So $(\tau_{ij}^k)_{k\in\mathbb{N}}$ admits a convergent subsequence whose limit is denoted by τ_{ij}.

For each $i \in \{1, ..., N\}$ and $j \in \{1, ..., n\}$ with $D_{2R}(x_i^k) \cap V_j \neq \emptyset$, there exists an isometry $\Psi_i^k : D_{2R}(x_i^k) \longrightarrow B \subset \mathbb{H}^2$ and $\psi_j : V_j \longrightarrow \mathbb{H}^2_{\alpha_j}$. As $\psi_i(D_{2R}(x_i^k) \cap V_j)$ is a simply connected subset of $\mathbb{H}^2_{\alpha_j}$, it is isometric to a subset of $B \subset \mathbb{H}^2$ by an isometry denoted Φ_j.

Pick-up a point $y^k \in D_{2R}(x_i^k) \cap V_j$. The map $\alpha_{ij}^k := \Phi_j \circ \psi_j \circ (\Psi_i^k)^{-1}$ (see Figure 4.1) is a positive local isometry of $\mathbb{H}^2$ which uniquely extends to an element of $PSL(2, \mathbb{R})$. Moreover, α_{ij}^k sends $\Psi_i^k(y)$ to $\Phi \circ \psi_j(y)$ which are both in the compact set $\overline{B} \subset \mathbb{H}^2$ (the closure of B). Then, by the same argument as before, $\alpha_{ij}^k \longrightarrow \alpha_{ij} \in PSL(2, \mathbb{R})$ (up to a subsequence).

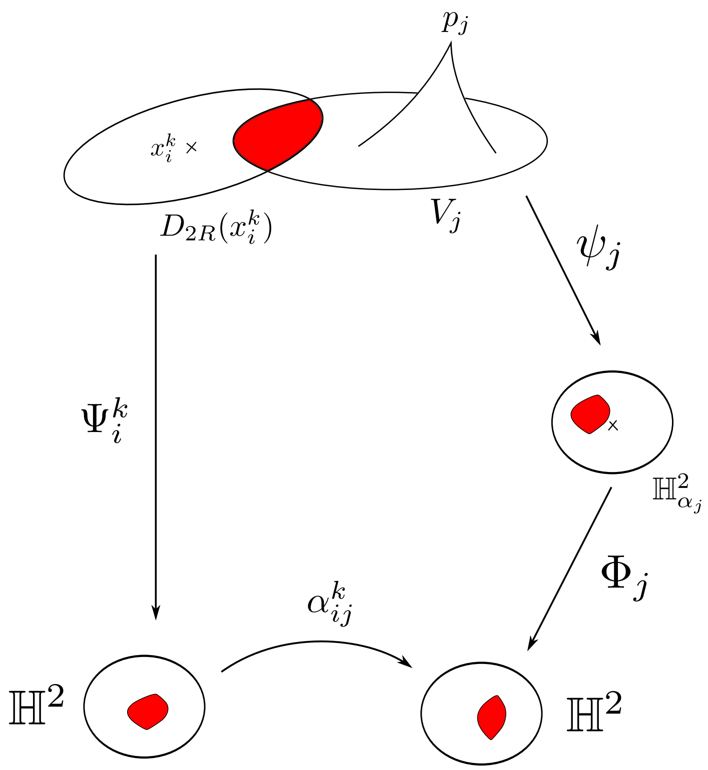

Figure 4.1: The map α_{ij}^k

Now, define

$$M := (B_1 \sqcup ... \sqcup B_N \sqcup \psi_1(V_1) \sqcup ... \sqcup \psi_n(V_n)) / \sim,$$

where $B_i = B \subset \mathbb{H}^2$ for each i and $\sim$ identifies:

- $x_i \in B_i$ with $x_j \in B_j$ whenever τ_{ij} exists and $\tau_{ij}(x_j) = x_i$.

- $x_i \in B_i$ with $x_j \in \psi_j(V_j)$ whenever α_{ij} exists and $\alpha_{ij}(x_i) = \Phi(x_j)$.

Obviously, M is an hyperbolic surface with cone singularities and defines a point $g \in \mathscr{M}_{-1}^{\alpha}(\Sigma_{\mathfrak{p}})$.

Now, we claim that there exist diffeomorphisms $f_k : M \longrightarrow (\Sigma_{\mathfrak{p}}, g_k)$ with $f_k(B_j) \subset D_R(x_j^k)$, $f_k(V_i) \subset V_i$ and such that

$$\Psi_j^k \circ f_k \underset{\mathscr{C}^2}{\longrightarrow} id \text{ on each } B_j, \text{ and } \psi_i \circ f_k \underset{\mathscr{C}^2}{\longrightarrow} id \text{ on each } \mathbb{H}^2_{\alpha_i}.$$

The proof of this claim is exactly analogous to the proof of [Tro92, Lemma C4 p.188] and will not be repeated here.

Hence, on each B_j, we have

$$f_k^* \Psi_j^{k*} g_P \underset{\mathscr{C}^2}{\longrightarrow} g_P,$$

(where g_P is the Poincaré metric) and on each V_i

$$f_k^* \psi_i^* g_{\alpha_i} \underset{\mathscr{C}^2}{\longrightarrow} g_{\alpha_i}.$$

But, as Ψ_j^k and ψ_i are isometries, we get:

$$f_k^* g_k \underset{\mathscr{C}^2}{\longrightarrow} g.$$

□

Now we are able to prove the properness of $\mathscr{E}_{g_0}$. Let $(\mathfrak{c}_k)_{k\in\mathbb{N}} \subset \mathscr{T}(\Sigma_{\mathfrak{p}})$ such that $(\mathscr{E}_{g_0}(\mathfrak{c}_k))_{k\in\mathbb{N}}$ is convergent. For each $k \in \mathbb{N}$, choose a point $g_k \in \mathscr{M}_{-1}^{\alpha}(\Sigma_{\mathfrak{p}})$ such that the conformal class of g_k is $\mathfrak{c}_k$. It follows that $E(u_{g_k,g_0}) \leq K$ for all $k \in \mathbb{N}$.

Let $\gamma \subset \Sigma_{\mathfrak{p}}$ be a simple closed curve. For each $k \in \mathbb{N}$, denote by γ_k the unique geodesic homotopic to γ in $(\Sigma_{\mathfrak{p}}, g_k)$.

First, note that there exists no geodesic homotopic to a cone point on a hyperbolic surface $\Sigma_{\mathfrak{p}}$. If fact, if γ would be such a geodesic, consider the surface obtained by taking two times the connected component of $\Sigma_{\mathfrak{p}} \setminus \gamma$ containing the cone point and glue them along γ. The remaining surface would be a hyperbolic sphere with two punctures, but it is well-know that such a hyperbolic surface does not exist.

It follows that if γ is not homotopic to a marked point, the distance between γ_k and the cone points is strictly positive. Hence we can lift a neighborhood of γ_k to the neighborhood of a piece of geodesic in $\mathbb{H}^2$. Applying [Tro92, Theorem 3.2.4] (in fact, we only need to apply collar lemma to get [Tro92, Theorem 3.2.4]), we get that:

$$l(\gamma_k) > \frac{C}{K}$$

for some constant $C > 0$.

In particular, the lenght of geodesics in $(\Sigma_{\mathfrak{p}} \setminus V, g_k)$ is uniformly bounded from below by $\frac{C}{K}$ and we can use Proposition 4.1.5. We get a family $(f_k)_{k\in\mathbb{N}} \subset \mathrm{Diff}(\Sigma_{\mathfrak{p}})$ such that $f_k^* g_k \underset{\mathscr{C}^2}{\longrightarrow} g$.

For all $k \in \mathbb{N}$, denote by $u_k : (\Sigma_{\mathfrak{p}}, \mathfrak{c}_k) \longrightarrow (\Sigma_{\mathfrak{p}}, g_0)$ the harmonic diffeomorphism isotopic to the identity. The result [Tro92, Lemma 3.2.3] easily extends to the case of singularities and implies that the sequence $(u_k)_{k\in\mathbb{N}}$ is equicontinuous. It follows that the classes of

$(f_k)_{k\in\mathbb{N}}$ in $\mathrm{Diff}(\Sigma_{\mathfrak{p}})/\mathrm{Diff}_0(\Sigma_{\mathfrak{p}})$ takes only a finite set of values. In fact, as

$$E(u_{\mathfrak{c}_k,g_0}) = E(u_{f_k^*\mathfrak{c}_k,g_0}) = E(f_k^* u_{\mathfrak{c}_k,g_0}) < K,$$

the sequence $(f_k^* u_k)_{k\in\mathbb{N}}$ is equicontinuous and admits a convergent subsequence by Arzelá-Ascoli. As $\mathrm{Diff}(\Sigma_{\mathfrak{p}})/\mathrm{Diff}_0(\Sigma_{\mathfrak{p}})$ is discrete, there exists a subsequence of $(f_k)_{k\in\mathbb{N}}$ (still denoted $(f_k)_{k\in\mathbb{N}}$ so that for k big enough, $[f_k] \in \mathrm{Diff}_0(\Sigma_{\mathfrak{p}})$ is constant. It follows that, up to a subsequence, $([f_k^*\mathfrak{c}_k])_{k\in\mathbb{N}}$ converges in $\mathscr{T}(\Sigma_{\mathfrak{p}})$.

4.1.2 Weil-Petersson gradient of $\mathscr{E}_{g_0}$

Let $\mathfrak{c} \in \mathscr{T}(\Sigma_{\mathfrak{p}})$. We are going to use real coordinates (x, y) on $(\Sigma_{\mathfrak{p}}, \mathfrak{c})$. From now on, denote by $\partial_1 := \partial_x$ and $\partial_2 := \partial_y$ and by (dx_1, dx_2) the dual framing. Denote by $u := u_{\mathfrak{c},g_0}$ and fix $\widetilde{g} \in \mathscr{M}_{-1}^{\alpha}(\Sigma_{\mathfrak{p}})$ such that the conformal class of $\widetilde{g}$ is $\mathfrak{c}$. In local coordinates, we have the following expression:

$$du = \sum_{i,j,\alpha,\beta=1}^{2} \partial_i u^\alpha dx_i \otimes \partial_{u^\alpha},$$

where (u^1, u^2) are the coordinates of u on $(\Sigma_{\mathfrak{p}}, g_0)$. Assume that (u^1, u^2) are isothermal coordinates for g_0, so

$$g_0 = \sum_{\alpha,\beta=1}^{2} \sigma^2(u)\delta_{\alpha\beta} du^\alpha du^\beta,$$

(here $\delta_{\alpha\beta}$ is the Kronecker symbol). Writing $\widetilde{g}$ in coordinates and using the Einstein convention, we have the following expression:

$$E(u) = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}}} \|du\|^2 dv_{\widetilde{g}} = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}}} \sigma^2 \delta_{\alpha\beta}\widetilde{g}^{ij}\partial_i u^\alpha \partial_j u^\beta vol_{\widetilde{g}}.$$

Here, $vol_{\widetilde{g}}$ is the volume form of $(\Sigma_{\mathfrak{p}}, \widetilde{g})$ and $\widetilde{g}^{ij}$ are the coefficients of the metric dual to $\widetilde{g}$ in $T^*\Sigma_{\mathfrak{p}}$.

For $h \in T_{\mathfrak{c}}\mathscr{T}(\Sigma_{\mathfrak{p}})$, denote by $\widetilde{h}$ the horizontal lift of $d\Theta_\alpha(h)$ in $T_{\widetilde{g}}\mathscr{M}_{-1}^{\alpha}(\Sigma_{\mathfrak{p}})$ (recall that Θ_α is the application given by the uniformization). So $\widetilde{h}$ is a zero trace divergence-free symmetric $2-$tensor on $(\Sigma_{\mathfrak{p}}, \widetilde{g})$.

We are going to compute the differential of $\widetilde{\mathscr{E}}_{g_0}$ at $\widetilde{g}$ in the direction $\widetilde{h}$. Note that the differential of $\widetilde{g} \longmapsto (\widetilde{g}^{ij})$ is given by $\widetilde{h} \longmapsto (-\widetilde{h}^{ij})$ and the differential of $\widetilde{g} \longmapsto vol_{\widetilde{g}}$ is $\widetilde{h} \longmapsto (\frac{1}{2}\mathrm{tr}_{\widetilde{g}}\widetilde{h})vol_{\widetilde{g}}$. So one gets:

$$d\widetilde{\mathscr{E}}_{g_0}(\widetilde{g})(\widetilde{h}) = -\frac{1}{2}\int_{\Sigma_{\mathfrak{p}}} \sigma^2 \widetilde{h}^{ij}\partial_i u^\alpha \partial_j u^\alpha vol_{\widetilde{g}} + \frac{1}{4}\int_{\Sigma_{\mathfrak{p}}} \sigma^2 \widetilde{g}^{ij}\partial_i u^\alpha \partial_j u^\alpha (tr_{\widetilde{g}}\widetilde{h}) vol_{\widetilde{g}} + R(\widetilde{h}),$$

where the term $R(\widetilde{h})$ is obtained by fixing $\widetilde{g}$ and $dvol_{\widetilde{g}}$ and varying the rest. It follows that $R(\widetilde{h})$ correspond to the first order variation of $E(u)$ in the direction $\widetilde{h}$. But as u is harmonic, $R(\widetilde{h}) = 0$.

Moreover, the second term is zero because we have chosen a horizontal lift of h, hence $tr_{\widetilde{g}}\widetilde{h} = 0$.

Writing $u = u^1 + iu^2$ and using the fact that $\widetilde{h}^{11} = -\widetilde{h}^{22}$ and $\widetilde{h}^{12} = \widetilde{h}^{21}$ (see Section

2.1), we get the following expression:

$$\begin{aligned} d\mathscr{E}_{g_0}(\widetilde{g})(\widetilde{h}) &= -\frac{1}{2}\int_{\Sigma_\mathfrak{p}} \sigma^2 \left(\widetilde{h}^{11}\left(|\partial_1 u|^2 - |\partial_2 u|^2\right) + 2\widetilde{h}^{12}\Re(\partial_1 u \partial_2 \overline{u})\right) vol_{\widetilde{g}} \\ &= \langle \widetilde{h}, \varphi \rangle_{S^2(\Sigma_\mathfrak{p})}, \end{aligned}$$

where

$$\varphi = -\frac{1}{2}\sigma^2(u)\left((|\partial_1 u|^2 - |\partial_2 u|^2)(dx^2 - dy^2) + 2\Re(\partial_1 u \partial_2 \overline{u})(dxdy + dydx)\right).$$

Note that, by definition, φ is the Weil-Petersson gradient $\nabla\mathscr{E}(\mathfrak{c})$ of $\mathscr{E}$ at the point $\mathfrak{c} \in \mathscr{T}(\Sigma_\mathfrak{p})$. On the other hand,

$$\begin{aligned} \Re(\Phi(u)) &= \Re(\sigma^2(u)\partial_z u \partial_z \overline{u} dz^2) \\ &= \Re\left(\frac{1}{4}\sigma^2(u)(\partial_1 u - i\partial_2 u)(\partial_1 \overline{u} - i\partial_2 \overline{u})(dx^2 - dy^2 + i(dxdy + dydx))\right) \\ &= \frac{1}{4}\sigma^2(u)\left((|\partial_1 u|^2 - |\partial_2 u|^2)(dx^2 - dy^2) + 2\Re(\partial_1 u \partial_2 \overline{u})(dxdy + dydx)\right). \end{aligned}$$

So $\nabla\mathscr{E}(\mathfrak{c}) = -2\Re(\Phi(u))$.

4.2 Minimal diffeomorphisms between hyperbolic cone surfaces

In this section, we prove the Main Theorem by studying the PDE satisfied by harmonic diffeomorphisms.

4.2.1 Existence

Proposition 4.2.1. *For each $\alpha, \alpha' \in \left(0, \frac{1}{2}\right)^n$, $g_1 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$ and $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_\mathfrak{p})$, there exists a minimal diffeomorphism $\Psi : (\Sigma_\mathfrak{p}, g_1) \longrightarrow (\Sigma_\mathfrak{p}, g_2)$ isotopic to the identity.*

Proof. Let $g_1 \in \mathscr{F}_\alpha(\Sigma_\mathfrak{p})$, $g_2 \in \mathscr{F}_{\alpha'}(\Sigma_\mathfrak{p})$ and consider $M := (\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 \oplus g_2)$. Given a conformal structure $\mathfrak{c} \in \mathscr{T}(\Sigma_\mathfrak{p})$, one can consider the map

$$f_\mathfrak{c} := (u_1, u_2) : (\Sigma_\mathfrak{p}, \mathfrak{c}) \longrightarrow M,$$

where $u_i : (\Sigma_\mathfrak{p}, \mathfrak{c}) \longrightarrow (\Sigma_\mathfrak{p}, g_i)$ is the harmonic diffeomorphism isotopic to the identity $(i = 1, 2)$.

Clearly, $E(f_\mathfrak{c}) = E(u_1) + E(u_2)$. From Section 4.1, the functional $\mathscr{E} := \mathscr{E}_{g_1} + \mathscr{E}_{g_2} : \mathscr{T}(\Sigma_\mathfrak{p}) \longrightarrow \mathbb{R}$ is proper. Let $\mathfrak{c}_0$ be a critical point of $\mathscr{E}$, so the map $\Psi := f_{\mathfrak{c}_0} : (\Sigma, \mathfrak{c}_0) \longrightarrow M$ is a harmonic immersion. We claim that Ψ is also conformal. In fact, $\Psi = (u_1, u_2)$, so

$$\begin{aligned} \Psi^*(g_1 \oplus g_2) &= u_1^* g_1 \oplus u_2^* g_2 \\ &= \Phi(u_1) + \Phi(u_2) + \rho^2(z)(e(u_1) + e(u_2))|dz|^2 + \overline{\Phi(u_1)} + \overline{\Phi(u_2)}, \end{aligned}$$

where z is a local holomorphic coordinates on $(\Sigma_\mathfrak{p}, \mathfrak{c}_0)$ such that $\Theta_\alpha(\mathfrak{c}_0) = \rho^2(z)|dz|^2$.

Now, as $\mathfrak{c}_0$ is a minimum of $\mathscr{E}$, $\nabla\mathscr{E}(\mathfrak{c}_0) = -2\Re\left(\Phi(u_1) + \Phi(u_2)\right) = 0$, so $\Phi(u_1) + \Phi(u_2) = 0$ and Ψ is conformal. It follows that Ψ is a conformal harmonic immersion, hence $\Psi(\Sigma_\mathfrak{p})$ is a minimal surface in M (see [ES64, Proposition p. 119]).

Denoting by $p_i : M \longrightarrow \Sigma_{\mathfrak{p}}$ the projection on the i-th factor ($i = 1, 2$) and $\Gamma = \Psi(\Sigma_{\mathfrak{p}})$, we get that $u_i = p_{i_{|\Gamma}}$ and $\Gamma = \mathrm{graph}(p_{2_{|\Gamma}} \circ p_{1_{|\Gamma}}^{-1})$. It follows that

$$p_{2_{|\Gamma}} \circ p_{1_{|\Gamma}}^{-1} : (\Sigma_{\mathfrak{p}}, g_1) \longrightarrow (\Sigma_{\mathfrak{p}}, g_2)$$

is a minimal diffeomorphism isotopic to the identity. □

4.2.2 Uniqueness

Before proving the rest of the Main Theorem, let's recall some results about the harmonic diffeomorphisms provided by [Gel10]. We use the same notations as in the proof above. Let z be conformal coordinates on Γ such that

$$g_\Gamma = \rho^2(z)|dz|^2, \; g_i = \sigma_i^2(u_i(z))|du_i|^2.$$

For $i = 1, 2$, set ∂u_i (respectively $\overline{\partial} u_i$) the $\mathbb{C}$-linear (respectively $\mathbb{C}$-antilinear) part of du_i. Their norms are given by

$$\begin{cases} \|\partial u_i\|^2(z) = \rho^{-2}(z)\sigma_i^2(u_i(z))|\partial_z u_i|^2 \\ \|\overline{\partial} u_i\|^2(z) = \rho^{-2}(z)\sigma_i^2(u_i(z))|\overline{\partial}_z u_i|^2. \end{cases}$$

Then we have the following expressions (cf. Section 4.1):

$$\begin{cases} \|\Phi(u_i)\| = \|\partial u_i\|\|\overline{\partial} u_i\| \\ e(u_i) = \|\partial u_i\|^2 + \|\overline{\partial} u_i\|^2 \\ J(u_i) = \|\partial u_i\|^2 - \|\overline{\partial} u_i\|^2. \end{cases}$$

Note that, as u_i is orientation preserving, $J(u_i) > 0$ and in particular $\|\partial u_i\| \neq 0$.

It is well-known that these functions satisfy Bochner type identities everywhere they are defined (see [SY78])

$$\begin{cases} \Delta \ln \|\partial u_i\| = \|\partial u_i\|^2 - \|\overline{\partial} u_i\|^2 - 1 \\ \Delta \ln \|\overline{\partial} u_i\| = -\|\partial u_i\|^2 + \|\overline{\partial} u_i\|^2 - 1, \end{cases} \tag{4.2}$$

where $\Delta = \Delta_{g_\Gamma} = \delta\delta^*$.

Note that, as $\Phi(u_i)$ is holomorphic outside $\mathfrak{p}$, the singularities of $\ln \|\overline{\partial} u_i\|$ on $\Sigma_{\mathfrak{p}}$ are isolated and have the form $c \ln r$ for some $c > 0$. In fact, as $J(u_i) > 0$, $\|\partial u_i\| \neq 0$. Because $\|\Phi(u_i)\| = \|\partial u_i\|\|\overline{\partial} u_i\|$, the singularities of $\ln \|\overline{\partial} u_i\|$ correspond to zeros of $\Phi(u_i)$.

Now, let's describe the behavior of $\|\partial u_i\|$ and $\|\overline{\partial} u_i\|$ around a puncture. Let z be a conformal coordinates system on $(\Sigma_{\mathfrak{p}}, g_\Gamma)$ centered at p. From [Gel10, Section 2.3], the map u_i has the following form around a puncture of angle $2\pi\alpha$:

$$u_i(z) = \lambda_i z + r^{1+\epsilon} f_i(z),$$

where $\lambda_i \in \mathbb{C}^*$, $r = |z|$, $\epsilon > 0$ and f is in some Banach space $\chi_b^{2,\gamma}(U)$ (where U is a open neighborhood of the puncture). We use the characterization (see [Gel10, Section 2.2]):

$$f \in \chi_b^{0,\gamma}(U) \Longleftrightarrow \sup_U |f| + \sup_{z,z' \in U} \frac{|f(z) - f(z')|}{|\theta - \theta'|^\gamma + \frac{|r-r'|^\gamma}{|r+r'|^\gamma}} < +\infty,$$

(here $z = re^{i\theta}, \; z' = r'e^{i\theta'}$) and $f \in \chi_b^{2,\gamma}(U)$ if $\varphi(f) \in \chi_b^{0,\gamma}(U)$ for all linear second order differential operator φ. Note that in particular, $f \in \mathscr{C}^2(U)$. Using

$$\begin{cases} \partial_z = & \frac{1}{2z}(r\partial_r - i\partial_\theta) \\ \overline{\partial}_z = & \frac{1}{2\overline{z}}(r\partial_r + i\partial_\theta) \end{cases}$$

we get that

$$\begin{cases} \partial_z u_i = & \lambda_i + r^\epsilon L(f_i) \\ \overline{\partial}_z u_i = & r^\epsilon \overline{L}(f_i) \end{cases}$$

where

$$\begin{cases} L = \frac{r}{2z}\left((1+\epsilon)Id + \partial_r - i\partial_\theta\right) \\ \overline{L} = \frac{r}{2\overline{z}}\left((1+\epsilon)Id + \partial_r + i\partial_\theta\right). \end{cases}$$

Let α (resp. α') be the cone angle of the singularity of g_1 (resp. g_2) at p. So, from Section 2.1, there exists some bounded non vanishing functions c_1 and c_2 so that

$$\begin{cases} \sigma_1^2(u_1) = c_1^2|u_1|^{2(\alpha-1)} \\ \sigma_2^2(u_2) = c_2^2|u_2|^{2(\alpha'-1)}. \end{cases}$$

It follows that

$$\begin{cases} \|\partial u_1\|^2 & = \rho^{-2}(z)c_1^2|\lambda_1 z + r^{1+\epsilon}f_1|^{2(\alpha-1)}|\lambda_1 + r^\epsilon L(f_1)|^2 \\ & = \rho^{-2}(z)c_1^2|\lambda_1|^{2\alpha}r^{2(\alpha-1)}\left(1+O(r^\epsilon)\right) \\ \|\overline{\partial}u_1\|^2 & = \rho^{-2}(z)c_1^2|\lambda_1|^{2(\alpha-1)}r^{2(\alpha-1)+2\epsilon}|\overline{L}(f_1)|^2(1+O(r^\epsilon)). \end{cases} \tag{4.3}$$

Proposition 4.2.2. *If $\alpha_i < \alpha_i'$ for all $i \in \{1, ..., n\}$, the minimal diffeomorphism $\Psi : (\Sigma_\mathfrak{p}, g_1) \longrightarrow (\Sigma_\mathfrak{p}, g_2)$ of Proposition 4.2.1 is unique.*

The proof follows from the stability of Γ.

Lemma 4.2.3. *Under the same conditions as in Proposition 4.2.2, a minimal graph $\Gamma \in (\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 \oplus g_2)$ is stable.*

Proof. Let Γ be a minimal graph in $(\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 \oplus g_2)$, and denote by u_i the i^{th} projection from Γ to (Σ, g_i) (for $i = 1, 2$). As Γ is minimal, the u_i are harmonic and $\Phi(u_1)+\Phi(u_2) = 0$.

Stability of minimal graph in products of surfaces has been studied for the classical case in [Wan97]. We have the following lemma:

Lemma 4.2.4. *Let Γ be a minimal graph in $(\Sigma_\mathfrak{p} \times \Sigma_\mathfrak{p}, g_1 \oplus g_2)$, then the second variation of the area functional under a deformation of Γ fixing its intersection with the singular loci is given by:*

$$A''(\Gamma) = E''(u_1) + E''(u_2) - 4\int_\Gamma \frac{\|\Phi'(u_1) + \Phi'(u_2)\|^2}{e(u_1) + e(u_2)} dv_\Gamma, \tag{4.4}$$

where E_2'' is the second variation of the energy of u_2 and $\Phi'(u_2)$ is the variation of the Hopf differential of u_2.

Proof. By definition, the area of Γ is given by:

$$A = \int_\Gamma \left(\det(u_1^* g_1 \oplus u_2^* g_2)\right)^{1/2} |dz|^2.$$

But we have:

$$\begin{aligned}\det(u_1^*g_1 \oplus u_2^*g_2) &= \det\Big(\rho^2(e(u_1)+e(u_2))|dz|^2 + 2\Re(\Phi(u_1)+\Phi(u_2))\Big)\\ &= \det\begin{pmatrix} \rho^2(e_1+e_2)+2\Re(\Phi(u_1)+\Phi(u_2)) & -2\Im(\Phi(u_1)+\Phi(u_2))\\ -2\Im(\Phi(u_1)+\Phi(u_2)) & \rho^2(e_1+e_2)-2\Re(\Phi(u_1)+\Phi(u_2))\end{pmatrix}\\ &= \rho^4(e(u_1)+e(u_2))^2 - 4|\phi(u_1)+\phi(u_2)|^2,\end{aligned}$$

where $\Phi(u_i) = \phi(u_i)dz^2$. It follows that

$$A = \int_\Gamma \Big(e(u_1)+e(u_2))^2 - 4\|\Phi(u_1)+\Phi(u_2)\|^2\Big)^{1/2} dv_\Gamma.$$

Writing

$$a := (e(u_1)+e(u_2))^2 - 4\|\Phi(u_1)+\Phi(u_2)\|^2,$$

we get

$$A = \int_\Gamma a^{1/2} dv_\Gamma.$$

Recall that, for $i=1,2$, we have

$$E(u_i) = \int_{\Sigma_\mathfrak{p}} e(u_i) dv_\Gamma.$$

Denote by $v_{1,t}$ and $v_{2,t}$ be the variations of u_1 and u_2 respectively corresponding to a variation Γ_t of Γ. Set $\psi_i := \frac{d}{dt}_{|t=0} v_{i,t}$ which is a section of $u_i^* T\Sigma_\mathfrak{p}$. Denote by ∇^{u_i} the pull-back by u_i of the Levi-Civita connection on $(\Sigma_\mathfrak{p}, g_i)$. In particular, we have:

$$\frac{d}{dt}_{|t=0} dv_{i,t} = \nabla^{u_i}\psi_i.$$

Now we have:

$$A''(\Gamma) = \frac{d^2}{dt^2}_{|t=0}\int_\Gamma a_t^{1/2} dv_\Gamma = \frac{1}{2}\int_\Gamma (a^{-1/2}a'' - \frac{1}{2}a^{-3/2}a'^2) dv_\Gamma.$$

But

$$\begin{aligned} a' &= \frac{d}{dt}_{|t=0}\Big((e(v_{1,t})+e(v_{2,t}))^2 - 4(\|\Phi(v_{1,t})+\Phi(v_{2,t})\|^2\Big)\\ &= 2(e(u_1)+e(u_2))(e'(u_1)+e'(u_2)) - 8\langle\Phi'(u_1)+\Phi'(u_2), \Phi(u_1)+\Phi(u_2)\rangle\\ &= 2(e(u_1)+e(u_2))(e'(u_1)+e'(u_2)),\end{aligned}$$

and

$$\begin{aligned} a'' &= \frac{d^2}{dt^2}_{|t=0}\Big((e(v_{1,t})+e(v_{2,t}))^2 - 4(\|\Phi(v_{1,t})+\Phi(v_{2,t})\|^2\Big)\\ &= 2(e'(u_1)+e'(u_2))^2 + 2(e(u_1)+e(u_2))(e''(u_1)+e''(u_2)) - 8\|\Phi'(u_1)+\Phi'(u_2)\|^2.\end{aligned}$$

Hence,

$$a^{-1/2}a'' - \frac{1}{2}a^{-3/2}a'^2 = 2(e''(u_1)+e''(u_2)) - 8\frac{\|\Phi'(u_1)+\Phi'(u_2)\|^2}{e(u_1)+e(u_2)}.$$

It follows

$$A''(\Gamma) = E''(u_1) + E''(u_2) - 4\int_\Gamma \frac{\|\Phi'(u_1) + \Phi'(u_2)\|^2}{e(u_1)+e(u_2)} dv_\Gamma.$$

□

Now, as pointed out in [Wan97], such a variation can be realized as a variation of u_2 only since the variation of u_1 can be interpreted as a change of coordinates which does not change the area functional. So, setting $\psi_1 = 0$, we get

$$A''(\Gamma) = E''(u_2) - 4\int_\Gamma \frac{\|\Phi'(u_2)\|^2}{e(u_1)+e(u_2)} dv_\Gamma.$$

Writing $w_i := \ln \frac{\|\partial u_i\|}{\|\overline{\partial} u_i\|}$ and using equation (4.2), we obtain:

$$\begin{aligned}
\Delta w_i &= \Delta \ln \|\partial u_i\| - \Delta \ln \|\overline{\partial} u_i\| \\
&= 2\|\partial u_i\|^2 - 2\|\overline{\partial} u_i\|^2 \\
&= 2\|\Phi\| \left(\frac{\|\partial u_i\|}{\|\overline{\partial} u_i\|} - \left(\frac{\|\partial u_i\|}{\|\overline{\partial} u_i\|}\right)^{-1} \right) \\
&= 4\|\Phi\| \sinh w_i,
\end{aligned}$$

where $\|\Phi\| = \|\Phi(u_1)\| = \|\Phi(u_2)\|$. That is, w_1 and w_2 satisfy the same equation.

As $\|\Phi\| = \|\partial u_1\|\|\overline{\partial} u_1\| = \|\partial u_2\|\|\overline{\partial} u_2\|$, then $\frac{\|\partial u_2\|}{\|\partial u_1\|} = \frac{\|\overline{\partial} u_1\|}{\|\overline{\partial} u_2\|}$. Moreover, as $J(u_i) = \|\partial u_i\|^2 - \|\overline{\partial} u_i\|^2 > 0$, then $\|\partial u_i\| > 0$ and $\frac{\|\partial u_2\|}{\|\partial u_1\|}\frac{\|\overline{\partial} u_1\|}{\|\overline{\partial} u_2\|}$ does not vanish. It follows that $w_2 - w_1$ is a regular function on $\Sigma_{\mathfrak{p}}$ satisfying:

$$\Delta(w_2 - w_1) = 4\|\Phi\|(\sinh w_2 - \sinh w_1). \tag{4.5}$$

Let's study the behavior of $w_1 - w_2$ at a singularity $p \in \mathfrak{p}$. Using the same notation as above, the norm of the Hopf differentials satisfy:

$$\begin{aligned}
\rho^2(z)\|\Phi(u_1)\|(z) &= \sigma_1^2(u_1)|\partial_z u_1||\partial_z \overline{u}_1| \\
&= c_1^2|\lambda_1 z + r^{1+\epsilon} f_1|^{2(\alpha-1)}|\lambda_1 + r^\epsilon L(f_1)||r^\epsilon \overline{L}(f_1)| \\
&= c_1^2|\overline{L}(f_1)||\lambda_i|^{2\alpha-1} r^{2(\alpha-1)+\epsilon}(1+O(r^\epsilon))
\end{aligned}$$

and

$$\begin{aligned}
\rho^2(z)\|\Phi(u_2)\|(z) &= \sigma_1^2(u_2)|\partial_z u_2||\partial_z \overline{u}_2| \\
&= c_2^2|\lambda_2 z + r^{1+\epsilon} f_2|^{2(\alpha'-1)}|\lambda_2 + r^\epsilon L(f_2)||r^\epsilon \overline{L}(f_2)| \\
&= c_2^2|\overline{L}(f_2)||\lambda_i|^{2\alpha'-1} r^{2(\alpha'-1)+\epsilon}(1+O(r^\epsilon)).
\end{aligned}$$

Hence, using $\|\Phi(u_1)\| = \|\Phi(u_2)\|$,

$$\left|\frac{\overline{L}(f_1)}{\overline{L}(f_2)}\right| = r^{2(\alpha'-\alpha)} C,$$

where C is a non-vanishing bounded function. Now, using equation (4.3), we obtain:

$$w_i = \ln\left(\frac{|\lambda_i|}{r^\epsilon|\overline{L}(f_i)|}(1+O(r^\epsilon))\right) = \ln\left(\frac{|\lambda_i|}{r^\epsilon|\overline{L}(f_i)|}\right) + O(r^\epsilon).$$

In particular,

$$w_2 - w_1 = 2(\alpha - \alpha')\ln r + C', \tag{4.6}$$

where C' is a bounded function. As $\alpha - \alpha' > 0$, $w_2 - w_1$ tends to $-\infty$ at the singularities.

So we can apply the maximum principle to equation (4.5) (recall that the Laplace-Beltrami operator in the equation has negative spectrum hence is negative at a local maximum), and we obtain that $w_2 \leq w_1$. Using $\|\Phi(u_1)\| = \|\Phi(u_2)\| = \|\Phi\|$, we finally obtain:

$$\|\partial u_2\| \leq \|\partial u_1\|.$$

Let's consider the function $f(x) = x + \|\Phi\|^2 x^{-1}$ defined on $\mathbb{R}_{>0}$. Its derivative is $f'(x) = 1 - \|\Phi\|^2 x^{-2}$, so f is increasing for $x \geq \|\Phi\|$. As $J(u_2) > 0$,

$$\|\partial u_2\|^2 \geq \|\partial u_2\|\|\overline{\partial} u_2\| = \frac{\|\Phi\|}{2}.$$

Applying f to $\|\partial u_2\|^2 \leq \|\partial u_1\|^2$, we get

$$e(u_2) \leq e(u_1).$$

So, from equation (4.4), we obtain:

$$A'' \geq E_2'' - 2\int_\Omega \frac{\|\Phi'(u_2)\|^2}{e(u_2)} vol_\Gamma.$$

Let $\psi := \frac{d}{dt}_{|t=0} v_t$ be a deformation of u_2 (so ψ is a section of $u_2^* T\Sigma_{\mathfrak{p}}$). We have the following expression (see e.g [Smi75, Equation 2]):

$$E''(u_2) = \int_\Gamma \left(\langle \nabla^{u_2}\psi, \nabla^{u_2}\psi\rangle - tr_{g_\Gamma} R^{g_2}(du_2, \psi, \psi, du_2)\right) dv_\Gamma,$$

where R^{g_2} is the curvature tensor on $(\Sigma_{\mathfrak{p}}, g_2)$, ∇^{u_2} is the pull-back by u_2 of the Levi-Civita connection on $(\Sigma_{\mathfrak{p}}, g_2)$ and the scalar product is taken with respect to the metric $g_\Gamma^* \otimes u_2^* g_2$ on $T^*\Gamma \otimes u_2^* T\Sigma_{\mathfrak{p}}$. Computing Φ', we get:

$$\begin{aligned}
\Phi' &= \frac{d}{dt}_{|t=0} v_t^* g_2(\partial_z, \partial_z) dz^2 \\
&= \frac{d}{dt}_{|t=0} g_2(dv_t(\partial_z), dv_t(\partial_z)) dz^2 \\
&= 2 g_2(\nabla^{u_2}\psi(\partial_z), du_2(\partial_z)) dz^2.
\end{aligned}$$

That is

$$\|\Phi'\|^2 = 4\sigma^2(u_2)|\langle \nabla^{u_2}\psi(\partial_z), du_2(\partial_z)\rangle|^2,$$

(where $\langle .,.\rangle$ is the scalar product with respect to g_2). By Cauchy-Schwarz and equation (4.1), we get

$$\begin{aligned}
\|\Phi'\|^2 &\leq 4\sigma^2(u)\left|\langle \nabla^{u_2}\psi(\partial_z), \overline{\nabla^{u_2}\psi(\partial_z)}\rangle\right|\left|\langle du_2(\partial_z), \overline{du_2(\partial_z)}\rangle\right| \\
&\leq \frac{1}{4}\|\nabla^{u_2}\psi\|^2\|du_2\|^2.
\end{aligned}$$

Hence,

$$\int_\Gamma \frac{\|\Phi'\|^2}{e(u_2)} vol_\Gamma \leq \frac{1}{2}\int_\Gamma \langle \nabla^u \psi, \nabla^u \psi\rangle vol_\Gamma.$$

Finally, we obtain:

$$A'' \geq -\int_\Gamma tr_{g_\Gamma} R^{g_2}(du, \psi, \psi, du) dv_\Gamma.$$

But as the sectional curvature of $(\Sigma_{\mathfrak{p}}, g_2)$ is -1, the right-hand side of the last equation is strictly positive (for a non zero ψ). So Γ is strictly stable. □

Now, using the classical estimates (see [ES64, Proposition p.126] or the proof of lemma 4.2.4),

$$\text{Area}(\Gamma) \leq E(\Psi)$$

and equality holds if and only if Ψ is a minimal immersion. It follows from the stability of Γ that the critical points of $\mathscr{E}_{g_1} + \mathscr{E}_{g_2}$ can only be minima. But a proper function whose unique extrema are minima with non-degenerate Hessian admits a unique minimum. So Ψ is the unique minimal diffeomorphism isotopic to the identity.

Chapter 5

Perspectives and Future Work

This thesis brings a set of natural questions and generalizations that we enumerate here. Some of them are related or extracted from [BBD$^+$12]:

5.1 CMC foliation

Given an AdS GHM manifold with particles (M, g), is it foliated by Constant Mean Curvature space-like surfaces?

In the classical case, it has been proved by Barbot, Béguin and Zeghib in [BBZ07].

5.2 Spin-particles AdS geometry

Given $\alpha \in \left(0, \frac{1}{2}\right)$ and $\sigma > 0$, one can consider the space obtained by cutting AdS^3 along two time-like half-plane intersecting along the central time-like curve and making an angle $2\pi\alpha$. Then, glue the two wedges by the elliptic transformation $\varphi_{\sigma,\alpha}$ composed by the rotation r_α of angle $2\pi(1-\alpha)$ fixing the central curve and the translation t_σ of (causal) length σ parallel to the central axis. We call this model the **local model for AdS space-times with spin-particles** and denote it $\mathrm{AdS}^3_{\alpha,\sigma}$.

One can define an AdS manifold with spin-particles as a Lorentz manifold of constant curvature -1 outside a singular set which is locally modelled on $\mathrm{AdS}^3_{\alpha,\sigma}$. It follows that the holonomy of an AdS manifold with spin-particles around a singular line is given (up to conjugation) by the elliptic transformation $\varphi_{\sigma,\alpha}$ described above. To compute the right and left action of $\varphi_{\sigma,\alpha}$ on $\mathbb{RP}^1$, we first fix P_0 to be the totally geodesic plane dual to the point x_∞ lying at infinity on the central axis in the Klein model of AdS^3 (see Section 1.3.1). P_0 provides an identification of the boundary $\partial\mathrm{AdS}^3$ with $\mathbb{RP}^1 \times \mathbb{RP}^1$ and, in this identification, the boundary ∂P_0 embeds diagonally in $\mathbb{RP}^1 \times \mathbb{RP}^1$.

- The action r_α on $\partial\mathrm{AdS}^3$ sends a point $(x, y) \in \mathbb{RP}^1 \times \mathbb{RP}^1$ to $(x + 2\pi(1-\alpha), y + 2\pi(1-\alpha))$, so the right and left part of r_α are two rotations on angle $2\pi(1-\alpha)$ in $PSL_2(\mathbb{R})$.
- To compute the action of t_σ on $\partial\mathrm{AdS}^3$, let $l_r \subset \partial\mathrm{AdS}^3$ be a line of the right family foliating $\partial\mathrm{AdS}^3$ (see Section 1.3.1) intersecting ∂P_0 at x. It follows that the line $t_\sigma(l_r)$ belongs to the right family and intersects ∂P_0 at $x + \sigma$. In the same way, if l_l is a line of the left family intersecting P_0 at x, its image $t_\sigma(l_l)$ intersects P_0 at $x - \sigma$ (see Figure 5.1). We obtain that the action of t_σ on $\mathbb{RP}^1 \times \mathbb{RP}^1$ is given by $t_\sigma(x, y) = (x - \sigma, y + \sigma)$.

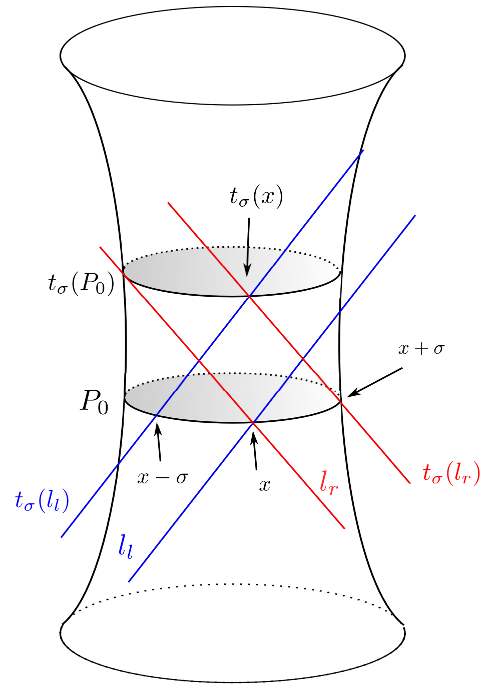

Figure 5.1: Action of t_σ

> We finally get that the left and right part of the holonomy $\varphi_{\sigma,\alpha}$ are two rotations of angles $2\pi(1-\alpha)-\sigma$ and $2\pi(1-\alpha)+\sigma$ respectively.

This kind of singularities has been studied by Barbot and Meusburger [BM12] in the flat case. In particluar, they defined a good notion of global hyperbolicity.

It is quite natural to wonder if we can interpret the minimal diffeomorphism of Main Theorem 3 as some "maximal surface" in a Globally Hyperbolic AdS space-time with spin-particles. It seems possible that the defect of angles between the two hyperbolic metrics of Main Theorem 3 is reflected in the spin of the particles.

5.3 One-harmonic maps between singular surfaces

A natural generalization of Main Theorem 3 would be to consider a pair of negatively curved metrics on $\Sigma_{\mathfrak{p}}$ with (possibly different) conical singularities of angle α and $\alpha' \in \left(0, \frac{1}{2}\right)^n$. One can ask, as in [TV95], if there exists a global minimaizer of the L^1-norm of the $\mathbb{C}$-linear part of the differential of diffeomorphisms isotopic to the identity. Such a minimizer would correspond to minimal diffeomorphisms preserving the curvature form.

It seems possible that the same kind of arguments as in [TV95] could be used to answer this question. We thank Francesco Bonsante for suggesting this question.

5.4 Surfaces of constant Gauss curvature in singular hyperbolic ends

Let $\alpha = (\alpha_1, ..., \alpha_n) \in \left(0, \frac{1}{2}\right)^n$. A **hyperbolic end with cone singularities of angle** α is a (singular) metric on $\Sigma_{\mathfrak{p}} \times [0, +\infty)$ which is hyperbolic outside the lines $d_i := p_i \times [0, +\infty)$

where $p_i \in \mathfrak{p}$ and carries a conical singularity of angle $2\pi\alpha_i$ at the d_i. Such a metric has to be complete at infinity and its restriction to the boundary component $\Sigma_{\mathfrak{p}} \times \{0\}$ is a concave pleated surface (with cone singularities). Natural examples for these singular hyperbolic ends are the complement of the convex core in a quasi-Fuchsian manifold with particles (see e.g. [KS07, LS14, MS09]).

For $k > 0$ and $g_1, g_2 \in \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}})$, it follows from Main Theorem 1 that there exists a unique $b \in \Gamma(\mathrm{End}(T\Sigma_{\mathfrak{p}}))$ self-adjoint Codazzi operator whose determinant is equal to 1 and such that $g_2 \cong g_1(b., b.)$. It follows that the operator $kb \in \Gamma(\mathrm{End}(T\Sigma_{\mathfrak{p}}))$ is self-adjoint, Codazzi, and so corresponds to the shape operator of a surface S embedded in a hyperbolic end. Such a surface has constant Gauss curvature $K = -1 - k^2$ and its first and third fundamental forms are conformal to g_1 and g_2 respectively.

It follows that we constructed a map

$$\Phi_k : \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}}) \times \mathscr{F}_\alpha(\Sigma_{\mathfrak{p}}) \longrightarrow \mathscr{E}_\alpha(\Sigma_{\mathfrak{p}}).$$

We can wonder is this map is one-to-one. In the classical case (that is without conical singularities), it has been proved by Labourie [Lab92].

5.5 Maximal surfaces in AdS space-times with interacting particles

Another interesting case of AdS manifolds with particles is the one of *"interacting particles"* as studied in [BBS11, BBS14]. When we allow the particles in a AdS space-time to have cone-angles in $[\pi, 2\pi]$, then the distance between two particles is not bounded from below anymore. In particular, we have a phenomenon of interaction and the singular locus is not a disjoint set of time-like lines but a graph.

A natural question is about the existence of a maximal surface in such AdS GHM space-times.

Notations

- Σ: closed oriented connected surface of genus $g > 1$.
- $\mathscr{U}\Sigma$ the unit tangent bundle of Σ.
- $\mathscr{M}_{-1}(\Sigma)$: set of metrics on Σ of constant curvature -1.
- $\mathscr{D}_0(\Sigma)$: set of diffeomorphisms of Σ isotopic to the identity.
- $\mathscr{F}(\Sigma) = \mathscr{M}_{-1}(\Sigma)/\mathscr{D}_0(\Sigma)$: the Fricke space of Σ, that is the space of marked hyperbolic structures on Σ.
- $\mathscr{T}(\Sigma)$: the Teichmüller space of Σ, that is the space of marked conformal structures on Σ.
- $\mathscr{A}(\Sigma)$: the moduli space of AdS GHM structures on $\Sigma \times \mathbb{R}$ (see Section 1.3.2).
- $\mathscr{H}(\Sigma)$: the moduli space of maximal AdS germs on Σ (see Section 1.3.4).
- $\Sigma_{\mathfrak{p}} := \Sigma \setminus \mathfrak{p}$ where $\mathfrak{p} = (p_1, ..., p_n) \subset \Sigma$.
- $\mathscr{M}_{-1}^{\alpha}(\Sigma_{\mathfrak{p}})$: set of hyperbolic metrics on $\Sigma_{\mathfrak{p}}$ with cone singularities of angle $\alpha \in \left(0, \frac{1}{2}\right)^n$ (see Section 2.1.2).
- $\mathscr{D}_0(\Sigma_{\mathfrak{p}})$: set of diffeomorphisms of $\Sigma_{\mathfrak{p}}$ isotopic to the identity (where the isotopies fix $\mathfrak{p}$ pointwise).
- $\mathscr{F}_\alpha(\Sigma_{\mathfrak{p}}) = \mathscr{M}_{-1}^{\alpha}(\Sigma_{\mathfrak{p}})/\mathscr{D}_0(\Sigma_{\mathfrak{p}})$: the Fricke space with cone singularities of angles α (see Definition 2.1.3).
- $\mathscr{T}(\Sigma_{\mathfrak{p}})$: the Teichmüller space of $\Sigma_{\mathfrak{p}}$.
- $\mathscr{A}_\alpha(\Sigma_{\mathfrak{p}})$: the moduli space of AdS GHM structures on $\Sigma_{\mathfrak{p}} \times \mathbb{R}$ with particles of angle α (see Section 2.2).
- $\mathscr{H}_\alpha(\Sigma_{\mathfrak{p}})$: the moduli space of maximal AdS germs with particles on $\Sigma_{\mathfrak{p}}$ (see Section 2.2.3).

For $E \longrightarrow (M, g)$ a vector bundle over a Riemannian manifold with connection ∇, we denote:

- $\Gamma(E)$: the space of smooth sections of E.
- $\Omega^k(M, E)$: the space of k-forms on M with value in E (note that $\Omega^k(M, E) = \Gamma(\bigwedge^k T^*M \otimes E)$).
- $d_\nabla : \Omega^k(M, E) \longrightarrow \Omega^{k+1}(M, E)$ the differential of vector-valued forms on M.

- $d_{\nabla}^{*} : \Omega^{k+1}(M, E) \longrightarrow \Omega^{k}(M, E)$ the dual of d_{∇}.
- $S^k(M)$: the bundle of symmetric k-tensors on M.
- $\mathrm{End}(E)$: the bundle of endomorphisms of E.

Bibliography

[AAW00] R. Aiyama, K. Akutagawa, and T. Y. H. Wan. Minimal maps between the hyperbolic discs and generalized Gauss maps of maximal surfaces in the anti-de Sitter 3-space. *Tohoku Math. J. (2)*, 52(3):415–429, 2000.

[ABB+07] L. Andersson, T. Barbot, R. Benedetti, F. Bonsante, W. M. Goldman, F. Labourie, K. P. Scannell, and J.-M. Schlenker. Notes on: "Lorentz spacetimes of constant curvature" [Geom. Dedicata **126** (2007), 3–45; mr2328921] by G. Mess. *Geom. Dedicata*, 126:47–70, 2007.

[ABBZ12] L. Andersson, T. Barbot, F. Béguin, and A. Zeghib. Cosmological time versus CMC time in spacetimes of constant curvature. *Asian J. Math.*, 16(1):37–87, 2012.

[Al′68] S. I. Al′ber. Spaces of mappings into a manifold of negative curvature. *Dokl. Akad. Nauk SSSR*, 178:13–16, 1968.

[BBD+12] T. Barbot, F. Bonsante, J. Danciger, W.M. Goldman, F. Guéritaud, F. Kassel, K. Krasnov, J.-M. Schlenker, and A. Zeghib. Some open questions on Anti-de Sitter geometry. *arXiv:1205.6103*, 2012.

[BBS11] T. Barbot, F. Bonsante, and J.-M. Schlenker. Collisions of particles in locally AdS spacetimes I. Local description and global examples. *Comm. Math. Phys.*, 308(1):147–200, 2011.

[BBS14] T. Barbot, F. Bonsante, and J.-M. Schlenker. Collisions of particles in locally AdS spacetimes II. Moduli of globally hyperbolic spaces. *Comm. Math. Phys.*, 327(3):691–735, 2014.

[BBZ07] T. Barbot, F. Béguin, and A. Zeghib. Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS_3. *Geometriae Dedicata*, 126(1):71–129, 2007.

[BG00] R. Benedetti and E. Guadagnini. Geometric cone surfaces and $(2+1)$-gravity coupled to particles. *Nuclear Phys. B*, 588(1-2):436–450, 2000.

[BM12] T. Barbot and C. Meusburger. Particles with spin in stationary flat spacetimes. *Geom. Dedicata*, 161:23–50, 2012.

[BS09] F. Bonsante and J.-M. Schlenker. AdS manifolds with particles and earthquakes on singular surfaces. *Geom. Funct. Anal.*, 19(1):41–82, 2009.

[Car98] S. Carlip. *Quantum gravity in* $2+1$ *dimensions*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1998.

[CBG69] Y. Choquet-Bruhat and R. Geroch. Global aspects of the cauchy problem in general relativity. *Communications in Mathematical Physics*, 14(4):329–335, 1969.

[Che80] J. Cheeger. On the Hodge theory of Riemannian pseudomanifolds. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 91–146. Amer. Math. Soc., Providence, R.I., 1980.

[Cor88] K. Corlette. Flat G-bundles with canonical metrics. *J. Differential Geom.*, 28(3):361–382, 1988.

[Don87] S. K. Donaldson. Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc. (3)*, 55(1):127–131, 1987.

[ES64] J. J. Eells and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.

[FT84] A. E. Fischer and A. J. Tromba. On the Weil-Petersson metric on Teichmüller space. *Trans. Amer. Math. Soc.*, 284(1):319–335, 1984.

[Gel10] J. Gell-Redman. Harmonic maps into conic surfaces with cone angles less than 2π. *arXiv:1010.4156*, 2010.

[Ger83] C. Gerhardt. H-surfaces in Lorentzian manifolds. *Comm. Math. Phys.*, 89(4):523–553, 1983.

[Gol88] W. Goldman. Topological components of spaces of representations. *Invent. Math.*, 93(3):557–607, 1988.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[Har67] P. Hartman. On homotopic harmonic maps. *Canad. J. Math.*, 19:673–687, 1967.

[Hit92] N. J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449–473, 1992.

[Hop51] H. Hopf. Über Flächen mit einer Relation zwischen den Hauptkrümmungen. *Math. Nachr.*, 4:232–249, 1951.

[JMR11] T. D Jeffres, R. Mazzeo, and Y. A Rubinstein. Kähler-Einstein metrics with edge singularities. *arXiv:1105.5216*, 2011.

[KS07] K. Krasnov and J.-M. Schlenker. Minimal surfaces and particles in 3-manifolds. *Geom. Dedicata*, 126:187–254, 2007.

[KS08] K. Krasnov and J.-M. Schlenker. On the renormalized volume of hyperbolic 3-manifolds. *Comm. Math. Phys.*, 279(3):637–668, 2008.

[KS12] K. Krasnov and J.-M. Schlenker. The Weil-Petersson metric and the renormalized volume of hyperbolic 3-manifolds. In *Handbook of Teichmüller theory. Volume III*, volume 17 of *IRMA Lect. Math. Theor. Phys.*, pages 779–819. Eur. Math. Soc., Zürich, 2012.

[Kuw96] E. Kuwert. Harmonic maps between flat surfaces with conical singularities. *Math. Z.*, 221(3):421–436, 1996.

[Lab92] F. Labourie. Surfaces convexes dans l'espace hyperbolique et $\mathbb{CP}^1$-structures. *J. London Math. Soc. (2)*, 45(3):549–565, 1992.

[Lab08] F. Labourie. Cross ratios, Anosov representations and the energy functional on Teichmüller space. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(3):437–469, 2008.

[Lam12] E. J. Lamb. The Hopf differential and harmonic maps between branched hyperbolic structures. *Thesis from Rice University*, 2012.

[LS14] C. Lecuire and J.-M. Schlenker. The convex core of quasifuchsian manifolds with particles. *Geom. Topol.*, 18(4):2309–2373, 2014.

[McO88] R. C. McOwen. Point singularities and conformal metrics on Riemann surfaces. *Proc. Amer. Math. Soc.*, 103(1):222–224, 1988.

[Mes07] G. Mess. Lorentz spacetimes of constant curvature. *Geom. Dedicata*, 126:3–45, 2007.

[Mon05a] G. Montconquiol. Déformation de métriques Einstein sur des variétés à singularité conique. *Thèse de l'Université Paul Sabatier*, 2005.

[Mon05b] G. Montcouquiol. On the rigidity of hyperbolic cone-manifolds. *C. R. Math. Acad. Sci. Paris*, 340(9):677–682, 2005.

[MRS13] R. Mazzeo, Y. A Rubinstein, and N. Sesum. Ricci flow on surfaces with conic singularities. *arXiv:1306.6688*, 2013.

[MS09] S. Moroianu and J.-M. Schlenker. Quasi-Fuchsian manifolds with particles. *J. Differential Geom.*, 83(1):75–129, 2009.

[MW93] M. J. Micallef and J. G. Wolfson. The second variation of area of minimal surfaces in four-manifolds. *Math. Ann.*, 295(2):245–267, 1993.

[Sam78] J. H. Sampson. Some properties and applications of harmonic mappings. *Ann. Sci. École Norm. Sup. (4)*, 11(2):211–228, 1978.

[Sch93] R. M. Schoen. The role of harmonic mappings in rigidity and deformation problems. In *Complex geometry (Osaka, 1990)*, volume 143 of *Lecture Notes in Pure and Appl. Math.*, pages 179–200. Dekker, New York, 1993.

[Sch98] J.-M. Schlenker. Métriques sur les polyèdres hyperboliques convexes. *J. Differential Geom.*, 48(2):323–405, 1998.

[Sch12] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2012.

[Smi75] R. T. Smith. The second variation formula for harmonic mappings. *Proc. Amer. Math. Soc.*, 47:229–236, 1975.

[Spi79] M. Spivak. *A comprehensive introduction to differential geometry. Vol. III.* Publish or Perish, Inc., Wilmington, Del., second edition, 1979.

[ST11] G. Schumacher and S. Trapani. Weil-Petersson geometry for families of hyperbolic conical Riemann surfaces. *Michigan Math. J.*, 60(1):3–33, 2011.

[SY78] R. Schoen and S. T. Yau. On univalent harmonic maps between surfaces. *Invent. Math.*, 44(3):265–278, 1978.

[Tau04] C. H. Taubes. Minimal surfaces in germs of hyperbolic 3-manifolds. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 69–100 (electronic). Geom. Topol. Publ., Coventry, 2004.

[Tei40] O. Teichmüller. Extremale quasikonforme Abbildungen und quadratische Differentiale. *Abh. Preuss. Akad. Wiss. Math.-Nat. Kl.*, 1939(22):197, 1940.

[tH93] G. 't Hooft. The evolution of gravitating point particles in $2+1$ dimensions. *Classical Quantum Gravity*, 10(5):1023–1038, 1993.

[tH96] G. 't Hooft. Quantization of point particles in $(2+1)$-dimensional gravity and spacetime discreteness. *Classical Quantum Gravity*, 13(5):1023–1039, 1996.

[Tou13] J. Toulisse. Maximal surfaces in anti-de sitter 3-manifolds with particles. *arXiv:1312.2724*, 2013.

[Tou14] J. Toulisse. Minimal diffeomorphism between hyperbolic surfaces with cone singularities. *arXiv:1411.2656*, 2014.

[Tro91] M. Troyanov. Prescribing curvature on compact surfaces with conical singularities. *Trans. Amer. Math. Soc.*, 324(2):793–821, 1991.

[Tro92] A. J. Tromba. *Teichmüller theory in Riemannian geometry.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992. Lecture notes prepared by Jochen Denzler.

[TV95] S. Trapani and G. Valli. One-harmonic maps on Riemann surfaces. *Comm. Anal. Geom.*, 3(3-4):645–681, 1995.

[Wan97] T. Y. H. Wan. Stability of minimal graphs in products of surfaces. In *Geometry from the Pacific Rim (Singapore, 1994)*, pages 395–401. de Gruyter, Berlin, 1997.

[Wol89] M. Wolf. The Teichmüller theory of harmonic maps. *J. Differential Geom.*, 29(2):449–479, 1989.