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Contents

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Abstract: Diffusion of innovations within simultaneous processes examined as univariate models of separate trajectories cannot take into account and properly explain systematic perturbations due to competition-substitution effects. This inability is quite evident in special product categories such as pharmaceutical drugs based upon equivalent or similar active compounds. A second relevant aspect in multiple competition is represented by the choice to model the word-of-mouth effect either at the category level (*balanced* model) or at the brand level, separating within-brand effect from cross-brand one (*unbalanced* model). The choice has to be grounded on the features of the products to be described. In this paper, balanced models will be studied, while the companion article is devoted to unbalanced ones. A third relevant aspect in simultaneous competing diffusions is the separation between synchronic and diachronic market entries. In the latter case, the proposed model is further extended in order to detect whether the beginning of competition alters the first entrant's diffusion parameters. The resulting differential system has a closed-form solution that allows an empirical validation through sales data of the assumptions underlying the model structure. An application to pharmaceutical drugs competition is discussed. Finally, we approach here the topic of agent heterogeneity by introducing a multivariate Cellular Automata representation which allows a feasible description of Complex Systems of this type with a direct specification of substitution effect between competing products.

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1 Introduction

The diffusion of innovations in a social system has been examined in the past four decades from many points of view. Sociologists (in particular, Rogers, 2003), mathematicians, physicists, quantitative marketing experts, statisticians, systems engineers, biologists, evolutionary economists, epidemiologists and experts in ecological systems have given deep contributions to theory and have realized widespread applications in different fields.

The main effort expressed in the corresponding literature is usually concentrated on univariate versions of such processes with a limited attention to multiple mathematical modelling and related multivariate statistical inference which highlight competition and interaction effects. The pioneering work by Bass (1969) and its subsequent extensions are valuable modelling tools for forecasting separate diffusion of innovations processes. One of the main assumptions in those models is the *aggregate* level of description. In other words, emphasis is focused on the behaviour of a special subpopulation to avoid modelling individual preferences and decisions for which data availability is unfeasible because of economic, ethical, or physical constraints. The Bass model is monodimensional in nature and relates to homogeneous category-level sales growth, or more extensively, to specific brand-level sales whose diffusion does not depend upon competing brands.

An outstanding advance in introducing external control variables in diffusion of innovation dynamics is the Bass et al. (1994) definition of the GBM (Generalized Bass Model). This extension allowed the introduction of relative prices and relative advertising effects as a first relevant example for marketing and management sciences. Nevertheless, market potential m is assumed constant over the life cycle and, even in this case, the focus is on a single product or category neglecting influence by competing actors in the marketplace.

Multi-product growth models were first examined in the marketing literature by Peterson and Mahajan (1978) under the assumption of *synchronic* launch of corresponding processes. They classify co-existing products in the marketplace in four categories: independent, complementary, contingent and substitute products. Only substitute products generate competition, which is modelled through the introduction of within-brand and cross-brand word-of-mouth (w.o.m.) effects related to brand specific residual markets. This choice partially contradicts the definition of a *common* category product where substitution and related competition are generated.

Category level diffusions originate mainly from a monopolistic diffusion process. New entrants define competition and substitution effects which may be described, following Parker and Gatignon (1994) in two classes: 1) diffusion processes which are product class driven (i.e., there is a common residual market potential); and 2) diffusion processes that are specific to the individual brand (i.e., each competitor detains its own market) as in Peterson and Mahajan (1978). Moreover, interpersonal influence of adopters may be either a function of product class relative knowledge or brand specific.

Further contributions to the multi-product growth models may be found in Mahajan et al. (1993). The restricted version of their model is based on diffusion processes that are product class driven, that is based upon a common residual market. This assumption correctly represents the situation of multiple substitute products in the same marketplace. In that model, interpersonal influence is, however, brand specific partially contradicting previous product class concepts. Moreover, the innovative effect is constrained to the specific residual market. Similar models are presented in Teng and Thompson (1983), Horsky and Mate (1988), and Kalish et al. (1995). A common feature of the above cited papers is the limited attention devoted to the “regime change” problem. In diachronic cases, it is not unusual that the new entrant affects the diffusion processes of previously existing products.

An old attempt to model simultaneous (synchronic) competition between two brands within the same market was performed in Bonaldo (1991). We refer to it in the sequel as GB model. The proposed model (Eqs. (10), Section 3) splits the rate sales into two separate equations under the hypothesis that the category is sufficiently homogeneous, and therefore, the relative knowledge of the product class is common knowledge driving word-of-mouth. Obviously, parameters that modulate access to that knowledge may be different for the two products.

Krishnan et al. (2000) rediscover the GB model for the synchronic case and introduce a new representation for the diachronic case, KBKD, which considers the late introduction of a third competitor with respect to two previously existing synchronic actors in the same category. The relevant assumptions of the model are the following. Market potential may change from m_a to m_c after the late entrance. It may be both an expansion or a reduction. The innovative parameters of the first entrants are not modified after the introduction of the third competitor (while imitative ones may change). The innovative parameter of the late entrant is forced to be zero (while the imitative parameter is free). The published closed-form solution presents some minor problems probably due to typing errors. We argue that the relevant constraints on the constancy of innovative parameters of old competitors and the assumed non-existence of an innovative effect for the late entrant may be avoided in order to properly test these assumptions. In this paper, the extended diachronic version, GBD, will be examined and solved, obtaining a closed-form representation including KBKD results as special cases.

We underline here the essential role of a closed-form solution. The crucial question of the model choice can be supported by available data, through an efficient estimation method that exploits cumulative observations and avoids the cumbersome procedure needed to fit instantaneous data to the differential equations. Moreover, the analysis of the closed-form solutions allows a correct interpretation of the competition and substitution effects. This would not be possible through the qualitative study of the differential equations, because that method describes only the asymptotic behavior of the system and not the competition dynamics that led to the equilibrium solution. Conversely, in this paper, the solutions of GBD highlight that the sales of each product are composed by a share of the whole category sales, corrected by two further departures. These ones take into account the role both of the level already reached by the first product when competition starts and of the relative skills in the diffusing of the two products along the whole life cycle. The relative sizes of these components are evaluated through parameter estimates and their interpretation enables analysis of the interactions between the competitors.

In this paper, we focus on diffusion processes that are product-class driven, and interpersonal influence is not specific. In particular, in this paper we assume a common (*balanced*) effect for within-brand and cross-brand w.o.m. components. A further perspective, examined for example in Savin and Terwiesch (2005), is not handled here. It is treated in the companion article Guseo and Mortarino (2010), which proposes the decomposition of the category relative knowledge in brand specific (*unbalanced*) components (with different access parameters in order to separate within-brand effects from cross-brand effects in corresponding parallel equations in synchronic and diachronic cases). In this paper, we do not examine different perspectives in model building, which rest on a brand specific dynamic residual market that generalizes the Lotka-Volterra framework (Abramson and Zanette, 1998, and

Morris and Pratt, 2003).

The introduction emphasizes the common choice based on *aggregate* dynamic modelling, which ignores individual preferences and attitudes. The limited attention paid to the individual level has been perceived as a strong restriction, especially in the economic and social contexts. In order to recognize different *local* diffusions, Cellular Automata models (CA) and Network Automata (NA) are a recent example of the trial to take into account the heterogeneity of adopters. Boccara et al. (1997), Boccara and Fuks (1998) and Boccara (2004) among others proposed interesting representations of special Cellular Automata models within the theory of Complex Systems. Nevertheless, such models give rise to simulative frameworks that do not allow a stable and well-characterized statistical inference. The proposed Genetic Algorithms in this area are not very efficient, and in the past, their performances were questioned. A much stronger argument is that, in applied contexts, it is much more common to work with aggregate adoption data that are cheaper, more reliable, and more suitable from a managerial point of view.

In Guseo and Guidolin (2008), it has been proved, for a univariate case and under a mean-field approximation, that there exists a differential dual representation of a particular CA driven by a Riccati equation that can be solved in a closed form. This property allows the use of well-founded statistical inference in this nonlinear context. A confirmation of this property is explicitly attained in Guseo and Guidolin (2009) under a more general framework where a dynamic information network allows the implementation of a dynamic market potential extending popular Bass models, BM and GBM. Here we propose a special CA where an agent may select, at time t, at most one between two competing innovations. This *bivariate* automaton is then simplified under a mean-field approximation, obtaining a continuous representation that gives rise to the GB duopolistic model.

The paper is organized as follows. In Section 2, we introduce a deterministic one-dimensional CA following Boccara (2004) and Boccara and Fuks (1998), and we extend it to a suitable probabilistic version. As a second step, we find a bivariate CA generating the GB duopolistic model summarized in Section 3. Moreover, in Section 4 we study some characterizations of the more complex twofold *diachronic* case, GBD, and compare it with KBKD model. In Section 5, we consider a specific application of the proposed twofold diachronic model, GBD, with reference to competing pharmaceutical drugs. Final remarks and discussion are presented in Section 6. Moreover, in the Appendix, we consider the problem of competition and environmental intervention, for both synchronic and diachronic cases, in order to extend our results following the inspiration of Bass et al. (1994).

2 Multivariate Cellular Automata

A deterministic univariate Cellular Automaton (CA) is characterized by three elements: a *population of agents* (cells), Z, a *state function* $s(i, t)$ and a *local evolutionary rule*, $f(\cdot)$.

The population of agents, Z, biunivocally corresponds to a set of labels for agents' identification. We assume Z as the set of all integers. The state function $s(i, t) \in Q$

denotes, for each agent $i \in Z$ at time $t \in I\!N^*$ (the set of all positive integers), a level within the class $Q = \{0,1\}$ of possible states: $s(i;t) = 1$ denotes the adoption of a particular innovation by agent i and, conversely, $s(i;t) = 0$ depicts the neutral state.

The local evolutionary rule (transition rule) is a function $f : Q^{r_\ell + r_r + 1} \to Q$, such that

$$s(i;t+1) = f(s(i-r_\ell;t), s(i-r_\ell+1;t), \cdots, s(i-1+r_r;t), s(i+r_r;t)), \qquad (1)$$

where the integers, r_ℓ, r_r, are the *radii* of the rule. The function $S_t : i \to s(i,t),\ i \in Z$ denotes the global state of the CA or *configuration* at time t. The space of all configurations is $\mathcal{S} = Q^Z$ so that $S_t \in \mathcal{S}, \forall t \in I\!N^*$. The configuration at time $t+1$, S_{t+1}, is univocally determined by the state S_t and by the rule f so that there exists a unique application $F_f : \mathcal{S} \to \mathcal{S}$ for which

$$S_{t+1} = F_f(S_t), \qquad (2)$$

where F_f is the *evolutionary operator* induced by the local rule f.

Let us define for $i \in Z$ a kind of *local pressure* of the system, $0 \le \sigma_s(i;t) \le 1$, depending on a flexible probability measure, $p_n \ge 0$, that allows a more general description of a neighboring stimulating effect towards adoption:

$$\sigma_s(i;t) = \sum_{n=-\infty}^{\infty} s(i+n;t)p_n\,; \qquad \sum_n p_n = 1. \qquad (3)$$

If local pressure is based on a common radius $r = r_\ell = r_r$ and is *translational invariant*, we may consider the mean-field approximation. This reduction excludes the local effect of distribution p_n,

$$\sigma_s(i;t) = \lim_{r\to\infty} \sum_{j=-r}^{r} \frac{s(i+j;t)}{2r+1} \simeq \nu(t) = \frac{z(t)}{m}, \qquad (4)$$

where $\nu(t)$ depicts the "density" of the adoption process or the normalized ratio $\nu(t) = z(t)/m$ with m the assumed constant market potential and $z(t)$ the cumulative product sales at time t.

Let us define a special rule $f(\cdot)$ under $Q = \{0,1\}$, through a partially probabilistic specification,

$$\begin{aligned} s(i;t+1) &= s(i;t) + [Bi(1,p) \oplus Bi(1, q\,\sigma_s(i;t))]\, I_{(s(i;t)=0)} - Bi(1,w)\, I_{(s(i;t)=1)} \\ &= s(i;t) + Bi(1, p + q\,\sigma_s(i;t))\, I_{(s(i;t)=0)} - Bi(1,w)\, I_{(s(i;t)=1)}, \end{aligned} \qquad (5)$$

where $\oplus$ denotes a *selection rule* between two *mutually exclusive* components. The first innovative component of equation (5), $Bi(1,p)$, depends upon a binomial experiment, with parameter p, which is realizable only if indicator function $I_{(s(i;t)=0)}$ is set to one, that is proposition $(s(i;t) = 0)$ is true. The meaning of such a first component may be linked to the effect of mass media communication channels. The change of state is possible, with probability p only if such "institutional communication" reaches the susceptible agent i which supports the initializing aspects of an adoption process. We consider such an agent to be an *innovator*. The second innovative component of equation (5), $Bi(1, q\,\sigma_s(i;t)))$, considers the joint probability $q\sigma_s(i;t)$, that depicts the local pressure effect of a neighboring social practice to adopt, $\sigma_s(i;t)$, combined with the intrinsic attitude to pure imitative response pushed by a parameter q. This second experiment is an opportunity strictly referred to standard agents and expresses the commonly perceived fact that imitative behaviour is twofold: an individual attitude combined with a local pressure due to the neighboring environment. We consider such an agent to be an *imitator*. Notice that activation of the previous two components is strictly alternative. In other words, under condition $I_{(s(i;t)=0)} = 1$, an agent selects only one binomial experiment, if any, at his free choice, so that the resulting framework is a special binomial experiment for the external observer, $Bi(1, p + q\sigma_s(i;t))$. In particular, the binomial parameter $p + q\sigma_s(i;t)$ represents the marginal adoption probability (mixture) based on the averaging of the corresponding group conditional adoption probabilities, 1 for innovators, $\sigma_s(i;t)$ for imitators and zero for neutral agents. The group weights are p, q and $1-p-q$ respectively so that we attain $p\cdot 1+q\cdot\sigma_s(i;t)+(1-p-q)\cdot 0 = p+q\sigma_s(i;t)$. As mentioned above, we denote with the term *selection rule* such a composition rule. Finally, the third component in equation (5) is a decay effect driven by a binomial $Bi(1, w)$ under the control of the correct state, $I_{(s(i;t)=1)}$, and describes a possible withdrawal from an active state.

The stochastic rule (5) defines a simple evolution of a CA and may be simulated on the basis of a specification of the involved parameters p, q, w and m. A more interesting problem is obviously the estimation of these unknown parameters with available data. Usually, *individual* data are not available at time t, due to privacy constraints or costs, while *aggregate* information about the general state of the system may be easier to handle.

Let us consider, therefore, the average behaviour of rule (5), under a mean-field approximation expressed by equation (4) followed by the sum of all states indicators $s(i;t)$ within Z divided by $2r + 1$. If the limit exists we have

$$\nu(t+1) = \nu(t) + (p + q\nu(t))(1 - \nu(t)) - w\nu(t). \tag{6}$$

We can approximate previous discrete time equation with a continuous Riccati equation, namely,

$$\nu'(t) = -q\nu^2(t) + (q - p - w)\nu(t) + p, \tag{7}$$

and if we exclude the exit rule component, $w = 0$, we have a standard Bass (1969) model. Solution $\nu(t)$ of equation (7) is described in Guseo and Guidolin (2008). We underline that $z(t) = m\nu(t)$ defines an absolute aggregate temporal evolution of the proposed Cellular Automaton.

Previous approach can be extended to a competitive market. We represent an automaton where an agent at time t may remain neutral, or select, without loss of generality, at most only one between two competing innovations. As we did before, we define with an indicator function $I_{(s(i;t)=0)} = I_{(s_1(i;t)+s_2(i;t)=0)}$ the condition under which agent i has performed no adoption at time t, where $s_1(i;t)$ and $s_2(i;t)$ denote the state functions related to two different innovations, namely, innovation 1 and innovation 2. Their sum, $s(i;t) = s_1(i;t) + s_2(i;t)$, is at most one. If we exclude the exit rule component, $w = 0$, we may generalize equation (5), simultaneously representing the transitions rules for both state functions, as follows:

$$\begin{aligned} s_1(i;t+1) &= s_1(i;t) + Bi(1, p_1 + q_1\sigma_s(i;t))I_{(s(i;t)=0)} \\ s_2(i;t+1) &= s_2(i;t) + Bi(1, p_2 + q_2\sigma_s(i;t))I_{(s(i;t)=0)}, \end{aligned} \tag{8}$$

where $\sigma_s(i;t) = \sum_{n=-\infty}^{\infty} s(i+n;t)p(n)$ and $\sum_n p_n = 1$. In particular, $\sigma_s(i;t)$ is a common pressure towards adoption based on the knowledge of the product category and does not depend on the specific brand/product.

Note that if we sum previous synchronic processes in equations (8) we obtain a category transition rule,

$$s(i;t+1) = s(i;t) + Bi(1, p + q\,\sigma_s(i;t))I_{(s(i;t)=0)}, \tag{9}$$

where $p = p_1 + p_2$ and $q = q_1 + q_2$. The reason for this result is that the sum of the binomial experiments in equations (8) follows the *selection rule* (the agent selects at most *one* brand). This explains, with a different terminology, the source of competition.

We may approximate the aggregate discrete time system (8) with a continuous representation, under the mean-field approximation described in equation (4),

$$z_1'(t) \simeq \sum_i [s_1(i;t+1) - s_1(i;t)] \quad \text{and} \quad z_2'(t) \simeq \sum_i [s_2(i;t+1) - s_2(i;t)],$$

where, in particular, $z(t) = \sum_i s(i;t) = \sum_i [s_1(i;t) + s_2(i;t)] = z_1(t) + z_2(t)$. These positions give rise to the differential counterpart, the GB model described in Section 3, and confirm analogous results expressed in Guseo and Guidolin (2008) and Guseo and Guidolin (2009) that establish a dualism between Complex Systems representation based on Cellular Automata and the corresponding mean-field aggregate versions based on traditional differential equations systems.

3 GB system and twofold synchronic competition

We start here with the simpler situation of a synchronic competition, where the competitors enter simultaneously into the market. For reasons of simplicity, we summarize the duopolystic case, but the system may be naturally extended to larger systems with more than two competitors. In that case, the GB model (Bonaldo, 1991) is:

$$\begin{aligned} z_1'(t) &= m\left[p_1 + q_1\frac{z(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right] \\ z_2'(t) &= m\left[p_2 + q_2\frac{z(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right], \end{aligned} \tag{10}$$

where $z(t) = z_1(t) + z_2(t)$ depicts the sum of two cumulative diffusion processes (aggregate sales) and m denotes the limiting state of $z(t)$, as far as $t \to +\infty$, the constant aggregate carrying capacity or market potential. Parameters p_i (q_i), $i = 1, 2$, refer to innovators (imitators) as in the standard Bass Model.

In the system (10), the residual market $m[1 - z(t)/m]$ is represented by the whole category market potential m from which the sales of both brands have to be subtracted. This model is adequate to describe competition between perfect substitute products competing for the same group of adopters. Coherently, the mechanism which governs the interpersonal influence is assumed to be a *common driver*, that is a common relative knowledge $z(t)/m$ and it is a typical property of a particular competitive *niche*: a competitive environment or a competitive market based on a common class of substitutes that "cooperate" in defining the agents' awareness towards the product category. Notice that the fraction of common driver, $q_i z(t)/m$, is the specific imitative characteristic of ith diffusion process. Nevertheless, the joint presence of two competitors gives rise to a non null substitution effect.

We notice that the well-known Givon et al. (1995) paper about piracy studies legal and illegal trajectories of a common product. Only the legal series is observable; the illegal one is latent. That model may be included as a special case in GB with the following constraints: $p_1 = p_\ell$, $p_2 = 0$, $q_1 = \alpha q_\ell$, $q_2 = (1 - \alpha)q_\ell$. In this case, direct estimation of all involved parameters, m, p_ℓ, q_ℓ, and α may be carried out through the legal observable series.

The solution of system (10) is based on the simple property of the aggregate process, $z(t)$, for which the standard differential equation of the Bass model is satisfied (see Bass, 1969)

$$z'(t) = m \left[p + q\frac{z(t)}{m}\right] \left[1 - \frac{z(t)}{m}\right] \tag{11}$$

where $p = p_1 + p_2$ and $q = q_1 + q_2$. The solution of previous equation (11), under initial condition $z(0) = 0$, is well–known

$$z(t) = m \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}}, \tag{12}$$

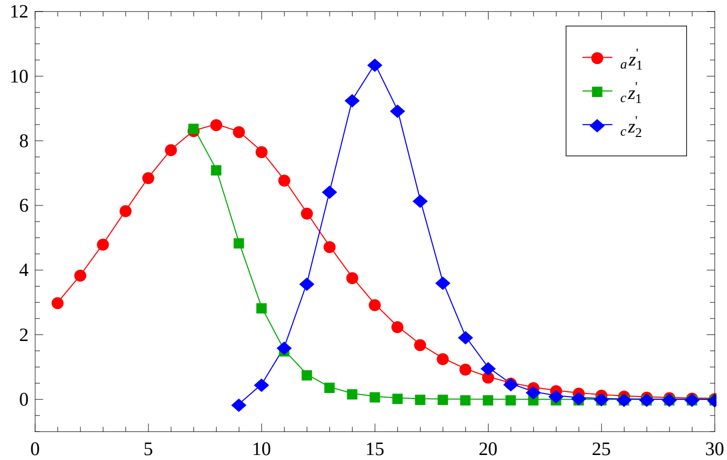
and the integration of system (10) gives rise to

$$\begin{aligned} z_1(t) &= m\, \frac{q_1}{q} \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} + m\, \frac{p}{q} \left(\frac{p_1}{p} - \frac{q_1}{q}\right) \ln \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \\ z_2(t) &= m\, \frac{q_2}{q} \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} + m\, \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q}\right) \ln \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}}. \end{aligned} \tag{13}$$

The proof is omitted here, since in the next section details will be given for the more general diachronic situation, including system (10) as a special case.

We remark that system (10) falls into Case 2 of Parker and Gatignon (1994), where it was defined as a situation of "non competitive interpersonal influence." However, the closed-form solution (13), also available in Bonaldo (1991) and Krishnan et al. (2000), highlights a direct effect of competition among brands based on a compensating dynamic deterministic perturbation (the parametric functions $(p_1/p - q_1/q)$ and $(p_2/p - q_2/q)$ are opposite values so that their sum is zero). In this

Figure 1: Twofold diachronic competition, GBD: rate or instantaneous observations. Stand alone first entrant sales, ${}_a z'_1$. First entrant sales under competition, ${}_c z'_1$. Second entrant sales under competition, ${}_c z'_2$.



case, the common (*balanced*) influence coefficient q_i multiplied by the relative category knowledge z/m does not imply a competition absence. An *unbalanced* influence of interpersonal relative knowledge of specific brands within a category, denoted as cross-brand or competitive interpersonal influence (see Peterson and Mahajan, 1978) may further emphasize competition or even allow for a market's leadership change. Moreover, we notice that Case 4 of Parker and Gatignon (1994), described in their equation

$$z'_i(t) = [a_i + b_i(z_i/m_i) + c_i(z - z_i)/(m - z_i)](m - z), \tag{14}$$

leads to a debatable model. In this case, the ratio $(z - z_i)/(m - z_i)$ does not represent a relative knowledge of the complementary brands with respect to z_i. In the light of previous reasoning, $(m - m_i)$ could probably be the coherent denominator. Moreover, the simultaneous presence of brand specific market potential, m_i, and category potential, m, appears as contradictory.

4 Twofold diachronic competition, GBD

Infrequently observed are two or more diffusion processes that are exactly synchronic and with direct effects of substitution between competitors within the same environment. On the contrary, it is a common experience to observe the late entrance of new diffusion processes. Let us consider the twofold simpler case with the late entrance of the second competitor at time $t = c_2$ with $c_2 > 0$ where $t = 0$ denotes the time origin for the first competitor (see Figure 1).

We propose here a modified GB that includes such a time lag on the birth of a second competitor, GBD:

$$
\begin{aligned}
z_1'(t) &= m\left\{\left[p_{1a} + q_{1a}\frac{z(t)}{m}\right](1 - I_{t>c_2}) + \left[p_{1c} + q_{1c}\frac{z(t)}{m}\right] I_{t>c_2}\right\}\left[1 - \frac{z(t)}{m}\right] \\
z_2'(t) &= m\left[p_2 + q_2\frac{z(t)}{m}\right]\left[1 - \frac{z(t)}{m}\right] I_{t>c_2} \\
m &= m_a(1 - I_{t>c_2}) + m_c I_{t>c_2} \\
z(t) &= z_1(t) + z_2(t) I_{t>c_2},
\end{aligned}
\tag{15}
$$

where usual parameters p_1 and q_1 for the first competitor may be different in the "stand alone" situation (subscript a) and in the competitive situation (subscript c). This special feature of our model allows dealing with the not unusual "regime change" problem. The transition from a monopolistic market to a duopolistic one is likely to upset the diffusion's structure of the first entrant and gives rise to different parameters for the first competitor and (or) to a new carrying capacity. As previously mentioned, the KBKD model by Krishnan et al. (2000) is a special case of the GBD model obtained when $p_{1a} = p_{1c}$ and $p_2 = 0$ (see Table 1).

The system (15) has the following closed-form solution:

$$
z_1(t) = {}_a z_1(t)(1 - I_{t>c_2}) + {}_c z_1(t) I_{t>c_2} \tag{16}
$$

$$
z_2(t) = 0 \cdot (1 - I_{t>c_2}) + {}_c z_2(t) I_{t>c_2} = {}_c z_2(t) I_{t>c_2}, \tag{17}
$$

where $p = p_{1c} + p_2$, $q = q_{1c} + q_2$,

$$
{}_a z_1(t) = m_a \frac{1 - e^{-(p_{1a}+q_{1a})t}}{1 + \frac{q_{1a}}{p_{1a}} e^{-(p_{1a}+q_{1a})t}} \tag{18}
$$

$$
{}_c z_1(t) = m_c \frac{q_{1c}}{q} w(t) + \frac{q_2}{q} z_s + m_c \frac{p}{q}\left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q}\right)\ln y(t) \tag{19}
$$

$$
{}_c z_2(t) = m_c \frac{q_2}{q} w(t) - \frac{q_2}{q} z_s + m_c \frac{p}{q}\left(\frac{p_2}{p} - \frac{q_2}{q}\right)\ln y(t) \tag{20}
$$

Table 1: Diachronic models. GBD=current model, KBKD= Krishnan et al. (2000) model. PR1= first entrant product, PR2=second entrant product, AGG= aggregate model.

Diachronic Models		Before competition $t < c_2$			Under competition $t \geq c_2$		
		inn	im	mkt pot	inn	im	mkt pot
GBD	PR1	p_{1a}	q_{1a}		p_{1c}	q_{1c}	
(# p. 8)	PR2	0	0	m_a	p_2	q_2	m_c
(# a.p. 6)	AGG	p_{1a}	q_{1a}		$p_{1c} + p_2$	$q_{1c} + q_2$	
KBKD	PR1	p_{1a}	q_{1a}		p_{1a}	q_{1c}	
(# p. 6)	PR2	0	0	m_a	0	q_2	m_c
(# a.p. 5)	AGG	p_{1a}	q_{1a}		p_{1a}	$q_{1c} + q_2$	

$$w(t) = \frac{1+\frac{q}{p}\frac{z_s}{m_c} - \left(1-\frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}}{1+\frac{q}{p}\frac{z_s}{m_c} + \frac{q}{p}\left(1-\frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}} \quad (21)$$

$$y(t) = \frac{1+\frac{q}{p}}{1+\frac{q}{p}\frac{z_s}{m_c} + \frac{q}{p}\left(1-\frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}}. \quad (22)$$

Proof. We try to determine $z(t)$ within the two different regimes, namely $z(t) = {}_az(t)(1-I_{t>c_2}) + {}_cz(t)I_{t>c_2}$. Until $t = c_2$ we observe no competition and then the global equation is the local one for $z_1(t)$,

$${}_az'(t) = {}_az_1'(t) = m_a\left[p_{1a} + q_{1a}\frac{z_1(t)}{m_a}\right]\left[1-\frac{z_1(t)}{m_a}\right], \quad t \le c_2, \quad (23)$$

so that the aggregate solution, under initial condition ${}_az(0) = {}_az_1(0) = 0$, is

$${}_az(t) = {}_az_1(t) = z_1(t)I_{t\le c_2} = m_a\frac{1-e^{-(p_{1a}+q_{1a})t}}{1+\frac{q_{1a}}{p_{1a}}e^{-(p_{1a}+q_{1a})t}}. \quad (24)$$

The final cumulative condition at time $t = c_2$ is $z_s = {}_az(c_2) = {}_az_1(c_2)$. After c_2, we observe competition and the aggregate equation is

$${}_cz'(t) = z'(t) = m_c\left[p + q\frac{z(t)}{m_c}\right]\left[1-\frac{z(t)}{m_c}\right], \quad t > c_2, \quad (25)$$

with initial condition z_s at time $t = c_2$ and $p = p_{1c} + p_2$, $q = q_{1c} + q_2$. Equation (25) may be solved under previous condition (see, for instance, Bass, 1969, p. 218),

$${}_cz(t) = m_c\frac{pe^{(p+q)c_2}\left(\frac{z_s}{m_c}-1\right) + \left(p+q\frac{z_s}{m_c}\right)e^{(p+q)t}}{-qe^{(p+q)c_2}\left(\frac{z_s}{m_c}-1\right) + \left(p+q\frac{z_s}{m_c}\right)e^{(p+q)t}}I_{t>c_2} \quad (26)$$

or, equivalently,

$${}_cz(t) = m_c\frac{1+\frac{q}{p}\frac{z_s}{m_c} - \left(1-\frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}}{1+\frac{q}{p}\frac{z_s}{m_c} + \frac{q}{p}\left(1-\frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}}I_{t>c_2} = m_cw(t)I_{t>c_2}. \quad (27)$$

We try to determine now the $z_1(t)$ component under the competitive situation for $t > c_2$ and, coherently with the previous notation, we denote such a solution with a special subscript c, ${}_cz_1(t) = z_1(t)I_{t\ge c_2}$.

We consider, preliminary, a new position,

$$E = \left[\left(1-\frac{z_s}{m_c}\right)/\left(1+\frac{q}{p}\frac{z_s}{m_c}\right)\right]e^{-(p+q)(t-c_2)}.$$

Equation (27) is equivalent, for $f = q/p$, to

$$W = w(t) = \frac{{}_cz(t)}{m_c} = \frac{1-E}{1+fE}$$

and, in particular, we observe $W(c_2) = z_s/m_c$.

Since we are considering $t > c_2$, integration of the first equation in (15) gives rise to

$$ {}_cz_1(t) = \frac{m_c}{p} \int \frac{p_{1c} + q_{1c}W}{1 + fW} dW, \tag{28} $$

and its general solution is

$$ H_1(t) = \frac{m_c}{pf} \left[\left(p_{1c} - \frac{q_{1c}}{f} \right) \ln(fW + 1) + q_{1c}W \right] + K, \tag{29} $$

with K a generic undetermined constant.

We may compute the definite integral within the range $[w(c_2), W]$ and, therefore, we obtain

$$ \begin{aligned} {}_cz_1(t) &= H_1(t) - H_1(c_2) + z_s \\ &= \frac{m_c}{pf} \left\{ \left(p_{1c} - \frac{q_{1c}}{f} \right) \ln \left[\frac{fW + 1}{fW(c_2) + 1} \right] + q_{1c}[W - W(c_2)] \right\} + z_s \\ &= m_c \left[\frac{qp_{1c} - q_{1c}p}{q^2} \ln \left(\frac{fW + 1}{f\frac{z_s}{m_c} + 1} \right) + \frac{q_{1c}}{q} W \right] + \frac{q_2}{q} z_s. \end{aligned} \tag{30} $$

Notice that, for $c_2 = 0$ we have $z_s = {}_az_1(c_2) = m_c w(c_2) = 0$ and then we attain the synchronic solution (13) by backward substitution.

In order to write equation (30) as an explicit function of t, we further develop it starting from the argument of the logarithm:

$$ \frac{fW + 1}{f\frac{z_s}{m_c} + 1} = \frac{1 + \frac{q}{p}}{1 + \frac{q}{p}\frac{z_s}{m_c} + \frac{q}{p}\left(1 - \frac{z_s}{m_c}\right) e^{-(p+q)(t-c_2)}} = y(t). \tag{31} $$

Substituting expression (31) into (30) gives, after simple simplifications, expression (19).

The profile of second or late entrant is very simple. For $t \leq c_2$ we have ${}_az_2(t) = {}_az(t) - {}_az_1(t) = 0$. For $t > c_2$ we obtain ${}_cz_2(t) = {}_cz(t) - {}_cz_1(t)$ or equivalently,

$$ \begin{aligned} {}_cz_2(t) &= m_c w(t) - m_c \left[\frac{q_{1c}}{q} w(t) + \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln y(t) \right] - \frac{q_2}{q} z_s \\ &= m_c \frac{q_2}{q} w(t) - \frac{q_2}{q} z_s + m_c \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln y(t). \end{aligned} \qquad \square $$

We remark that the aggregated sales resulting from the GBD model $z(t) = {}_az(t)(1 - I_{t>c_2}) + {}_cz(t)I_{t>c_2}$, $t \in [0, +\infty)$ are not described by a pure Bass model. It is a two regimes function based on local Bass models: $BM(m_a, p_{1a}, q_{1a})$ for $t \in [0, c_2]$ and $BM(m_c, p, q)$ for $t \in (c_2, +\infty)$ with a continuity condition $z(c_2) = {}_az(c_2) = z_s$ for $t = c_2$.

In (19) we can recognize three components: a "baseline" process

$$ b_1(t) = m_c \frac{q_{1c}}{q} w(t); $$

a time dependent "perturbation"

$$r_1(t) = m_c \, \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln y(t),$$

whose sign and size depend upon parameter values; a constant term, $\frac{q_2}{q} z_s$. In a similar way, in (20) we can recognize three components: the "baseline" process

$$b_2(t) = m_c \frac{q_2}{q} w(t);$$

the time dependent "perturbation"

$$r_2(t) = -r_1(t) = m_c \, \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln y(t);$$

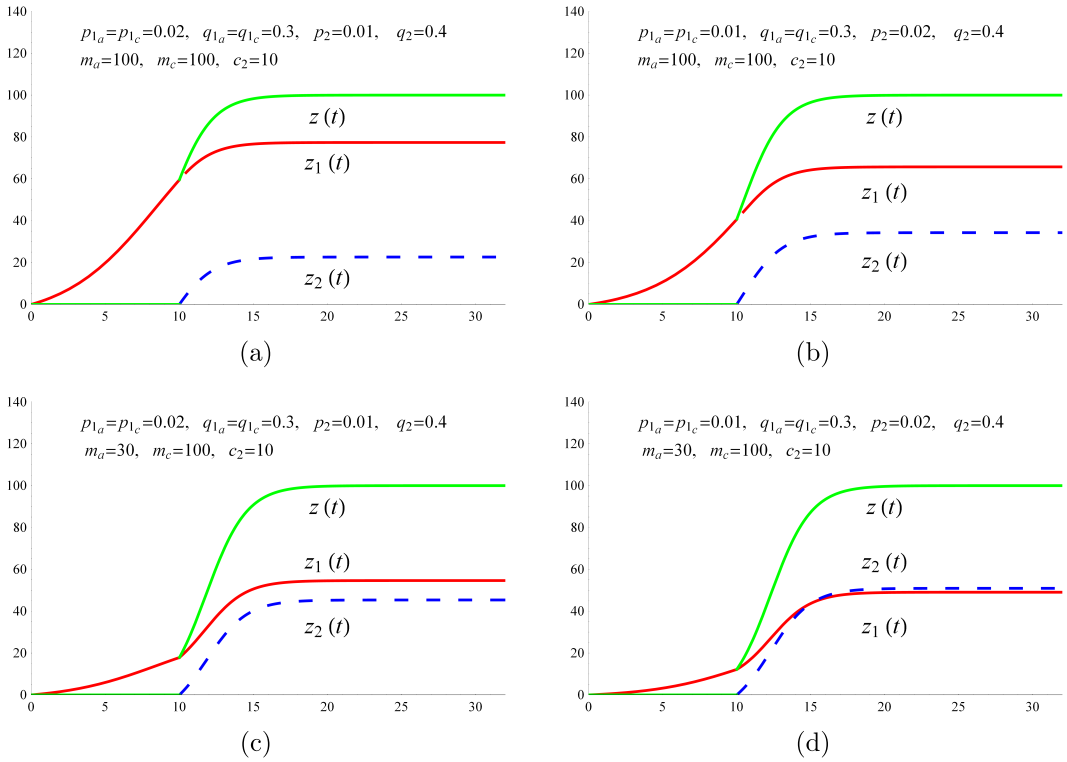
the constant term, $-\frac{q_2}{q} z_s$. The last two components exactly compensate the corresponding ones seen in ${}_c z_1(t)$. For each product, the "baseline" process represents a share of the whole category sales. The time dependent departures, $r_i(t)$, $i = 1, 2$, highlight competition/substitution effects and, in particular, these perturbations tell us whether competition generates advantage $(q_{1c}/p_{1c} > q_2/p_2)$ or drawback $(q_{1c}/p_{1c} < q_2/p_2)$ to the first competitor. Notice that, if local parameters are proportional, i.e., $q_1/p_1 = q_2/p_2$, the competition effects, $r_i(t)$, $i = 1, 2$, vanish. Finally, the constant term $\frac{q_2}{q} z_s$ represents the advantage of the first entrant.

The asymptotic behaviour of equations (19) and (20) is straightforward and highlights a non intuitive splitting of aggregate carrying capacity m_c, namely,

$$\begin{aligned}
\lim_{t \to +\infty} {}_c z_1(t) &= m_c \, \frac{q_{1c}}{q} + m_c \, \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln \left(\frac{1 + q/p}{1 + (q/p) \frac{z_s}{m_c}} \right) + \frac{q_2}{q} z_s \\
\lim_{t \to +\infty} {}_c z_2(t) &= m_c \, \frac{q_2}{q} + m_c \, \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln \left(\frac{1 + q/p}{1 + (q/p) \frac{z_s}{m_c}} \right) - \frac{q_2}{q} z_s.
\end{aligned} \tag{32}$$

The closed-form solution also allows simple graphical analyses of competition structures consistent with the GBD model. Figure 2 summarizes, with a qualitative description, some possible relationships. For the sake of simplicity, the situations here depicted describe the case of the unmodified first competitor's parameters after the second competitor's entrance. In Figure 2(a) we consider a new entrant at time $c_2 = 10$ with no incremental market potential, $m_a = m_c = 100$, with $p_{1a} = p_{1c} = 0.02$, $p_2 = 0.01$, $q_{1a} = q_{1c} = 0.3$ and $q_2 = 0.4$. We observe a diminished ceiling of the first competitor, $z_1(+\infty) \simeq 78$. If we interchange the values of parameters $p_{1a} = p_{1c}$ and p_2, i.e., $p_{1a} = p_{1c} = 0.01$ and $p_2 = 0.02$, we note a worse situation for the first entrant (see Figure 2(b)). We may be interested in the effect of a market potential expansion. In Figure 2(c) we notice, for $m_a = 30$ and $m_c = 100$, an interesting benefit for the first entrant with an asymptotic potential $z_1(+\infty) \simeq 54$, as compared with the stand alone asymptotic level (30). If we interchange the innovator parameters of the two competitors, that is $p_{1a} = p_{1c} = 0.01$ and $p_2 = 0.02$, we observe a more limited benefit for the first entrant (Figure 2(d)). In this situation the second entrant becomes the market leader.

Figure 2: Twofold diachronic competition, GBD: comparison among different situations.



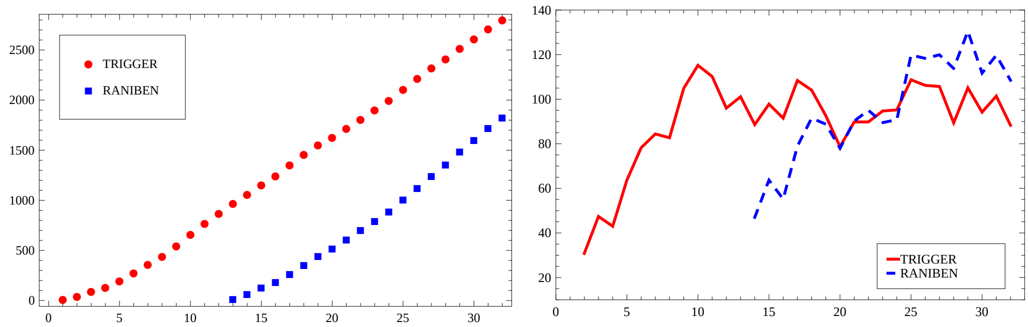
It is useful to examine more deeply the relationship between m_a and m_c. In particular, in the light of (32), strong closeness between m_a and m_c, due to the perfect knowledge of a mature environment, generates a substitutive turbulent competition. Conversely, a large divergence between m_a and m_c (with $m_a \ll m_c$) is a synonym of a first explorative situation, within which the competition effect is a stimulating causal precursor with a possible benefit for the first entrant competitor. These considerations are in agreement with similar ones developed in Givon et al. (1995).

As a final remark, we underline that also the GBD model can be easily extended to deal with more than two competitors and that closed-form solutions exist for the most complex cases where the entrance of each new competitor generates a regime change.

5 An application to pharmaceutical drugs

Ranitidine was introduced to the Italian pharmaceutical market in 1981, fourth quarter. The active compound acts as a non-imidazole blocker of those histamine receptors (H2 receptors) that mediate gastric secretion. It is used to treat gastrointestinal ulcers and related pathologies. In particular, the typical covered diseases are duodenal ulcer, mild to moderate reflux oesophagitis, gastric ulceration, and

Figure 3: Cumulative and rate sales (quarterly data). Source: IMS-Health , Italy.



peptic ulcer. It is equally safe and effective in preventing or reducing symptoms of heartburn. Competing therapeutical active compounds constitute a wide class: for instance, *Cimetidine, Famotidine, Omeprazole, Nizatidine,* and, among others, *Misoprostol, Sucralfate, Lansoprazole.* Since their launch, 1981/4, Zantac and Ranidil dominated the Italian *Ranitidine* market (90% in 1991). Here we examine two subsequent further entrants, Trigger in 1983/4 and Raniben in 1986/4.

Our data, provided by IMS-Health, Italy, consist of the cumulative quarterly number of packages sold in Italy by Trigger and Raniben. Data are available until the third quarter of 1991 (32 observations for Trigger and 20 observations for Raniben, see Figure 3).

The equations stemming from system (15), i.e., (18), (19) (which together give rise to (16)) and (20), were fitted simultaneously applying the Beauchamp and Cornell (1966) technique (in the first step the models (16) and (20) pertaining to the two products are fitted separately to their own series in order to estimate the covariance matrix for the two responses through residuals; at the second step, the models are fitted jointly with weighted nonlinear least squares using the covariance matrix as weight). Parameter estimates are summarized in Table 2 and the agreement between observed and fitted values is shown in Figure 4. The most impressive feature is a huge increase in market potential after Raniben's launch. Moreover, we observe that differences between $\hat{p}_{1a}$ and $\hat{p}_{1c}$ and $\hat{q}_{1a}$ and $\hat{q}_{1c}$ are significant: both parameters are reduced after c_2. Finally, we notice that $\hat{p}_2 \simeq 0$, i.e., the innovative quote for the second competitor is negligible.

In this situation, the fit of a simplified model with p_2 constrained to zero seemed a natural choice. Estimates and marginal confidence intervals for the other parameters were almost identical to those presented in Table 2, with the only exception of m_c : the new estimate is $\hat{m}_c = 9463.52$ with a much smaller confidence interval, (8630.41,10296.6).

The effect of competition between the two products can be seen in Figure 5. Here the first competitor reached a smaller part of the overall market ($\frac{q_{1c}}{q} = 0.293$), but gained through a positive increasing perturbation $\delta(t)$ and, obviously, through $(q_2/q)z_s$. In this case, Trigger was facing a declining phase of its diffusion process when Raniben entered the market. This upheaval increased the market of the whole

Table 2: Multivariate estimation results for model GBD.

	Estimate	Standard Error	95% Confidence Interval
m_a	1330.09	105.950	(1117.85, 1542.33)
m_c	8696.34	984.832	(6723.49, 10669.2)
p_{1a}	0.01419	0.00067	(0.01285, 0.01554)
q_{1a}	0.29802	0.02303	(0.25189, 0.34416)
p_{1c}	0.00922	0.00059	(0.00803, 0.01041)
q_{1c}	0.02364	0.00486	(0.01391, 0.03337)
p_2	-0.00063	0.00093	(-0.00248, 0.00122)
q_2	0.05698	0.00627	(0.04443, 0.06953)

$R^2 = 0.999872$

Figure 4: GBD. Comparison between observed and fitted values.

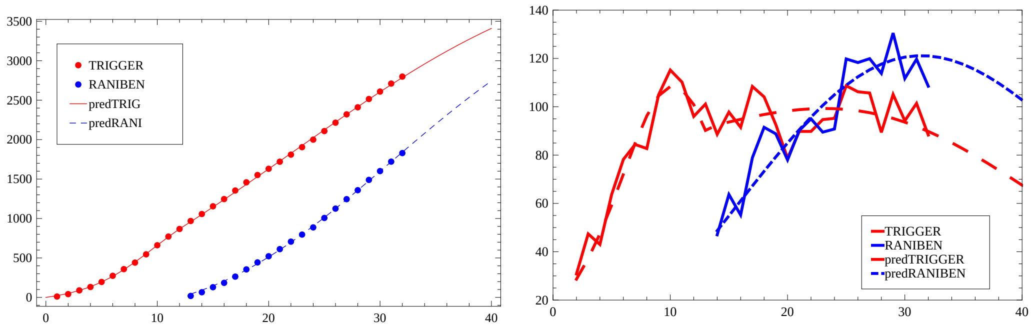


Figure 5: GBD. Model components.

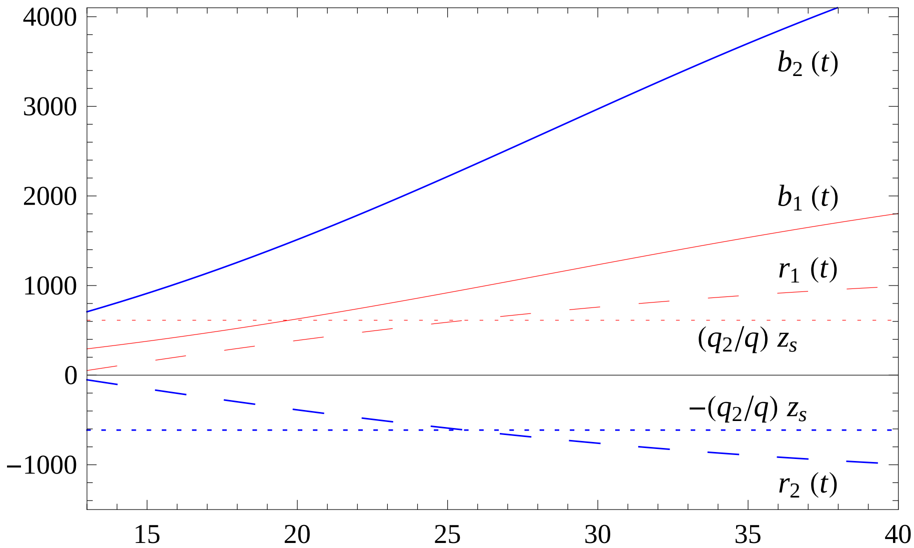


Table 3: Squared Pearson correlation coefficient between observed and fitted values for alternative models.

ρ^2	INDEP. MODELS (BM)	GBD MODEL (B&C)	GBD MODEL (DIRECT)	KBKD MODEL (B&C)
TRIGGER (n=32)	0.998023	0.999823	0.999837	0.999739
RANIBEN (n=20)	0.999485	0.999578	0.999528	0.999619
#parameters	6	8	8	6

category and kept Trigger vital.

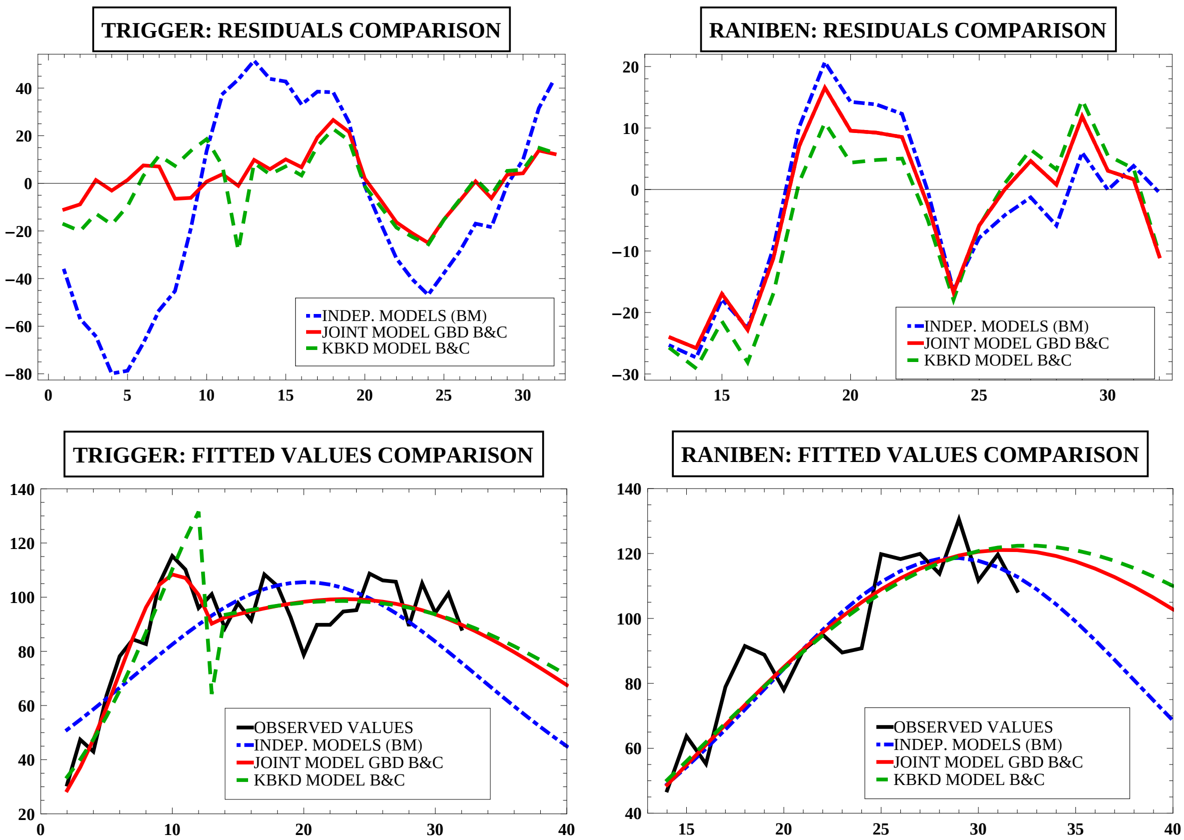
In order to compare the previous model with alternative solutions, the same data were used to fit two independent Bass models (one for each series), and the joint model (KBKD) by Krishnan et al. (2000).

Moreover, in order to control for noise due to model misspecification, the joint estimates of our model were performed without weighting as required by Beauchamp and Cornell (1966) (we will refer to this technique as a "direct" estimation).

The comparison among different models is performed through a simple measurement, the squared Pearson correlation coefficient between observed and fitted values. Results are proposed in Table 3. There are small differences between the Beauchamp and Cornell (1966) technique and the direct (unweighted) least squares estimates (2nd and 3rd column of Table 3). The joint model performs much better than independent models (1st column) for the first competitor. Raniben's trend, conversely, is well described also by an independent model that essentially ignores competition.

The KBKD model (4th column) has values similar to (15) joint model (as reported in Section 4, it is a special case of model (15) obtained when $p_2 = 0$ and $p_{1c} = p_{1a}$). With reference to the first constraint in our application, the innovative component of the second competitor was negligible ($\hat{p}_2 \simeq 0$), but we deduced this information from sales data, instead of imposing it through a model choice. The choice of a common innovative parameter for the first competitor both before and after the beginning of competition is, conversely, not well supported by our data: in Table 2, we see that $\hat{p}_{1a}$ significantly differs from $\hat{p}_{1c}$; moreover, from Figure 6 we see that the stand alone fit of the KBKD model is worse than the fit of our model (in detail, for the KBKD model, the Squared Pearson correlation coefficient between observed and fitted values is 0.997459 for the stand alone period and 0.999433 for the competitive period; for our model, the corresponding values are, respectively, 0.999612 and 0.999424). An F test to detect whether the gain from the KBKD model to the more complex model (15) is significant assigns the value 10.93, denoting the relevance of the extended model GBD (for details about the F test, see Section 5 in the companion article, Guseo and Mortarino, 2010). The analysis of residuals and fitted values (see Figure 6) confirms that the joint model (15), GBD, is essential in order to catch the features of the first competitor, while for the second competitor differences among alternative models are, in this case, less appreciable.

Figure 6: Comparisons among residuals and fitted values of independent models, joint model GBD and KBKD model with the Beauchamp and Cornell (1966) technique.



6 Final remarks and discussion

Diffusion of innovation methodologies and the corresponding models have faced and are facing new challenges in order to incorporate, in a parsimonious model building, the major effects that can modify their evolutionary shapes over time. The seminal paper by Bass (1969) has originated a wide set of contributions, including the possibility of an external control effect through the intervention function within the GBM framework, Bass et al. (1994). That function allows a time domain control, expanding or reducing sales over time under a fixed market potential. This useful re-allocation tool depends on market-mix policies and strategic interventions. Nevertheless, further actions due to management policies, regulatory contexts, and network externality effects, may modify dynamically the market potential (Guseo and Guidolin, 2009, Guseo and Guidolin, 2010).

A complementary direction in modelling and explaining observed systematic perturbations in a diffusion may be described via a multi-innovation diffusion model that explicitly takes into account competition among substitute products through the simultaneous study of the competitors' diffusions. The proposed GBD model considers a diachronic competition that extends the Krishnan et al. (2000) model, KBKD, in a natural way avoiding unnecessary restrictions on parameters. The basic GB model is obtained through a multivariate Cellular Automata representation restricted by a mean-field approximation confirming a recent methodology that avoids direct simulative approaches.

The application to a case study concerning two competing drugs based on Ranitidine's active compound allows valuable interpretations regarding interactions induced by the late entrant which, in this case, delays the declining behaviour of the first entrant and expands the whole category.

The main assumption of the present diachronic model GBD is related to the balanced word-of-mouth effects that do not take into account separate influences induced by the within-brand relative knowledge as opposed to the parallel cross-brand one. This assumption may be adequate for a wide range of applications, where differences between brands are narrow. For applications where that assumption is questionable, we refer to Guseo and Mortarino (2010), where the unbalanced version of this model is analyzed.

From an inferential point of view, it is surprising to observe quite systematically a substantial equivalence in terms of efficiency of the well-known re-weighted methodology by Beauchamp and Cornell (B&C) that induces regularity conditions on multivariate residuals (homoscedasticity and incorrelation) and a more robust multivariate NLS procedure without previous correction. Experience with parallel cases suggests that B&C is an efficient method if there is no error due to model specification. Its fragility is evident when some limited model deviation is interpreted as a stochastic residual that (erroneously) contributes to the regularization. In this case, a simple multivariate NLS procedure may be more robust and efficient. A detailed comparison of the two techniques will be studied in the future.

Appendix. Competition and environmental intervention

The GBD model may be generalized if we introduce a common intervention function $x(t)$ following the ideas developed by Bass et al. (1994) for the Generalized Bass Model. Function $x(t)$ is positive definite and locally integrable. This function, which may depend upon exogenous variables, can modify the velocity of time elapsing within the *niche* that includes the two competitors. We underline here that we assume a common function $x(t)$ in order to express common dynamical properties of the competitive environment affecting both competitors. These properties may include only exogenous political or macroeconomic effects as well as general effects due to expansion–competition interaction between the competitors.

If competition arises at time $c_2 > 0$ we have to follow the way designed within Section 4 with an updated model such that the existence of a common intervention function $x(t)$ is represented,

$$\begin{aligned} z_1'(t) &= m\left\{\left[p_{1a}+q_{1a}\frac{z(t)}{m}\right](1-I_{t>c_2})+\left[p_{1c}+q_{1c}\frac{z(t)}{m}\right]I_{t>c_2}\right\}\left[1-\frac{z(t)}{m}\right]x(t) \\ z_2'(t) &= m\left[p_2+q_2\frac{z(t)}{m}\right]\left[1-\frac{z(t)}{m}\right]x(t)I_{t>c_2} \\ m &= m_a(1-I_{t>c_2})+m_cI_{t>c_2} \\ z(t) &= z_1(t)+z_2(t)I_{t>c_2}, \end{aligned} \tag{33}$$

If $t \le c_2$ the generalized version of aggregate (and first entrant) cumulative function is

$$_{ag}z(t) = {}_{ag}z_1(t) = m_a \frac{1-e^{-(p_{1a}+q_{1a})\int_0^t x(\tau)d\tau}}{1+\frac{q_{1a}}{p_{1a}}e^{-(p_{1a}+q_{1a})\int_0^t x(\tau)d\tau}}, \quad t \le c_2, \tag{34}$$

where we denote the final condition with ${}_gz_s = {}_{ag}z_1(c_2)$.

The aggregate cumulative function ${}_{cg}z(t)$ for $t > c_2$ is a Generalized Bass Model with initial condition ${}_gz_s$ for $t = c_2$ and has the following shape

$$_{cg}z(t) = m_c \frac{1-\left[\left(1-\frac{_gz_s}{m_c}\right)/\left(1+\frac{q}{p}\frac{_gz_s}{m_c}\right)\right]e^{-G(t)}}{1+\frac{q}{p}\left[\left(1-\frac{_gz_s}{m_c}\right)/\left(1+\frac{q}{p}\frac{_gz_s}{m_c}\right)\right]e^{-G(t)}}I_{t\ge c_2}, \tag{35}$$

with $G(t) = (p+q)\int_{c_2}^t x(\xi)d\xi$ and $p = p_{1c}+p_2$, $q = q_{1c}+q_2$.

The new function, ${}_{cg}z_1(t)$, that depicts the behaviour of $z_1(t)$ for $t \ge c_2$ has a new form,

$$_{cg}z_1(t) = {}_gH_1(t) - {}_gH_1(c_2) + {}_gz_s, \tag{36}$$

where

$$_gH_1(t) = \frac{m_c}{pf}\left[\left(p_{1c}-\frac{q_{1c}}{f}\right)\ln(f\,{}_gW+1)+q_{1c}\,{}_gW\right]+K, \tag{37}$$

with ${}_gE = \left[\left(1-\frac{_gz_s}{m_c}\right)/\left(1+\frac{q}{p}\frac{_gz_s}{m_c}\right)\right]e^{-(p+q)\int_{c_2}^t x(\tau)d\tau}$, ${}_gW = \frac{_{cg}z(t)}{m_c} = (1-{}_gE)/(1+f_gE)$ and $f = q/p$, so that equation (36) reduces to

$$_{cg}z_1(t) = m_c\left[\frac{q_{1c}p-qp_{1c}}{q^2}\ln\left(\frac{f\frac{_gz_s}{m_c}+1}{f_gW+1}\right)+\frac{q_{1c}}{q}{}_gW\right]+\frac{q_2}{q}{}_gz_s. \tag{38}$$

Analogously, for the second competitor we have

$$_{cg}z_2(t) = m_c\left[\frac{q_2p-qp_2}{q^2}\ln\left(\frac{f\frac{_gz_s}{m_c}+1}{f_gW+1}\right)+\frac{q_2}{q}{}_gW\right]-\frac{q_2}{q}{}_gz_s, \quad t > c_2, \tag{39}$$

where ${}_{cg}z_2(t) = 0$ for $t < c_2$. Under $x(t) = 1, t \ge 0$, ${}_gz_s = z_s$, ${}_gE = E$ and ${}_gW = W$. In that case, equations (38) and (39) reduce to (19) and (20).

In the simpler synchronic case, the solutions of the generalized GB model under competition are

$$z_1(t) = m\frac{q_1}{q}\frac{1-e^{-(p+q)\int_0^t x(\tau)d\tau}}{1+\frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}} + m\frac{p}{q}\left(\frac{p_1}{p}-\frac{q_1}{q}\right)\ln\frac{1+\frac{q}{p}}{1+\frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}}$$

$$z_2(t) = m\frac{q_2}{q}\frac{1-e^{-(p+q)\int_0^t x(\tau)d\tau}}{1+\frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}} + m\frac{p}{q}\left(\frac{p_2}{p}-\frac{q_2}{q}\right)\ln\frac{1+\frac{q}{p}}{1+\frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}}.$$

From the comparison of the previous solutions with system (13), it is easy to see how the function $x(t)$ affects the diffusion of the two brands.

References

Abramson, G. and Zanette, D. (1998). "Statistics of extinction and survival in Lotka-Volterra systems". *Physical Review E*, 57, 4, pp. 4572–4577.

Bass, F. (1969). "A new product growth model for consumer durables". *Management Science*, 15, pp. 215–227.

Bass, F., Krishnan, T., and Jain, D. (1994). "Why the Bass model fits without decision variables". *Marketing Science*, 13, pp. 203–223.

Beauchamp, J. and Cornell, R. (1966). "Simultaneous nonlinear estimation". *Technometrics*, 8, 2, pp. 319–326.

Boccara, N. (2004). *Modeling Complex Systems*. Springer–Verlag, New York.

Boccara, N. and Fuks, H. (1998). Modeling diffusion of innovations with probabilistic cellular automata. In *Lecture Notes of the Spring School on Cellular Automata*. Dordrecht, Kluwer.

Boccara, N., Fuks, H., and Geurten, S. (1997). "A new class of automata networks". *Physica, D*, 103, pp. 145–154.

Bonaldo, D. (1991). Competizione tra prodotti farmaceutici: strumenti di previsione. Thesis. University of Padova, Italy.

Givon, M., Mahajan, V., and Müller, E. (1995). "Software piracy: Estimation of the lost sales and the impact on software diffusion". *Journal of Marketing*, 59, 1, pp. 29–37.

Guseo, R. and Guidolin, M. (2008). "Cellular automata and Riccati equation models for diffusion of innovations". *Statistical Methods and Applications*, 17, 3, pp. 291–308.

Guseo, R. and Guidolin, M. (2009). "Modelling a dynamic market potential: A class of automata networks for diffusion of innovations". *Technological Forecasting and Social Change*, 76, 6, pp. 806–820.

Guseo, R. and Guidolin, M. (2010). "Cellular automata with network incubation in information technology diffusion". *Physica A: Statistical Mechanics and its Applications*, 389, pp. 2422–2433.

Guseo, R. and Mortarino, C. (2010). *Competition Modelling in Multi-Innovation Diffusions. Part II: Unbalanced Models*. Technical Report N. 10/2010, Department of Statistical Sciences, University of Padova.

Horsky, D. and Mate, K. (1988). "Dynamic advertising strategies of competing durable good producers". *Marketing Science*, 7, 4, pp. 356–367.

Kalish, S., Mahajan, V., and Muller, E. (1995). "Waterfall and sprinkler new-product strategies in competitive global markets". *International Journal of Research in Marketing*, 12, pp. 105–119.

Krishnan, T., Bass, F., and Kumar, V. (2000). "Impact of a late entrant on the diffusion of a new product/service". *Journal of Marketing Research*, XXXVII, pp. 269–278.

Mahajan, V., Sharma, S., and Buzzell, R. (1993). "Assessing the impact of competitive entry on market expansion and incumbent sales". *Journal of Marketing*, 57, 3, pp. 39–52.

Morris, S. and Pratt, D. (2003). "Analysis of the Lotka-Volterra competition equations as a technological substitution model". *Technological Forecasting and Social Change*, 70, 2, pp. 103–133.

Parker, P. and Gatignon, H. (1994). "Specifying competitive effects in diffusion models: An empirical analysis". *International Journal of Research in Marketing*, 11, pp. 17–39.

Peterson, R. and Mahajan, V. (1978). Multi-product growth models. In J. Sheth (Ed.), *Research in Marketing* pp. 201–231. Greenwich, CT: JAI Press. Vol. 1.

Rogers, E. (2003). *Diffusion of Innovations, 5th ed.* Free Press, New York.

Savin, S. and Terwiesch, C. (2005). "Optimal product launch times in a duopoly: Balancing life-cycle revenues with product cost". *Operations Research*, 53, 1, pp. 26–47.

Teng, J. and Thompson, G. (1983). "Oligopoly models for optimal advertising when production costs obey a learning curve". *Management Science*, 29, 9, pp. 1067–1101.

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