Competition Modelling in Multi-Innovation Diffusions. Part II: Unbalanced Models

Renato Guseo, Cinzia Mortarino
Department of Statistical Sciences
University of Padua
Italy

Department of Statistical Sciences

University of Padua

Italy

Abstract: Competition between rival brands within the same category gives rise to special competition/substitution effects of great interest for involved firms. In the companion article, we studied balanced models that are adequate to describe a homogeneous category for which within-brand and cross-brand word-of-mouth effects are indistinguishable. Conversely, in this paper we propose an unbalanced model that, besides separating these two imitative sources, also allows for a change in the parameter values of the first entrant as soon as the second one enters the market. We prove that our model has a closed-form solution allowing parameters to be estimated with sales data. Moreover, we compare our model with other unbalanced models, both from a theoretical point of view and from an empirical one, by comparing their performance on pharmaceutical drug data.

Keywords: multivariate cellular automata, multivariate diffusion process, generalized Bass model, competition and intervention

UNIVERSITA
DEGLI STUDI
DI PADOVA
DIPARTIMENTO
DI SCIENZE
STATISTICHE

Contents

Department of Statistical Sciences
Via Cesare Battisti, 241
35121 Padova
Italy

tel: +39 049 8274168
fax: +39 049 8274170
http://www.stat.unipd.it

Corresponding author:
Cinzia Mortarino
tel: +39 049 827 4184
mortarino@stat.unipd.it
http://www.stat.unipd.it/~mortarino

Competition Modelling in Multi-Innovation Diffusions. Part II: Unbalanced Models

Renato Guseo, Cinzia Mortarino
Department of Statistical Sciences
University of Padua
Italy

Abstract: Competition between rival brands within the same category gives rise to special competition/substitution effects of great interest for involved firms. In the companion article, we studied balanced models that are adequate to describe a homogeneous category for which within-brand and cross-brand word-of-mouth effects are indistinguishable. Conversely, in this paper we propose an unbalanced model that, besides separating these two imitative sources, also allows for a change in the parameter values of the first entrant as soon as the second one enters the market. We prove that our model has a closed-form solution allowing parameters to be estimated with sales data. Moreover, we compare our model with other unbalanced models, both from a theoretical point of view and from an empirical one, by comparing their performance on pharmaceutical drug data.

Keywords: multivariate cellular automata, multivariate diffusion process, generalized Bass model, competition and intervention

1   Introduction

The competition among different brands within a common category may give rise to significant interaction effects in the diffusions of each product. As in the companion article, Guseo and Mortarino (2010a), we consider multi-product growth models that refer to a category based on substitute products. In this sense, we assume that category level diffusions generate specific diffusion processes that are product class driven (Parker and Gatignon, 1994). In other words, we assume that there is a common residual market for both products, which is obtained as the difference between the initial market potential and the past *category* sales. This assumption differs, for instance, from Peterson and Mahajan (1978), where brand specific residual markets are assumed.

A relevant aspect of multi-product growth modelling is related to the interpersonal communication effects due to word-of-mouth. In the companion article, we analyzed a *balanced* model, that is a model where the word-of-mouth (w.o.m.) effect does not separate the adoptions of each brand from those of the competitor. However, the relative knowledge may be decomposed in brand specific factors (Peterson and Mahajan, 1978; Kalish et al., 1995; Mahajan et al., 1993). The components of interpersonal communication may be focused on two main contributions which

are proportional to the adoptions of each brand, both within-brand and cross-brand effects. Within-brand component depicts the contribution of relative knowledge that is specific to the current brand. Cross-brand component is a contribution (positive or negative) of interpersonal communication on current sales, which depends upon the relative knowledge of competing products in the category. In this paper, we focus on *unbalanced* models, that is models that distinguish within-brand from cross-brand w.o.m. effects.

A further crucial aspect in multi-product growth modelling is the entrance timing of different competitors. In current literature, much effort is concentrated in simultaneous or synchronic modelling, a special case. Only a few papers deal with diachronic models. An important contribution in this context is developed in Savin and Terwiesch (2005). In addition, a necessary adjustment has been proposed in Guseo and Mortarino (2010b). Savin and Terwiesch propose an unbalanced model, with exogenously determined parameters, that is applied in order to estimate the optimal launch timing of a competitor. In that work, a very interesting result pertains to the equivalence of any diachronic competition with a corresponding synchronic one. However, perhaps due to the different aim of their work, Savin and Terwiesch (2005) do not deal with the diffusion of the first entrant before competition. For a reduced application of Savin and Terwiesh modelling see Libai et al. (2009).

A more general approach takes into account entrance differences and regime changes in the parameters, which may describe, in particular, a non-uniform behaviour of the first entrant over time. Recent approaches emphasize the rational claim that simultaneous multi-product growth modelling had to identify and jointly estimate parameters of different and parallel regimes in order to detect and test specific perturbation effects due to a late entrant. This aim is strongly simplified if the corresponding complex system's equations give rise to closed-form solutions for the corresponding multiple regimes.

In this article, we start from a general duopolistic diachronic model, UNID, which assumes category level diffusions and totally free within-brand and cross-brand effects that may change after the late entrance. This general framework does not have a closed-form solution. With a weak restriction on the equivalence of "discrimination" between specific within-brand and cross-brand effects in interpersonal communication, we obtain a duopolistic multi-regime model, UNCD, with a closed-form solution. This model includes, as special cases, the Savin and Terwiesch (2005) model, STD (and consequently the more restricted one by Libai et al., 2009, LMPD) and also includes the balanced GBD model described in Guseo and Mortarino (2010a) (and, therefore, the nested model KBKD by Krishnan et al., 2000).

The paper is organized as follows. In Section 2, we introduce a duopolistic or twofold diachronic competition model, UNID, and discuss the first weak restriction, which leads to UNCD between "discrimination" parameters δ and γ. These parameters describe specific separation between within-brand and cross-brand effects in interpersonal communication. In Section 3, we report the main results concerning the UNCD model and its components. Moreover, we obtain explicit forms for the corresponding synchronic case, UNC. Section 4 discusses the relative generality of the proposed model UNCD (or UNC) with respect to other relevant constrained

models: STD, LMPD, GBD and KBKD or related synchronic versions, ST, LMP, GB, KBK. Section 5 depicts an application to two pharmaceutical drugs and compares the obtained results with corresponding nested models. Section 6 is devoted to final remarks and discussion. In the Appendix, we report the details of the UNCD solution, and its coherence with the solutions of the GBD model presented in the companion article, Guseo and Mortarino (2010a).

2 Twofold diachronic competition: the UNID model

Consider a situation in which two brands of a common category compete for the same customers. We will analyze here a case in which the brands are similar enough to each other to have a common market potential, m, and, correspondingly, a common residual market, $m - z(t)$, where $z(t) = z_1(t) + z_2(t)$ denotes the cumulative category sales, $z_i(t)$, $i = 1, 2$, the cumulative sales of product i.

Competition might start since the common launch (synchronic competition) or it may arise after a monopolistic period of time (diachronic competition). Since it is a common experience to observe the late entrance of a new product and exact synchrony is rather infrequent, we will focus here on diachronic competition and evaluate synchronic solutions as a special case. Let us consider a twofold case with the late entrance of the second competitor at time $t = c_2$ with $c_2 > 0$ where $t = 0$ denotes the time origin for the first competitor:

$$
\begin{aligned}
z_1'(t) &= m \left\{ \left[p_{1a} + q_{1a} \frac{z(t)}{m} \right] (1 - I_{t>c_2}) + \right. \\
&\quad \left. + \left[p_{1c} + (q_{1c} + \delta) \frac{z_1(t)}{m} + q_{1c} \frac{z_2(t)}{m} \right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m} \right] \\
z_2'(t) &= m \left[p_2 + (q_2 - \gamma) \frac{z_1(t)}{m} + q_2 \frac{z_2(t)}{m} \right] \left[1 - \frac{z(t)}{m} \right] I_{t>c_2} \qquad (1) \\
m &= m_a (1 - I_{t>c_2}) + m_c I_{t>c_2}, \\
z(t) &= z_1(t) + z_2(t) I_{t>c_2}.
\end{aligned}
$$

Here $z_i'(t)$, $i = 1, 2$, represents instantaneous adoptions of product i.

The previous system describes a unified twofold model, UNID (Unified totally Independent Diachronic Model), that takes into account different aspects of diachronic diffusion. It is assumed that the first entrant product is characterized by three parameters: external influence p_{1a}, internal influence q_{1a}, and market potential m_a, during the stand alone period, $t \leq c_2$. The competition period, for $t > c_2$, is characterized by a free behavior of all involved parameters in order to describe situations in which a new product entering the market may change the older product's diffusion. Market potential assumes a new value, m_c, that may be lower or greater than the stand alone state, m_a. The first entrant product presents a new external influence p_{1c} and two w.o.m. effects. The first one takes into account imitation due to information conveyed by adopters of the same brand (modulated through the *within-brand* imitative coefficient $(q_{1c} + \delta)$); the second one describes imitation due to relevant information conveyed by adopters of the competing product (modelled through the

cross-brand imitative coefficient, q_{1c}). The second entrant has three corresponding free parameters: p_2 for external influence, q_2 for the specific within-brand effect and $(q_2 - \gamma)$ for cross-brand communication. We denote here the difference between the within-brand effect and the cross-brand effect of each product with the term "discrimination." Both discriminations δ and γ may be either positive or negative according to specific categories with within-brand effect either stronger or weaker than cross-brand one.

This structure can be interpreted also in a different way, if the first two equations of system (1), for $t > c_2$, are rearranged as follows:

$$
\begin{aligned}
z_1'(t) I_{t>c_2} &\propto p_{1c} + q_{1c} \frac{z_1(t) + z_2(t)}{m} + \delta \frac{z_1(t)}{m} \\
z_2'(t) I_{t>c_2} &\propto p_2 + (q_2 - \gamma) \frac{z_1(t) + z_2(t)}{m} + \gamma \frac{z_2(t)}{m}
\end{aligned}
$$

Previous equations describe a system where each product is characterized by an imitation coefficient at the *category* level (q_{1c}, for the first entrant, or $q_2 - \gamma$, for the second one) and a different imitation coefficient specific for the *brand* level (δ or γ).

Model (1) was also stated in Savin and Terwiesch (2005) and Libai et al. (2009) assuming that competition does not modify the parameters of the first competitor. Moreover, as explained in detail in Section 4, the models effectively used in these papers and for which a closed-form solution is given, are further restricted with additional constraints.

Notice that the last equation in (1) denotes the aggregate diffusion of the category product. It is characterized by a closed-form solution if the constraint

$$
q_{1c} + \delta + q_2 - \gamma = q_{1c} + q_2 \quad \Leftrightarrow \quad \gamma = \delta \tag{2}
$$

is satisfied, i.e., the local discriminations, in terms of different sources of w.o.m., are equivalent. For this situation, we obtain,

$$
\begin{aligned}
z'(t) &= m \left\{ \left[p_{1a} + q_{1a} \frac{z(t)}{m} \right] (1 - I_{t>c_2}) + \right. \\
&\quad \left. + \left[(p_{1c} + p_2) + (q_{1c} + q_2) \frac{z_1(t)}{m} + (q_{1c} + q_2) \frac{z_2(t)}{m} \right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m} \right] \\
&= m \left\{ \left[p_{1a} + q_{1a} \frac{z(t)}{m} \right] (1 - I_{t>c_2}) + \right. \tag{3} \\
&\quad \left. + \left[(p_{1c} + p_2) + (q_{1c} + q_2) \frac{z(t)}{m} \right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m} \right].
\end{aligned}
$$

The aggregate equation is a two–regime Bass model: $BM(m_a, p_{1a}, q_{1a})$ with the initial condition $z(0) = 0$ for the first part, and $BM(m_c, p_{1c} + p_2, q_{1c} + q_2)$ with the initial condition $z_s = z(c_2) = z_1(c_2)$ for the second one (Guseo and Mortarino, 2010b). We denote system (1) under restriction (2) the UNCD (UNified Constrained Diachronic) model.

3 Analysis of the UNCD model

In expanded form, model (1) with the constraint (2) gives

$$
z_1'(t) = m \left\{ \left[p_{1a} + q_{1a} \frac{z(t)}{m} \right] (1 - I_{t>c_2}) + \right. \tag{4}
$$
$$
\left. + \left[p_{1c} + (q_{1c} + \delta) \frac{z_1(t)}{m} + q_{1c} \frac{z_2(t)}{m} \right] I_{t>c_2} \right\} \left[1 - \frac{z(t)}{m} \right]
$$
$$
z_2'(t) = m \left[p_2 + (q_2 - \delta) \frac{z_1(t)}{m} + q_2 \frac{z_2(t)}{m} \right] \left[1 - \frac{z(t)}{m} \right] I_{t>c_2} \tag{5}
$$
$$
m = m_a(1 - I_{t>c_2}) + m_c I_{t>c_2}
$$
$$
z(t) = z_1(t) + z_2(t) I_{t>c_2}.
$$

The constraint (2) imposes that the discriminations δ and γ between the w.o.m. effects on a product due to adopters of the same product and adopters of the competitor one are the same for the two products: $\delta = \gamma$. This restriction is obviously weaker than the assumption of common imitative parameters for both products for which $\delta = \gamma = 0$. As defined above, discrimination represents the difference between within-brand w.o.m. effect and cross-brand w.o.m. effect. In the borderline case where products are completely identical to each other, brands can not be distinguished. Information conveyed by an adoption of any brand is equivalent to information conveyed by an adoption of the competitor (null discrimination). Conversely, in the less unusual case where substitute products do somehow differ, each product receives a different feedback from adoptions of its own brand and adoptions of its competitor (the discrimination is either positive or negative, according to the level of satisfaction/dissatisfaction of adopters). In these terms, discrimination can be interpreted as a measure of difference between the two information sources. It may be reasonable to model it symmetrically, $\delta = \gamma$.

If we study model (4), we observe that before competition, the first entrant's sales follow a Bass model:

$$
z_1(t) I_{t \leq c_2} = m_a \frac{1 - e^{-(p_{1a}+q_{1a})t}}{1 + \frac{q_{1a}}{p_{1a}} e^{-(p_{1a}+q_{1a})t}}. \tag{6}
$$

As explained above, after the beginning of the competition, the aggregate process is a standard Bass model $BM(m_c, p_{1c} + p_2, q_{1c} + q_2)$, with the initial condition $z_s = z(c_2) = z_1(c_2)$, where

$$
z_s = m_a \frac{1 - e^{-(p_{1a}+q_{1a})c_2}}{1 + \frac{q_{1a}}{p_{1a}} e^{-(p_{1a}+q_{1a})c_2}}.
$$

Let $p = p_{1c} + p_2$ and $q = q_{1c} + q_2$. It follows that the aggregate cumulative sales $_c z(t)$ are

$$
_c z(t) = m_c \frac{1 + \frac{q}{p} \frac{z_s}{m_c} - \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}}{1 + \frac{q}{p} \frac{z_s}{m_c} + \frac{q}{p} \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}} I_{t>c_2}. \tag{7}
$$

In the Appendix it is proven that, for $\delta \neq q$,

$$z_1(t) = m_c \frac{q_{1c}}{(q-\delta)} w(t) + \frac{q_2 - \delta}{q - \delta} z_s +$$
$$+ \left[m_c \frac{p}{\delta} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q-\delta} \right) + \frac{q_2 - \delta}{q-\delta} z_s \right] \left[y(t)^{\frac{\delta}{q}} - 1 \right] \qquad (8)$$

$$z_2(t) = m_c \left(\frac{q_2 - \delta}{q - \delta} \right) w(t) - \frac{q_2 - \delta}{q - \delta} z_s + \qquad (9)$$
$$+ \left[m_c \frac{p}{\delta} \left(\frac{p_2}{p} - \frac{q_2 - \delta}{q-\delta} \right) - \frac{q_2 - \delta}{q-\delta} z_s \right] \left[y(t)^{\frac{\delta}{q}} - 1 \right], \qquad (10)$$

where

$$w(t) = \frac{1 + \frac{q}{p} \frac{z_s}{m_c} - \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}}{1 + \frac{q}{p} \frac{z_s}{m_c} + \frac{q}{p} \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}} \qquad (11)$$

$$y(t) = \frac{1 + \frac{q}{p}}{1 + \frac{q}{p} \frac{z_s}{m_c} + \frac{q}{p} \left(1 - \frac{z_s}{m_c} \right) e^{-(p+q)(t-c_2)}}. \qquad (12)$$

Observe that $w(t)$ represents the relative aggregate sales, $_c z(t)/m_c$. In the special case when $\delta = q$, the solution of the system (4) is:

$$z_1(t) = \left[m_c \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) + z_s \frac{q}{p} \frac{1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q}}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right] w(t) + z_s \frac{\left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right)}{\left(1 + \frac{q}{p} \frac{z_s}{m_c} \right)} +$$
$$+ m_c \frac{q_{1c} p}{q^2} \left(1 + \frac{q}{p} \frac{z_s}{m_c} \right) y(t) \ln y(t) \qquad (13)$$

$$z_2(t) = \left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right) \left(m_c - z_s \frac{q}{p} \frac{1}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right) w(t) - z_s \frac{\left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right)}{\left(1 + \frac{q}{p} \frac{z_s}{m_c} \right)} +$$
$$- m_c \frac{q_{1c} p}{q^2} \left(1 + \frac{q}{p} \frac{z_s}{m_c} \right) y(t) \ln y(t). \qquad (14)$$

The three terms in equation (8) can be interpreted respectively as,

1. a baseline process, i.e., a fraction of the aggregate process: $b_1(t) = \dfrac{q_{1c}}{q - \delta} {}_c z(t)$;

2. a constant departure: $D_1 = \dfrac{q_2 - \delta}{q - \delta} z_s$;

3. a time dependent departure, $r_1(t)$, (monotonically increasing or decreasing, according to the parameters' values).

In a similar way, equation (10) is composed by the baseline process, i.e., $b_2(t) = \frac{q_2 - \delta}{q - \delta} {}_c z(t)$, and by departures $D_2 = -D_1$ and $r_2(t) = -r_1(t)$. The baseline components denote that the aggregate sales are split between the competitors according

to the factors' shares $q_{1c}/(q - \delta)$ and $(q_2 - \delta)/(q - \delta)$, respectively. The departure components allow a deeper analysis of the competition dynamics and quantify the extra amount of sales achieved by one of the competitors at the other one's expense. The relative size of the three components can be easily interpreted in real applications, once model fitting to sales data produces parameters' estimates (see, for an example, Sect. 5).

Asymptotically, the solutions (8) and (10) converge to:

$$\lim_{t \to \infty} z_1(t) = m_c \frac{q_{1c}}{q - \delta} + \frac{q_2 - \delta}{q - \delta} z_s +$$
$$+ \left[m_c \frac{p}{\delta} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q - \delta} \right) + \frac{q_2 - \delta}{q - \delta} z_s \right] \left[\left(\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right)^{\frac{\delta}{q}} - 1 \right] \quad (15)$$

$$\lim_{t \to \infty} z_2(t) = m_c \frac{q_2 - \delta}{q - \delta} - \frac{q_2 - \delta}{q - \delta} z_s +$$
$$+ \left[m_c \frac{p}{\delta} \left(\frac{p_2}{p} - \frac{q_2 - \delta}{q - \delta} \right) - \frac{q_2 - \delta}{q - \delta} z_s \right] \left[\left(\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right)^{\frac{\delta}{q}} - 1 \right]. \quad (16)$$

For the sake of completeness, we observe that in the special case of a synchronous competition, the original system (4) reduces to:

$$z_1'(t) = m \left[p_1 + (q_1 + \delta) \frac{z_1(t)}{m} + q_1 \frac{z_2(t)}{m} \right] \left[1 - \frac{z(t)}{m} \right] \quad (17)$$

$$z_2'(t) = m \left[p_2 + q_2 \frac{z_2(t)}{m} + (q_2 - \delta) \frac{z_1(t)}{m} \right] \left[1 - \frac{z(t)}{m} \right]. \quad (18)$$

The solutions' expressions simplify, for $\delta \neq q$, to:

$$z_1(t) = m \frac{q_1}{q - \delta} \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} + m \frac{p}{\delta} \left(\frac{p_1}{p} - \frac{q_1}{q - \delta} \right) \left\{ \left[\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \right]^{\frac{\delta}{q}} - 1 \right\} \quad (19)$$

$$z_2(t) = m \frac{q_2 - \delta}{q - \delta} \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} + m \frac{p}{\delta} \left(\frac{p_2}{p} - \frac{q_2 - \delta}{q - \delta} \right) \left\{ \left[\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \right]^{\frac{\delta}{q}} - 1 \right\}. \quad (20)$$

On the other hand, for $\delta = q \neq 0$, we obtain

$$z_1(t) = m \left(\frac{p_1}{p} - \frac{q_1}{q} \right) \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} + m \frac{q_1 p}{q^2} \left[\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \right] \ln \left[\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \right]$$

$$z_2(t) = m \left(1 + \frac{p_2}{p} - \frac{q_2}{q} \right) \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} +$$
$$+ m \frac{p}{q} \left(\frac{q_2}{q} - 1 \right) \left[\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \right] \ln \left[\frac{1 + \frac{q}{p}}{1 + \frac{q}{p} e^{-(p+q)t}} \right].$$

Table 1: Synchronic models. UNI=unified model, UNC=unified model with a one–dimensional constraint, GB=I part model, ST=Savin and Terwiesch (2005) model, KBK= Krishnan et al. (2000) model, LMP= Libai et al. (2009) model. PR1= first entrant product, PR2=second entrant product, AGG= aggregate model.

SYNCHRONIC MODELS		INN	IM WITHIN	IM CROSS	MKT POT	CONSTRAINTS IMPOSED ON UNI
UNI	PR1	p_1	$q_1 + \delta$	q_1		
(# p. 7)	PR2	p_2	q_2	$q_2 - \gamma$	m	
(# a.p. 4)	AGG	$p_1 + p_2$	$q_1 + q_2$	$(\delta - \gamma)^{(*)}$		
UNC	PR1	p_1	$q_1 + \delta$	q_1		
(# p. 6)	PR2	p_2	q_2	$q_2 - \delta$	m	$\gamma = \delta$
(# a.p. 3)	AGG	$p_1 + p_2$	$q_1 + q_2$			
GB	PR1	p_1	q_1	q_1		$\delta = 0$
(# p. 5)	PR2	p_2	q_2	q_2	m	$\gamma = 0$
(# a.p. 3)	AGG	$p_1 + p_2$	$q_1 + q_2$			
ST	PR1	p_1	q_1	$q_1 - \delta$		
(# p. 6)	PR2	p_2	q_2	$q_2 - \delta$	m	$\gamma = \delta$
(# a.p. 3)	AGG	$p_1 + p_2$	$q_1 + q_2 - \delta$			
KBK	PR1	p_1	q_1	q_1		$\delta = 0$
(# p. 5)	PR2	p_2	q_2	q_2	m	$\gamma = 0$
(# a.p. 3)	AGG	$p_1 + p_2$	$q_1 + q_2$			
LMP	PR1	p_1	q_1	$q_1 - \delta$		$q_2 = q_1 + \delta$
(# p. 5)	PR2	p_2	q_1	$q_1 - \delta$	m	$q_2 - \gamma = q_1$
(# a.p. 3)	AGG	$p_1 + p_2$	$2q_1 - \delta$			$(\Rightarrow \gamma = \delta)$

$^{(*)}$(w.r.t. $z_1(t)$)

4   A comparison with other models

Condition (2) is satisfied by three different models within UNCD in the sense that they add further restrictions. We examine comparatively STD (ST) by Savin and Terwiesch (2005), KBKD (or KBK) by Krishnan et al. (2000), LMPD (or LMP) by Libai et al. (2009) and GBD (or GB) described in the companion paper. The parametric characterization of above mentioned models is specified in Table 1 (for the synchronic case) and in Table 2 (for the diachronic case, D suffix).

We observe that GBD is a simple reduction imposed on UNCD by a further restriction of equivalence between within-brand and cross-brand effects. In GBD we assume $\delta = \gamma = 0$. There is no differential contribution to the knowledge of competing products arising from different past adoptions (or adopters). We remark that equations (8) and (10) generalize the solutions of the GBD model found in Guseo and Mortarino (2010a), since the latter represent the limit, as $\delta \to 0$, of the former (see proof in the Appendix). We notice that KBKD is a nested model within GBD

Table 2: Diachronic models. UNID=unified model, UNCD=unified model with a one–dimensional constraint, GBD= I part model, STD=Savin and Terwiesch (2005) model, KBKD= Krishnan et al. (2000) model, LMPD= Libai et al. (2009) model. PR1= first entrant product, PR2=second entrant product, AGG= aggregate model.

DIACHRONIC MODELS		BEFORE COMPETITION $t < c_2$			UNDER COMPETITION $t \geq c_2$				CONSTRAINTS IMPOSED ON UNID
		INN	IM	MKT POT	INN	IM WITHIN	IM CROSS	MKT POT	
UNID	PR1	p_{1a}	q_{1a}		p_{1c}	$q_{1c}+\delta$	q_{1c}		
(# p. 10)	PR2	0	0	m_a	p_2	q_2	$q_2-\gamma$	m_c	
(# a.p. 7)	AGG	p_{1a}	q_{1a}		$p_{1c}+p_2$	$q_{1c}+q_2$	$(\delta-\gamma)^{(*)}$		
UNCD	PR1	p_{1a}	q_{1a}		p_{1c}	$q_{1c}+\delta$	q_{1c}		$\gamma=\delta$
(# p. 9)	PR2	0	0	m_a	p_2	q_2	$q_2-\delta$	m_c	
(# a.p. 6)	AGG	p_{1a}	q_{1a}		$p_{1c}+p_2$	$q_{1c}+q_2$			
GBD	PR1	p_{1a}	q_{1a}		p_{1c}	q_{1c}	q_{1c}		$\delta=0$
(# p. 8)	PR2	0	0	m_a	p_2	q_2	q_2	m_c	$\gamma=0$
(# a.p. 6)	AGG	p_{1a}	q_{1a}		$p_{1c}+p_2$	$q_{1c}+q_2$			
STD	PR1	p_{1a}	q_{1a}		p_{1a}	q_{1a}	$q_{1a}-\delta$		$p_{1c}=p_{1a}$
(# p. 6)	PR2	0	0	m	p_2	q_2	$q_2-\delta$	m	$q_{1c}+\delta=q_{1a}; \ \gamma=\delta$
(# a.p. 5)	AGG	p_{1a}	q_{1a}		$p_{1a}+p_2$	$q_{1a}+q_2-\delta$			$m_c=m_a\ (=m)$
KBKD	PR1	p_{1a}	q_{1a}		p_{1a}	q_{1c}	q_{1c}		$p_{1c}=p_{1a}; \ \delta=0$
(# p. 6)	PR2	0	0	m_a	0	q_2	q_2	m_c	$p_2=0; \ \gamma=0$
(# a.p. 5)	AGG	p_{1a}	q_{1a}		p_{1a}	$q_{1c}+q_2$			
LMPD	PR1	p_{1a}	q_{1a}		p_{1a}	q_{1a}	$q_{1a}-\delta$		$p_{1c}=p_{1a}; \ q_{1c}+\delta=q_{1a}$
(# p. 5)	PR2	0	0	m	p_2	q_{1a}	$q_{1a}-\delta$	m	$q_2=q_{1c}+\delta; \ q_2-\gamma=q_{1c}$
(# a.p. 5)	AGG	p_{1a}	q_{1a}		$p_{1a}+p_2$	$2q_{1a}-\delta$			$(\Rightarrow \gamma=\delta); \ m_c=m_a\ (=m)$

$^{(*)}$(w.r.t. $z_1(t)$)

because it adds two further constraints: no possible variation in external influence parameter for the first entrant during competition, $p_{1c}=p_{1a}$; and the exclusion of an external effect for the second entrant, $p_2=0$.

On the other hand, model STD is based on a different class of reductions imposed on UNCD. In particular, it is assumed that there is no variation in the external influence parameter during competition for the first entrant, $p_{1c}=p_{1a}$, and that the within-brand effect for the first entrant, $q_{1c}+\delta$ does not change under competition, $q_{1c}+\delta=q_{1a}$. A further restriction, which may be weakened, refers to the assumed constant market potential in the two regimes, $m_a=m_c$. We underline that the solution to the UNCD model is coherent also with the solution of the STD model,

Figure 1: Cumulative and rate sales (quarterly data). Source: IMS-Health, Italy.
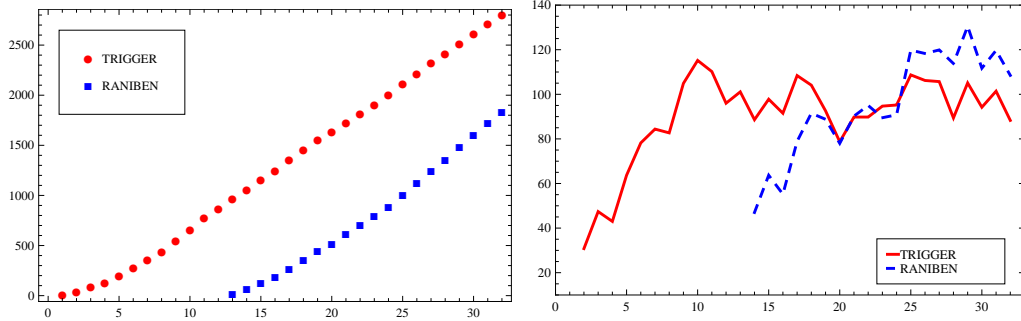
if equation (15) of Savin and Terwiesch (2005) is corrected as explained in Guseo and Mortarino (2010b). The special case $\delta = q$ corresponds in Savin and Terwiesch (2005) notation to $\alpha_1 = \alpha_2 = 1$, although, probably due to a typo, in the cited equation the condition $\alpha_1 + \alpha_2 = 2$ is used.

In their paper, Savin and Terwiesch (2005) write that closed-form solutions can still be obtained also for the case $m_a \neq m_c$. In the rest of their paper, however, the authors assume that $m_a = m_c$. For this reason, we will denote by STD their model with that constraint, and by STDE (extended) the model obtained when that assumption is relaxed. Both versions of Savin and Terwiesch (2005) model will be applied to our data in the next section, in order to compare them with the UNCD model. We notice that LMPD is a nested model within STD because it adds a further constraint. In particular, the second entrant does not have a specific within-brand effect during competition, but duplicates the corresponding one of the first entrant, $q_2 = q_{1a} = q_{1c} + \delta$.

A correct comparison between GBD and STD, and correspondingly between KBKD and LMPD, may not be addressed theoretically: it is a function of observed data. Preference depends upon the real-world situation. Obviously, the UNCD includes monotonically two different branches GBD ($\supset$ KBKD) and STD ($\supset$ LMPD), and must have, therefore, a better performance.

Previous monotonicity conditions for the diachronic case are obviously confirmed in the synchronic case summarized in Table 1. For this case we observe the natural equivalences UNC $\leftrightarrow$ ST and GB $\leftrightarrow$ KBK.

5   An application

Our data, provided by IMS-Health, Italy, consist of the cumulative quarterly number of packages sold in Italy by Trigger and Raniben. Data are available until the third quarter of 1991 (32 observations for Trigger and 20 observations for Raniben. See also the companion article, Guseo and Mortarino (2010a).

The equations stemming from model (4), i.e., (6), (8) and (10), were fitted simultaneously by applying the Beauchamp and Cornell (1966) technique. Parameter estimates are summarized in Table 3 and the agreement between observed and fitted

Table 3: Estimation results for UNCD model, (4).

|	Estimate	Standard Error	95% Confidence Interval
m_a	1330.09	60.6351	(1208.57, 1451.61)
m_c	9792.73	899.989	(7989.11, 11596.3)
p_{1a}	0.01419	0.00038	(0.01342, 0.01496)
q_{1a}	0.29802	0.01318	(0.27161, 0.32444)
p_{1c}	0.01708	0.00167	(0.01373, 0.02044)
q_{1c}	0.10684	0.00959	(0.08762, 0.12605)
p_2	-0.00855	0.00160	(-0.01175, -0.00535)
q_2	-0.03547	0.00920	(-0.05390, -0.01704)
δ	-0.16917	0.01719	(-0.20362, -0.13471)

$R^2 = 0.999959$

Figure 2: Comparison between observed and fitted values, UNCD model.
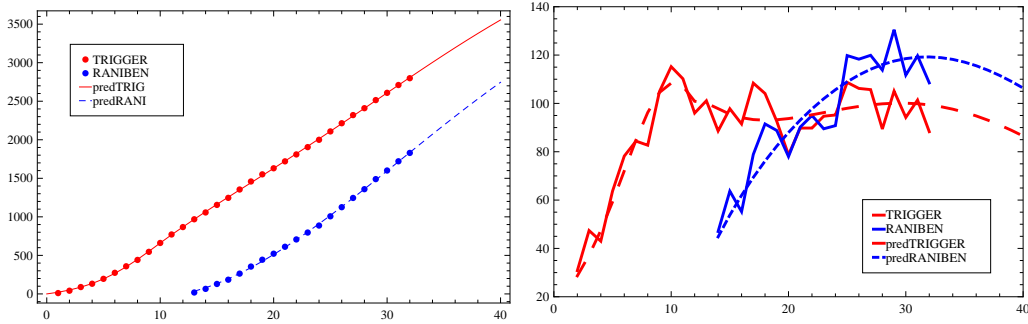
values is shown in Figure 2. We observe a noteworthy increase in market potential after the beginning of competition. If we focus on innovation parameters, we see that after Raniben's launch innovators increase their trust in Trigger ($\hat{p}_{1c} > \hat{p}_{1a}$) and hold the new entrant back ($\hat{p}_2 < 0$). This is not a surprising behavior in the drugs market: the two products are based upon the same active compound, and the older one may be considered safer since it is not at risk for unknown counterindications. Imitative parameters have to be interpreted with reference to the proposed model (e.g., the comparison between $\hat{q}_{1c}$ and $\hat{q}_{1a}$ is meaningless because they measure something different; moreover, the analysis of $\hat{q}_2$ may be misleading, if we do not interpret it in the light of $\hat{\delta}$). If we substitute the obtained estimates in the model (4), we have, after Raniben's launch, that

$$
\begin{aligned}
z_1'(t) &\propto 0.0171 - 0.0623\frac{z_1(t)}{m_c} + 0.1068\frac{z_2(t)}{m_c} \\
z_2'(t) &\propto -0.0086 + 0.1337\frac{z_1(t)}{m_c} - 0.0355\frac{z_2(t)}{m_c}.
\end{aligned}
$$

We observe that not only the cross-brand effect is much stronger than the within-brand effect, but also the within-brand effect has a negative sign. That situation

Figure 3: Model components.
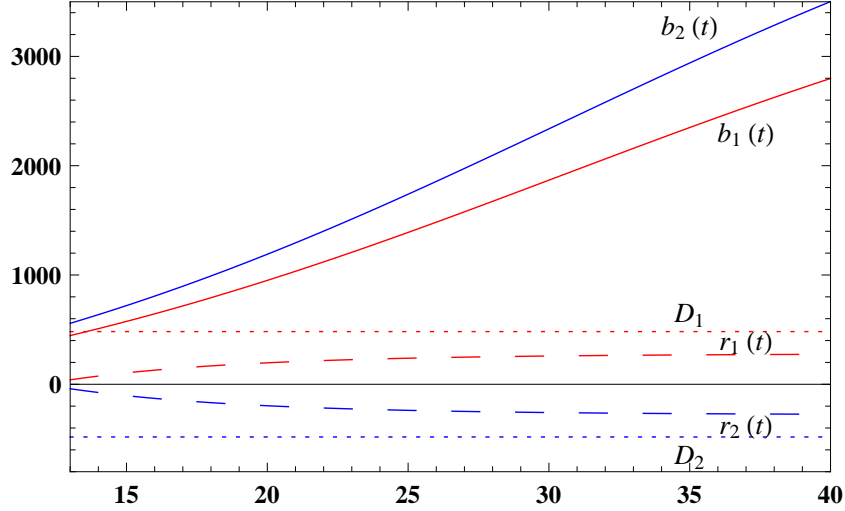
Table 4: Squared Pearson correlation coefficient between observed and fitted values for alternative models.

ρ^2	UNCD model (B&C)	STDE model (B&C)	STD model (B&C)	LMP model (B&C)
TRIGGER (n=32)	0.999924	0.999076	0.997828	0.995353
RANIBEN (n=20)	0.999646	0.999589	0.998847	0.996632
#parameters	9	7	6	5
$\widetilde{R}^2$ w.r.t. UNCD	–	0.926745	0.968493	0.988439
F test	–	347.9	563.552	1175.57

describes a market where consumers of both products are substantially unsatisfied with their drug and turn to the alternative one. The new buyers of Raniben use the knowledge of the active principle spread through Trigger's diffusion. Conversely, Trigger is kept vital because it is based on an active compound that is perceived as efficacious enough to generate a new drug.

The effect of competition between the two products can be studied also with the help of Figure 3 where the three components described in Section 3 are plotted. Here the first competitor reached less than a half of the aggregate sales ($\frac{q_{1c}}{q-\delta} = 0.444$), but gained through small positive departures D_1 and $r_1(t)$.

In order to compare the previous model with alternative solutions, the same data were also used to fit the model of Savin and Terwiesch (2005), both in the standard version (STD) and in the extended version (STDE), in which the market potential after competition, m_c, is not constrained to be equal to the market potential before competition, m_a, and the joint model (LMPD) of Libai et al. (2009). The model

Figure 4: Comparisons among residuals of UNCD, STDE, STDE and LMPD models with the Beauchamp and Cornell (1966) technique.
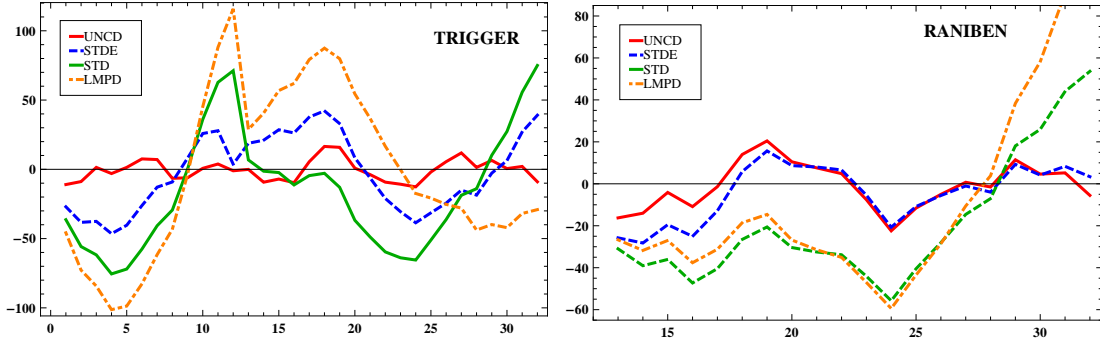
KBKD of Krishnan et al. (2000), which is presented in Table 2 was not included in this comparison because it does not separate cross-brand from within-brand effect. The results of the application of KBKD model to these data are presented in the companion article, Guseo and Mortarino (2010a).

The comparison among different models is performed through a simple measurement, the squared Pearson correlation coefficient between observed and fitted values. Results are proposed in Table 4. Moreover since LMPD, STD, and STDE are nested models in the UNCD model, the squared multiple partial correlation coefficient

$$\widetilde{R}^2 = (R^2_{UNCD} - R^2_M)/(1 - R^2_M) \tag{21}$$

was calculated (here R^2_M denotes the determination index of the reduced model that in turn has to be compared to the UNCD). A possible test to verify the significance of the s parameters of the UNCD that are not included in model M may be given by

$$F = [\widetilde{R}^2(N - k)]/[(1 - \widetilde{R}^2)s], \tag{22}$$

where N denotes the number of observations used to fit the model and k is the number of parameters included in the UNCD. Under the null hypothesis of equivalence between model M and the UNCD, (22) is distributed as a Snedecor's F with $(s, N - k)$ degrees of freedom, if $\varepsilon(t)$ is normal i.i.d. Nevertheless, F-ratio (22) can be used as an approximate robust criterion to compare model M nested in UNCD (Guseo et al., 2007), by considering the well-known common threshold 4. Statistics $\widetilde{R}^2$ and F are calculated for the three alternative models to UNCD and the results are summarized in Table 4. Although the distribution of F is only approximately equal to the nominal one, all values in the final row are incredibly higher than the standard threshold of 4, giving strong support to the use of UNCD to model our data.

The analysis of residuals (see Figure 4) confirms that the UNCD model (4) is essential to catch the features of the first competitor, and, to a smaller extent, of the second competitor. The LMPD model fit is quite lower both for the first and

Figure 5: Comparison among observed and fitted values for GBD and UNCD models.
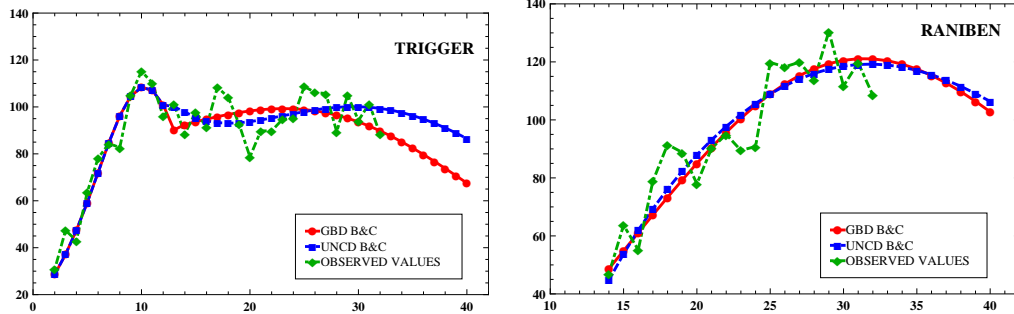
the second competitor. This fact essentially tells us that assumptions required to implement the LMPD model (equality of parameters of the two competitors and unmodified market potential after competition's beginning) are not adequate to our application. Also the STD model does not capture the features of the competition. The fact that the UNCD, for this dataset, is distinctly superior to STDE, which is the more complex alternative, denotes that Raniben's launch modified substantially the evolution of Trigger. With reference to the UNCD, we note that STDE introduces two more constraints: $p_{1c} = p_{1a}$ and $q_{1c} + \delta = q_{1a}$. That fact is obviously not an assumption supposed to hold in any application, but we think that as a first step, the more general UNCD model should be applied. Further restrictions might be subsequently added in order to reduce the model's complexity, *only if* data support them.

As a concluding remark, we notice that the UNCD model proved to be superior to the GBD (see Table 5 and Figure 5). In particular, the UNCD model seems to prevail on the GBD because it is flexible enough to "follow" very well the profile of Trigger's sales. The proposed unbalanced application highlights the utility of a separation between within-brand and cross-brand effects with reference to the balanced approach GBD as denoted by the F-ratio, $F = 118$ (see Table 5). At the same time, competing unbalanced models like STDE, STD, LMPD present a stronger departure in terms of F-ratios, that is 348, 564 and 1176, respectively. In particular, LMPD is characterized by a poor performance, and this aspect is clearly confirmed by the graphical analysis of residuals.

6   Final remarks and discussion

In this paper we introduced, for a duopolistic setting, a UNID model based on diffusions accessing a common (or category) residual market. Interpersonal communication effects are separated into natural components in order to examine and test separately for each competitor, within-brand and cross-brand effects. The main emphasis is on a diachronic setting, which may be reduced to a synchronic case. The product specific within-brand and cross-brand discriminations, δ and γ, which are free in UNID general model, are then restricted to be equivalent for each competitor,

$\delta = \gamma$, thus obtaining a closed-form solution for the corresponding system, UNCD. The theoretical comparisons between rival models GBD, STD, KBKD and LMPD reveal that they are locally nested.

The UNCD model deals with competition between products that arises after a monopolistic regime, and it is particularly suitable to describe situations when the start of competition upsets the diffusion that the first entrant was facing before that timepoint. Moreover, we remark that the UNCD model can be extended to deal with more than two competitors with parameter constraints similar to the ones described in this paper.

The theoretical generality of the proposed model UNCD, with reference to hierarchical-included competing models, does not automatically warrant a systematic weak departure from the more general UNID model for which there is no closed-form solution useful to correctly compare performances. Nevertheless, the global high value placed on the determination index in current application suggests a very limited space for the natural improvement. On the other hand, a different modelling of within-brand and cross-brand components based on an overcoming of linear (constrained) assumption based on an increased order (second order, etc.) may deserve theoretical interest. However, in an applied perspective, the natural improvement may appear of limited size if we look, for example, at the present case study.

Appendix. Proofs

PROOF OF EQUATIONS (8) AND (10) (CASE $\delta \neq q$). We denote by

$$E = \left[\left(1 - \frac{z_s}{m_c} \right) \Big/ \left(1 + \frac{q}{p} \frac{z_s}{m_c} \right) \right] e^{-(p+q)(t-c_2)}.$$

with $W = w(t) = {}_c z(t)/m_c = \dfrac{1 - E}{1 + (q/p)E}$, $w(c_2) = z_s/m_c$. Conversely the expression of t as a function of W is equal to:

$$t = \frac{1}{p+q} \ln \left(\frac{1 + \frac{q}{p} W}{1 - W} \right) + c_2 + \frac{1}{p+q} \ln \left(\frac{1 - \frac{z_s}{m_c}}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right)$$

Table 5: Comparison between GBD and UNCD.

ρ^2	UNCD model (B&C)	UNCD model (DIRECT)	GBD model (B&C)	GBD model (DIRECT)
TRIGGER (n=32)	0.999924	0.999924	0.999823	0.999837
RANIBEN (n=20)	0.999646	0.999651	0.999578	0.999528
#parameters	9	9	8	8
$\widetilde{R}^2$ w.r.t. UNCD	–	–	0.681198	0.447068
F test	–	–	117.521	44.4697

and

$$dt = \frac{1}{p(1-W)(1+\frac{q}{p}W)} dW. \tag{23}$$

The first equation of the system (4) (after competition) can be rewritten as

$$z_1'(t)I_{t>c_2} = {}_c z_1'(t) = m_c \left[p_{1c} + q_{1c}w(t) + \delta \frac{z_1(t)}{m_c} \right] [1 - w(t)],$$

whose solution, with the initial condition ${}_c z_1(c_2) = z_s$, is

$${}_c z_1(t) = z_s e^{-\Delta(t)} + e^{-\Delta(t)} \int_{c_2}^t Q(\tau)e^{\Delta(\tau)}d\tau, \qquad \text{with } \Delta(t) = \int_{c_2}^t P(\tau)d\tau, \tag{24}$$

and

$$Q(t) = m_c[p_{1c} + q_{1c}w(t)][1 - w(t)] \tag{25}$$
$$P(t) = -\delta[1 - w(t)]. \tag{26}$$

It is easy to prove, through equation (23), that

$$\Delta(t) = -\frac{\delta}{p} \int_{\frac{z_s}{m_c}}^W \frac{1}{(1+\frac{q}{p}\omega)} d\omega = -\frac{\delta}{q} \ln \left(\frac{1+\frac{q}{p}W}{1+\frac{q}{p}\frac{z_s}{m_c}} \right). \tag{27}$$

It may be useful to denote by k the ratio δ/q. Equation (27) entails that

$$e^{-\Delta(t)} = \left(\frac{1+\frac{q}{p}W}{1+\frac{q}{p}\frac{z_s}{m_c}} \right)^k. \tag{28}$$

The first integral in equation (24) can be solved again through the substitution of the integration variable:

$$\int_{c_2}^t Q(\tau)e^{\Delta(\tau)}d\tau = \int_{c_2}^t m_c[p_{1c} + q_{1c}w(\tau)][1 - w(\tau)] \left[\frac{1+\frac{q}{p}\frac{z_s}{m_c}}{1+\frac{q}{p}w(\tau)} \right]^k d\tau$$

$$= \frac{m_c}{p} \left(1 + \frac{q}{p}\frac{z_s}{m_c} \right)^k \int_{\frac{z_s}{m_c}}^W (p_{1c} + q_{1c}\omega) \left(1 + \frac{q}{p}\omega \right)^{-k-1} d\omega. \tag{29}$$

In order to solve the general case, we have to assume that $k \neq 0$, and subsequently evaluate the limit of the solution as $k \to 0$ (this result will be equal to the solution found for $\delta = 0$ in Guseo and Mortarino, 2010a). Moreover we have to separate from the rest of the analysis the case $k = 1(\delta = q)$.

For $k \neq 1$, the integral in (29) has the following solution:

$$\int (p_{1c} + q_{1c}\omega) \left(1 + \frac{q}{p}\omega \right)^{-k-1} d\omega = -\frac{\left(1 + \frac{q}{p}\omega \right)^{-k} \left\{ q_{1c} + \frac{q}{p}[(k-1)p_{1c} + kq_{1c}\omega] \right\}}{k(k-1)(q/p)^2} + C$$

$$= -\frac{p^2}{\delta(\delta-q)} \left(1 + \frac{q}{p}\omega \right)^{-k} \left(q_{1c} + \frac{\delta-q}{p}p_{1c} + \frac{\delta}{p}q_{1c}\,\omega \right) + C. \tag{30}$$

By equations (29) and (30) it follows that

$$\int_{c_2}^{t} Q(\tau)e^{\Delta(\tau)}d\tau = \frac{m_c p}{\delta(\delta - q)}\left[q_{1c} + \frac{\delta - q}{p}p_{1c}+\right.$$

$$\left. +\frac{\delta}{p}q_{1c}\frac{z_s}{m_c} - \left(\frac{1 + \frac{q}{p}\frac{z_s}{m_c}}{1 + \frac{q}{p}W}\right)^k \left(q_{1c} + \frac{\delta - q}{p}p_{1c} + \frac{\delta}{p}q_{1c}W\right)\right]. \qquad (31)$$

Finally, by using equations (28) and (31), we obtain:

$$
\begin{aligned}
z_1(t) &= z_s e^{-\Delta(t)} + e^{-\Delta(t)}\int_{c_2}^{t} Q(\tau)e^{\Delta(\tau)}d\tau \\[6pt]
&= z_s\left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}}\right)^k + \frac{m_c p}{\delta(\delta - q)}\left[\left(q_{1c} - \frac{q - \delta}{p}p_{1c} + \frac{\delta}{p}q_{1c}\frac{z_s}{m_c}\right)\left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}}\right)^k +\right. \\[6pt]
&\quad \left. -\left(q_{1c} - \frac{q - \delta}{p}p_{1c} + \frac{\delta}{p}q_{1c}W\right)\right] \\[6pt]
&= m_c\frac{q_{1c}}{q - \delta}W + z_s\left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}}\right)^k\left(1 - \frac{q_{1c}}{q - \delta}\right) + \\[6pt]
&\quad +\frac{m_c}{\delta(q - \delta)}[q_{1c}p - (q - \delta)p_{1c}]\left[1 - \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}}\right)^k\right] \\[6pt]
&= m_c\frac{q_{1c}}{q - \delta}W - z_s\frac{q_2 - \delta}{q - \delta}\left[1 - \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}}\right)^k\right] + \frac{q_2 - \delta}{q - \delta}z_s + \\[6pt]
&\quad +m_c\frac{p}{\delta}\left(\frac{q_{1c}}{q - \delta} - \frac{p_{1c}}{p}\right)\left[1 - \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}}\right)^k\right] \\[6pt]
&= m_c\frac{q_{1c}}{q - \delta}W + \frac{q_2 - \delta}{q - \delta}z_s \\[6pt]
&\quad +\left[m_c\frac{p}{\delta}\left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q - \delta}\right) + \frac{q_2 - \delta}{q - \delta}z_s\right]\left[\left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}}\right)^k - 1\right]. \qquad (32)
\end{aligned}
$$

Through

$$W = w(t) = \frac{1 - E}{1 + \frac{q}{p}E} = \frac{1 + \frac{q}{p}\frac{z_s}{m_c} - \left(1 - \frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}}{1 + \frac{q}{p}\frac{z_s}{m_c} + \frac{q}{p}\left(1 - \frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}}, \qquad (33)$$

and

$$1 + \frac{q}{p}w(t) = \frac{1 + \frac{q}{p}\frac{z_s}{m_c} + \frac{q}{p} + \frac{q^2}{p^2}\frac{z_s}{m_c}}{1 + \frac{q}{p}\frac{z_s}{m_c} + \frac{q}{p}\left(1 - \frac{z_s}{m_c}\right)e^{-(p+q)(t-c_2)}} = \left(1 + \frac{q}{p}\frac{z_s}{m_c}\right)y(t), \qquad (34)$$

the final expression for $z_1(t)$ can be evaluated:

$$z_1(t) = m_c \frac{q_{1c}}{q-\delta} w(t) + \frac{q_2 - \delta}{q-\delta} z_s + \left[m_c \frac{p}{\delta} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q-\delta} \right) + \frac{q_2 - \delta}{q-\delta} z_s \right] \left[y(t)^{\frac{\delta}{q}} - 1 \right],$$

which is equal to equation (8). The expression for $z_2(t)$ in (10) can be derived as the difference between equation (7) and equation (8):

$$z_2(t) = m_c \left(\frac{q_2 - \delta}{q-\delta} \right) w(t) - \frac{q_2 - \delta}{q - \delta} z_s +$$
$$+ \left[m_c \frac{p}{\delta} \left(\frac{p_2}{p} - \frac{q_2 - \delta}{q-\delta} \right) - \frac{q_2 - \delta}{q-\delta} z_s \right] \left[y(t)^{\frac{\delta}{q}} - 1 \right]. \qquad \square$$

PROOF OF EQUATIONS (13) AND (14) (CASE $\delta = q$). We start from equation (29). For $k = 1$, the integral has the following solution:

$$\int (p_{1c} + q_{1c}\omega) \left(1 + \frac{q}{p}\omega \right)^{-2} d\omega = \frac{-\frac{q}{p}p_{1c} + q_{1c}}{\left(1 + \frac{q}{p}\omega \right) (q/p)^2} + \frac{q_{1c} \ln \left(1 + \frac{q}{p}\omega \right)}{(q/p)^2} + C.$$

Hence

$$\int_{c_2}^{t} Q(\tau)e^{\Delta(\tau)} = m_c \frac{p}{q^2} \left(1 + \frac{q}{p}\frac{z_s}{m_c} \right) \left[\frac{-\frac{q}{p}p_{1c} + q_{1c}}{1 + \frac{q}{p}W} - \frac{-\frac{q}{p}p_{1c} + q_{1c}}{1 + \frac{q}{p}\frac{z_s}{m_c}} + q_{1c} \ln \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}} \right) \right]$$
$$= \frac{m_c}{q} \left(-\frac{q}{p}p_{1c} + q_{1c} \right) \frac{\frac{z_s}{m_c} - W}{1 + \frac{q}{p}W} + m_c \frac{q_{1c}p}{q^2} \left(1 + \frac{q}{p}\frac{z_s}{m_c} \right) \ln \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}} \right)$$

and the solution for $z_1(t)$ is the following:

$$z_1(t) = z_s e^{-\Delta(t)} + e^{-\Delta(t)} \int_{c_2}^{t} Q(\tau)e^{\Delta(\tau)} d\tau$$
$$= z_s \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}} \right) + \frac{m_c}{q} \left(-\frac{q}{p}p_{1c} + q_{1c} \right) \frac{\frac{z_s}{m_c} - W}{1 + \frac{q}{p}\frac{z_s}{m_c}} +$$
$$+ m_c \frac{q_{1c}p}{q^2} \left(1 + \frac{q}{p}W \right) \ln \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}} \right)$$
$$= \frac{z_s \frac{q}{p} + m_c \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right)}{1 + \frac{q}{p}\frac{z_s}{m_c}} W + \frac{z_s - \frac{p_{1c}}{p}z_s + \frac{q_{1c}}{q}z_s}{1 + \frac{q}{p}\frac{z_s}{m_c}} +$$
$$+ m_c \frac{q_{1c}p}{q^2} \left(1 + \frac{q}{p}W \right) \ln \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}} \right)$$
$$= \left[m_c \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) + \frac{z_s \frac{q}{p} + m_c \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) - m_c \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right)\left(1 + \frac{q}{p}\frac{z_s}{m_c}\right)}{1 + \frac{q}{p}\frac{z_s}{m_c}} \right] W +$$
$$+ z_s \frac{\left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right)}{1 + \frac{q}{p}\frac{z_s}{m_c}} + m_c \frac{q_{1c}p}{q^2} \left(1 + \frac{q}{p}W \right) \ln \left(\frac{1 + \frac{q}{p}W}{1 + \frac{q}{p}\frac{z_s}{m_c}} \right)$$

$$= \left[m_c \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) + z_s \frac{q}{p} \frac{1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q}}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right] W +$$

$$+ z_s \frac{\left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right)}{1 + \frac{q}{p} \frac{z_s}{m_c}} + m_c \frac{q_{1c}p}{q^2} \left(1 + \frac{q}{p} W \right) \ln \left(\frac{1 + \frac{q}{p} W}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right)$$

$$\overset{(33)+(34)}{=} \left[m_c \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) + z_s \frac{q}{p} \frac{1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q}}{1 + \frac{q}{p} \frac{z_s}{m_c}} \right] w(t) + z_s \frac{\left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right)}{\left(1 + \frac{q}{p} \frac{z_s}{m_c} \right)} +$$

$$+ m_c \frac{q_{1c}p}{q^2} \left(1 + \frac{q}{p} \frac{z_s}{m_c} \right) y(t) \ln y(t).$$

The solution for $z_2(t)$ is:

$$z_2(t) = \left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right) \left[m_c - z_s \frac{q}{p} \frac{1}{1 + \frac{q}{p} \frac{z_c}{m_c}} \right] w(t) - z_s \frac{\left(1 - \frac{p_{1c}}{p} + \frac{q_{1c}}{q} \right)}{\left(1 + \frac{q}{p} \frac{z_s}{m_c} \right)} +$$

$$- m_c \frac{q_{1c}p}{q^2} \left(1 + \frac{q}{p} \frac{z_s}{m_c} \right) y(t) \ln y(t). \qquad \square$$

PROOF OF COHERENCE WITH THE RESULTS IN GUSEO AND MORTARINO (2010A). It is easy to show that the limit as $\delta \to 0$ of the equations (8) and (10) are equal, respectively, to the solutions found in Guseo and Mortarino (2010a) for the special case of cross-brand effect equal to within-brand effect ($\delta = 0$). By l'Hôpital's rule,

$$\lim_{\delta \to 0} z_1(t) = m_c \frac{q_{1c}}{q} w(t) + \frac{q_2}{q} z_s + m_c p \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \lim_{\delta \to 0} [y(t)]^{\frac{\delta}{q}} \cdot \ln y(t) \frac{1}{q} + z_s \frac{q_2}{q} \cdot 0$$

$$= m_c \frac{q_{1c}}{q} w(t) + \frac{q_2}{q} z_s + m_c \frac{p}{q} \left(\frac{p_{1c}}{p} - \frac{q_{1c}}{q} \right) \ln(y(t)).$$

If we remind that the last two terms in equation (8) are the opposite of the corresponding terms in equation (10), it is straightforward to see that

$$\lim_{\delta \to 0} z_2(t) = m_c \frac{q_2}{q} w(t) - \frac{q_2}{q} z_s + m_c \frac{p}{q} \left(\frac{p_2}{p} - \frac{q_2}{q} \right) \ln y(t). \qquad \square$$

References

Beauchamp, J. and Cornell, R. (1966). "Simultaneous nonlinear estimation". *Technometrics*, 8, 2, pp. 319–326.

Guseo, R., Dalla Valle, A., and Guidolin, M. (2007). "World oil depletion models: Price effects compared with strategic or technological interventions". *Technological Forecasting and Social Change*, 74, 4, pp. 452–469.

Guseo, R. and Mortarino, C. (2010a). *Competition Modelling in Multi-Innovation Diffusions. Part I: Balanced Models.* Technical Report N. 09/2010, Department of Statistical Sciences, University of Padova.

Guseo, R. and Mortarino, C. (2010b). "Correction to the paper "Optimal product launch times in a duopoly: Balancing life-cycle revenues with product cost"". *Operations Research*. (doi:10.1287/opre.1100.0811), (in press).

Kalish, S., Mahajan, V., and Muller, E. (1995). "Waterfall and sprinkler new-product strategies in competitive global markets". *International Journal of Research in Marketing*, 12, pp. 105–119.

Krishnan, T., Bass, F., and Kumar, V. (2000). "Impact of a late entrant on the diffusion of a new product/service". *Journal of Marketing Research*, XXXVII, pp. 269–278.

Libai, B., Muller, E., and Peres, R. (2009). "The role of within-brand and cross-brand communications in competitive growth". *Journal of Marketing*, 73, pp. 19–34.

Mahajan, V., Sharma, S., and Buzzell, R. (1993). "Assessing the impact of competitive entry on market expansion and incumbent sales". *Journal of Marketing*, 57, 3, pp. 39–52.

Parker, P. M. and Gatignon, H. (1994). "Specifying competitive effects in diffusion models: An empirical analysis". *International Journal of Research in Marketing*, 11, 1, pp. 17–39.

Peterson, R. A. and Mahajan, V. (1978). Multi-product growth models. In J. Sheth (Ed.), *Research in marketing, Vol. 1* pp. 201–231. JAI Press.

Savin, S. and Terwiesch, C. (2005). "Optimal product launch times in a duopoly: Balancing life-cycle revenues with product cost". *Operations Research*, 53, 1, pp. 26–47.

Working Paper Series
Department of Statistical Sciences, University of Padua

You may order paper copies of the working papers by emailing wp@stat.unipd.it

Most of the working papers can also be found at the following url: http://wp.stat.unipd.it

Department of Statistical Sciences
University of Padua
Italy