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Keywords: Network externalities, Threshold diffusion models, Chilling effect, Cellular automata, Riccati equation, Generalized Bass Model, NLS.

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1   Introduction

Innovation diffusion dynamics are strongly determined by the physical and technological character of a good or a service. Normal goods or *stand–alone goods* create benefits for a buyer that do not depend on adoption by other potential or actual buyers.

Network goods, i.e., telephone, electricity, natural gas, roadways, Internet, railroads, facsimiles, etc., define expanding physical networks. The increase in sales of network goods (network terminals or network services) determines external benefits for all network participants, positively modifying their utility function. Such external benefits are not market mediated and are usually termed *direct network externalities*.

Network goods exhibit delayed full benefits so that sales are depressed for long

periods, and successful industries, constrained by network externalities, highlight a collective threshold, a *positive critical mass*, which precedes a rapid take–off. Such a *critical mass* may be seen at the same time as both cause and effect of the aggregate behaviour of heterogeneous agents oriented, in their purchase decisions, by individual complex thresholds. Nevertheless, hedonic utility, which may explain a personal assessment motivating adoptions, is a latent function: we only observe final adoptions.

The issue of network externalities modelling has been addressed by many authors in different areas. In economics, we mention, among others, Katz and Shapiro (1985), Cabral (1990), Katz and Shapiro (1986), Economides and Himmelberg (1995),and Grajek (2002). In marketing science there are interesting integrated contributions by Goldenberg et al. (2005), Srinivasan et al. (2004), Rahmandad and Sterman (2004), and Shuster (1998), and general review agendas by Hauser et al. (2006) and Muller et al. (2007). In sociological and statistical sciences there are interesting advances that emphasise some social and multiphase aspects of diffusion under network effects. See, for instance, Rogers (2003), Granovetter (1978), Granovetter and Soong (1986), Mahler and Rogers (1999), Seber and Wild (1989), Snijders et al. (2006), Guseo and Guidolin (2008), Guseo and Guidolin (2009), and Young (2005).

In Complex System Analysis, modelling emphasises micro level aspects like agents' heterogeneity that generate an emerging macro behaviour. Some important references, among others, are Young (2003), Moldovan and Goldenberg (2004), Newman (2003), Windrum and Birchenhall (2005), Boccara (2004), Boccara et al. (1997), and Wolfram (1983). In particular, Cellular Automata are special models within Complex Systems theories. For some specific references see, for instance Boccara and Fukś (1999), Ganguly et al. (2003), Goldenberg et al. (2001), and Guinot (2002).

The aim of this paper is to present a model which takes into account the long–lasting effects of network externalities in the early part of a network good's diffusion process.

In order to consider the incubation or *chilling effect* of network externalities (see, for instance, Goldenberg et al. (2005)) we propose a general model based on a Cellular Automata representation which describes an adoption process depending on a dynamic market potential generated by heterogeneous individual thresholds. For successful network goods we may argue, at the aggregate level, that the market potential is characterised by a local depression which describes a *change–point* time, $\hat{t}$, separating two different regimes: a long preliminary incubation period followed by a sudden *take–off* associated with the attainment of a *positive critical mass*.

Our Cellular Automata model is transformed into an aggregate form by a *mean field approximation* which allows for a differential description. The proposed representation, a Riccati equation, gives rise to a closed form solution. Some statistical features are involved in the inferential process based on different contiguous regimes. In particular, special weighting is essential within NLS (Nonlinear Least Squares) techniques.

The paper is organised as follows. Section 2 is devoted to establishing an important link between the economic theory representation of *willingness to pay* under network externalities and an individual threshold device compared with cumulative

share of sales. This allows the definition of a dynamic market potential. Section 3 illustrates Cellular Automata and introduces the specific CA model, its mean field approximation, and the corresponding solution, which is described in the final Appendix. Section 4 examines a direct application to USA fax machines time series with a special comparison between reduced models without network effects and the new one. Section 5 is devoted to final remarks and discussion.

2   Economic and social modelling: a common key rule

Innovation diffusion is usually explained through information contagion, which establishes some relationships between agents included in the residual market and agents that are adopters by time t. Agents simply hear about an existing innovation: The Bass model (see Bass (1969)) is a basic way to represent such a concept. A quite different approach in Young (2005) is based on learning. An agent adopts if the perceived gain due to adoption, based on decisions of other adopters, exceeds a threshold which is a function of his/her beliefs. Heterogeneity of beliefs and related thresholds may be described by a distribution. Young proposes a sharp separation between contagion and learning, where the latter may produce super–exponential growth rates in the early stages of adoption, contrary to standard contagion models. This immediately raises a question: why not consider sub–exponential growth, which is so typical in network goods' diffusion? We certainly agree about the complexity of the learning process that involves at least two separate effects: 1) increasing adoptions that generate a positive information useful to persuade the residual agents to adopt and 2) the remaining agents who are intrinsically more sceptical and surely harder to persuade due to a stronger *threshold* system. It is not so obvious whether the author's modelling can qualitatively and formally express the above mentioned aspects in order to test empirical situations. Continuous time representation is based on a rigid equation, $\dot{p}(t) = \lambda[F(p(t)) - p(t)]$, where $p(t)$ denotes the proportion of the group (market) who have adopted the product or the idea by time t, and $F(\cdot)$ is the cumulative distribution of resistance or threshold which is equated to the current proportion of adopters. Parameter $\lambda > 0$ expresses the well–known fact that only a part of the proportion of individuals, who are prepared to adopt, $F(p(t)) - p(t)$, really do that at time t. The previous equation is very simple and allows a closed form solution, namely, $t = p^{-1}(x)$. Nevertheless, considering a non–constant function, $F(p(t))$, opposed to the usual assumption based on a unitary potential, 1, is not sufficient. There is no effort to extend such an idea outside the monomolecular (or exponential) vision of diffusion simply related to a proportion λ of the residual normalised market. Why not introduce parallel contagion effects with a greater attention to the commonly perceived fact that learning and contagion are not sharply separable effects in real–world diffusions?

This aspect is jointly conceived in the present paper by considering that learning and contagion both affect the dynamic market potential definition and the corresponding adoption process. In particular, in our equation, the market potential $m(t)$ is a free function, so that we may include flexible distributive forms governing the heterogeneous thresholds, which are time dependent in order to take into account

not only previously mentioned point 1) about the complexity of a learning process, but also point 2), which is not examined in Young's paper.

Let us consider first a basic economic representation of critical mass effect due to heterogeneous thresholds that characterise different customers in the marketplace. We refer to Economides and Himmelberg (1995) with some terminological adaptations in symbolic notation.

The authors assume that consumers expect a normalised network size ν^e such that $0 \le \nu^e \le 1$ and define a *network externalities function* which represents the overall value of the good

$$v(\nu^e) = k + \delta f(\nu^e),\tag{1}$$

where k denotes the *value of the good* in the absence of network effects, δ is an indicator function taking the value 1 if there are *network externalities*, and zero elsewhere. Function $f(\cdot)$ is monotone increasing with initial condition $f(0) = 0$, $f'(\cdot) > 0$ and $f''(\cdot) \le 0$ in order to describe an increasing utility under larger expected sizes of networks. This network externalities function describes a commonly perceived value of the good, which is individual independent.

The authors assume a special definition of the *willingness to pay* for one unit of the good in a network of expected size ν^e, i.e.,

$$w(h, \nu^e) = hv(\nu^e),\tag{2}$$

where $h \in [0,1]$ is a penalising index characterising a specific consumer, *consumer index* in the sequel, and $P(h)$ is a corresponding cumulative distribution function over the population of interest. This proposed multiplicative specification of the willingness to pay by Economides and Himmelberg (1995) "allows different types of consumers to receive differing values of network externality from the same network". This specification is extremely interesting for the purposes of the present paper and diverges from the more common additive one (see, for instance, Katz and Shapiro (1985), Cabral (1990)), under which all consumers receive the same benefit from the same network.

Given expectations ν^e and price p, the authors define the index h^* of marginal consumer as a solution of equation $p = w(h, \nu^e)$ so that, in particular,

$$h^* = \frac{p}{v(\nu^e)}.\tag{3}$$

Under given expectations and price, all consumers with index $h \ge h^*$ buy the good, so that the complementary normalised network size at price p is

$$d = 1 - P(h^*),\tag{4}$$

defining the *demand* for the network good: in this way they are able to write the willingness to pay for the last consumer in a network of size $1 - d$ with expectations ν^e, i.e.,

$$p(d, \nu^e) = v(\nu^e)P^{-1}(1 - d).\tag{5}$$

Equation (5) is a basic tool for a systematic discussion of a critical mass under different market structure hypotheses. Economides and Himmelberg (1995) define critical mass as the smallest network size, ν^0, that can be sustained in equilibrium.

In Equation (2) the authors consider $v(\nu^e)$ as a common benchmark for individual evaluations and h summarises heterogeneity, i.e., all individual specificities without any reference to personal comparisons. In their assumptions, a generic consumer filters $v(\nu^e)$ with a *passe–partout* penalising coefficient h in order to derive his or her willingness to pay.

We propose a different definition of consumer index h not focusing on the marginal consumer at time t but with reference to a generic consumer. We emphasise the personal evaluation of the ratio

$$h = \frac{p}{\tilde{v}(\nu, \nu^e)} \tag{6}$$

expressing a kind of *resistance* to adoption, i.e., a threshold. In this case h high values denote a strong resistance to buy, while the definition by Economides and Himmelberg (1995) has an opposite meaning.

This ratio in the population may be interpreted as a random variable H_t at time t representing an individual dependent assessment where p is the public price and $\tilde{v}(\nu, \nu^e)$ is a personal evaluation of the "value of the good", i.e., a network externalities function more general than $v(\nu^e)$, as expressed in Equation (1). It is based, at least, on the share of adoptions $\nu(t) = y(t)/m(t)$, i.e., a density ratio between $y(t)$, the absolute cumulative adoptions and $m(t)$, the absolute dynamic market potential. Unlike the static definition of demand d for network goods given by Economides and Himmelberg (1995) in Equation (4), we consider a dynamic perspective. This assumption is essential for identifying the incubation effect that penalises the potential precisely in the first part of the corresponding life cycle. Other components of function $\tilde{v}(\cdot)$ include, explicitly, the personal expectations $\nu^e(t)$ and, implicitly, a personal basic value k of the good which may be different among current potential consumers, due to different knowledge levels, different perceived performances, and different incomes.

Following our proposal, we argue that $h \in H_t$ is a plausible individual expression of divergence between the current price p and a personal evaluation of the "value of the good" at time t. Notice that the ratio (6) is a dimensionless pure number and it is less than one for potential adopters.

Dynamic market potential definition

We assume that a consumer may become a potential buyer if his or her ratio h is lower than the expressed preferences summarised by the share of adoptions or density $\nu(t)$. Accordingly, we define the *normalised dynamic market potential* by a susceptibility probability,

$$n(t) = P(H_t \leq \nu(t)). \tag{7}$$

If all potential consumers are characterised by a personal low threshold h as compared with $\nu(t)$, then they are possible adopters, i.e., $P(H_t \leq \nu(t)) \simeq 1$. Otherwise, penalisation operates especially with $P(H_t \leq \nu(t)) << 1$ during the network good incubation period.

The *absolute dynamic market potential* $m(t)$ is defined as a function of the asymptotic market potential U and the susceptibility probability (7),

$$m(t) = U\, P\left(H_t \leq \nu(t)\right) = U n(t) = E[Bi(U, P(H_t \leq \nu(t)))] = U\, E\left(I_{(H_t \leq \nu(t))}\right). \tag{8}$$

Such a definition is useful to represent a possible depression of dynamic potential during a long initial incubation period because it combines the density $\nu(t)$ and a non–stationary preference distribution, H_t, which is characterised by its moments and, in particular, by its location parameter, $\mu(t) = E(H_t)$, termed *resistance* in the sequel.

Dynamics and control in individual threshold distribution

We may study the mean value of the threshold effect under a normality assumption, $H_t \sim \mathcal{N}(\mu(t), \sigma)$,

$$m(t) = U\, P\left(H_t \leq \nu(t)\right) = U\, \Phi\left(\frac{\nu(t) - \mu(t)}{\sigma}\right). \tag{9}$$

We model the mean $\mu(t)$ in order to avoid a stationarity assumption of threshold distribution during the *incubation period* as compared with the successive regular one. A mild positive evolution of the adoption process depicts a lower value of μ as far as time elapses from the origin. The function

$$\mu_1(t) = a + bt^{-c}, \qquad a, b, c > 0, \tag{10}$$

represents the average *resistance* at time t because low levels of $\mu(t)$ with reference to $\nu(t)$ imply high levels of $n(t)$ or $m(t)$. For $t \to 0$ this effect is infinite, at time $t = 1$ the level is $a + b$ and asymptotically converges to a. Parameter c depicts the decay speed of resistance during evolution.

An alternative resistance evolution may be an exponential one, i.e.,

$$\mu_2(t) = a + be^{-ct}, \qquad a, b, c > 0. \tag{11}$$

At time $t = 0$, the resistance is $a + b$ and it decays to level a as far as t diverges. Parameter c represents the decay speed in resistance evolution.

A flexible alternative in resistance description is based on a polynomial function,

$$\mu_3(t) = a + bt + ct^2. \tag{12}$$

This function may be used only within a limited incubation period and, under suitable circumstances, it may be unexpectedly increasing.

Critical mass

The definition of critical mass in our approach is an operational notion, related to the joint combination of the share of adoptions $\nu(t)$ and the average resistance $\mu(t)$, which controls the location of individual threshold distribution H_t. The critical mass, c_m, is the level of cumulative sales, a collective threshold, corresponding to a suitable *change–point* $\hat{t}$ which strictly separates the incubation period from a sudden

take–off, $c_m = y(\hat{t})$. After $\hat{t}$, the potential market $m(t)$ is increasing and indicates a regular growth of the corresponding adoption process. The technical identification of $\hat{t}$ is based on a minimum of $P(H_t \leq \nu(t))$.

In the subsequent Section 3 we jointly model the adoption process $y(t)$ in the presence of a dynamic market potential $m(t)$ in order to take into account shrinking and expanding effects due to heterogeneity of agents over time. We do this through a Complex System representation.

3   Network Incubation Period in a Cellular Automaton

Cellular Automata, some notations

Agent–based models and Complexity theory dedicate considerable attention to the micro level in system analysis in order to take into account detailed local heterogeneity of individuals in achieving a bottom–up macro behaviour. This is internally characterised by simple local *transition rules*, $g(\cdot)$, and partially controllable intervention tools, $x(t)$, acting on system environment.

An Automaton $W = \{1, 2, \ldots, i, \ldots, U\}$ is a set of cells i that may vary their status on the basis of local and general transition rules. Each cell i in W may assume, at time t, a special *state* denoted by the indicator function $s(i; t)$. A cell may be active, i.e., has adopted, if $s(i; t) = 1$.Viceversa, $s(i; t) = 0$ represents a neutral cell or $i \in W$. Here we follow some notations expressed for Cellular Automata in Boccara et al. (1997) and Boccara and Fukś (1999), and in Guseo and Guidolin (2008).

Local pressure and transition rules

Let us define a kind of *local pressure* (probability) of the system, $\sigma_s(i; t)$, upon cell i to turn it from a neutral status, 0, towards an active one, 1. This pressure depends on a flexible probability measure, $p_r \geq 0$, that allows a more general description of a neighborhood, possibly i–dependent.

$$\sigma_s(i; t) = \sum_{r=-\infty}^{\infty} s(i+r; t) p_r; \qquad \sum_r p_r = 1. \qquad (13)$$

If the local pressure is *translational invariant*, we may consider the "mean field approximation" that excludes the local effect of distribution p_r (see, for instance Boccara et al. (1997), Boccara and Fukś (1999), Wolfram (1983) and Guseo and Guidolin (2008)),

$$\sigma_s(i; t) \simeq \nu(t) = \sum_{j \in W} \frac{s(j; t)}{m(t)} = \frac{y(t)}{m(t)}, \qquad m(t) > 0, \qquad (14)$$

where $y(t)$ denotes the cumulative observed adoptions (or active states) at time t and $\nu(t)$ is the *density* of the adoption process at time t with reference to the dynamic potential $m(t)$. For $m(t) = 0$ we assume $\nu(t) = 0$.

The transition rule $g(\cdot)$, governing the state dynamics, must be properly specified in order to recover possible effects of an *incubation period* due to *network*

externalities that affect diffusion. We suppose that the change of state of i–th unit, $i \in W = \{1, 2, ..., U\}$, is driven by two major processes: the first one refers to innovative and imitative contributions to a *perturbed* residual market penalised by a threshold or resistance effect; the second one is a global effect due to the relative variation of dynamic market potential at time t,

$$
\begin{aligned}
s(i; t+1) &= s(i; t) + \\
&+ I_{(h_i \leq \nu(t))} \cdot [Bi(1, p + q\, \sigma_s(i; t))] \frac{x(t)}{m(t)} I_{(s(i;t)=0)} + \\
&+ s(i; t) \cdot \frac{m'(t)}{m(t)}.
\end{aligned}
\tag{15}
$$

H_t is a random variable expressing the individual threshold for susceptibility, $h_i \in H_t$, and $\nu(t)$ represents the density of the adoption process. If the individual threshold is lower than reference value $\nu(t)$, then the indicator function $I_{(h_i \leq \nu(t))}$ is set to 1, depicting the admissibility of the *inductive experiments.*

Binomial inductive experiment $Bi(1, p + q\, \sigma_s(i; t))$ represents the contributions of innovative (p) and imitative (q) components and may performed if $I_{(s(i;t)=0)} = 1$.

Integrable function $x(t)$, namely the *intervention function*, allows a modification of the perceived residual market, $(m(t) - y(t))$, and dynamically represents environmental pressure variations, economic choices, political regulations, firms strategies effects, or marketing mix policies. Its equilibrium value is 1. This factor is very important as a contrasting tool against the delay of adoptions due to *network externalities.*

Parameter U denotes the asymptotic market, and the dynamic market potential, $m(t)$, is defined following Equation (8).

The component $s(i; t) . \frac{m'(t)}{m(t)}$ in Equation (15) describes an infinitesimal variational contribution to the individual state, a self–reinforcement effect, due to the dynamic structure of the market potential.

Transition rule and macro effects

The average behaviour of Equation (15) followed by a summation of all cell states $s(i; t)$, $i \in W$ is a discrete time *cumulative evolutionary model* under a mean field approximation, i.e.,

$$
y(t+1) = y(t) + m(t) \left\{ \left(p + q \frac{y(t)}{m(t)} \right) \frac{(m(t) - y(t))}{m(t)} x(t) \right\} + y(t) \frac{m'(t)}{m(t)}.
\tag{16}
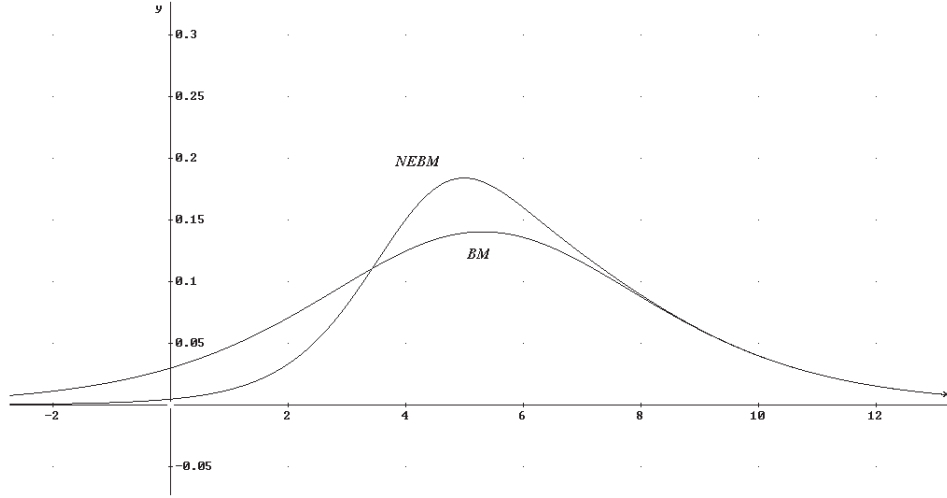$$

A continuous approximation of the previous equation is

$$
y'(t) = m(t) \left\{ \left(p + q \frac{y(t)}{m(t)} \right) \frac{(m(t) - y(t))}{m(t)} x(t) \right\} + y(t) \frac{m'(t)}{m(t)}
\tag{17}
$$

or

$$
\left(\frac{y(t)}{m(t)} \right)' = \left\{ \left(p + q \frac{y(t)}{m(t)} \right) \left(1 - \frac{y(t)}{m(t)} \right) x(t) \right\}.
\tag{18}
$$

Let us recall position, $\nu(t) = y(t)/m(t)$, so that we attain

$$
\nu' = (p + q\nu)(1 - \nu)x(t).
\tag{19}
$$

Figure 1: Network externality effects: standard Bass profile (BM) vs. moving effects due to externalities (NEBM): $p = 0.03$, $q = 0.5$, $m = 1$, $H \sim \mathcal{N}(\mu = 0.2, \sigma = 0.2)$.

The general solution for the previous equation is discussed in Guseo and Guidolin (2009) and briefly summarised in Appendix A, so that we obtain a simple aggregate representation of our CA under network externalities effects, i.e.,

$$y(t) = m(t) \cdot \nu(t) = U \, P\left(H_t \leq \nu(t)\right) \cdot \nu(t), \tag{20}$$

where,

$$\nu(t) = \frac{1 - e^{-(p+q)\int_0^t x(\tau)d\tau}}{1 + \frac{q}{p}e^{-(p+q)\int_0^t x(\tau)d\tau}}. \tag{21}$$

The explicit closed form solution, as expressed by Equations (20 – 21), is impressive for its simplicity. It is an extension of the Generalized Bass Model, GBM (see Bass et al. (1994)), with a variable market potential (see, for instance, Guseo (2004) and Guseo and Guidolin (2009)). The simultaneous description of network externalities that affect dynamic market potential, and the formal presence of an explicit intervention function $x(t)$ that allows measurability of the effects of convenient strategic marketing or management contrasting actions, are strength elements of the present class of models. Notice that, by examining Equations (20 – 21), the exponent $\int_0^t x(\tau)d\tau$ modifies with compensative actions the geometry of time, and not the asymptotic potential U, which is under the control of $m(t)$.

In Figure 1 we represent two different non cumulative diffusion frameworks under a limited life cycle hypothesis. The BM case considers a standard Bass diffusion model (see, Bass (1969)) with parameter $U = 1$, $p = 0.03$ and $q = 0.5$. The NEBM case exhibits the behaviour of the newly proposed model, with network externalities effects producing a network incubation period necessary for critical mass attainment. In the latter case, $H \sim \mathcal{N}(\mu = 0.2, \sigma = 0.2)$ introduces a stationary delay in Equation (20) with dynamic market potential described by Equation (9).
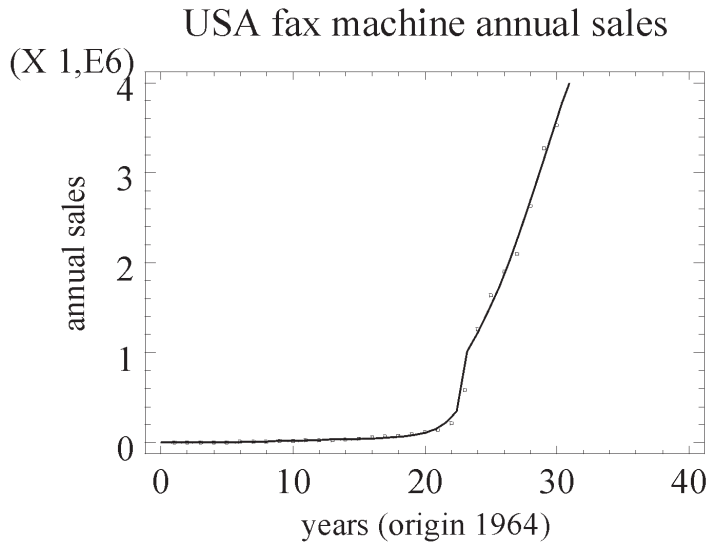
Figure 2: USA Fax Machines. Bass Model and network externalities: normal threshold with linear resistance. Observed data and predictions under Norton–Bass–like perspective.

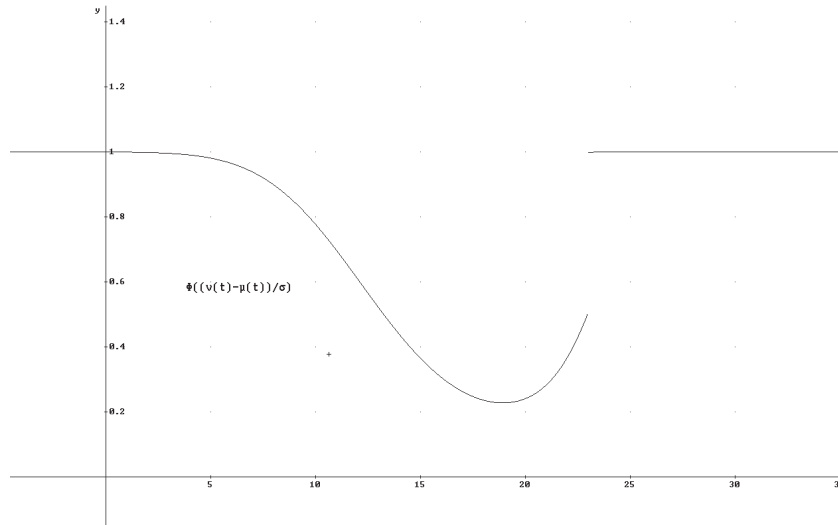## 4 USA Fax Machines: network externalities vs marketing efforts

In order to examine the performance of the new model defined by Equations (20 – 21) we consider the well–known example of USA fax machines (1965–94). The original source is CBEMA (1998) (*Information Technology Industry Data Book*). This particular series starts in the mid–1960s with a long incubation period which lasts for about twenty years, followed by a fast *take–off*. This long incubation period is quite rare, since today the majority of durables and commodities present a shortened life cycle so that, sometimes, the presence in the series of significant incubation periods preceding the take–off is not so easy to identify statistically.

A good reason for a long left tail in USA fax machine evolution is the network externalities effect balanced with high price/quality ratios. Nevertheless, we suppose that such a device was supported with a long and efficient marketing and management effort by producing firms under evolving technologies.

In the sequel, we consider two strongly different evolutionary hypotheses.

4.1 Normal Network Externalities and Norton–Bass–like Evolution

The first hypothesis assumes a Norton and Bass (1987) like perspective of USA fax machines by describing non cumulative sales as a change of regime from a lower level (zero) to an upper stationary level. The main hypothesis is that the market will reach a stable level in fax machines annual sales with an equilibrium not affected

Figure 3: USA Fax Machines. Bass Model and network externalities: normal threshold with linear resistance. Normalized dynamic market potential (Norton–Bass–like perspective).

by successive absorbing generations. The previous hypothesis may make sense in a suitable right neighborhood of observed data. We found a satisfactory performance of NLS procedure under a simple weighting function, i.e., $w(t) = 1/y(t)$ and a limited action of threshold distribution during the incubation period. We noticed a good performance of the model by assuming an extended incubation period between 1965 and 1987. Constant, power and exponential average resistance $\mu(t)$ did not exhibit an acceptable behaviour for USA fax machines under normal threshold distribution. On the contrary, the linear case is quite satisfactory. The estimation results are summarized in Table 1.

Table 1: USA fax machine diffusion. Parameters estimates of a Bass model with network externalities: normal threshold with linear resistance and weighted NLS. Norton–Bass–like perspective. ( ) indicates marginal linearised asymptotic 95% confidence limits

U	p	q	a	b	σ	R_1^2	$D - W$
6464290	0.0000773	0.28078	-0.17579	0.01409	0.051295	0.994979	2.0046
(4684260)	(0.0000410)	(0.24827)	(-0.27661)	(0.00816)	(0.022734)	SSE :	
(8244320)	(0.0001135)	(0.31330)	(-0.07496)	(0.02004)	(0.079856)	[88498]	

We note a good global fitting, $R_1^2 = 0.994979$, with a good performance in linearised asymptotic marginal confidence intervals. The graphical representations of both annual sales and Bass model under network externalities, with an assumed average linear resistance and weighted NLS, are depicted in Figure 2. We note, in particular, that the chilling effect during the incubation period is properly estimated.

Under previous results, we can examine some details of market potential evolu-
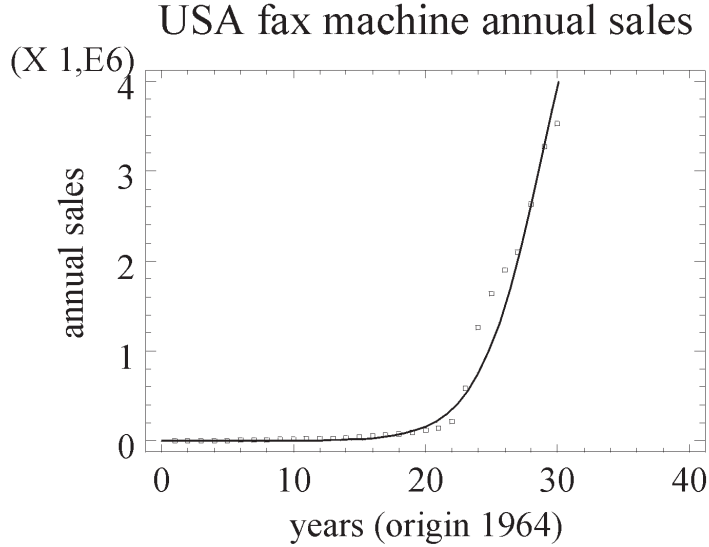
Figure 4: USA Fax Machines. Standard Bass Model, no threshold. Norton–Bass–like perspective.

tion $m(t) = U\Phi((\nu(t) - \mu(t))/\sigma)$ under a linear hypothesis about resistance $\mu(t)$.

In Figure 3 we represent, for USA fax machines, the corresponding normalised version. Such a probability depicts a non–monotone behaviour of receptiveness to innovation during the incubation period. In particular, network externalities increase resistance, with a decreasing probability, from 1965 till 1983 (1964+19) where a minimum in $\Phi(\cdot)$ is attained for $\hat{t} = 19$. Such a time $\hat{t}$ may denote the temporal maturity of "social awareness" due to a perceived *critical mass* presence of the new technology. Starting from 1983, the normalised potential, $\Phi(\cdot)$, rapidly emerges to its stationary level (complete receptiveness) in only a few years, by 1987. The critical mass evaluated at 1983 is about 550000 fax machines.

Table 2: USA fax machine diffusion. Parameters estimates of a Bass model without network externalities under weighted NLS. Norton–Bass–like perspective. ( ) indicates marginal linearised asymptotic 95% confidence limits

U	p	q	R_2^2	$D - W$
6457100	0.00000266	0.413402	0.949555	0.558276
(3032120)	(0.00000039)	(0.377258)	$SSE:$	
(9882080)	(0.00000492)	(0.449547)	[889089]	

In order to appreciate the strength of network externalities during the incubation period in USA fax machine technology, we may apply a standard Bass model, following the Norton and Bass approach, to the annual sales data 1965–1994. We avoid an improper (zero) underestimation during the incubation period by assuming

USA fax machine annual sales
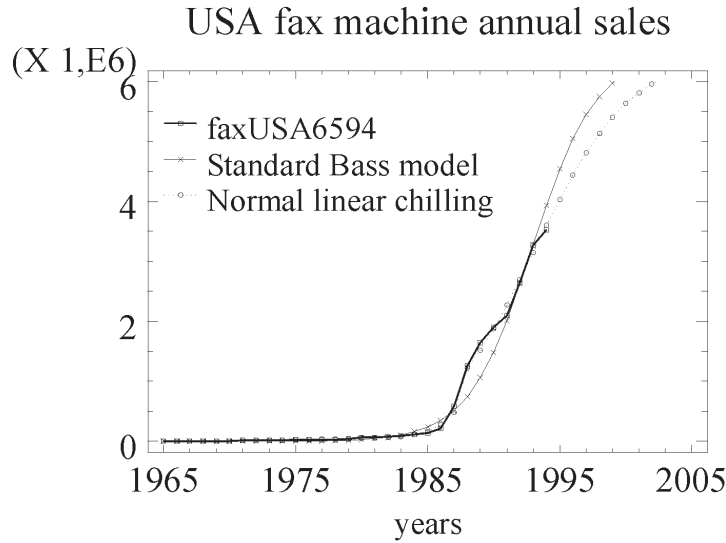
Figure 5: USA Fax Machines. Standard Bass Model vs. a Bass model evolution with dynamic normal threshold based on linear resistance. Norton–Bass–like perspective.

weighted NLS with weight $w(t) = 1/y(t)$. Table 2 summarises the obtained results.

The incubation period is partially recognised, via weighted NLS (see, in particular, Figure 4). Nevertheless, determination index $R_2^2 = 0.949555$ is very poor if compared with the previous one, and the low level of Durbin–Watson statistic must be interpreted correctly. It is not a departure from the i.i.d. error structure towards an autocorrelated one. Rather it denotes a systematic model omission. See, in particular, Figure 5, where we can appreciate the departure between the standard Bass model without network externalities and the proposed Bass model with a dynamic normal threshold based on a linear resistance.

Following previous ideas, we can evaluate whether the *chilling effect* may be properly described by a more precise function within the incubation period. In Table 3 we report a simple extension of the proposed model with a quadratic resistance, $\mu(t) = a + bt + ct^2$.

Table 3: USA fax machine diffusion. Parameters estimates of a Bass model with network externalities: normal threshold with quadratic resistance. Norton–Bass–like perspective. ( ) indicates marginal linearised asymptotic 95% confidence limits

U	p	q	a	b	c	σ	R_3^2	$D-W$
8206410	0.0001303	0.24277	-0.00244	-0.002706	0.0003612	0.02943	0.99542	2.3682
(1270020)	(-0.0000026)	(0.15036)	(-0.12241)	(-0.011905)	(0.000021)	(0.00122)	SSE :	
(15142800)	(0.00026312)	(0.33518)	(0.11752)	(0.006493)	(0.0007010)	(0.05764)	[80725]	

We observe that $R_3^2 = 0.99542$ denotes a very limited improvement with refer-
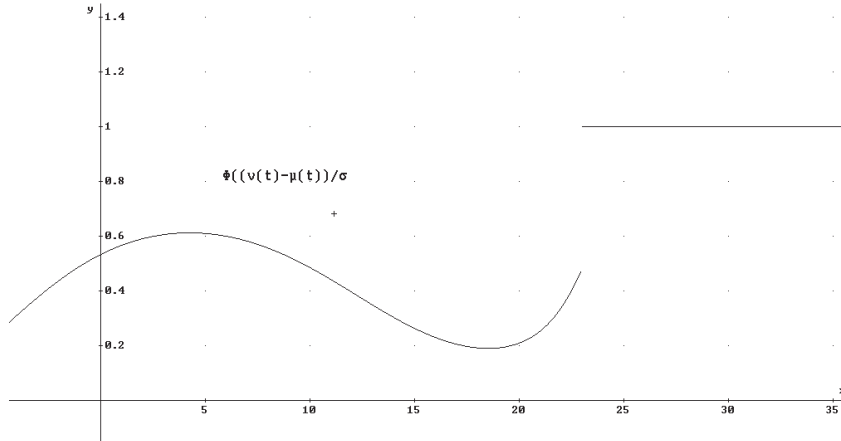
Figure 6: USA Fax Machines. Bass Model and network externalities, normal threshold probability with quadratic resistance. Norton–Bass–like perspective.

ence to the linear case and, in correspondence, marginal linearised asymptotic 95% confidence limits highlight some instabilities.

Normalised potential plotting (see Figure 6), under normal threshold probability and quadratic evolution of resistance $\mu(t)$, gives rise to a similar pattern if compared with the linear case described in Figure 3.

4.2   Normal Network Externalities and Bass–like Cumulative Evolution

An alternative way to determine USA fax machine technology dynamics is based on a different evolutionary approach following a Bass standard vision as a function of a limited life cycle. In this case annual sales are not interpreted as a change of regime from a low level to an upper stationary level, but as an evolving increasing performance from a low level to the top level suddenly followed by a decreasing behaviour. The selection between these alternatives cannot be anticipated in the very early stage of evolution without further contextual information. A good discrimination between competing models is given after peak sales attainment.

Here we examine this second hypothesis by using cumulative sales data for the estimation of the Bass model under quadratic normal network externalities within a limited (23 years) incubation period and a properly weighted NLS estimation procedure, $w(t) = 1/y(t)$.

In Table 4 we report the obtained results that are sufficiently satisfactory even if with some instabilities in marginal linearised asymptotic 95% confidence limits. Figure 7 illustrates the good performance of the model and Figure 8 represents the normalised potential over the incubation period. We note a quite similar behaviour if compared with the non–cumulative case.
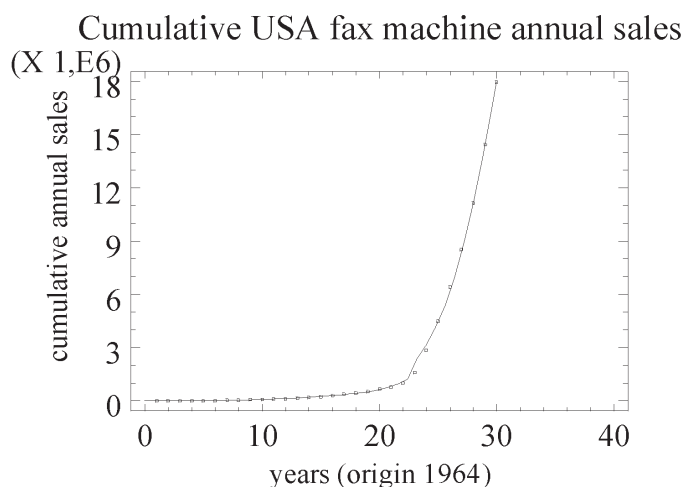
5   Final Remarks and Discussion

Our final remarks and discussion are devoted to understanding the main dif-

Table 4: USA fax machine diffusion. Parameters estimates of a Bass model with network externalities: normal threshold with quadratic resistance. Bass–like perspective. ( ) indicates marginal linearised asymptotic 95% confidence limits

U	p	q	a	b	c	σ	R^2_4	$D-W$
84926400	0.0000169	0.28017	-0.47077	0.04095	-0.000819	0.0637185	0.999158	2.1051
(76420200)	(0.0000149)	(0.27407)	(-0.47577)	(-4.46845E6)	(-1.84157E8)	(-502200)	$SSE:$	
(93432500)	(0.0000190)	(0.28628)	(-0.46576)	(4.46845E6)	(1.84157E8)	(502200)	[60819]	

Cumulative USA fax machine annual sales

Figure 7: Cumulative USA Fax Machines. Bass Model and network externalities, normal threshold with quadratic resistance. Bass–like perspective.

ferences between some well–known key papers in economics about diffusion under network externality effects and the new proposed approach.

Positive consumption externalities are classified by Katz and Shapiro (1985) into three main categories: direct, indirect, and service network. Nevertheless, there are a number of further effects that may be included: information that is more available for popular brands, market share as a signal of product quality, bandwagon effects, etc.. What matters is the dependence of a utility function upon the number (and structure) of users who are in the same network. The scope of the relevant network depends on whether the products of different brands may be used together. A small network reduces the consumer's willingness to pay, which is based, in turn, on expectations regarding the size of innovative competing networks. In other words, some demand–side economies of scale may be observed where multiple fulfilled expectations equilibria may exist due to different costs and utility functions. The basic assumptions in Katz and Shapiro (1985) are two: a) there are no income effects and b) consumers act to maximize their surplus defined as a scale dependent difference between a willingness to pay, $r + v(y^e)$, and a current price, p. Willingness to pay is the sum of consumer type, r, "uniformly" distributed within an unlimited range, and

$\Phi((v(t)-\mu(t))/\sigma$
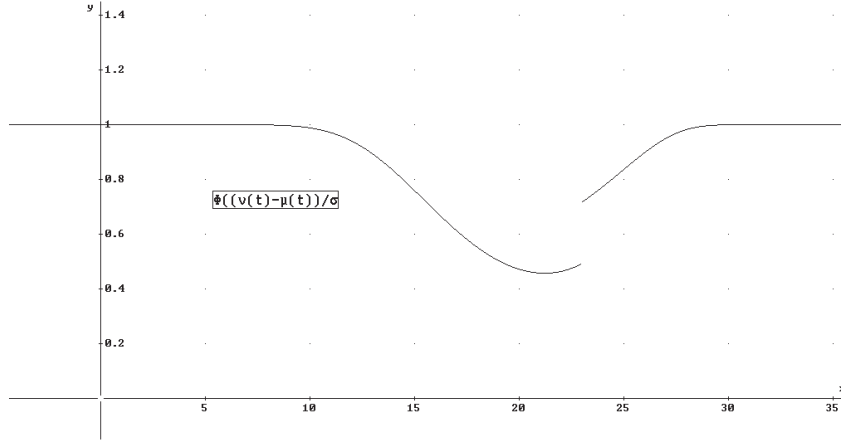
Figure 8: Cumulative USA Fax Machines. Bass Model and network externalities, normal threshold with quadratic resistance. Bass–like perspective.

a common externality effect, $v(y^e)$, dependent upon an expected network size, y^e. We observe that agents' heterogeneity is generally more complex and multidimensional in nature so that the assumptions based upon consumers' identical expectations of network sizes and consumers' basic willingness to pay, independent upon personal assessments, and uniformly distributed over relevant network members, seem quite debatable.

Cabral (1990) emphasises distributed preferences ν which are compared through a probability distribution F with observed normalised adoption decisions x at time t. In equilibrium at time t, the benefit flow satisfies equation $B(\nu, x, t) = 0$, and for the indifferent adopter's level ν the corresponding representation is $B(g(x,t), x, t) = 0$. The author defines function $H(x,t) = 1 - F(g(x,t))$, which depicts the fraction of adopters at time t that have a preference parameter larger than $g(x,t)$. A static equilibrium at time t is that x for which $x = H(x,t)$. The author obtains $x = 1 - F(g(x,t))$ or equivalently $F^{-1}(1-x) = g(x,t)$. For fixed x we get $y = \tilde{g}(t)$. If $\tilde{g}(\cdot)$ is a function, we have $\tilde{g}^{-1}(y) = t$, i.e., $h(x) = t$. The converse is not generally true, so that the correspondence $x = H(x,t) = \phi(t)$ is not always a function and gives rise to multiple equilibria. The graph of $x = \phi(t)$ is a smooth one–dimensional manifold with singular points. The connection of such a theory with smooth s–shaped diffusion models describing network externality effects is not so immediate.

In Grajek (2002) the approach is similar to the corresponding one by Cabral (1990). Instead of a benefit (net benefit flow) that incorporates prices, Grajek considers a utility function $u(\nu, x_i(t-\delta))$ with preference ν for brand i and $x_i(t-\delta)$ a normalised extension of network at time $t - \delta$. Indifferent consumer ν at time t satisfies equation $u(\nu^*_{i,t}, x_i(t-\delta)) = p_i(t)$, where $p_i(t)$ is brand i price. Assuming no multiplicity, he defines $H_i(\nu^*)$ as the number of consumers willing to buy brand i within time t, $H_i(\nu^*) = 1 - F(\nu^*)$, and F is the distribution function of a uniform random variable. The author then equates the proposed theoretical model with sales: $y_i(t) = H_i(\nu^*)$. Further specifications refer to the polynomial definition of the utility function, $u(\nu, x_i(t-\delta)) = a\nu + bx_i(t-\delta)) + cx_i^2(t-\delta)$. Within this

logic there is no control over saturating effects due to the limited extension of the market potential, and there is no memory of previous decisions. In its current specification, externality effects do not modify market potential over time. The final equation governing cumulative adoptions is assumed to be a polynomial regression: $y_{i,t} = \alpha + \beta p_{i,t} + \gamma_1 x_{i,t-1} + \gamma_2 x_{i,t-1}^2 + \varepsilon_{i,t}$. Notice that $y_{i,t}$ are simply the cumulative sales of brand i, while $x_{i,t-1}$ denotes the global relevant network at time $t-1$ for brand i. If brands are incompatible, then we have that $x_{i,t-1} = y_{i,t-1}$.

The paper by Moldovan and Goldenberg (2004) is designed to uncover the influence of resistance of leaders that primarily inhibits the diffusion process and may reduce market potentials. There are some similarities between resistance to innovations and negative word–of–mouth. Resistance to change often occurs because new technology is unfamiliar or complex. But complexity may involve the limited diffusion and corresponding externality effects in network goods. Dissatisfaction may arise from an expected inadequate performance of the network over time and space and from a perceived high price. "Neutrality" to innovation and possible negative word–of–mouth are common responses to the early diffusion of network goods.

The starting point of this paper is that network externalities depress sales in the first part of a network good's life cycle. We have captured this effect with a multiplicative model simultaneously describing a dynamic market potential and the corresponding adoption process.

We propose a dynamic market potential $m(t)$ which depends on a heterogeneous individual threshold derived from a simple reinterpretation of Economides and Himmelberg's consumer index h in willingness to pay definition. This dynamic market potential is embedded in a Cellular Automata model which takes into account threshold effects and innovative and imitative forces in adoptions under exogenous interventions (marketing mix strategies). The heterogeneity of the local transition rules effect is then simplified with a *mean field approximation* that allows a reduction to a solvable Riccati equation.

Under the normality assumption on threshold customer index distribution, the corresponding dynamic market potential exhibits, in a successful case (USA fax machines), a joint representation of an incubation period followed by a subsequent take–off, due to a sufficient critical mass. Probably, a failure in network goods diffusion is characterised by a non–reversing monotony of market potential after a long period of incubation so that a critical mass is not attained.

Our dynamic market potential is useful for managerial purposes. It allows the identification of two important contiguous periods: the first one with a zero second derivative, which depicts the starting point of an "awareness process" (see, for example, Figure 3, point $x = t = 12$), and the second one with a zero first derivative, which establishes the maturity of the awareness process, i.e., the reversion in market potential dynamics (see, for example, Figure 3, point $x = t = 19$).

In the USA fax machine case, we have examined a different distributive approach, not reported here, by implementing the beta family in market potential definition. In spite of a possible *a priori* more flexible behaviour, the applied results do not confirm such a hypothesis. Our results, were more efficient with a normal assumption under a polynomial evolution of $\mu(t)$.

The proposed dynamic market potential cannot be estimated accurately at a very early stage of diffusion when there are no symptoms, in the data, of an actual take–off. This aspect is not a special feature of the proposed model, but is a common character of models with change-point events between different regimes.

Moreover, we highlight that the assumed evolution of the adoption process of interest, following a Bass–like or a Norton–Bass–like perspective, implies different theoretical and pragmatical choices that are external to the involved incubation effect of network externalities and certainly not easy to determine in the early stages.

Appendix A

A Riccati Equation

Let us consider the following special non–autonomous Riccati equation in (X,Y) real space

$$y'_x = a\frac{f(x)}{g(x)}y^2 + \left(bf(x) + \frac{g'(x)}{g(x)}\right)y + cf(x)g(x), \tag{22}$$

where $a, b, c \in R$, $D = \sqrt{b^2 - 4ac} > 0$ and $g(x) \neq 0$, $f(x)$ are real functions. Its general discussion may be found in Guseo and Guidolin (2009). Here we report the final results concerning its closed form solution.

Let us consider the real roots of equation $az^2 + bz + c = 0$, i.e., $r_i = (-b \pm D)/2a \in R$, $i = 1, 2$, where $D = a(r_2 - r_1) = \sqrt{b^2 - 4ac} > 0$.

The general solution of Equation (22) is,

$$y(x) = g(x)\frac{r_1 r_2(1 - G(x)) - C(r_1 - r_2 G(x))}{r_2 - r_1 G(x) - C(1 - G(x))}, \tag{23}$$

where $G(x) = e^{D\int_0^x f(\tau)d\tau}$, and C is an arbitrary constant of integration.

If the initial condition is set to zero, $y(0) = 0$, we obtain $C = 0$ and, therefore,

$$y(x) = g(x)\frac{1 - e^{-D\int_0^x f(\tau)d\tau}}{\frac{1}{r_2} - \frac{1}{r_1}e^{-D\int_0^x f(\tau)d\tau}}. \tag{24}$$

If $\lim_{x\to\infty}\int_0^x f(\tau)d\tau = +\infty$, we obtain an interesting limiting behaviour of $y(x)$, i.e., $\lim_{x\to\infty} y(x) = r_2 \lim_{x\to\infty} g(x)$.

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