Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Corso di Laurea Magistrale in Matematica

Lie algebras and triple systems

Tesi di Laurea in Algebra

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III Sessione
Anno accademico 2013/2014

A chi non c'è più...

Introduction

Jordan algebras first appeared in a 1933 paper by P. Jordan on the foundations of quantum mechanics. The classification of simple finite-dimensional Jordan algebras over an algebraically closed field of characteristic different from two was obtained by Albert, [1], in 1947 but a much easier proof of this classification was given in the 60's, thanks to the discovery of the Tits-Kantor-Koecher ($\mathcal{TKK}$) construction, [10] [5] [7]. This is based on the observation that if $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie algebra with a short $\mathbb{Z}$-grading and f lies in $\mathfrak{g}_1$, then the formula

$$a \bullet b = [[a, f], b]$$

defines a structure of a Jordan algebra on $\mathfrak{g}_{-1}$. This leads to a bijective correspondence between simple unital Jordan algebras and simple Lie algebras with an $\mathfrak{sl}_2$-triple $\{f, h, e\}$ whose semisimple element h, with eigenvalues $0, -1, 1$, defines a short grading of $\mathfrak{g}$.

Over the years the $\mathcal{TKK}$ construction has revealed more and more relevant, due to its many generalizations.

The first natural generalization is to Jordan triple systems, whose algebraic study was initiated by K. Meyberg in 1969. A Jordan triple system is a 3-algebra whose product $\{\cdot, \cdot, \cdot\}$ satisfies the following identities:

$$\{x, y, z\} = \{z, y, x\}$$

$$\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} + [x, y, [u, v, z]]$$

Another natural generalization is to superalgebras: using the $\mathcal{TKK}$ construction V. Kac, [4], obtained in 1977 the classification of simple finite-dimensional Jordan superalgebras over a field of characteristic zero, from the classification of simple finite-dimensional Lie superalgebras. More recently, the same ideas were generalized by N. Cantarini and V. Kac, [2], in order to establish the equivalence of the category of unital linearly compact Jordan superalgebras and the category of linearly compact Lie superalgebras with a short subalgebra. This equivalence lead to the classification of infinite-dimensional linearly compact simple Jordan superalgebras.

At the same time, J. Palmkvist, [8] [9], studied how to extend the $\mathcal{TKK}$ construction to the so-called Kantor triple systems. These are a class of triple systems including Jordan triple systems. In this case a $\mathbb{Z}$-graded Lie algebra of length 5, $\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$, is associated to a Kantor triple system. This construction is undoubtedly more complicated, both from a conceptual and a technical point of view. It is worth mentioning that in the latest years triple systems have found several applications to different branches of physics, in particular to 3-dimensional supersymmetric gauge theories. For this reason the physicists community has shown great interest in these algebraic structures.

The thesis is divided into three chapters. In the first chapter the preliminary material on Jordan and $\mathbb{Z}$-graded Lie algebras is presented. The second chapter is dedicated to the Tits-Kantor-Koecher construction which is described in all details. In the third chapter the generalization of the $\mathcal{TKK}$ construction to triple systems is given. Also in this case, all details are provided. In Chapter 3, some examples are given, namely, the $\mathcal{TKK}$ construction is described in the case of $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{sl}_4$ and $\mathfrak{sp}_4$ (with short gradings induced by $\mathfrak{sl}_2$-triples).

Introduzione

Le algebre di Jordan fanno la loro prima apparizione nel 1933 in un articolo di P. Jordan sui fondamenti della meccanica quantistica. La classificazione delle algebre di Jordan semplici finito dimensionali su un campo algebricamente chiuso di caratteristica diversa da due viene ottenuta da Albert, [1], nel 1947 ma una dimostrazione meno complicata di questa classificazione viene data solo negli anni sessanta grazie alla scoperta della costruzione di Tits-Kantor-Koecher ($\mathcal{TKK}$), [10] [5] [7].
Essa si basa sull'osservazione che se $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ è un'algebra di Lie con una $\mathbb{Z}$-graduazione corta ed f appartiene a $\mathfrak{g}_1$, allora il prodotto

$$a \bullet b = [[a, f], b]$$

definisce una struttura di algebra di Jordan su $\mathfrak{g}_{-1}$. Ne deriva una corrispondenza biunivoca tra algebre di Jordan semplici con unità e algebre di Lie con una $\mathfrak{sl}_2$-tripla $\{f, h, e\}$ il cui elemento semisemplice h, con autovalori $0, -1, 1$, definisce una $\mathbb{Z}$-graduazione corta su $\mathfrak{g}$.

Nel corso degli anni la costruzione $\mathcal{TKK}$ si è rivelata sempre più importante, grazie alle sue molteplici generalizzazioni.

Una prima naturale generalizzazione è ai Jordan triple systems, il cui studio viene cominciato da K. Meyberg nel 1969. Un Jordan triple system è una 3-algebra il cui prodotto $\{\cdot, \cdot, \cdot\}$ soddisfa le seguenti relazioni:

$$\{x, y, z\} = \{z, y, x\}$$

$$\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} + [x, y, [u, v, z]]$$

Un'altra generalizzazione è alle superalgebre: usando la costruzione $\mathcal{TKK}$ V. Kac, [4], ottiene nel 1977 la classificazione delle superalgebre di Jordan semplici finito dimensionali su un campo di caratteristica zero, tramite la classificazione delle superalgebre di Lie semplici di dimensione finita. Di recente, generalizzando la stessa idea N. Cantarini e V. Kac, [2], dimostrano l'equivalenza tra la categoria delle superalgebre di Jordan unitarie linearmente compatte e la categoria delle superalgebre di Lie linearmente compatte con una $\mathbb{Z}$-graduazione corta. Grazie a questa equivalenza viene ottenuta la classificazione delle superalgebre di Jordan semplici linearmente compatte infinito dimensionali.

Contemporaneamente, J. Palmkvist, [8] [9], estende la costruzione $\mathcal{TKK}$ ai Kantor triple systems. Questi ultimi costituiscono una classe di triple systems contenente i Jordan triple systems. In questo caso viene associata ad un Kantor triple system un'algebra di Lie $\mathbb{Z}$-graduata di lunghezza 5, $\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$. Nel caso dei Kantor triple systems la costruzione si rivela senza dubbio più complicata, sia concettualmente che tecnicamente.

Vale la pena di sottolineare che negli ultimi anni i triple systems hanno trovato numerose applicazioni a branche diverse della fisica, in particolare alle teorie di gauge tridimensionali supersimmetriche. Per questo motivo la comunità fisica ha rivolto un grande interesse a queste stutture algebriche.

La tesi si divide in tre capitoli. Nel primo vengono introdotti definizioni ed esempi di algebre di Jordan e di algebre di Lie $\mathbb{Z}$-graduate. Il secondo capitolo è dedicato alla costruzione di Tits-Kantor-Koecher descritta in ogni dettaglio. Nel terzo capitolo la costruzione $\mathcal{TKK}$ viene estesa ai triple systems. Anche in questo caso vengono forniti tutti i dettagli. Inoltre, nel terzo capitolo vengono trattati gli esempi $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{sl}_4$ e $\mathfrak{sp}_4$ (con $\mathbb{Z}$-graduazione corta indotta da una $\mathfrak{sl}_2$-tripla).

Contents

Chapter 1

Preliminary notions on algebras

1.1 Jordan algebras and Lie algebras

In what follows we will denote by $\mathbb{F}$ the base field. We will always assume $char\, \mathbb{F} = 0$.

Definition 1.1 (Algebra)**.** An *algebra* $(A, \cdot\ )$ is an $\mathbb{F}$-vector space A with a product, i.e., a bilinear map

$$\cdot\ : A \times A \to A.$$

We will say that A is associative if the product satisfies the following relation

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{1.1}$$

Definition 1.2 (Jordan algebra)**.** A *Jordan algebra* is an algebra $(A, \cdot\ )$ whose product satisfies the following axioms:

$$\begin{array}{cr} x \cdot y = y \cdot x & (commutativity) \\ (x^2 \cdot y) \cdot x - x^2 \cdot (y \cdot x) = 0 & (Jordan\ identity) \end{array} \tag{1.2}$$

Definition 1.3 (Lie algebra)**.** A *Lie algebra* is an algebra $(A\,, [\,,\,]\ )$ whose product satisfies the following axioms:

$$\begin{array}{cr} [x, y] = -[y.x] & (anticommutativity) \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 & (Jacobi\ identity) \end{array} \tag{1.3}$$

Example 1.1.1. Let $(A, \cdot)$ be an associative algebra.
Then:

a) $A^+ = (A, \bullet)$, where $x \bullet y = \frac{1}{2}(x \cdot y + y \cdot x)$, is a Jordan algebra.
Indeed $\bullet$ is commutative and it satisfies the Jordan identity since $\cdot$ is associative:

$$(x^2 \bullet y) \bullet x - x^2 \bullet (y \bullet x) = \frac{1}{2}(\frac{1}{2}(x^2 \cdot y + y \cdot x^2) \cdot x + x \cdot \frac{1}{2}(x^2 \cdot y + y \cdot x^2)) -$$
$$-\frac{1}{2}(x^2 \cdot \frac{1}{2}(x \cdot y + y \cdot x) + \frac{1}{2}(x \cdot y + y \cdot x) \cdot x^2) =$$
$$= \frac{1}{4}(x^2 \cdot y \cdot x + y \cdot x^3 + x^3 \cdot y + x \cdot y \cdot x^2) -$$
$$-\frac{1}{4}(x^3 \cdot y + x^2 \cdot y \cdot x + x \cdot y \cdot x^2 + y \cdot x^3) = 0$$

b) $A^- = (A, [\ ,\ ])$ where $[x, y] = \frac{1}{2}(x \cdot y - y \cdot x)$ is a Lie algebra.
The product $[\ ,\ ]$ is anti-commutative and satisfies the Jacobi identity.
Indeed:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = \frac{1}{2}(x \cdot \frac{1}{2}(y \cdot z - z \cdot y) - \frac{1}{2}(y \cdot z - z \cdot y) \cdot x) +$$
$$+\frac{1}{2}(y \cdot \frac{1}{2}(z \cdot x - x \cdot z) - \frac{1}{2}(z \cdot x - x \cdot z) \cdot y) + \frac{1}{2}(z \cdot \frac{1}{2}(x \cdot y - y \cdot x) - \frac{1}{2}(x \cdot y - y \cdot x) \cdot z) =$$
$$= \frac{1}{4}(x \cdot y \cdot z - x \cdot z \cdot y - y \cdot z \cdot x + z \cdot y \cdot x) +$$
$$+\frac{1}{4}(y \cdot z \cdot x - y \cdot x \cdot z - z \cdot x \cdot y + x \cdot z \cdot y) +$$
$$+\frac{1}{4}(z \cdot x \cdot y - z \cdot y \cdot x - x \cdot y \cdot z + y \cdot x \cdot z) = 0$$

Definition 1.4 ($gl(V)$)**.** Let $(End(V), \circ)$ be the associative algebra of endomorphisms of the vector space V, with product given by the composition of endomorphisms. We set $gl(V) = End(V)^-$, i.e. $gl(V)$ is the Lie algebra obtained from $End(V)$ as shown in Example 1.1.1 (b. If the dimension of V is n we will use $\mathfrak{gl}_n(\mathbb{F}) \cong gl(V)$.

Definition 1.5 ($gl(V)^+$)**.** Let $(End(V), \circ)$ be the associative algebra of endomorphism of the vector space V. We define $gl(V)^+ = End(V)^+$ to be the Jordan algebra obtained from $End(V)$ as shown in Example 1.1.1 (a. If the dimension of V is n we will use $\mathfrak{gl}_n(\mathbb{F})^+ \cong gl(V)^+$.

Definition 1.6 (Subalgebras, Ideals and Simple algebras)**.** Let $(A, \cdot)$ be an algebra. A *subalgebra* B of A is a subspace of A which is closed under multiplication, i.e. $B \cdot B \subseteq B$. An *ideal* I of A is a subspace of A which is closed under multiplication by A, i.e. $A \cdot I \subseteq I, I \cdot A \subseteq I$. An algebra A is *simple* if it has no proper ideal, i.e. the only ideals of A are 0 and A itself.

Example 1.1.2 ($\mathfrak{sl}_2(\mathbb{F})$)**.** We denote by $\mathfrak{sl}_2(\mathbb{F})$ the simple Lie subalgebra of $\mathfrak{gl}_2(\mathbb{F})$ given by the elements of trace 0.
Note that

$$\mathfrak{sl}_2(\mathbb{F}) =< f, h, e >,$$

where $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
We have $[e, f] = h$, $[h, f] = -2f$ and $[h, e] = 2e$.

Example 1.1.3 ($\mathfrak{B}_n(\beta, \mathbb{F})$)**.** Let β be a non-degenerate symmetric bilinear form on V and suppose $dim(V) = n$. We denote by $\mathfrak{B}_n(\beta, \mathbb{F})$ the following, simple, Jordan subalgebra of $\mathfrak{gl}_n(\mathbb{F})^+$

$$\mathfrak{B}_n(\beta, \mathbb{F})) = \{a \in \mathfrak{gl}_n(\mathbb{F})^+ | \beta(a(x), y) = \beta(x, a(y)), \forall x, y \in V\}.$$

Example 1.1.4 ($\mathfrak{B}_2(\sigma, \mathbb{F})$)**.** Let $x = (x_1, x_2)^t, y = (y_1, y_2)^t \in \mathbb{F}^2$ and $\sigma(x, y) = x_1 y_2 + x_2 y_1$. Then

$$\mathfrak{B}_2(\sigma, \mathbb{F})) =< F,\ Id,\ E >,$$

where $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
We have $Id \bullet Id = Id$, $Id \bullet E = E$, $Id \bullet F = F$, $F \bullet E = \frac{1}{2} Id$ and $E \bullet E = F \bullet F = 0$.

Definition 1.7 (Morphism of algebras)**.** A *morphism of algebras* $\phi : (A, \cdot) \to (A', \cdot')$ is a linear map $\phi : A \to A'$ of vector spaces such that $\phi(x \cdot y) = \phi(x) \cdot' \phi(y)$. A bijective morphism is called an *isomorphism*.

In the case of unital algebras we will suppose that the morphism sends unit to unit, i.e. $\phi(e) = e'$.

Definition 1.8 ($\mathbb{Z}$-Graded algebra)**.** An algebra $(A, \cdot)$ is called $\mathbb{Z}$*-graded* if:

$$A = \bigoplus_{i \in \mathbb{Z}} A_i \qquad A_i \cdot A_j \subseteq A_{i+j} \tag{1.4}$$

with A_i subspaces of A. We will say that a $\mathbb{Z}$-graded algebra is *shortly graded* if $A_i = 0 \ \ \forall i$ such that $|i| > 1$:

$$A = A_{-1} \oplus A_0 \oplus A_1 \tag{1.5}$$

We will say that a $\mathbb{Z}$-graded algebra is *5-graded* if $A_i = 0 \ \ \forall i$ such that $|i| > 2$:

$$A = A_{-2} \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus A_2 \tag{1.6}$$

Remark 1. If $\mathfrak{g}$ is a simple graded Lie algebra we have $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.

Definition 1.9 (Graded involution)**.** Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie algebra. A graded involution τ is an automorphism of $\mathfrak{g}$ such that $\tau(\tau(g)) = g,\ \forall g \in \mathfrak{g}$ and $\tau(\mathfrak{g}_i) = \mathfrak{g}_{-i}$.

Definition 1.10. An sl_2-triple is an isomorphic copy of the Lie algebra sl_2.

Definition 1.11. Let $\mathfrak{g}$ be a shortly graded Lie algebra. The grading is said to be induced by an sl_2-triple $\{f, h, e\}$ if

$$[h, g] = -g \iff g \in \mathfrak{g}_{-1}, \quad [h, g] = 0 \iff g \in \mathfrak{g}_0, \quad [h, g] = g \iff g \in \mathfrak{g}_1.$$

Definition 1.12 (Lie algebra module)**.** Let $(A, [\ ,\ ])$ be a Lie algebra and let V be a vector space. V is an A-*module* if there is a bilinear operation $\ .\ : A \times V \to V$ such that

$$[a, b].v = a.(b.v) - b.(a.v) \quad \forall a, b \in A, v \in V.$$

Definition 1.13 (Submodule)**.** A subspace W of an A-module V is an $A-$*submodule* of V if $A.w \in W\ \forall w \in W$. V is said irreducible if it has no proper submodule.

Chapter 2

The Tits-Kantor-Koecher construction

In this chapter we will show a way to 'embed' a Jordan algebra into a Lie algebra via the so-called Tits-Kantor-Koecher construction, [10], [5], [7].
We start from a Jordan algebra $\mathfrak{J}$ in order to obtain a Lie algebra $Lie(\mathfrak{J})$ containing $\mathfrak{J}$.

2.1 A construction of shortly graded Lie algebras

In this section we will show how a triple of vector fields $(\mathfrak{h}; \mathfrak{a}, \mathfrak{b})$ can be turned into a shortly graded Lie Algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.
Let us consider a shortly-graded vector space $\mathfrak{a} \oplus \mathfrak{h} \oplus \mathfrak{b}$ where

(1) $\mathfrak{h}$ is a Lie algebra ;

(2) $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{h}$-modules;

(3) there exists a bilinear map $\Box : \mathfrak{a} \times \mathfrak{b} \to \mathfrak{h}$.

Definition 2.1. Let $(\mathfrak{h}; \mathfrak{a}, \mathfrak{b})$ be a triple of spaces satisfying (1), (2) and (3). We define on $\mathfrak{L} = \mathfrak{a} \oplus \mathfrak{h} \oplus \mathfrak{b}$ the following product: for $x_1 = a_1 + h_1 + b_1$, $x_2 = a_2 + h_2 + b_2 \in \mathfrak{L}$ we set $[x_1, x_2] = a + h + b$ with

$$h = [h_1, h_2] + a_1 \Box b_2 - a_2 \Box b_1; \quad a = h_1.a_2 - h_2.a_1; \quad b = h_1.b_2 - h_2.b_1; \qquad (2.1)$$

where $[h_1, h_2]$ is the product of $\mathfrak{h}$. We will use the notation $[\,,\,]$ for both the product in $\mathfrak{h}$ and in $\mathfrak{L}$ since the restriction of the latter to $\mathfrak{h} \times \mathfrak{h}$ coincides with the first.

Remark 2. If $h \in \mathfrak{h}, a \in \mathfrak{a}, b \in \mathfrak{b}$, we have $[h,a] = -[a,h] = h.a$, $[h,b] = -[b,h] = h.b$ and $[a,b] = -[b,a] = a\Box b$, so that $\mathfrak{L}$ is *anti-commutative*. The following inclusions follow from Definition 2.1

$$[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h};\ [\mathfrak{h},\mathfrak{a}] \subseteq \mathfrak{a};\ [\mathfrak{h},\mathfrak{b}] \subseteq \mathfrak{b};\ [\mathfrak{a},\mathfrak{a}] = [\mathfrak{b},\mathfrak{b}] = 0;\ [\mathfrak{a},\mathfrak{b}] = a\Box b \subseteq \mathfrak{h} \tag{2.2}$$

Therefore $(\mathfrak{L}, [\ ,\ ])$ is a shortly graded algebra.

Theorem 2.1.1. *Let $(\mathfrak{h};\mathfrak{a},\mathfrak{b})$ satisfies conditions (1), (2) and (3). Then $\mathfrak{L} = \mathfrak{a} \oplus \mathfrak{h} \oplus \mathfrak{b}$, as in Definition 2.1, is a Lie algebra if and only if the following relations hold:*

$$[h, a\Box b] = h.a\Box b + a\Box h.b \tag{2.3}$$

$$(a_1\Box b).a_2 = (a_2\Box b).a_1, \qquad (a\Box b_1).b_2 = (a\Box b_2).b_1 \tag{2.4}$$

for $h \in \mathfrak{h},\ a, a_1, a_2 \in \mathfrak{a}$ and $b, b_1, b_2 \in \mathfrak{b}$.

Proof. By Remark 2 we know that $(\mathfrak{L}, [\ ,\ ])$ is anti-commutative. Therefore we shall prove that the product $[\ ,\ ]$ satisfies the Jacobi identity if and only if relations (2.3) and (2.4) hold.
For this purpose we introduce the jacobian

$$J(x_1, x_2, x_3) = [x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]]$$

for $x_1, x_2, x_3 \in \mathfrak{L}$.
We have

$$J(x_1, x_1, x_2) = [x_1, [x_1, x_2]] + [x_1, [x_2, x_1]] + [x_2, [x_1, x_1]] =$$

$$= [x_1, [x_1, x_2]] - [x_1, [x_1, x_2]] = 0,$$

and the same holds for $J(x_1, x_2, x_1)$ and $J(x_2, x_1, x_1)$, thus J vanishes if two arguments coincide. This imply, thanks to the linearity of the jacobian,

$$J(x_1, x_2, x_3) + J(x_2, x_1, x_3) = J(x_1 + x_2, x_2 + x_1, x_3) = 0$$

therefore J is antisymmetric with respect to the first two variables and we can show similarly that it is antisymmetric with respect to any two variables.
Now $J(\mathfrak{h},\mathfrak{h},\mathfrak{h}) = 0$ since $\mathfrak{h}$ is a Lie algebra. Besides

$$J(\mathfrak{h},\mathfrak{a},\mathfrak{a}) = [\mathfrak{h}, [\mathfrak{a},\mathfrak{a}]] + [\mathfrak{a}, [\mathfrak{a},\mathfrak{h}]] + [\mathfrak{a}, [\mathfrak{h},\mathfrak{a}]] = 0$$

since, by Remark 2, $[\mathfrak{h},\mathfrak{a}] \subseteq \mathfrak{a}$ and $[\mathfrak{a},\mathfrak{a}] = 0$. In a similar way we obtain

$$J(\mathfrak{h},\mathfrak{b},\mathfrak{b}) = J(\mathfrak{a},\mathfrak{a},\mathfrak{a}) = J(\mathfrak{b},\mathfrak{b},\mathfrak{b}) = 0.$$

For $h_1, h_2 \in \mathfrak{h}, a \in \mathfrak{a}$, we have:
$J(h_1, h_2, a) = h_1.h_2.a - h_2.h_1.a - [h_1, h_2].a = 0$, since $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{h}$-modules, hence $J(\mathfrak{h}, \mathfrak{h}, \mathfrak{a}) = 0$ and similarly $J(\mathfrak{h}, \mathfrak{h}, \mathfrak{b}) = 0$. Now we note that for $h \in \mathfrak{h}$, $a \in \mathfrak{a}$, $b \in \mathfrak{b}$

$$J(h, a, b) = [h, a\Box b] - a\Box h.b - h.a\Box b$$

$$J(a_1, a_2, b) = -(a_2\Box b).a_1 + (a_1\Box b).a_2 \quad J(b_1, b_2, a) = (a\Box b_2).b_1 - (a\Box b_1).b_2.$$

Therefore $J(\mathfrak{L}, \mathfrak{L}, \mathfrak{L}) = 0$ if and only if (2.3) and (2.4) hold. □

Remark 3. Condition (2.3) is satisfied by every $h \in \mathfrak{h}$ if it is satisfied by a set of Lie-generators of $\mathfrak{h}$.

2.1.1 Ideals of $\mathfrak{L}$

Let $\pi_{\mathfrak{h}}$ (resp. $\pi_{\mathfrak{a}}, \pi_{\mathfrak{b}}$) be the projection of $\mathfrak{L}$ onto $\mathfrak{h}$ (resp. $\mathfrak{a}, \mathfrak{b}$).

Proposition 2.1.2. *Let $\mathfrak{M}$ be an ideal of $\mathfrak{L}$. Then*

$$\tilde{\mathfrak{M}} = \pi_{\mathfrak{a}}\mathfrak{M} \oplus \pi_{\mathfrak{h}}\mathfrak{M} \oplus \pi_{\mathfrak{b}}\mathfrak{M}$$

is an ideal of $\mathfrak{L}$.
Conversely, for any ideal $\mathfrak{h}_0$ of $\mathfrak{h}$ and $\mathfrak{h}$-submodules $\mathfrak{a}_0 \subseteq \mathfrak{a}$, $\mathfrak{b}_0 \subseteq \mathfrak{b}$ satisfying

$$\mathfrak{h}_0\mathfrak{a} \subset \mathfrak{a}_0,\ \mathfrak{h}_0\mathfrak{b} \subset \mathfrak{b}_0,\ \mathfrak{a}_0\Box\mathfrak{b} \subset \mathfrak{h}_0,\ \mathfrak{a}\Box\mathfrak{b}_0 \subset \mathfrak{h}_0, \tag{2.5}$$

$\mathfrak{M}_0 = \mathfrak{a}_0 \oplus \mathfrak{h}_0 \oplus \mathfrak{b}_0$ *is an ideal of $\mathfrak{L}$.*

Proof. Let $\mathfrak{M}$ be an ideal of $\mathfrak{L}$. We have

$$\begin{gathered}[\pi_{\mathfrak{h}}m, h] = \pi_{\mathfrak{h}}[m, h] \subseteq \pi_{\mathfrak{h}}\mathfrak{m},\\ [\pi_{\mathfrak{a}}m, h] = \pi_{\mathfrak{a}}[m, h] \subseteq \pi_{\mathfrak{a}}\mathfrak{m},\ [\pi_{\mathfrak{b}}m, h] = \pi_{\mathfrak{b}}[m, h] \subseteq \pi_{\mathfrak{h}}\mathfrak{m}\end{gathered} \tag{2.6}$$

for all $m \in \mathfrak{M}, h \in \mathfrak{h}$. Thus $\pi_{\mathfrak{h}}\mathfrak{M}$ is an ideal of $\mathfrak{h}$, and both $\pi_{\mathfrak{a}}\mathfrak{M}$ and $\pi_{\mathfrak{b}}\mathfrak{M}$ are $\mathfrak{h}$-submodules.
Moreover, if $m = h' + a' + b' \in \mathfrak{M}$, $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$, then

$$[m, a + b] = (a\Box b' - a'\Box b) + h'.a + h'.b\,.$$

As a consequence the following inclusions hold

$$\begin{gathered}[(\pi_{\mathfrak{h}}\mathfrak{M}), \mathfrak{a}] \subseteq \pi_{\mathfrak{a}}\mathfrak{M} \quad [(\pi_{\mathfrak{h}}\mathfrak{M}), \mathfrak{b}] \subseteq \pi_{\mathfrak{b}}\mathfrak{M}\\ [(\pi_{\mathfrak{a}}\mathfrak{M}), \mathfrak{b}] \subseteq \pi_{\mathfrak{h}}\mathfrak{M} \quad [(\pi_{\mathfrak{b}}\mathfrak{M}), \mathfrak{a}] \subseteq \pi_{\mathfrak{h}}\mathfrak{M}\end{gathered} \tag{2.7}$$

This, together with (2.6), proves that $\tilde{\mathfrak{M}} = \pi_{\mathfrak{a}}\mathfrak{M} \oplus \pi_{\mathfrak{h}}\mathfrak{M} \oplus \pi_{\mathfrak{b}}\mathfrak{M}$ is an ideal of $\mathfrak{L}$.
Conversely, if $\mathfrak{h}_0, \mathfrak{a}_0, \mathfrak{b}_0$ satisfy relations (2.5), their direct sum is, by definition of product on $\mathfrak{L}$, an ideal. □

Remark 4. If $\mathfrak{M}$ is an ideal of $\mathfrak{L}$ we always have $\mathfrak{M} \subseteq \tilde{\mathfrak{M}}$.

Definition 2.2. Let $\mathfrak{M}$ be an ideal of $\mathfrak{L}$. We say that $\mathfrak{M}$ is split if

$$\mathfrak{M} = \tilde{\mathfrak{M}} = \pi_{\mathfrak{a}}\mathfrak{M} \oplus \pi_{\mathfrak{h}}\mathfrak{M} \oplus \pi_{\mathfrak{b}}\mathfrak{M}.$$

In what follows we will be interested in triples $(\mathfrak{h}; \mathfrak{a}, \mathfrak{b})$ satisfying, in addition of properties (1),(2) and (3), some of the following conditions:

(i) $\exists h_0 \in \mathfrak{h}$ such that $h_0.a = a$, $h_0.b = -b$ and $[h_0, h] = 0$ for all $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $h \in \mathfrak{h}$;

(ii) the map $\square$ admits only trivial annihilators, i.e. if $a\square b = 0, \forall b \in \mathfrak{b}$ then $a = 0$ and if $a\square b = 0, \forall a \in \mathfrak{a}$ then $b = 0$;

(iii) $h.\mathfrak{a} = 0$ or $h.\mathfrak{b} = 0$ for some $h \in \mathfrak{h}$ implies $h = 0$;

(iv) $\mathfrak{h}$ is generated, as an algebra, by the subspace $\mathfrak{a}\square\mathfrak{b}$.

Lemma 2.1.3. *Let* $(\mathfrak{h}; \mathfrak{a}, \mathfrak{b})$ *be a triple satisfying condition (i). Then any ideal of* $\mathfrak{L}$ *is split.*

Proof. Let $h_0 \in \mathfrak{h}$ be an element as in (i) and let $\mathfrak{M}$ be an ideal of $\mathfrak{L}$. For any $x = a + h + b \in \mathfrak{M}$, we have $[h_0, x] = [h_0, a] + [h_0, h] + [h_0, b] = a - b \in \mathfrak{M}$ since $\mathfrak{M}$ is an ideal and $[h_0, x + a - b] = 2a \in \mathfrak{M}$, meaning that $\pi_{\mathfrak{a}}\mathfrak{M} \subseteq \mathfrak{M}$. Applying h_0 to $x - a \in \mathfrak{M}$ we get $\pi_{\mathfrak{b}}\mathfrak{M} \subseteq \mathfrak{M}$ so that we can conclude that $\pi_{\mathfrak{h}}\mathfrak{M} \subseteq \mathfrak{M}$ too. This ends the proof because $\tilde{\mathfrak{M}} = \pi_{\mathfrak{a}}\mathfrak{M} \oplus \pi_{\mathfrak{h}}\mathfrak{M} \oplus \pi_{\mathfrak{b}}\mathfrak{M} \subseteq \mathfrak{M}$ together with Remark 4. $\square$

Proposition 2.1.4. *Let* $(\mathfrak{h}; \mathfrak{a}, \mathfrak{b})$ *be a triple satisfying conditions (i) and (ii). If* $\mathfrak{h}$ *is simple then* $\mathfrak{L}$ *is simple.*

Proof. Let $\mathfrak{M} \neq 0$ be an ideal of $\mathfrak{L}$. By (ii) we must have $\pi_{\mathfrak{h}}\mathfrak{M} \neq 0$. In fact, suppose $\pi_{\mathfrak{h}}\mathfrak{M} = 0$, then we should have, by (2.7), $(\pi_{\mathfrak{a}}\mathfrak{M})\square\mathfrak{b} \subseteq \pi_{\mathfrak{h}}\mathfrak{M} = 0$ but by (ii) this would mean $\pi_{\mathfrak{a}}\mathfrak{M} = 0$ and similarly we would have $\pi_{\mathfrak{b}}\mathfrak{M} = 0$ leading to $\tilde{\mathfrak{M}} = 0$, a contradiction since $\mathfrak{M}$ is different from 0.
By hypothesis $\mathfrak{h}$ is simple, so it must be $\pi_{\mathfrak{h}}\mathfrak{M} = \mathfrak{h}$. From (2.7) we get $\mathfrak{h}.\mathfrak{a} \subseteq \pi_{\mathfrak{a}}\mathfrak{M}$ and $\mathfrak{h}.\mathfrak{b} \subseteq \pi_{\mathfrak{b}}\mathfrak{M}$. Due to (i) there exists an element $h_0 \in \mathfrak{h}$ such that $h_0.a = a$, $h_0.b = -b$ and $[h_0, h] = 0$, hence $\mathfrak{h}.\mathfrak{a} = \mathfrak{a}$ and $\mathfrak{h}.\mathfrak{b} = \mathfrak{b}$.
Summing up we have shown that $\pi_{\mathfrak{a}}\mathfrak{M} = \mathfrak{a}$, $\pi_{\mathfrak{h}}\mathfrak{M} = \mathfrak{h}$, $\pi_{\mathfrak{b}}\mathfrak{M} = \mathfrak{b}$, hence, by Lemma 2.1.3, $\mathfrak{M} = \tilde{\mathfrak{M}} = \mathfrak{L}$, i.e. the only ideal of $\mathfrak{L}$ different from 0 is $\mathfrak{L}$ itself. $\square$

We will now prove the main result about the ideals of $\mathfrak{L}$, namely:

Theorem 2.1.5. *Let* $(\mathfrak{h}; \mathfrak{a}, \mathfrak{b})$ *be a triple satisfying (i)-(iv). If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are irreducible* $\mathfrak{h}$*-modules then* $\mathfrak{L}$ *is simple.*

Proof. Let $\mathfrak{M}$ be an ideal of $\mathfrak{L}$. Then, by (2.6), $\pi_{\mathfrak{a}}\mathfrak{M}$ is an $\mathfrak{h}$-submodule of $\mathfrak{a}$ so it is either 0 or $\mathfrak{a}$.
Suppose $\pi_{\mathfrak{a}}\mathfrak{M} = 0$. By (2.7) it must be $(\pi_{\mathfrak{h}}\mathfrak{M}).\mathfrak{a} = 0$ hence, by condition (iii), $\pi_{\mathfrak{h}}\mathfrak{M} = 0$. As we showed in the proof of Proposition 2.1.4, if condition (ii) is satisfied then $\pi_{\mathfrak{h}}\mathfrak{M} = 0$ implies $\pi_{\mathfrak{a}}\mathfrak{M} = \pi_{\mathfrak{b}}\mathfrak{M} = 0$. This being the case, it would be $\mathfrak{M} = 0$. In the same way $\pi_{\mathfrak{b}}\mathfrak{M} = 0$ leads to $\mathfrak{M} = 0$.
We are left with the case $\pi_{\mathfrak{a}}\mathfrak{M} = \mathfrak{a}$ and $\pi_{\mathfrak{b}}\mathfrak{M} = \mathfrak{b}$. Again from relation (2.7) we have $\mathfrak{a}\Box\mathfrak{b} \subseteq \pi_{\mathfrak{h}}\mathfrak{M}$ hence, by (iv), we can conclude $\pi_{\mathfrak{h}}\mathfrak{M} = \mathfrak{h}$ since $\pi_{\mathfrak{h}}\mathfrak{M}$ is a subalgebra of $\mathfrak{h}$. This leads to $\mathfrak{M} = \mathfrak{L}$. □

2.2 The Tits-Kantor-Koecher construction

Let $\mathfrak{J}$ be an algebra. For an element $a \in \mathfrak{J}$ we denote by L_a the left multiplication by a.

Definition 2.3. Let $\mathfrak{J}$ be a Jordan algebra. We will denote by $\mathfrak{h}(\mathfrak{J})$ the Lie subalgebra of $gl(\mathfrak{J})$ generated by L_a with $a \in \mathfrak{J}$:

$$\mathfrak{h}(\mathfrak{J}) = L(\mathfrak{J}) + [L(\mathfrak{J}), L(\mathfrak{J})] + \cdots \ .$$

Remark 5. Here we denote by $L(\mathfrak{J})$ the span of the L_a's with $a \in \mathfrak{J}$. Note that $(\alpha L_a + \beta L_b)(c) = L_{\alpha a+\beta b}(c)$.

Proposition 2.2.1. *Let* J *be a unital Jordan algebra. Then*

$$\mathfrak{h}(\mathfrak{J}) = L(\mathfrak{J}) \oplus [L(\mathfrak{J}), L(\mathfrak{J})].$$

Proof. First we show that the sum $L(\mathfrak{J}) + [L(\mathfrak{J}), L(\mathfrak{J})]$ is direct.
Let $T \in L(\mathfrak{J}) \cap [L(\mathfrak{J}), L(\mathfrak{J})]$. Then $T = L_a = [L_b, L_c]$,
for some $a, b, c \in \mathfrak{J}$. By applying T to the unit e of $\mathfrak{J}$, we obtain

$$T(e) = L_a(e) = [L_b, L_c](e) \Rightarrow a = bc - cb = 0, \quad \text{i.e. } T = 0$$

In order to prove that every higher-order commutator of elements in $L(\mathfrak{J})$ is a sum of elements of $L(\mathfrak{J})$ and $[L(\mathfrak{J}), L(\mathfrak{J})]$ we will show that if $\mathfrak{J}$ is a Jordan algebra then the following relations hold:

$$[L_a, L_{bc}] + [L_b, L_{ca}] + [L_c, L_{ab}] = 0. \tag{2.8}$$

In fact, if $\mathfrak{J}$ satisfies 2.8, then $\mathfrak{J}$ satisfies also equation

$$[[L_a, L_b], L_c] = L_{[L_a,L_b](c)} \tag{2.9}$$

The left-hand side of equation (2.9) applied to an element $d \in \mathfrak{J}$ reads

$$a(b(cd)) - b(a(cd)) - c(a(bd)) + c(b(ad)). \tag{2.10}$$

Equation (2.8) is equivalent to:

$$a((bc)d) = (bc)(ad) - b((ac)d) + (ac)(bd) - c((ab)d) + (ab)(cd). \tag{2.11}$$

Thanks to commutativity, $a((bc)d) = a(d(cb))$, hence if we swap b and d in (2.11) and insert the result in (2.10), we get:

$$\begin{aligned} (dc)(ab) - d((ac)b) + (ac)(bd) - c((ad)b) + (ad)(cb)- \\ -b(a(cd)) - c(a(bd)) + c(b(ad)) = \\ = -b(a(cd)) - c(a(bd)) + (ad)(cb) + (ac)(bd) + (dc)(ab) - d((ac)b). \end{aligned} \tag{2.12}$$

Once again, the first term of the right-hand side of equation (2.12) is the left-hand side of equation (2.10) under the permutation $a \to b,\ b \to c,\ c \to d,\ d \to a$, hence

$$\begin{aligned} -(cd)(ba) + c((bd)a) - (bd)(ca) + d((bc)a) - (bc)(da)- \\ -c(a(bd)) + (ad)(cb) + (ac)(bd) + (dc)(ab) - d((ac)b) = \\ = (a(bc))d - (b(ac))d = L_{[L_a,L_b](c)}(d). \end{aligned} \tag{2.13}$$

In order to prove equation (2.8) we linearize the Jordan identity.
Let $a, b, c, d \in \mathfrak{J}$. Then, taking $x = a + b, y = d$ in the Jordan identity and using the commutativity of the product, we get

$$\begin{aligned} &0 = (a + b)((a + b)^2 d) - (a + b)^2((a + b)d) = \\ &= a(a^2 d) + a(b^2 d) + 2a((ab)d) + b(a^2 d) + b(b^2 d) + 2b((ab)d)- \\ &-a^2(ad) - b^2(ad) - 2(ab)(ad) - a^2(bd) - b^2(bd) - 2(ab)(bd) \end{aligned} \tag{2.14}$$

Setting $x = a - b, y = d$ in the Jordan identity, we have

$$\begin{aligned} &0 = (a - b)((a - b)^2 d) - (a - b)^2((a - b)d) = \\ &= a(a^2 d) + a(b^2 d) - 2a((ab)d) - b(a^2 d) - b(b^2 d) + 2b((ab)d)- \\ &-a^2(ad) - b^2(ad) + 2(ab)(ad) + a^2(bd) + b^2(bd) - 2(ab)(bd) \end{aligned} \tag{2.15}$$

Taking into account equations (2.14) and (2.15) we find

$$0 = 4a((ab)d) + 2b(a^2 d) - 4(ab)(ad) - 2a^2(bd). \tag{2.16}$$

By taking $a = a + c$ and $a = a - c$ in (2.16) we have the two equations

$$
\begin{aligned}
&0 = 4(a+c)(((a+c)b)d) + 2b((a+c)^2 d)- \\
&-4((a+c)b)((a+c)d) - 2(a+c)^2(bd) = \\
&= 4(\ a((ab)d) + a((cb)d) + c((ab)d) + c((cb)d)\ )+ \\
&+2(\ b(a^2 d) + 2b((ac)d) + b(c^2 d)\ )- \\
&-4(\ (ab)(ad) + (ab)(cd) + (cb)(ad) + (cb)(cd)\ )- \\
&-2(\ a^2(bd) + 2(ac)(bd) + c^2(bd)\ ) \\
&0 = 4(a-c)(((a-c)b)d) + 2b((a-c)^2 d)- \\
&-4((a-c)b)((a-c)d) - 2(a-c)^2(bd) = \\
&= 4(\ a((ab)d) - a((cb)d) - ((ab)d) + c((cb)d)\ )+ \\
&+2(\ b(a^2 d) - 2b((ac)d) + b(c^2 d)\ )- \\
&-4(\ (ab)(ad) - (ab)(cd) - (cb)(ad) + (cb)(cd)\ )- \\
&-2(\ a^2(bd) - 2(ac)(bd) + c^2(bd)\ )
\end{aligned}
\tag{2.17}
$$

Hence, subtracting the second equation from the first of (2.17),

$$0 = a((bc)d) - (bc)(ad) + b((ac)d) - (ac)(bd) + c((ab)d) - (ab)(cd) \tag{2.18}$$

which is exactly equation (2.8) applied to the element d.
Summing up, equation (2.8) is satisfied, thus equation (2.9) is satisfied too. Now, equation (2.9) implies that

$$[[L(\mathfrak{J}), L(\mathfrak{J})], L(\mathfrak{J})] \subseteq L(\mathfrak{J}).$$

Due to the Jacobi identity we have

$$[[L(\mathfrak{J}), L(\mathfrak{J})], [L(\mathfrak{J}), L(\mathfrak{J})]] \subseteq [[[L(\mathfrak{J}), L(\mathfrak{J})], L(\mathfrak{J})], L(\mathfrak{J})] \subseteq [L(\mathfrak{J}), L(\mathfrak{J})].$$

By an analogous argument every higher-order commutator of elements of $L(\mathfrak{J})$ is an element of either $L(\mathfrak{J})$ or $[L(\mathfrak{J}), L(\mathfrak{J})]$. $\square$

Thanks to Proposition 2.2.1 we can extend the identity map of $L(\mathfrak{J})$ to an antiautomorphism $* : \mathfrak{h}(\mathfrak{J}) \to \mathfrak{h}(\mathfrak{J})$ by setting $T^* = -T$ for $T \in [L(\mathfrak{J}), L(\mathfrak{J})]$.
We recall that an antiautomorphism of an algebra A is an automorphism A such that $\phi(xy) = \phi(y)\phi(x)$.

Definition 2.4. Let $\mathfrak{J}$ be a Jordan algebra and let $\overline{\mathfrak{J}}$ be an isomorphic copy of $\mathfrak{J}$. We define

$$Lie(\mathfrak{J}) = \mathfrak{J} \oplus \mathfrak{h}(\mathfrak{J}) \oplus \overline{\mathfrak{J}}$$

where $T \in \mathfrak{h}(\mathfrak{J})$ acts on $a \in \mathfrak{J}$ via $T.a = T(a)$ and on $\bar{a} \in \overline{\mathfrak{J}}$ as $T.\bar{a} = -\overline{T^*(a)}$.
We also define the following map $\square : \mathfrak{J} \times \overline{\mathfrak{J}} \to \mathfrak{h}(\mathfrak{J})$,

$$a \square \bar{b} = 2(L_{ab} + [L_a, L_b]). \tag{2.19}$$

Lemma 2.2.2. *Let $\mathfrak{J}$ be a Jordan algebra and let $Lie(\mathfrak{J})$ be as in Definition 2.4. Then the algebra $Lie(\mathfrak{J})$ satisfies conditions (1),(2) and (3) of Section 2.1. Moreover, $Lie(\mathfrak{J})$ is a Lie algebra.*

Proof. Conditions (1) and (3) are obviously satisfied since $\mathfrak{h}(\mathfrak{J})$ is a Lie algebra and the map $\Box$ is bilinear. In order to show that $\mathfrak{J}$ is an $\mathfrak{h}(\mathfrak{J})$-module we compute

$$[L_a, L_b].c = L_a(L_b c) - L_b(L_a c) = L_a.(L_b.c) - L_a.(L_b.c),$$

similarly,

$$\begin{aligned}[L_a, L_b].\bar{c} = L_a(-\overline{L_b(c)}) - L_b(-\overline{L_a c}) = \\ = \overline{L_a(L_b c)} - \overline{L_b(L_a c)} = L_a.(L_b.\bar{c}) - L_a.(L_b.\bar{c}),\end{aligned}$$

and $\overline{\mathfrak{J}}$ is an $\mathfrak{h}(\mathfrak{J})$-module too.
In order to prove the second part of the lemma we use Theorem 2.1.1. Indeed $\mathfrak{h}(\mathfrak{J})$ is a Lie algebra. We have:

$$\begin{aligned}
\tfrac{1}{2}(a_1\Box\bar{b}).a_2 &= (a_1 b)a_2 + a_1(ba_2) - b(a_1 a_2)\\
\tfrac{1}{2}(a_2\Box\bar{b}).a_1 &= (a_2 b)a_1 + a_2(ba_1) - b(a_2 a_1)\\
\tfrac{1}{2}(a\Box\bar{b}_1).\bar{b}_2 &= -(ab_1)b_2 + a(b_1 b_2) - b_1(ab_2)\\
\tfrac{1}{2}(a\Box\bar{b}_2).\bar{b}_1 &= -(ab_2)b_1 + a(b_2 b_1) - b_2(ab_1)
\end{aligned}$$

where we have exploited the fundamental fact that $[L_a, L_b]^* = -[L_a, L_b]$. Applying the commutativity of $\mathfrak{J}$ and reordering properly we see that conditions (2.4) are satisfied by $\mathfrak{J}$.
Furthermore, condition (2.3) follows from (2.8) and (2.9) which are satisfied by any Jordan algebra hence by $\mathfrak{J}$. Indeed:

$$\begin{aligned}
&\tfrac{1}{2}(\ [L_a, b\Box\bar{c}] - (L_a.b)\Box\bar{c} - b\Box\overline{(L_a.\bar{c})}\ ) = \\
&= [L_a, L_{bc}] + [L_a, [L_b, L_c]] - \\
&- L((ab)c) - [L_{ab}, L_c] + L(b(ac)) + [L_b, L_{ac}] = \\
&= [L_a, L_{bc}] + [L_b, L_{ac}] + [L_c, L_{ab}] + \\
&+ [L_a, [L_b, L_c]] - L([L_a, L_b].c) = 0.
\end{aligned}$$

This ends the proof. □

Proposition 2.2.3. *Let $\mathfrak{J}$ be a Jordan algebra with unit element e. Then conditions (i)-(iv) of Section 2.1 are satisfied by $Lie(\mathfrak{J})$.*

Proof. Condition (i) is satisfied with $h_0 = L(e)$. Obviously $L(e).a = a$, $L(e).\bar{b} = -\bar{b}$ and $[L(e), T] = 0$, $\forall a \in \mathfrak{J}, \bar{b} \in \overline{\mathfrak{J}}$ and $T \in \mathfrak{h}(\mathfrak{J})$.
To prove (ii), suppose $a\Box\bar{b} = 0$, $\forall a \in \mathfrak{J}$, then also $e\Box\bar{b} = 2L_b = 0$ hence $b = 0$. Moreover $a\Box\bar{b} = 0$, $\forall \bar{b} \in \overline{\mathfrak{J}}$ implies $a = 0$.

Condition (iii) is a consequence of the fact that each $h \in \mathfrak{h}(\mathfrak{J})$ is a linear transformation.
Finally, we have $a \Box \bar{e} = 2L_a$ and the elements L_a, with $a \in \mathfrak{J}$, generate $\mathfrak{h}(\mathfrak{J})$, hence (iv) is satisfied. $\square$

Remark 6. For any subset $\mathfrak{m} \subseteq \mathfrak{J}$, (resp. $\mathfrak{m} \subseteq \overline{\mathfrak{J}}$), $\mathfrak{m}$ is an $\mathfrak{h}(\mathfrak{J})$-submodule of $\mathfrak{J}$ if and only if $\mathfrak{m}$ is an ideal of $\mathfrak{J}$, (resp. $\overline{\mathfrak{J}}$).
Let $\mathfrak{m}$ be an $\mathfrak{h}(\mathfrak{J})$-submodule of $\mathfrak{J}$. Then we have $T.\mathfrak{m} \subseteq \mathfrak{m}$ for all $T \in \mathfrak{h}(\mathfrak{J})$, thus, since the L_a's generate $\mathfrak{h}(\mathfrak{J})$, $\mathfrak{m}$ is an ideal of $\mathfrak{J}$.

Theorem 2.2.4 (Tits-Kantor-Koecher construction)**.** *Let $\mathfrak{J}$ be a unital Jordan algebra. $Lie(\mathfrak{J})$ is simple if and only if $\mathfrak{J}$ is simple.*

Proof. Let $\mathfrak{J}$ be simple. Thanks to Remark 6 this implies that $\mathfrak{J}$ is $\mathfrak{h}(\mathfrak{J})$-irreducible and, by Proposition 2.2.3, we may apply Theorem 2.1.5. Therefore $Lie(\mathfrak{J})$ is simple.
Conversely, let $Lie(\mathfrak{J})$ be simple. Suppose that there exists an ideal of $\mathfrak{J}$, $\mathfrak{m} \subseteq \mathfrak{J}$ such that $\mathfrak{m} \neq 0, \mathfrak{J}$. We define

$$Lie(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{h}(\mathfrak{m}) \oplus \overline{\mathfrak{m}},$$

where $\mathfrak{h}(\mathfrak{m})$ is the ideal of $\mathfrak{h}(\mathfrak{J})$ generated by $L(\mathfrak{m})$,

$$\mathfrak{h}(\mathfrak{m}) = L(\mathfrak{m}) + [L(\mathfrak{m}), L(\mathfrak{J})] + \cdots .$$

We have $\mathfrak{m} \Box \overline{\mathfrak{J}} \subset Lie(\mathfrak{m})$ and $\mathfrak{J} \Box \overline{\mathfrak{m}} \subset Lie(\mathfrak{m})$, so that relations (2.5) are satisfied, hence $Lie(\mathfrak{m})$ is an ideal of $Lie(\mathfrak{J})$.
Since $\mathfrak{m} \neq 0, \mathfrak{J}$, $Lie(\mathfrak{m}) \neq 0, Lie(\mathfrak{J})$ which is absurd. $\square$

Remark 7. If $\mathfrak{J}$ is a Jordan algebra with unit 1 then the grading of $Lie(\mathfrak{J})$ is induced by the $\mathfrak{sl}_2$-triple given by $< 1,\ L_1,\ \bar{1} >$.
An isomorphism between the Lie algebra $< 1,\ L_1,\ \bar{1} >$ and $\mathfrak{sl}_2(\mathbb{F}) =< f, h, e >$ is given by $\phi(1) = -f,\ \phi(L_1) = \frac{1}{2}h,\ \phi(\bar{1}) = -e$.
Moreover $[-L_1, x] = -x, [-L_1, \bar{x}] = \bar{x}$ and $[-L_1, T] = 0$ for $x \in Lie(\mathfrak{J})_{-1}$, $T \in Lie(\mathfrak{J})_0$, $\bar{x} \in Lie(\mathfrak{J})_1$.

Chapter 3

Lie algebras and triple systems

In this chapter we will extend the Tits-Kantor-Koecher construction to Jordan triple systems and Kantor triple systems.

3.1 Triple Systems

Definition 3.1 (Triple system)**.** A *triple system* is a vector space A together with a triple product, i.e. a trilinear map

$$(\,\cdot\,,\,\cdot\,,\,\cdot\,) : A \times A \times A \to A$$

$$(x, y, z) \to (xyz).$$

Definition 3.2. Let S be a triple system. We define, for every $x, y \in S$, the linear operator $< x, y >: \; S \to S$ by setting

$$< x, y > (z) = (xzy) - (yzx).$$

Definition 3.3. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra with graded involution τ. We call $(\mathfrak{g}_{-1}, (\;,\;,\;))$, with product

$$(xyz) = [[x, \tau(y)], z], \tag{3.1}$$

the triple system *derived* from $\mathfrak{g}$.

Lemma 3.1.1. *Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra with graded involution τ and let $(\mathfrak{g}_{-1}, (\;,\;,\;))$ be the triple system derived from $\mathfrak{g}$. Then, for $x, y, z \in \mathfrak{g}_{-1}$, we have:*

$$< x, y > (z) = [[x, y], \tau(z)]. \tag{3.2}$$

Proof. Since $[\ ,\ ]$ satisfies the Jacobi identity we have

$$< x, y > (z) = [[x, \tau(z)], y] - [[y, \tau(z)], x] = [[x, \tau(z)], y] + [[\tau(z), y], x] =$$

$$= -[[y, x], \tau(z)] = [[x, y], \tau(z)]$$

□

Definition 3.4 (Lie triple system)**.** A *Lie triple system* is a triple system $(L, [,,])$ satisfying the following properties:

$$\begin{aligned} &[x, y, z] = -[y, x, z] && \text{(Anticommutativity)} \\ &[x, y, z] + [y, z, x] + [z, x, y] = 0 && \text{(Generalized Jacobi identity)} \\ &[u, v, [x, y, z]] = [[u, v, x], y, z]+ \\ &+[x, [u, v, y], z] + [x, y, [u, v, z]] && \text{(Principal identity.)} \end{aligned} \tag{3.3}$$

Definition 3.5 (Jordan triple system)**.** A *Jordan triple system* is a triple system $(J, \{,,\})$ whose triple product satisfies the following properties:

$$\begin{aligned} &\{x, y, z\} = \{z, y, x\} && \text{(Commutativity)} \\ &\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\}+ \\ &-\{x, \{v, u, y\}, z\} + \{x, y, \{u, v, z\}\} && \text{(Principal identity)} \end{aligned} \tag{3.4}$$

Definition 3.6 (Kantor triple system)**.** A *Kantor triple system* is a triple system $(K, \{\ ,\ ,\ \})$ whose triple product satisfies the following properties:

$$\begin{aligned} &\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\}+ \\ &-\{x, \{v, u, y\}, z\} + \{x, y, \{u, v, z\}\} && \text{(Principal identity)} \\ &<< u, v > (x), y >=< \{y, x, u\}, v > - < \{y, x, v\}, u > && \text{(Auxiliar identity)} \end{aligned} \tag{3.5}$$

Remark 8. By definition any Jordan triple system is a Kantor triple system with trivial auxiliary identity.

Example 3.1.1. Let $\mathfrak{L}$ be a Lie algebra. Then $\mathfrak{L}$ endowed with triple product defined by

$$[x, y, z] = [[x, y], z]$$

is a Lie triple system.

Indeed the anticommutativity and the generalized Jacobi identity are satisfied since $[\ ,\ ]$ is a Lie product. Moreover

$$[u, v, [x, y, z]] = [\ [u, v], [[x, y], z]\ ] = -[[[x, y], z], [u, v]] =$$

$$= [[z, [u, v]], [x, y]] + [[[u, v], [x, y]], z] = [[x, y], [[u, v], z]] - [[[x, y], [u, v]], z] =$$

$$= [x, y, [u, v, z]] + [[[y, [u, v]], x], z] + [[[[u, v], x], y], z] =$$

$$= [x, y, [u, v, z]] + [[x, [[u, v], y]], z] + [[[[u, v], x], y], z] =$$

$$= [x, y, [u, v, z]] + [x, [u, v, y], z] + [[u, v, x], y, z].$$

Example 3.1.2. Let $\mathfrak{J}$ be a Jordan algebra. Then $L(\mathfrak{J}) =< L_a; a \in \mathfrak{J} >$ is a Lie triple system with triple product defined by $[L_a, L_b, L_c] = [[L_a, L_b], L_c]$. This is a consequence of equation 2.9, which proves that $L(\mathfrak{J})$ is closed under triple product, and Example 3.1.1.

Example 3.1.3. Let $\mathfrak{g}$ be a shortly graded Lie algebra with graded involution τ. Then $\mathfrak{g}_{-1}$ with triple product

$$\{x, y, z\} = [[x, \tau(y)], z]$$

is a Jordan triple system.
Since $[\,,\,]$ is a Lie product and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$, we have

$$\{x, z, y\} - \{y, z, x\} =< x, z > (y) = [[x, z], \tau(y)] = 0.$$

Furthermore

$$\{u, v, \{x, y, z\}\} = [\ [u, \tau(v)], [[x, \tau(y)], z]\ ] = -[[[x, \tau(y)], z], [u, \tau(v)]] =$$

$$= [[z, [u, \tau(v)]], [x, \tau(y)]] + [[[u, \tau(v)], [x, \tau(y)]], z] =$$

$$= [[x, \tau(y)], [[u, \tau(v)], z]] - [[[x, \tau(y)], [u, \tau(v)]], z] =$$

$$= \{x, y, \{u, v, z\}\} + [[[\tau(y), [u, \tau(v)]], x], z] + [[[[u, \tau(v)], x], \tau(y)], z] =$$

$$= \{x, y, \{u, v, z\}\} - [[x, [[\tau(v), u], \tau(y)]], z] + [[[[u, \tau(v)], x], \tau(y)], z] =$$

$$= \{x, y, \{u, v, z\}\} - [[x, \tau(\ [[v, \tau(u)], y]], z]\ ) + [[[[u, \tau(v)], x], \tau(y)], z] =$$

$$= \{x, y, \{u, v, z\}\} + \{x, \{v, u, y\}, z\} + \{\{u, v, x\}, y, z\}.$$

Example 3.1.4. Let $\mathfrak{g}$ be a 5-graded Lie algebra with graded involution τ. Then $\mathfrak{g}_{-1}$ with triple product

$$\{x, y, z\} = [[x, \tau(y)], z]$$

is a Kantor triple system.
From Example 3.1.3 we know that $\mathfrak{g}_{-1}$ satisfies the principal identity. Thus, we just need to verify the auxiliary identity.
Thanks to Lemma 3.1.1, we have

$$<< u, v > (x), y >= [[[u, v], \tau(x)], y] = -[[\tau(x), y], [u, v]] - [[y, [u, v]], \tau(x)].$$

Since $\mathfrak{g}$ is 5-graded we have $[y,[u,v]] \in \mathfrak{g}_{-3} = 0$, hence

$$<< u, v > (x), y >= -[[\tau(x), y], [u, v]] = [[u, v], [\tau(x), y]] =$$

$$= -[[v, [\tau(x), y]], u] - [[[\tau(x), y], u], v] = [[[y, \tau(x)], u], v] - [[[y, \tau(x)], v], u] =$$

$$=< \{y, x, u\}, v > - < \{y, x, v\}, u > .$$

Remark 9. Let $\mathfrak{g}$ be as in Example 3.1.3. We make $\mathfrak{g}_1$ a Jordan triple system by setting

$$\{x, y, z\}_\tau = [[x, \tau(y)], z],$$

for all $x, y, z \in \mathfrak{g}_1$. This Jordan triple system is related to the one defined on $\mathfrak{g}_{-1}$ by

$$\{x, y, z\}_\tau = \tau(\ \{\tau(x), \tau(y), \tau(z)\}\ ).$$

3.1.1 Universal Lie algebras

Definition 3.7 (Operators of order p on U)**.** Let U be a vector space and let us consider a map $f : U \to U$. We call the map f an operator of order $p \geq 1$ whenever there exists a p-linear and symmetric map $F : U^p \to U$, i.e. $F(u_1, u_2, \dots, u_p) = F(u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(p)})$ for any permutation σ, such that

$$f(u) = F(u, \dots, u).$$

If f is constant then it is said to be of order 0.

Let f be an operator of order p and g an operator of order q. We define $f\Box g$ the operator of order $p + q - 1$.

$$(f\Box g)(u) = pF(g(u), u, \dots, u) =$$

$$= F(g(u), u, \dots, u) + F(u, g(u), u, \dots, u) + \dots + F(u, \dots, u, g(u)) =$$

$$= (F\Box G)(u, \dots, u)$$

In the case f is of order 0 we have $f\Box g := 0$.

We denote by $T_k(U)$ the vector space of operators of order $k+1$ if $k \geq -1$ and for $k < -1$ we set $T_k(U) = 0$.

Definition 3.8 (Operators of order p on $V \oplus W$)**.** Let V, W be vector spaces and let $f : V \oplus W \to V \oplus W$. We call the map f an operator of order p if:

there exists a map $F = F_V + F_W$, with $F_V : V^{p_V^V} \oplus W^{p_W^V} \to V$ a $(p_V^V + p_W^V)$-linear map and $F_W : V^{p_V^W} \oplus W^{p_W^W} \to W$ a $(p_V^W + p_W^W)$-linear map such that

$$p_V^V + 2p_W^V = p + 1 \quad \text{ and } p_V^W + 2p_W^W = p + 2.$$

If f is constant then it is said to be of order -2.
Let f be an operator of order p and g an operator of order q. We define $f\Box g$ the operator of order $p+q$.

$$(f\Box g)(v,w) = (f\Box g)_V(v,w) + (f\Box g)_W(v,w),$$

$$(f\Box g)_V(v,w) =$$

$$= F_V(g_V(v), v, \dots, v, w, \dots, w) + F_V(v, g_V(v), \dots, v, w, \dots, w) + \cdots$$

$$\cdots + F_V(v, v, \dots, g_V(v), w, \dots, w) +$$

$$+ F_V(v, \dots, v, g_W(w), w, \dots, w) + F_V(v, \dots, v, w, g_W(w), \dots, w) + \cdots$$

$$F_V(v, \dots, v, w, w, \dots, g_W(w)),$$

$$(f\Box g)_W(v,w) =$$

$$= F_W(g_V(v), v, \dots, v, w, \dots, w) + F_W(v, g_V(v), \dots, v, w, \dots, w) + \cdots$$

$$\cdots + F_W(v, v, \dots, g_V(v), w, \dots, w) +$$

$$+ F_W(v, \dots, v, g_W(w), w, \dots, w) + F_W(v, \dots, v, w, g_W(w), \dots, w) + \cdots$$

$$F_W(v, \dots, v, w, w, \dots, g_W(w)).$$

In the case f is constant we set $f\Box g := 0$.
We denote by $T_k(V,W)$ the vector space of operators of order k if $k \geq -2$. For $k < -2$ we set $T_k(V,W) = 0$.

Definition 3.9 (Universal Lie algebra)**.** Let U be a vector space. We define the *universal Lie algebra* $T(U)$ of U the $\mathbb{Z}$-graded Lie algebra

$$T(U) = \bigoplus_{k \geq -1} T_k(U)$$

with product defined by

$$[f,g] = f\Box g - g\Box f.$$

Let V, W be a pair of vector spaces. We define the *universal Lie algebra* $T(V,W)$ of $V \oplus W$ the $\mathbb{Z}$-graded Lie algebra

$$T(V,W) = \bigoplus_{k \geq -2} T_k(V,W)$$

with product

$$[f,g] = f\Box g - g\Box f.$$

Definition 3.10 (Realization of $\mathfrak{g}$)**.** Let $\mathfrak{g}$ be a Lie algebra and U, V, W vector spaces. A *realization* of $\mathfrak{g}$ on U, respectively $V \oplus W$, is a homomorphism $\phi : \mathfrak{g} \to T(U)$, respectively $\phi : \mathfrak{g} \to T(V, W)$.
Notice that if all elements are mapped to linear operators then ϕ is a representation of $\mathfrak{g}$.

More details about the construction of $T(V, W)$ can be found in [9].

3.2 The Meyberg Theorem

We shall now reverse the Tits-Kantor-Koecher construction we described in Section 2.2.

Theorem 3.2.1 (Meyberg's theorem)**.** *Let $\mathfrak{g}$ be a shortly graded Lie algebra with graded involution τ and let $w \in \mathfrak{g}_{-1}$. Then $\mathfrak{g}_{-1}$ with product defined by*

$$xy = \{x, w, y\} = [[x, \tau(w)], y]$$

is a Jordan algebra. We shall denote this algebra by $\mathfrak{J}_w(\mathfrak{g})$.

Proof. Let $x, y, z \in \mathfrak{g}_{-1}$ and let $f, g, h \in \mathfrak{g}_1$. We define $[x, g, y] = [[x, g], y] \in \mathfrak{g}_{-1}$ and $[g, x, h] = [[g, x], h] \in \mathfrak{g}_1$. Notice that whenever $g = \tau(w)$ we have $xy = \{x, w, y\} = [x, g, y]$.
In Example 3.1.3 we showed that $(\mathfrak{g}_{-1}, \{\ ,\ ,\ \})$ is a Jordan triple system, hence

$$xy = \{x, w, y\} = \{y, w, x\} = yx.$$

Moreover, the product $[\ ,\ ,\ ]$ satisfies the principal identity:

$$[x, f, [y, g, z]] - [y, g, [x, f, z]] = [[x, f, y], g, z] - [y, [f, x, g], z], \tag{3.6}$$

$$[f, x, [g, y, h]] - [g, y, [f, x, h]] = [[f, x, g], y, h] - [g, [x, f, y], h]. \tag{3.7}$$

If we take $y = x$, $g = f$ in equation (3.6) and $g = f$, $y = x$ in equation (3.7), we get:

$$[[x, f, x], f, z] = [x, [f, x, f], z], \tag{3.8}$$

$$[[f, x, f], x, h] = [f, [x, f, x], h]. \tag{3.9}$$

Now, if we exchange x, f and y, g in equation (3.6), we get

$$[y, g, [x, f, z]] - [x, f, [y, g, z]] = [[y, g, x], f, z] - [x, [g, y, f], z]. \tag{3.10}$$

If we sum equations (3.6) and (3.10) we obtain:

$$\begin{aligned} 0 &= [x, f, [y, g, z]] - [y, g, [x, f, z]] + [y, g, [x, f, z]] - [x, f, [y, g, z]] = \\ &= [[x, f, y], g, z] - [y, [f, x, g], z] + [[y, g, x], f, z] - [x, [g, y, f], z]. \end{aligned} \tag{3.11}$$

Using equations (3.8) and (3.9), if we exchange g with $[g,x,g]$ and take $f=g,\ y=x$, equation (3.11) implies the following:

$$\begin{aligned}0 = [[x,g,x],[g,x,g],z] - [x,[g,x,[g,x,g]],z]+\\ +[[x,[g,x,g],x],g,z] - [x,[[g,x,g],x,g],z] =\\ = [[x,g,x],[g,x,g],z] + [[x,[g,x,g],x],g,z] - 2[x,[[g,x,g],x,g],z] =\\ = [[x,g,x],[g,x,g],z] + [[[x,g,x,]g,x],g,z] - 2[x,[[g,x,g],x,g],z].\end{aligned} \tag{3.12}$$

To summarize, we have:

$$2[x,[[g,x,g],x,g],z] = [[x,g,x],[g,x,g],z] + [[[x,g,x,]g,x],g,z]. \tag{3.13}$$

Next, we set $x=[y,g,y],\ f=g$ in equation (3.6) and equation (3.13), hence

$$\begin{aligned}2[[y,g,y],g,[y,g,z]] - 2[y,g,[[y,g,y],g,z]] =\\ = 2[[[y,g,y],g,y],g,z] - 2[y,[g,[y,g,y],g],z] =\\ = 2[[[y,g,y],g,y],g,z] - [[y,g,y],[g,y,g],z] + [[[y,g,y,]g,y],g,z] =\\ = [[[y,g,y],g,y],g,z] - [[y,g,y],[g,y,g],z].\end{aligned} \tag{3.14}$$

By setting $x=y,\ f=g$ and replacing y with $[y,g,y]$ in equation (3.6) we have

$$\begin{aligned}[y,g,[[y,g,y],g,z]] - [[y,g,y],g,[y,g,z]] =\\ = [[[y,g,y],g,y],g,z] - [[y,g,y],[g,y,g],z].\end{aligned} \tag{3.15}$$

Finally, taking the difference between equations (3.14) and (3.15)

$$3[[y,g,y],g,[y,g,z]] - 3[y,g,[[y,g,y],g,z]] = 0. \tag{3.16}$$

Since equation (3.16) yields for every $y,z \in \mathfrak{g}_{-1}$ and $g \in \mathfrak{g}_1$,
we obtain

$$y^2(yz) = y(y^2z).$$

□

Corollary 3.2.2. *Let $\mathfrak{J}$ be a Jordan algebra with unit e. Then $\mathfrak{J}_{\frac{1}{2}e}(Lie(\mathfrak{J})) = \mathfrak{J}$.*

Proof. Let $\mathfrak{J}$ be a Jordan algebra with unit e. From Meyberg's theorem we already know that $\mathfrak{J} \cong \mathfrak{J}_{\frac{1}{2}e}(Lie(\mathfrak{J}))$ as vector spaces.
Let $x,y \in \mathfrak{J}$, denote by $[\ ,\ ]$ the product in $Lie(\mathfrak{J})$ defined in 2.4 and by $\{\ ,\ \}$ the product in $\mathfrak{J}_{\frac{1}{2}e}(Lie(\mathfrak{J}))$. Recall that $Lie(\mathfrak{J})$ has the graded involution given by $\tau(x) = \bar{x}$. We have

$$\{x,y\} = [[x,\frac{1}{2}\bar{e}],y] = [\frac{1}{2}(x\Box\bar{e}),y] = [L_{xe} + [L_x,L_e],y] = L_x(y) = xy.$$

□

Corollary 3.2.3. *Let $\mathfrak{g}$ be a simple shortly graded Lie algebras with grading induced by an $\mathfrak{sl}_2$-triple $\{f,h,e\}$. Suppose that τ is a graded involution of $\mathfrak{g}$. Then $\mathfrak{g}$ is isomorphic to Lie$(\mathfrak{J}_{\tau(e)}(\mathfrak{g}))$.*
An isomorphism is given by

$$\phi : \begin{array}{c|ccc} \mathfrak{g}_1 & \tau(y) & \to & \bar{y} \\ \mathfrak{g}_0 & [x,\tau(y)] & \to & 2(L_{xy}+[L_x,L_y]) \\ \mathfrak{g}_{-1} & x & \to & x \end{array} \tag{3.17}$$

Proof. Let $\mathfrak{g}$ be a Lie algebra satisfying the hypotheses. Then $\mathfrak{J}_{\tau(e)}(\mathfrak{g}) = \mathfrak{J}$ is a Jordan algebra with unit f. Indeed, if $x \in \mathfrak{g}_{-1}$, in $\mathfrak{J}$ we have

$$xf = fx = [[f,e],x] = [[f,e],x] = -[h,x] = x,$$

due to Definition 1.11. Hence, we can apply the Tits-Kantor-Koecher construction to $\mathfrak{J}$.
Now we show that ϕ is a Lie algebra homomorphism. By Definition 2.4, we have

$$\phi([\mathfrak{g}_{-1},\mathfrak{g}_{-1}]) = [\phi(\mathfrak{g}_{-1}),\phi(\mathfrak{g}_{-1})] = 0,\ \phi([\mathfrak{g}_1,\mathfrak{g}_1]) = [\phi(\mathfrak{g}_1),\phi(\mathfrak{g}_1)] = 0$$

and, for any $x,y \in \mathfrak{g}_{-1}$,

$$[\phi(x),\phi(\tau(y))] = [x,\bar{y}] = 2(L_{xy}+[L_x,L_y]) = \phi([x,\tau(y)]).$$

Since $\mathfrak{g}$ is simple, we have $[\mathfrak{g}_{-1},\mathfrak{g}_1] = \mathfrak{g}_0$. We use this property in order to extend ϕ to the whole $\mathfrak{g}$. This extension is well defined. Indeed, let $x,x',y,y' \in \mathfrak{g}_{-1}$ and $[x,y] = [x',y']$. Then, for any $z \in \mathfrak{g}_{-1}$, we have

$$[\phi([x,\tau(y)]),\phi(z)] = 2(L_{xy}(z)+[L_x,L_y](z))$$
$$[\phi([x',\tau(y')]),\phi(z)] = 2(L_{x'y'}(z)+[L_{x'},L_{y'}](z))$$

which implies

$$L_{xy}(f)+[L_x,L_y](f) = L_{x'y'}(f)+[L_{x'},L_{y'}](f)$$

and, since $[L_x,L_y](f) = [L_{x'},L_{y'}](f) = 0$,

$$L_{xy}(f) = xy = x'y' = L_{x'y'}(e).$$

Thus, if $[g,f] = [g',f']$ we have $L_{gf} = L_{g'f'}$ and, consequently, $[L_x,L_y] = [L_{x'},L_{y'}]$. Using the Jacobi identity we obtain the remaining necessary relations. This proves that ϕ is a homomorphism.
Since $\mathfrak{g}$ is simple and $\phi \neq 0$, ϕ is injective. Finally, in order to show that ϕ

is surjective, it is sufficient to show that $\phi(\mathfrak{g}_0) = Lie(\mathfrak{J})_0$. This follows from the relation $\phi([f,\tau(x)]) = 2L_x \in Lie(\mathfrak{J})_0$ and the fact that the L_x's generate $Lie(\mathfrak{J})_0$. This completes the proof. □

Remark 10. Let $\mathfrak{g}$ be as in Corollary 3.2.3. Then ϕ sends $\{f,h,e\}$, the $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, to $\{f, L_f, \bar{f}\}$ the $\mathfrak{sl}_2$-triple of $Lie(\mathfrak{J}_{\tau(e)}(\mathfrak{g}))$ which induces its grading.

Theorem 3.2.4. *There is a bijective correspondence between isomorphism classes of simple unital Jordan algebras and isomorphism classes of simple shortly graded Lie algebras with grading induced by an $\mathfrak{sl}_2$-triple $\{f,h,e\}$ and a graded involution.*

Proof. Let $\mathfrak{J}$ be a simple Jordan algebra with unit 1. Then, by Theorem 2.2.4, $Lie(\mathfrak{J})$ is simple. Moreover $\{1, L_1, \frac{1}{2}\bar{1}\}$ is an $\mathfrak{sl}_2$-triple in $Lie(\mathfrak{J})$ inducing its grading.
Conversely, if $\mathfrak{g}$ is a simple shortly graded Lie algebra with $\mathfrak{sl}_2$-triple $\{f,h,e\}$, $\mathfrak{J}_f(\mathfrak{g})$ is a unital Jordan algebra. Since, by Corollary 3.2.3, $Lie(\mathfrak{J}_f(\mathfrak{g}))$ is simple we have that $\mathfrak{J}_f(\mathfrak{g})$ is simple, due to Theorem 2.2.4. □

Example 3.2.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$ described in Example 1.1.2. We give $\mathfrak{g}$ the grading defined by

$$\mathfrak{g} =< f > \oplus < h > \oplus < e >= \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Notice that $\mathfrak{g}$ has a graded involution τ given by matrix transposition with negative sign, $\tau(f) = -f^t = -e,\ \ \tau(h) = -h^t = -h$.
The algebra $\mathfrak{J}_{-\frac{1}{2}f}(\mathfrak{sl}_2(\mathbb{F}))$ is $(< f >, \cdot\,)$ with product

$$f \cdot f = [[f, \frac{1}{2}e], f] = -\frac{1}{2}[h,f] = f.$$

This describes completely the algebra $\mathfrak{J}_{-\frac{1}{2}f}(\mathfrak{sl}_2(\mathbb{F}))$ which is isomorphic to the algebra $\mathfrak{gl}_1^+ \simeq \mathbb{F}$ described in Definition 1.5.

Example 3.2.2. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{F})$. We give $\mathfrak{g}$ the grading defined by

$$\mathfrak{g} =< f_{21}, f_2 > \oplus < f_1, h_1, h_2, e_1 > \oplus < e_2, e_{12} >,$$

where

$$f_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ e_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is the standard basis of $\mathfrak{sl}_3(\mathbb{F})$. The graded involution is again $\tau(x) = -x^t$.
In this case we can construct the corresponding Jordan algebra through two different, linearly independent, elements, f_2 and f_{21}.

- (f_2): The algebra $\mathfrak{J}_{-\frac{1}{2}f_2}(\mathfrak{sl}_3(\mathbb{F}))$ is $(< f_{21}, f_2 >, \cdot_2)$ where the product $\cdot_2$ is given by

$$f_2 \cdot_2 f_2 = [[f_2, \frac{1}{2}e_2], f_2] = \frac{1}{2}[-h_2, f_2] = 2f_2,$$
$$f_2 \cdot_2 f_{21} = [[f_2, \frac{1}{2}e_2], f_{21}] = \frac{1}{2}[-h_2, f_{21}] = f_{21},$$
$$f_{21} \cdot_2 f_{21} = [[f_{21}, \frac{1}{2}e_2], f_{21}] = \frac{1}{2}[-f_1, f_{21}] = 0.$$

- (f_{21}): The algebra $\mathfrak{J}_{-\frac{1}{2}f_{21}}(\mathfrak{sl}_3(\mathbb{F}))$ is $(< f_{21}, f_2 >, \cdot_{21})$ and

$$f_2 \cdot_{21} f_2 = [[f_2, \frac{1}{2}e_{12}], f_2] = \frac{1}{2}[-e_1, f_2] = 0,$$
$$f_2 \cdot_{21} f_{21} = [[f_2, \frac{1}{2}e_{12}], f_{21}] = \frac{1}{2}[-e_1, f_{21}] = f_2,$$
$$f_{21} \cdot_{21} f_{21} = [[f_{21}, \frac{1}{2}e_{12}], f_{21}] = \frac{1}{2}[-(h_1 + h_2), f_{21}] = 2f_{21}.$$

The two algebras just built are isomorphic. An isomorphism

$$\phi : \mathfrak{J}_{-\frac{1}{2}f_2}(\mathfrak{sl}_3(\mathbb{F})) \to \mathfrak{J}_{-\frac{1}{2}f_{21}}(\mathfrak{sl}_3(\mathbb{F}))$$

is given by $\phi(f_2) = f_{21}, \ \phi(f_{21}) = f_2$. Note that these two algebras are not simple. We have that, $< f_2 >$ is an ideal of $\mathfrak{J}_{-\frac{1}{2}f_{21}}(\mathfrak{sl}_3(\mathbb{F}))$.
This does not contradict Theorem 3.2.4 since the grading of $\mathfrak{sl}_3$ that we are considering is not induced by an $\mathfrak{sl}_2$-triple. In fact, if we take $h = \frac{1}{3}h_1 + \frac{2}{3}h_2$ we have $[h, f_2] = -f_2$ and $[h, f_{21}] = -f_{21}$ but there are no elements e, f with $e \in \mathfrak{g}_1$ and $f \in \mathfrak{g}_{-1}$ such that $[e, f] = h$.
Suppose such elements exist, then $f = af_2 + bf_{21}, \ e = ce_2 + de_{12}$ and

$$[af_2 + bf_{21}, \ ce_2 + de_{12}] =$$

$$= ac[f_2,\ e_2] + ad[f_2,\ e_{12}] + bc[f_{21},\ e_2] + bd[f_{21},\ e_{12}] =$$
$$= -(\ ach_2 + ade_1 + bcf_1 + bd(h_1 + h_2)\ ) = \frac{1}{3}h_1 + \frac{2}{3}h_2,$$

which imply $ad = 0$ and $bc = 0$. If $a = 0, b = 0$ then we should have

$$-bd(h_1 + h_2) = \frac{1}{3}h_1 + \frac{2}{3}h_2$$

a contradiction. Similarly, the other cases lead to a contradiction. Thus such elements e, f cannot exist.

Let us consider the algebra $\mathfrak{J}_f(\mathfrak{sl}_3(\mathbb{F}))$ with product $\cdot_f$ where f is a generic element of $\mathfrak{g}_{-1}$, set $f = \lambda f_2 + \mu f_{21}$ with $\lambda, \mu \neq 0$.

Since $[[x, \tau(f)], y]$ is linear in f we have

$$f_2 \cdot_f f_2 = 2\lambda f_2, \quad f_2 \cdot_f f_{21} = \lambda f_{21} + \mu f_2, \quad f_{21} \cdot_f f_{21} = 2\mu f_{21}.$$

Then $< (\mu f_2 - \lambda f_{21}) >$ is a proper ideal of $\mathfrak{J}_f(\mathfrak{sl}_3(\mathbb{F}))$. We have

$$f_2 \cdot (\mu f_2 - \lambda f_{21}) = 2\lambda\mu f_2 - \lambda^2 f_{21} - \lambda\mu f_2 = \lambda(\mu f_2 - \lambda f_{21})$$

$$f_{21} \cdot (\mu f_2 - \lambda f_{21}) = \lambda\mu f_{21} + \mu^2 f_2 - 2\mu^2 f_{21} = -\mu(\mu f_2 - \lambda f_{21}).$$

This computation shows that $\mathfrak{J}_f(\mathfrak{sl}_3(\mathbb{F}))$ is not simple for any $f \in \mathfrak{g}_{-1}$, according to Theorem 3.2.4. This result is consistent with the fact that $\mathfrak{sl}_3$ has no short grading induced by an $\mathfrak{sl}_2$-triple (see, for example, [2]).

Example 3.2.3 ($\mathfrak{sl}_4(\mathbb{F})$)**.** Let $\mathfrak{g} = \mathfrak{sl}_4$. We give $\mathfrak{g}$ the following grading

$$< f_{321}, f_{32}, f_{21}, f_2 > + < f_1, f_3, h_1, h_2, h_3, e_1, e_3 > + < e_2, e_{12}, e_{23}, e_{123} >$$

where

$$f_{321} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ f_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ f_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ f_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ h_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$e_i = f_i^t$, $e_{ij} = f_{ji}^t$ and $e_{123} = f_{321}^t$. On $\mathfrak{g}$ we take the graded involution given by $x \to -x^t$.

This grading of $\mathfrak{sl}_4$ is induced by the $\mathfrak{sl}_2$-triple

$$\{f = f_{32} + f_{21}, h = \frac{1}{2}(h_1 + 2h_2 + h_3), e = \frac{1}{2}(e_{12} + e_{23})\}.$$

Let $\mathfrak{J} = \mathfrak{J}_f(\mathfrak{g}) = (< f, f_{32} - f_{21}, f_2, f_{321} >, \cdot)$. We have that $\mathfrak{J}$ is isomorphic to gl_2^+ introduced in Definition 1.5.

An isomorphism is given by $\phi(f) = Id$,

$$\phi(f_{32} - f_{21}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \phi(f_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \phi(f_{321}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This comes directly from the multiplication rules in $\mathfrak{J}$, i.e.:

$$\begin{array}{ll} f \cdot v = v, & (f_{32} - f_{21}) \cdot (f_{32} - f_{21}) = f, \\ (f_{32} - f_{21}) \cdot f_2 = 0\ , & (f_{32} - f_{21}) \cdot f_{321} = 0, \\ f_2 \cdot f_2 = 0, & f_2 \cdot f_{321} = f, \\ f_{321} \cdot f_{321} = 0. & \end{array} \tag{3.18}$$

Example 3.2.4 ($\mathfrak{sp}_4(\mathbb{F})$)**.** Let $\mathfrak{g} = \mathfrak{sp}_4$. We give $\mathfrak{g}$ the following grading

$$< p_{21}, p_{22}, p_{11} > + < h_1, e_{12}, e_{21}, h_2 > + < n_{11}, n_{22}, n_{12} >$$

where

$$p_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ p_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ p_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ e_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_{21} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$n_{11} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ n_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ n_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The map $x \to -x^t$ is again a graded involution of $\mathfrak{g}$.
Let us take the elements $h = \frac{1}{2}(h_1 + h_2), \ f = p_{21}, \ e = \frac{1}{2}n_{12}$. Simple calculations show that $< f, h, e >$ is an $\mathfrak{sl}_2$-triple, hence the algebra $\mathfrak{J}_{-\tau^{-1}(e)}(\mathfrak{g}) = \mathfrak{J}_{-\frac{1}{2}p_{21}}(\mathfrak{g})$ is unital.
Let $\mathfrak{J} = \mathfrak{J}_{-\frac{1}{2}p_{21}}(\mathfrak{g}) = (< p_{21}, p_{22}, p_{11} >, \ \cdot \ )$. We have

$$\begin{aligned} p_{21} \cdot p_{21} &= [[p_{21}, \tfrac{1}{2}n_{12}], p_{21}] = -\tfrac{1}{2}[h_1 + h_2, p_{21}] = p_{21} \\ p_{21} \cdot p_{22} &= -\tfrac{1}{2}[h_1 + h_2, p_{22}] = p_{22} \\ p_{21} \cdot p_{11} &= -\tfrac{1}{2}[h_1 + h_2, p_{11}] = p_{11} \\ p_{22} \cdot p_{22} &= [[p_{22}, \tfrac{1}{2}n_{12}], p_{22}] = -\tfrac{1}{2}[e_{12}, p_{22}] = 0 \\ p_{22} \cdot p_{11} &= -\tfrac{1}{2}[e_{12}, p_{11}] = \tfrac{1}{2}p_{21} \\ p_{11} \cdot p_{11} &= [[p_{11}, \tfrac{1}{2}n_{12}], p_{11}] = -\tfrac{1}{2}[e_{21}, p_{11}] = 0. \end{aligned} \tag{3.19}$$

It is straight forward that p_{21} is the unit of $\mathfrak{J}$.
According to Theorem 3.2.4 $\mathfrak{J}$ is simple. In fact, suppose $I \neq \mathfrak{J}$ is an ideal and let $I \ni v$, $v = ap_{21} + bp_{22} + cp_{11}$. Then

$$(((v \cdot p_{22}) \cdot p_{22}) \cdot p_{11}) = ((((\ ap_{21} + bp_{22} + cp_{11} \ ) \cdot p_{22}) \cdot p_{22}) \cdot p_{11}) =$$

$$= (((\ ap_{22} + \frac{1}{2}cp_{21} \ ) \cdot p_{22}) \cdot p_{11}) = \frac{1}{4}cp_{22} \cdot p_{11} = \frac{1}{8}cp_{21} \in I,$$

this implies $c = 0$, otherwise we would have $p_{21} \in I$ and $I = \mathfrak{J}$ which is absurd, since we have set $I \neq \mathfrak{J}$. Thus $v = ap_{21} + bp_{22}$. Then $(v \cdot p_{22}) \cdot 4p_{11} = bp_{21} \in I$ implies $b = 0$, and $v = ap_{21}$. This leads to $v = 0$, which means that $\mathfrak{J}$ is simple.

The Jordan algebra $\mathfrak{J}_{-\tau^{-1}(e)}(\mathfrak{g})$ is isomorphic to the algebra $\mathfrak{B}_2(\sigma, \mathbb{F}) = \mathfrak{B}_2$ introduced in Example 1.1.4.

$$\mathfrak{B}_2 =< Id, E, F >\subseteq gl_2^+.$$

An isomorphism is given by

$$\phi(p_{21}) = Id, \ \phi(p_{22}) = E, \ \phi(p_{11}) = F.$$

3.3 Lie algebras and Jordan triple systems

Proposition 3.3.1. *Let $\mathfrak{g}$ be a shortly graded Lie algebra with graded involution τ. Then $J(\mathfrak{g})$ the triple system derived from $\mathfrak{g}$ is a Jordan triple system.*

Proof. The proof of this proposition is given by Example 3.1.3. □

Definition 3.11. Let $(\ J\,, (\ ,\ ,\ )\ )$ be a Jordan triple system and let $a, b \in J$. We denote by $u_a,\ s_{ab},\ \tilde{u}_a$ the operators of J defined by:

$$\begin{aligned} u_a(x) &= a, \\ s_{ab}(x) &= (abx), \\ \tilde{u}_a(x) &= -\tfrac{1}{2}(xax). \end{aligned} \tag{3.20}$$

Theorem 3.3.2. *Let $(\ J\,, (\ ,\ ,\ )\ )$ be a Jordan triple system. Then*

$$Lie(J) = < u_a > \oplus < s_{ab} > \oplus < \tilde{u}_a >$$

with commutator defined as in Definition 3.9, is a shortly graded Lie subalgebra of $T(J)$.
We will call $Lie(J)$ the Lie algebra associated to the Jordan triple system J.

Proof. In order to prove the theorem we need to show that $[Lie(J)_i, Lie(J)_j] \subseteq Lie(J)_{i+j}$. In particular, we will have that, for any $a, b, c, d, x \in J$

$$\begin{aligned} &[s_{ab}, s_{cd}] = s_{(abc)d} - s_{c(dab)}, && [s_{ab}, u_c] = u_{(abc)}, \\ &[s_{ab}, \tilde{u}_c] = -\tilde{u}_{(bac)}, && [u_a, \tilde{u}_b] = s_{ab}, \\ &[u_a, u_b] = 0, && [\tilde{u}_a, \tilde{u}_b] = 0. \end{aligned} \tag{3.21}$$

Let $a, b, c, d, x \in J$. Then

$$[u_a, u_b] = u_a \circ u_b - u_b \circ u_a = 0$$

since u_a and u_b are both order 0 operators.
Moreover, since the principal identity is satisfied and the product is commutative in the first and third variable, we have:

$$\begin{aligned} 2[\tilde{u}_a, \tilde{u}_b](x) &= 4\tilde{U}_a(\tilde{u}_b(x), x) - 4\tilde{U}_b(\tilde{u}_a(x), x) = \\ &= ((xbx)ax) - ((xax)bx) = (xa(xbx)) - (xb(xax)) = \\ &= ((xax)bx) - (x(axb)x) - (xb(xax)) = -(x(axb)x), \\ 4[\tilde{u}_a, \tilde{u}_b](x) &= 2[\tilde{u}_a, \tilde{u}_b] - 2[\tilde{u}_b, \tilde{u}_a] = \\ &= -(x(axb)x) + -(x(bxa)x) = 0; \end{aligned}$$

$$\begin{aligned} [s_{ab}, s_{cd}](x) &= s_{ab}(s_{cd}(x)) - s_{cd}(s_{ab}(x)) = \\ &= (ab(cdx)) - (cd(abx)) = ((abc)dx) - (c(bad)x) = \\ &= s_{(abc)d}(x) - s_{c(dab)}(x); \end{aligned}$$

$$
\begin{aligned}
[s_{ab}, u_c](x) &= s_{ab}(u_c(x)) = \\
&= (abc) = u_{(abc)}(x);
\end{aligned}
$$

$$
\begin{aligned}
[s_{ab}, \tilde{u}_c](x) = s_{ab}(\tilde{u}_c(x)) - 2\tilde{U}_c(s_{ab}(x), x) = \\
= -\tfrac{1}{2}(ab(xcx)) + ((abx)cx) = \\
= -(\ \tfrac{1}{2}(ab(xcx)) - \tfrac{1}{2}((abx)cx) - \tfrac{1}{2}(xc(abx))\ ) = \\
= -\tfrac{1}{2}(x(bac)x) = \tilde{u}_{bac};
\end{aligned}
$$

$$
\begin{aligned}
[u_a, \tilde{u}_b](x) &= -2\tilde{U}_b(u_a(x), x) = \\
&= (abx) = s_{ab}(x).
\end{aligned}
$$

□

Corollary 3.3.3. *Let $\mathfrak{g}$ be a shortly graded simple Lie algebra with graded involution τ and let $J = J(\mathfrak{g})$ be the Jordan triple system derived from $\mathfrak{g}$.*
Then $\mathfrak{g}$ is isomorphic to $Lie(J(\mathfrak{g}))$. Moreover, an isomorphism is given by

$$
\phi : \begin{array}{c|ccc}
\mathfrak{g}_1 & \tau(a) & \to & \tilde{u}_a \\
\mathfrak{g}_0 & [a, \tau(b)] & \to & [u_a, \tilde{u}_b] \\
\mathfrak{g}_{-1} & a & \to & u_a
\end{array} \tag{3.22}
$$

for $a, b \in \mathfrak{g}_{-1}$. We call $\phi : \mathfrak{g} \to T(\mathfrak{g}_{-1})$ the conformal realization of $\mathfrak{g}$ on $\mathfrak{g}_{-1}$.

Proof. The map ϕ is well defined. If $[a, \tau(b)] = [c, \tau(d)]$ then

$$
(s_{ab} - s(cd))(x) = (abx) - (cdx) = [[a, \tau(b)], x] - [[c, \tau(d)], x] =
$$

$$
= [([a, \tau(b)] - [c, \tau(d)]), x] = 0.
$$

Next, we show that the map ϕ is a homomorphism. Let $a, b, c, d \in \mathfrak{g}_{-1}$. Since $[a, b] = 0$ and $[\tau(a), \tau(b)] = 0$ we just need to show

$$
\begin{aligned}
\phi([[a, \tau(b)], c]) &= [s_{ab}, u_c] = u_{(abc)}, \\
\phi([[a, \tau(b)], \tau(c)]) &= [s_{ab}, \tilde{u}_c] = -\tilde{u}_{(bac)}, \\
\phi([[a, \tau(b)], [c, \tau(d)]]) &= s_{(abc)d} - s_{c(dab)}.
\end{aligned} \tag{3.23}
$$

The first equation follows directly from the fact that in $Lie(J(\mathfrak{g}))$ we have $[[a, \tau(b)], c] = (abc)$.
Furthermore, for the same reason,

$$
\phi([[a, \tau(b)], \tau(c)]) = -\phi([[\tau(b), a], \tau(c)]) = -\phi(\tau([[b, \tau(a)], c])) = -\tilde{u}_{bac}.
$$

We are then left to prove the third relation: using the fact that $[\,,\,]$ is a Lie product we obtain

$$
\phi([[a, \tau(b)], [c, \tau(d)]]) = -\phi([[c, \tau(d)], ([a, \tau(b)])]) =
$$

$$= \phi([[\tau(d),([a,\tau(b)])],c]) + \phi([[([a,\tau(b)]),c],\tau(d)]) =$$

$$= [u_(abc), \tilde{u}_d] - \phi([c,\tau([d,[\tau(a),b]]]) =$$

$$= s_{(abc)d} - [u_c, \phi(\tau([[b,\tau(a)],d])] =$$

$$= s_{(abc)d} - s_{c(dab)}.$$

Thus ϕ is a homomorphism of Lie algebras. It must be injective since $\mathfrak{g}$ is simple and $Ker(\phi) \neq \mathfrak{g}$ is an ideal, thus it must be $Ker(\phi) = 0$. Since $Lie(J(\mathfrak{g}))$ is spanned by the operators $u_a, s_{ab}, \tilde{u}_a$, ϕ is also surjective. □

3.4 Lie algebras and Kantor triple systems

Proposition 3.4.1. *Let $\mathfrak{g}$ be a 5-graded Lie algebra with graded involution τ. Then $K(\mathfrak{g})$ the triple system derived from $\mathfrak{g}$ is a Kantor triple system.*

Proof. Example 3.1.4 shows that $\mathfrak{g}_{-1}$, with the derived triple product, satisfies both the identities of equation (3.6). □

Definition 3.12. Let $(K, (\, , \, , \,))$ be a Kantor triple system, let $L = span\{< u, v >,\ u, v \in K\}$ and let $a, b \in K$. We define the following families of operators of $K \oplus L$:

$$\begin{gathered}
k_{ab}(x+X) = 2 < a, b >, \\
u_a(x+X) = a + < a, x >, \\
s_{ab}(x+X) = (abx) - < a, X(b) >, \\
\tilde{u}_a(x+X) = -\tfrac{1}{2}(xax) - \tfrac{1}{2}X(a) + \\
+\tfrac{1}{6} < (xax), x > -\tfrac{1}{2} < X(a), x >, \\
\tilde{k}_{ab}(x+X) = -\tfrac{1}{6}(x < a, b > (x)x) - \tfrac{1}{2}X(< a, b > (x)) + \\
+\tfrac{1}{12} < (x < a, b > (x)x), x > + \tfrac{1}{2} < X(a), X(b) >,
\end{gathered} \tag{3.24}$$

for $x \in K$ and $X \in L$.

Theorem 3.4.2. *Let $(K, (\, , \, , \,))$ be a Kantor triple system and let $L = span\{< u, v >,\ u, v \in K\}$. Then*

$$Lie(K) = < k_{ab} > \oplus < u_a > \oplus < s_{ab} > \oplus < \tilde{u}_a > \oplus < \tilde{k}_{ab} >$$

with commutator defined as in Definition 3.9 is a 5-graded Lie subalgebra of $T(K, L)$.

We will call $Lie(K)$ the Lie algebra associated to the Kantor triple system K.

Proof. In order to prove the theorem we need to show that for any $a, b, c, d, x \in K$ and $X \in L$ we have

$$\begin{array}{ll}
[k_{ab}, k_{cd}] = 0, & [k_{ab}, u_c] = 0, \\
[k_{ab}, \tilde{u}_c] = u_{<a,b>c}, & [k_{ab}, \tilde{k}_{cd}] = s_{<a,b>(c)d} - s_{<a,b>(d)c}, \\
[u_a, u_b] = k_{ab}, & [u_a, \tilde{u}_b] = s_{ab}, \\
[u_a, \tilde{k}_{cd}] = -\tilde{u}_{<c,d>(a)}, & [s_{ab}, k_{cd}] = k_{<c,d>(b)a}, \\
[s_{ab}, u_c] = u_{(abc)}, & [s_{ab}, s_{cd}] = s_{(abc)d} - s_{c(dab)}, \\
[s_{ab}, \tilde{u}_c] = -\tilde{u}_{(bac)}, & [s_{ab}, \tilde{k}_{cd}] = -\tilde{k}_{<c,d>(a)b}, \\
[\tilde{u}_a, \tilde{u}_b] = \tilde{k}_{ab}, & [\tilde{u}_a, \tilde{k}_{cd}] = [\tilde{k}_{ab}, \tilde{k}_{cd}] = 0.
\end{array} \tag{3.25}$$

It is immediate, by Definition 3.9, that

$$[k_{ab}, k_{cd}] = 0$$

and

$$[k_{ab}, u_c] = 0$$

since k_{ab} is constant, u_c is constant with respect to K and the fact that k_{ab} is L-valued.
Besides, it follows from the definition that:

$$[k_{ab}, \tilde{u}_c](x + X) =< a, b > (c)+ << a, b > (c), x >= u_{<a,b>c}(x + X),$$

$$[u_a, u_b](x + X) =< a, b > - < b, a >= 2 < a, b >= k_{ab}(x + X)$$

and

$$[s_{ab}, k_{cd}](x + X) = - < a, < c, d > (b) >= k_{<c,d>(b)a}(x + X)$$

Before we show the next identity, we first need to verify

$$< a, b > (< c, d > (x)) = (< a, b > (c)dx) - (< a, b > (d)cx). \tag{3.26}$$

Due to the principal identity, the auxiliar identity and the definition of $<,>$, we have

$$< a, b > (< c, d > (x)) - (< a, b > (c)dx) + (< a, b > (d)cx) =$$

$$= (a(cxd)b) - (a(dxc)b) - (b(cxd)a) + (b(dxc)a) -$$

$$-(<a,b>(c)dx)+(<a,b>(d)cx)=$$

$$=((xca)db)-(xc(adb))+(ad(xcb))-$$
$$-((xda)cb)+(xd(acb))-(ac(xdb))-$$
$$-((xcb)da)+(xc(bda))-(bd(xca))+$$
$$+((xdb)ca)-(xd(bca))+(bc(xda))-$$
$$-(<a,b>(c)dx)+(<a,b>(d)cx)=$$

$$=<(xca),b>(d)-<(xda),b>(c)-<(xcb),a>(d)+<(xdb),a>(c)+$$
$$+(xd<a,b>(c))-(xc<a,b>(d))-(<a,b>(c)dx)+(<a,b>(d)cx)=$$

$$=<(xca),b>(d)-<(xcb),a>(d)-<<a,b>(c),x>(d)-$$
$$-<(xda),b>(c)+<(xdb),a>(c)+<<a,b>(d),x>(c)=0.$$

Thanks to equation (3.26), it is easy to calculate

$$[k_{ab},\tilde{k}_{cd}](x+X)=<a,b>(\ <c,d>(x)\ )-$$
$$-<<a,b>(c),X(d)\ >-<\ X(c),<a,b>(d)\ >=$$
$$=(\ <a,b>(c)\,d\,x\ )-<<a,b>(c),X(d)\ >-$$
$$-(\ <a,b>(d)\,c\,x\ )+<<a,b>(d),X(c)\ >$$
$$=s_{<a,b>(c)d}-s_{<a,b>(d)c}.$$

Using the auxiliar identity, we get

$$[s_{ab},u_c](x+X)=(abc)-<a,<c,x>(b)>-<c,(abx)>=$$
$$=(abc)+<<c,x>(b),a>=$$
$$=(abc)+<(abc),x>-<(abx),c>+<(abx),c>=$$
$$=u_{(abc)}(x+X).$$

We have:

$$[u_a,\tilde{u}_b](x+X)=<\ a,-\frac{1}{2}(xbx)-\frac{1}{2}X(b)\ >+$$
$$+\frac{1}{2}(abx)+\frac{1}{2}(xba)+\frac{1}{2}<a,x>(b)-$$

$$-\frac{1}{6} < (abx), x > -\frac{1}{6} < (xba), x > -\frac{1}{6} < (xbx), a > +$$
$$+\frac{1}{2} << a, x > (b), x > +\frac{1}{2} < X(b), a >=$$

$$= \frac{1}{2}(abx) + \frac{1}{2}(xba) + \frac{1}{2}(abx) - \frac{1}{2}(xba)-$$
$$-\frac{1}{6} < (abx), x > -\frac{1}{6} < (xba), x > -\frac{1}{6} < (xbx), a > +$$
$$+\frac{1}{2} << a, x > (b), x > +\frac{1}{2} < X(b), a > -\frac{1}{2} < a, (xbx) > -\frac{1}{2} < a, X(b) >=$$

$$= (abx) - \frac{1}{2} < a, X(b) > +$$
$$+\frac{3}{6} << a, x > (b), x > -\frac{1}{6} < (abx), x > -\frac{1}{6} < (xba), x > +\frac{2}{6} < (xbx), a >=$$

$$= s_{ab}(x + X)+$$
$$+\frac{3}{6} << a, x > (b), x > -\frac{1}{6} < (abx), x > -\frac{1}{6} < (xba), x > +\frac{2}{6} < (xbx), a >,$$

hence, we need to prove

$$\begin{gathered} 3 << a, x > (b), x >= \\ < (abx), x > + < (xba), x > -2 < (xbx), a > . \end{gathered} \tag{3.27}$$

Using the auxiliar identity we get

$$3 << a, x > (b), x >= 2 << a, x > (b), x > + << a, x > (b), x >=$$
$$= 2 < (xba), x > -2 < (xbx), a > + < (abx), x > - < (xba), x >=$$
$$=< (abx), x > + < (xba), x > -2 < (xbx), a > .$$

Thus $[u_a, \tilde{u}_b] = s_{ab}$.

We have

$$[s_{ab}, \tilde{u}_c](x + X) = (ab(-\frac{1}{2}(xcx) - \frac{1}{2}X(c)))-$$
$$- < a, \frac{1}{6} < (xcx), x > (b) - \frac{1}{2} < X(c), x > (b) > +$$
$$+\frac{1}{2}((abx)cx) + \frac{1}{2}(xc(abx)) - \frac{1}{2} < a, X(b) > (c)-$$
$$-\frac{1}{6} < ((abx)cx), x > -\frac{1}{6} < (xc(abx)), x > -\frac{1}{6} < (xcx), (abx) > -$$
$$-\frac{1}{2} << a, X(b) > (c), x > +\frac{1}{2} < X(c), (abx) >=$$

$$\begin{gathered}
= \tfrac{1}{2}(\ -(ab(xcx)) + ((abx)cx) + (xc(abx))\ )+ \\
-\tfrac{1}{2} < a, X(b) > (c) - \tfrac{1}{2}(abX(c))- \\
-\tfrac{1}{6} < ((abx)cx), x > -\tfrac{1}{6} < (xc(abx)), x > - \\
-\tfrac{1}{6} < (xcx), (abx) > -\tfrac{1}{6} < a, < (xcx), x > (b) > + \\
+\tfrac{1}{2} < a, < X(c), x > (b) > - \\
-\tfrac{1}{2} << a, X(b) > (c), x > +\tfrac{1}{2} < X(c), (abx) > .
\end{gathered} \tag{3.28}$$

In order to show that equation (3.28) gives exactly $-\tilde{u}_{bac}$ we calculate:

$$\begin{gathered}
- < a, < u, v > (b) > (c) - (\ a\, b\, (< u, v > (c))\ ) = \\
=<< u, v > (b), a > (c) - (ab(ucv)) + (ab(vcu)) = \\
=< (abu), v > (c) - < (abv), u > (c) - (ab(ucv)) + (ab(vcu)) = \\
= ((abu)cv) - (vc(abu)) - ((abv)cu) + (uc(abv)) - (ab(ucv)) + (ab(vcu)) = \\
= -(ab(ucv)) + ((abu)cv) + (uc(abv)) + \\
+(ab(vcu)) - ((abv)cu) - (vc(abu)) = \\
(u(bac)v) - (v(bac)u) =< u, v > (bac),
\end{gathered}$$

since $X \in L$ is a linear combination of the $< u, v >$'s, we have just proved that

$$-\frac{1}{2} < a, X(b) > (c) - \frac{1}{2}(abX(c)) = \frac{1}{2}X(bac); \tag{3.29}$$

Besides we have

$$\begin{gathered}
- < (xcx), (abx) > - < a, < (xcx), x > (b) >= \\
=<< (xcx), x > (b), a > - < (xcx), (abx) >= \\
=< (ab(xcx)), x > - < (abx), (xcx) > - < (xcx), (abx) >= \\
=< (ab(xcx)), x >,
\end{gathered}$$

thus

$$\begin{gathered}
-\frac{1}{6} < ((abx)cx), x > -\frac{1}{6} < (xc(abx)), x > - \\
-\frac{1}{6} < (xcx), (abx) > -\frac{1}{6} < a, < (xcx), x > (b) >=
\end{gathered}$$

$$= -\frac{1}{6} < ((abx)cx), x > -\frac{1}{6} < (xc(abx)), x > +\frac{1}{6} < (ab(xcx)), x >=$$

$$= -\frac{1}{6} < (x(bac)x), x >; \tag{3.30}$$

finally, we have

$$< a, < X(c), x > (b) > - << a, X(b) > (c), x > + < X(c), (abx) >=$$

$$=< a, < X(c), x > (b) > - << a, X(b) > (c), x > -$$
$$- < (abx), X(c) > + < (abX(c)), x > - < (abX(c)), x >=$$

$$=< a, < X(c), x > (b) > - << a, X(b) > (c), x > -$$
$$- << x, X(c) > (b), a > - < (abX(c)), x >=$$

$$= - << a, X(b) > (c), x > - < (abX(c)), x > +$$
$$+ << X(c), x > (b), a > + < a, < X(c), x > (b) >=$$

$$=< (- < a, X(b) > (c) - (abX(c))), x >,$$

using equation (3.29) we get

$$\begin{gathered} < a, < X(c), x > (b) > - << a, X(b) > (c), x > + < X(c), (abx) >= \\ =< X(bac), x > . \end{gathered} \tag{3.31}$$

If we use equations (3.29),(3.30) and (3.31) in equation (3.28) and apply the principal identity to the first three terms of equation (3.28) we obtain

$$[s_{ab}, \tilde{u}_c](x + X) = (3.28) =$$
$$= \frac{1}{2}(x(bac)x) + \frac{1}{2}X((bac)) - \frac{1}{6} < (x(bac)x), x > + \frac{1}{2} < X((bac)), x >=$$
$$= -\tilde{u}_{(bac)}.$$

Now we prove

$$[\tilde{u}_a, \tilde{u}_b] = \tilde{k}_{ab}. \tag{3.32}$$

We have

$$[\tilde{u}_a, \tilde{u}_b](x + X) =$$

$$= -\frac{1}{2}((-\frac{1}{2}(xbx) - \frac{1}{2}X(b))ax) -$$

$$-\frac{1}{2}(xa(\ -\frac{1}{2}(xbx)-\frac{1}{2}X(b)\ ))-$$
$$-\frac{1}{2}(\ +\frac{1}{6}<(xbx),x>-\frac{1}{2}<X(b),x>\ )(a)+$$
$$+\frac{1}{6}<((\ -\frac{1}{2}(xbx)-\frac{1}{2}X(b)\ )ax),x>+$$
$$+\frac{1}{6}<(xa(\ -\frac{1}{2}(xbx)-\frac{1}{2}X(b)\ )),x>+$$
$$+\frac{1}{6}<(xax),(\ -\frac{1}{2}(xbx)-\frac{1}{2}X(b)\ )>-$$
$$-\frac{1}{2}<(\ +\frac{1}{6}<(xbx),x>-\frac{1}{2}<X(b),x>\ )(a),x>-$$
$$-\frac{1}{2}<X(a),(\ -\frac{1}{2}(xbx)-\frac{1}{2}X(b)\ )>+$$

$$+\frac{1}{2}((\ -\frac{1}{2}(xax)-\frac{1}{2}X(a)\ )bx)+$$
$$+\frac{1}{2}(xb(\ -\frac{1}{2}(xax)-\frac{1}{2}X(a)\ ))+$$
$$+\frac{1}{2}(\ +\frac{1}{6}<(xax),x>-\frac{1}{2}<X(a),x>\ )(b)-$$
$$-\frac{1}{6}<((\ -\frac{1}{2}(xax)-\frac{1}{2}X(a)\ )bx),x>-$$
$$-\frac{1}{6}<(xb(\ -\frac{1}{2}(xax)-\frac{1}{2}X(a)\ )),x>-$$
$$-\frac{1}{6}<(xbx),(\ -\frac{1}{2}(xax)-\frac{1}{2}X(a)\ )>+$$
$$+\frac{1}{2}<(\ +\frac{1}{6}<(xax),x>-\frac{1}{2}<X(a),x>\ )(b),x>+$$
$$+\frac{1}{2}<X(b),(\ -\frac{1}{2}(xax)-\frac{1}{2}X(a)\ )>=$$

$$=\frac{1}{4}((xbx)ax)+\frac{1}{4}(X(b)ax)+$$
$$+\frac{1}{4}(xa(xbx))+\frac{1}{4}(xaX(b))-$$
$$-\frac{1}{12}<(xbx),x>(a)+\frac{1}{4}<X(b),x>(a)-$$
$$-\frac{1}{12}<((xbx)ax),x>-\frac{1}{12}<(X(b)ax),x>-$$
$$-\frac{1}{12}<(xa(xbx)),x>-\frac{1}{12}<(xaX(b)),x>-$$
$$-\frac{1}{12}<(xax),(xbx)>-\frac{1}{12}<(xax),X(b)>-$$
$$-\frac{1}{12}<<(xbx),x>(a),x>+\frac{1}{12}<<X(b),x>(a),x>+$$

$$+\frac{1}{4} < X(a),(xbx) > +\frac{1}{4} < X(a),X(b) > -$$

$$-\frac{1}{4}((xax)bx) - \frac{1}{4}(X(a)bx) -$$
$$-\frac{1}{4}(xb(xax)) - \frac{1}{4}(xbX(a)) +$$
$$+\frac{1}{12} < (xax),x > (b) - \frac{1}{4} < X(a),x > (b) +$$
$$+\frac{1}{12} < ((xax)bx),x > +\frac{1}{12} < (X(a)bx),x > +$$
$$+\frac{1}{12} < (xb(xax)),x > +\frac{1}{12} < (xbX(a)),x > +$$
$$+\frac{1}{12} < (xbx),(xax) > +\frac{1}{12} < (xbx),X(a) > +$$
$$+\frac{1}{12} << (xax),x > (b),x > -\frac{1}{12} << X(a),x > (b),x > -$$
$$-\frac{1}{4} < X(b),(xax) > -\frac{1}{4} < X(b),X(a) >=$$

$$\begin{aligned} &= \tfrac{1}{4}((xbx)ax) + \tfrac{1}{4}(xa(xbx)) - \tfrac{1}{12} < (xbx),x > (a) - \\ &-\tfrac{1}{4}((xax)bx) - \tfrac{1}{4}(xb(xax)) + \tfrac{1}{12} < (xax),x > (b) + \end{aligned} \tag{3.33}$$

$$\begin{aligned} &+\tfrac{1}{4} < X(b),x > (a) + \tfrac{1}{4}(X(b)ax) + \tfrac{1}{4}(xaX(b)) - \\ &-\tfrac{1}{4} < X(a),x > (b) - \tfrac{1}{4}(xbX(a)) - \tfrac{1}{4}(X(a)bx) - \end{aligned} \tag{3.34}$$

$$\begin{aligned} &-\tfrac{1}{12} < ((xbx)ax),x > -\tfrac{1}{12} < (xa(xbx)),x > + \\ &+\tfrac{1}{12} < ((xax)bx),x > +\tfrac{1}{12} < (xb(xax)),x > + \\ &+\tfrac{1}{12} < (xbx),(xax) > +\tfrac{1}{12} << (xax),x > (b),x > - \\ &-\tfrac{1}{12} < (xax),(xbx) > -\tfrac{1}{12} << (xbx),x > (a),x > - \end{aligned} \tag{3.35}$$

$$\begin{aligned} &-\tfrac{1}{12} < (X(b)ax),x > -\tfrac{1}{12} < (xaX(b)),x > + \\ &+\tfrac{1}{12} < (X(a)bx),x > +\tfrac{1}{12} < (xbX(a)),x > + \\ &-\tfrac{1}{4} < X(b),(xax) > -\tfrac{1}{12} < (xax),X(b) > + \\ &+\tfrac{1}{4} < X(a),(xbx) > +\tfrac{1}{12} < (xbx),X(a) > + \\ &+\tfrac{1}{12} << X(a),x > (b),x > -\tfrac{1}{12} << X(b),x > (a),x > + \end{aligned} \tag{3.36}$$

$$+\tfrac{1}{4} < X(a),X(b) > -\tfrac{1}{4} < X(b),X(a) > . \tag{3.37}$$

In order to prove equation (3.32), we verify the two following identities, which will be quite useful:

$$((xbx)ax) + 2(xa(xbx)) - ((xax)bx) - 2(xb(xax)) = (x < b,a > (x)x), \tag{3.38}$$

$$\begin{aligned}(<x,y>(b)az)-(<x,y>(a)bz)=\\=(x<a,b>(y)z)-(y<a,b>(x)z),\end{aligned} \tag{3.39}$$

$$\begin{aligned}<<u,v>(a)\ ,\ <x,y>(b)\ >=\\=<(x<a,b>(y)u),v>-<(y<a,b>(x)u),v>+\\+<(y<a,b>(x)u),v>-<(x<a,b>(y)v),u>,\end{aligned} \tag{3.40}$$

for any $u,v,x,y,z\in K$.

Using the principal identity, we obtain

$$((xbx)ax)+2(xa(xbx))-((xax)bx)-2(xb(xax))=$$

$$=((xbx)ax)+(xa(xbx))+(xa(xbx))-$$

$$-((xax)bx)-(xb(xax))-(xb(xax))=$$

$$=(xb(xax))+(x(bxa)x)+(xa(xbx))-$$

$$-(xa(xbx))-(x(axb)x)-(xb(xax))=$$

$$=(x(bxa)x)-(x(axb)x)=(x<b,a>(x)x),$$

hence (3.38) holds.

By the same argument, we have

$$(<x,y>(b)az)-(<x,y>(a)bz)=$$

$$=((xby)az)-((ybx)az)-((xay)bz)+((yax)bz)=$$

$$=(xb(yaz))+(y(bxa)z))-(ya(xbz))-$$

$$-(yb(xaz))-(x(bya)z))+(xa(ybz))-$$

$$-(xa(ybz))-(y(axb)z))+(yb(xaz))-$$

$$+(ya(xbz))+(x(ayb)z))-(xb(yaz))=$$

$$=(x(ayb)z))-(x(bya)z))-(y(axb)z))+(y(bxa)z)=$$

$$=(x<a,b>(y)z))-(y<a,b>(x)z)),$$

and (3.39) holds too.

Due to 3.39 and the auxiliar identity, we have

$$<<u,v>(a)\ ,\ <x,y>(b)\ >=$$

$$=<\ (<x,y>(b)au),v\ \ >-<\ \ (<x,y>(b)av),u\ \ >=$$

$$=<(x<a,b>(y)u),v>-<(y<a,b>(x)u),v>+$$

$$+<(y<a,b>(x)u),v>-<(x<a,b>(y)v),u>.$$

If we use equation (3.38) in (3.33) we get

$$\frac{1}{4}((xbx)ax)+\frac{1}{4}(xa(xbx))-\frac{1}{12}<(xbx),x>(a)-$$
$$-\frac{1}{4}((xax)bx)-\frac{1}{4}(xb(xax))+\frac{1}{12}<(xax),x>(b)=$$

$$=\frac{3}{12}((xbx)ax)+\frac{3}{12}(xa(xbx))-\frac{1}{12}((xbx)ax)+\frac{1}{12}(xa(xbx))-$$
$$-\frac{3}{12}((xax)bx)-\frac{3}{12}(xb(xax))+\frac{1}{12}((xax)bx)-\frac{1}{12}(xb(xax))=$$

$$=\frac{1}{6}(\ ((xbx)ax)+2(xa(xbx))-((xax)bx)-2(xb(xax))\ )=$$

$$=\frac{1}{6}(x<b,a>(x)x)=-\frac{1}{6}(x<a,b>(x)x),$$

which is exactly the first term of $\tilde{k}_{ab}$.

Thanks to equation (3.29), we have

$$X(bxa)=(X(b)ax)-(xaX(b))-(xbX(a)),$$

thus, in (3.34), we get

$$\frac{1}{4}<X(b),x>(a)+\frac{1}{4}(X(b)ax)+\frac{1}{4}(xaX(b))-$$
$$-\frac{1}{4}<X(a),x>(b)-\frac{1}{4}(xbX(a))-\frac{1}{4}(X(a)bx)=$$

$$=\frac{1}{2}(\ (X(b)ax)-(X(a)bx)\ )=$$

$$=\frac{1}{2}(\ (X(b)ax)-(xaX(b))-(xbX(a))\ )-$$
$$-\frac{1}{2}(\ (X(a)bx)+(xbX(a))+(xaX(b))\ )=$$

$$=\frac{1}{2}(\ X(bxa)-X(axb)\ )=-\frac{1}{2}X(<a,b>(x)),$$

the second term of $\tilde{k}_{ab}$.

Due to equation (3.38) and the auxiliar identity, we have

$$
\begin{aligned}
(3.35) = &-\frac{1}{12} < ((xbx)ax), x > -\frac{1}{12} < (xa(xbx)), x > + \\
&+\frac{1}{12} < ((xax)bx), x > +\frac{1}{12} < (xb(xax)), x > + \\
&+\frac{1}{12} < (xbx), (xax) > +\frac{1}{12} << (xax), x > (b), x > - \\
&-\frac{1}{12} < (xax), (xbx) > -\frac{1}{12} << (xbx), x > (a), x >=
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{1}{12} < ((xbx)ax), x > -\frac{1}{12} < (xa(xbx)), x > + \\
&+\frac{1}{12} < ((xax)bx), x > +\frac{1}{12} < (xb(xax)), x > + \\
&+\frac{1}{12} < (xbx), (xax) > +\frac{1}{12} < (xb((xax)), x > -\frac{1}{12} < (xbx), (xax) > - \\
&-\frac{1}{12} < (xax), (xbx) > -\frac{1}{12} < (xa(xbx)), x > +\frac{1}{12} < (xax), (xbx) >=
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{1}{12}(< ((xbx)ax), x > +2 < (xa(xbx)), x >)+ \\
&+\frac{1}{12}(< ((xax)bx), x > +2 < (xb(xax)), x >) =
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{1}{12} < (\ ((xbx)ax) + 2(xa(xbx)) \ ), x > + \\
&+\frac{1}{12} < (\ ((xax)bx) + 2(xb(xax)) \ ), x >=
\end{aligned}
$$

$$
= -\frac{1}{12} < (x < b, a > (x)x), x >= \frac{1}{12} < (x < a, b > (x)x), x >,
$$

which equals the third summand of $\tilde{k}_{ab}$.

Now, we prove that $(3.36) = 0$. We have

$$
\begin{aligned}
&-\frac{1}{12} < (X(b)ax), x > -\frac{1}{12} < (xaX(b)), x > + \\
&+\frac{1}{12} < (X(a)bx), x > +\frac{1}{12} < (xbX(a)), x > + \\
&-\frac{1}{4} < X(b), (xax) > -\frac{1}{12} < (xax), X(b) > + \\
&+\frac{1}{4} < X(a), (xbx) > +\frac{1}{12} < (xbx), X(a) > + \\
&+\frac{1}{12} << X(a), x > (b), x > -\frac{1}{12} << X(b), x > (a), x >=
\end{aligned}
$$

$$
= -\frac{1}{12} < (X(b)ax), x > -\frac{1}{12} < (xaX(b)), x > +
$$

$$+\frac{1}{12} < (X(a)bx), x > +\frac{1}{12} < (xbX(a)), x > +$$
$$-\frac{1}{4} < X(b), (xax) > +\frac{1}{4} < X(b), (xax) > +$$
$$+\frac{1}{4} < X(a), (xbx) > -\frac{1}{4} < X(a), (xbx) > +$$
$$+\frac{1}{12} < (X(a)bx), x > +\frac{1}{12} < (xbX(a)), x > -$$
$$-\frac{1}{12} < (X(b)ax), x > +\frac{1}{12} < (xaX(b)), x >=$$

$$= -\frac{1}{6} < (X(b)ax), x > +\frac{1}{6} < (X(a)bx), x >=$$

$$= -\frac{1}{6} < \ (X(b)ax) - (X(a)bx),\ \ x > .$$

Since (3.34) $= -\frac{1}{2}X(< a, b > (x))$, we obtain

$$(3.36) = -\frac{1}{6} < \ (X(b)ax) - (X(a)bx),\ \ x >=$$
$$= -\frac{1}{3} < \ -\frac{1}{2}X(< a, b > (x)), x \ \ > .$$

If we suppose $X =< u, v >$, we get

$$\frac{1}{2} << u, v > (< a, b > (x)), x \ \ >=$$
$$= \frac{1}{2} < \ (x < a, b > (x)u), v \ \ > - < \ (x < a, b > (x)v), w \ \ >,$$

which is exactly equation (3.40) with $x = y$, which implies

$$\frac{1}{2} << u, v > (< a, b > (x)), x \ \ >=<< u, v > (a)\ ,\ < x, x > (b) \ \ >= 0.$$

By linearity this leads to

$$< \ X(< a, b > (x)), x \ \ >= 0$$

and, thereby, to (3.36) $= 0$.

Finally, we have that 3.37 is equal to the last term of $\tilde{k}_{ab}$.
Thus, we have proved that $[\tilde{u}_a, \tilde{u}_b] = \tilde{k}_{ab}$.

In order to obtain the remaining non zero commutators we just need to use the already proved ones and the Jacobi identity.

We have

$$[s_{ab}, \tilde{k}_{cd}] = [s_{ab}, [\tilde{u}_c, \tilde{u}_d]] = [[\tilde{u}_d, s_{ab}], \tilde{u}_c] + [[s_{ab}, \tilde{u}_c], \tilde{u}_d] =$$

$$= [\tilde{u}_{(bad)}, \tilde{u}_c] - [\tilde{u}_{(bac)}, \tilde{u}_d] = \tilde{k}_{(bad)c} - \tilde{k}_{(bac)d}.$$

If we look at the first three terms of $\tilde{k}_{ab}$, by their multi-linearity, we obtain that they depends linearly on $< a, b >$, actually this holds true also for the term $< X(a), X(b) >$, hence also $\tilde{k}_{ab}$ depends linearly on $< a, b >$, in the sense that $\tilde{k}_{ab} = \tilde{k}(< a, b >)$. More generally, we have that $< X(a), Y(b) >$ is a linear function of $< a, b >$. Indeed if $X =< u, v >$, $Y =< x, y >$, by equation (3.40), $< X(a), Y(b) >= F(< a, b >)$, for some linear function F, hence, using the bilinearity of $<, >$, it holds for any $X, Y \in L$. Thus we have

$$\tilde{k}_{(bad)c} - \tilde{k}_{(bac)d} = \tilde{k}(< (bad), c >) + \tilde{k}(- < (bac), d >) =$$

$$= \tilde{k}(< (bad), c > - < (bac), d >) = \tilde{k}(< < c, d > (a), d \ >) = \tilde{k}_{<c,d>(a)d}.$$

Similarly, we get

$$[u_a, \tilde{k}_{cd}] = [u_a, [\tilde{u}_c, \tilde{u}_d]] = [[\tilde{u}_d, u_a], \tilde{u}_c] + [[u_a, \tilde{u}_c], \tilde{u}_d] =$$

$$= -[\tilde{s}_{ad}, \tilde{u}_c] + [\tilde{s}_{ac}, \tilde{u}_d] = \tilde{u}_{(dac)} - \tilde{u}_{(cad)} = -\tilde{u}_{<c,d>(a)},$$

thanks to the linearly dependence of $\tilde{u}_a$ on a.

We also have

$$[s_{ab}, s_{cd}] = [s_{ab}, [u_c, \tilde{u}_d]] = [[\tilde{u}_d, s_{ab}], u_c] + [[s_{ab}, u_c], \tilde{u}_d] =$$

$$= [\tilde{u}_{(bad)}, u_c] + [u_{(abc)}], \tilde{u}_d] = s_{(abc)d} - s_{c(bad)}$$

Now, we are ready to show $[\tilde{k}_{ab}, \tilde{u}_c] = 0$. We have

$$[[\tilde{k}_{ab}, \tilde{u}_c], k_{xy}] = -[[\tilde{u}_c, k_{xy}], \tilde{k}_{ab}] - [[k_{xy}, \tilde{k}_{ab}], \tilde{u}_c] =$$

$$= [[u_{<x,y>}(c), \tilde{k}_{ab}] - [s_{<x,y>(a)b} - s_{<x,y>(b)a}, \tilde{u}_c] =$$

$$= -\tilde{u}_{<a,b>(<x,y>(c))} + \tilde{u}_{(b<x,y>(a)c)} - \tilde{u}_{(a<x,y>(b)c} =$$

$$= -\tilde{u}_{(\ <a,b>(<x,y>(c))-(b<x,y>(a)c)+(a<x,y>(b)c \ )}.$$

From the proof of equation (3.34) we have

$$X(< a, b > (c)) = (X(a)bx) - (X(b)ax). \tag{3.41}$$

Using (3.41) and (3.39) we get

$$(< a, b > (< x, y > (c)) - (b < x, y > (a)c) + (a < x, y > (b)c \ ) =$$

$$= (< a, b > (x)yc) - (< a, b > (y)xc) + (< a, b > (y)xc) - (< a, b > (x)yc) = 0,$$

hence $[[\tilde{k}_{ab}, \tilde{u}_c], k_{xy}] = 0$.

We also have $[[\tilde{k}_{ab}, \tilde{u}_c], u_z] = 0$. Indeed

$$[[\tilde{k}_{ab}, \tilde{u}_c], u_z] = -[[\tilde{u}_c, u_z], \tilde{k}_{ab}] - [[u_z, \tilde{k}_{ab}], \tilde{u}_c] =$$

$$= [s_{zc}, \tilde{k}_{ab}] + [\tilde{u}_{<a,b>(z)}, \tilde{u}_c] = -\tilde{k}_{<a,b>(z)c} + \tilde{k}_{<a,b>(z)c} = 0.$$

Since $[[\tilde{k}_{ab}, \tilde{u}_c], k_{xy}] = 0$ we have

$$0 = ([\tilde{k}_{ab}, \tilde{u}_c]\Box k_{x'y})(x + X) = ([\tilde{k}_{ab}, \tilde{u}_c])^{|x}(x, \cdots, x, 2 < x', y >)+$$

$$+([\tilde{k}_{ab}, \tilde{u}_c])^{|X}(x, \cdots, x, 2 < x', y >, X) + ([\tilde{k}_{ab}, \tilde{u}_c])^{|X}(x, \cdots, x, X, 2 < x', y >),$$

if we take $X = 2 < x', y >$ we get

$$0 = ([\tilde{k}_{ab}, \tilde{u}_c]\Box k_{xy})(x + 2 < x', y >) = 2[\tilde{k}_{ab}, \tilde{u}_c](x + 2 < x', y >),$$

by linearity we get $[\tilde{k}_{ab}, \tilde{u}_c](x + X) = 0, \ \forall x \in K, \ X \in L.$

From $[\tilde{k}_{ab}, \tilde{u}_c] = 0$, we get

$$[\tilde{k}_{ab}, \tilde{k}_{cd}] = [\tilde{k}_{ab}, [\tilde{u}_c, \tilde{u}_d]] = [[\tilde{u}_d, \tilde{k}_{ab}], \tilde{u}_c] + [[\tilde{k}_{ab}, \tilde{u}_c], \tilde{u}_d] = 0.$$

This completes the proof. □

Remark 11. If K is the Kantor triple system derived from a simple 5-graded Lie algebra $\mathfrak{g}$ with involution τ then K can be identified with $\mathfrak{g}_{-1}$ and L with $\mathfrak{g}_{-2}$ via $< u, v >= [u, v]$.

Corollary 3.4.3. *Let $\mathfrak{g}$ be a 5-graded simple Lie algebra with graded involution τ and let $K = K(\mathfrak{g})$ be the Kantor triple system derived from $\mathfrak{g}$.*
Then $\mathfrak{g}$ is isomorphic to $Lie(K(\mathfrak{g}))$. An isomorphism is given by

$$\phi : \begin{array}{c|ccccc} \mathfrak{g}_2 & [\tau(a), \tau(b)] & \to & [\tilde{u}_a, \tilde{u}_b] & = & \tilde{k}_{ab} \\ \mathfrak{g}_1 & \tau(a) & \to & \tilde{u}_a & & \\ \mathfrak{g}_0 & [a, \tau(b)] & \to & [u_a, \tilde{u}_b] & = & s_{ab} \\ \mathfrak{g}_{-1} & a & \to & u_a & & \\ \mathfrak{g}_{-2} & [a, b] & \to & [u_a, u_b] & = & k_{ab} \end{array} \tag{3.42}$$

for $a, b \in \mathfrak{g}_{-1}$. We call $\phi : \mathfrak{g} \to T(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2})$ the realization of $\mathfrak{g}$ on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$.

Proof. First of all we have to show that the map ϕ is well defined. To do that, we just need to show $s_{ab} = s_{cd}$ if $[a, \tau(b)] = [c, \tau(d)]$ and $\tilde{k}_{ab} = \tilde{k}_{cd}$ if $[\tau(a), \tau(b)] = [\tau(c), \tau(d)]$.

Indeed, since, as seen in the proof of Theorem 3.4.2, $\tilde{k}_{ab} = \tilde{k}_{<a,b>}$ and since $[u, v] =< u, v >$ in $\mathfrak{g}_{-2}$ we get

$$\tilde{k}_{ab} = \tilde{k}_{<a,b>} = \tilde{k}_{<c,d>} = \tilde{k}_{cd}$$

whenever $[\tau(a), \tau(b)] = [\tau(c), \tau(d)]$.
Furthermore, if we have $[a, \tau(b)] = [c, \tau(d)]$

$$s_{ab}(x + X) = (abx) - < a, X(b) >= [[a, \tau(b)], x] - < a, X(b) >=$$

$$= [[c, \tau(d)], x] - < a, X(b) > .$$

If we set $X =< u, v >$, then we have

$$< a, X(b) >=< a, < u, v > (b) >= - << u, v > (b), a >=$$

$$= - < (abu), v > + < (abv), u >= - < (cdu), v > + < (cdv), u >$$

which leads to $s_{ab} = s_{cd}$. Thus ϕ is well defined.
The fact that ϕ is a isomorphism of Lie algebras follows from the Jacobi identity and the fact that $\mathfrak{g}$ is simple, all the details are similar to those given in the proof of Theorem 3.3.3. $\square$

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