Working Paper Series, N. 9, May 2009

Department of Statistical Sciences
University of Padua
Italy

Simultaneous and Two-step Reconciliation of Systems of Time Series

Tommaso Di Fonzo
Department of Statistical Sciences
University of Padua
Italy

Marco Marini
Directorate of National Accounts
ISTAT
Italy

Abstract: The reconciliation of systems of time series subject to both temporal and contemporaneous constraints can be solved in such a way that the temporal profiles of the original series be preserved "at the best" (*movement preservation principle*). Thanks to the sparsity of the linear system to be solved, a feasible procedure can be developed to solve simultaneously the problem. A two-step strategy might be more suitable in the case of large systems: firstly, each series is aligned to the corresponding temporal constraints according to a movement preservation principle; secondly, all series are reconciled within each low-frequency period according to the given constraints. This work compares the results of simultaneous and two-step approaches for medium/large datasets from real-life and discusses conditions under which the two-step procedure can be a valid alternative to the simultaneous one.

Keywords: Reconciliation, Movement preservation.

UNIVERSITÀ DEGLI STUDI DI PADOVA
DIPARTIMENTO DI SCIENZE STATISTICHE

This version (2009-05-29)

Contents

Department of Statistical Sciences
Via Cesare Battisti, 241
35121 Padova
Italy

tel: +39 049 8274168
fax: +39 049 8274170
http://www.stat.unipd.it

Corresponding author:
Tommaso Di Fonzo
tel: +39 049 827 4158
difonzo@stat.unipd.it
http://www.stat.unipd.it/~difonzo

Simultaneous and Two-step Reconciliation of Systems of Time Series

Tommaso Di Fonzo
Department of Statistical Sciences
University of Padua
Italy

Marco Marini
Directorate of National Accounts
ISTAT
Italy

Abstract: The reconciliation of systems of time series subject to both temporal and contemporaneous constraints can be solved in such a way that the temporal profiles of the original series be preserved "at the best" (*movement preservation principle*). Thanks to the sparsity of the linear system to be solved, a feasible procedure can be developed to solve simultaneously the problem. A two-step strategy might be more suitable in the case of large systems: firstly, each series is aligned to the corresponding temporal constraints according to a movement preservation principle; secondly, all series are reconciled within each low-frequency period according to the given constraints. This work compares the results of simultaneous and two-step approaches for medium/large datasets from real-life and discusses conditions under which the two-step procedure can be a valid alternative to the simultaneous one.

Keywords: Reconciliation, Movement preservation.

1 Introduction

Economic statistics are often linked by a system of accounting relationships. Some linear restrictions originate from the economic theory (i.e. GDP as balance of the uses and resources account), while others are related to the level of disaggregation at which such statistics are compiled (i.e. value added for the total economy is the sum of value added of the 17 sections of the NACE classification).

However, the given constraints are rarely met by the observed variables: this happens, for example, because economic data are frequently collected by different methods, using different sample surveys or different pieces of measuring equipment. Then, data sets of real-world variables generally show discrepancies with respect to prior restrictions on their values. Such discrepancies are rarely accepted because they usually cause "confusion among users and criticism or embarrassment to the publishers" (Quenneville and Rancourt, 2005). The adjustment of a set of data in order to satisfy a number of accounting restrictions - and thus to remove any discrepancy - is generally known as *balancing* or *reconciliation* (Dagum and Cholette, 2006).

Data reconciliation problems are frequently faced by National Statistical Institutes (NSI) in the production of official statistics. NSI are often obliged to publish consistent sets of time series to fulfill legal regulations or common practices on statistics set by international institutions (UN, IMF, Eurostat, etc.). A typical example is the compilation process of National Accounts (NA). The updating and balancing of input-output or social and economic accounts matrices (Aceituno Puga, 2008), or the benchmarking of Quarterly National Accounts (QNA) series to annual aggregates (Bikker and Buijtenhek, 2006; Daalmans and Mushkudiani, 2009; other references in Di Fonzo and Marini, 2005a) are nowadays common practices followed by statistical agencies. In addition, reconciliation procedures may be used to restore additivity in systems of directly seasonally adjusted (SA) component series (e.g., either one or two-way classified), which are wished to be in line with both the SA marginal aggregates and the grand-total series (Cholette, 1988; Taillon, 1988; Di Fonzo and Marini, 2005b; Quenneville and Rancourt, 2005; Dagum and Cholette, 2006; Quenneville and Fortier, 2006).

It can be said, in summary, that many published economic statistics, set in the form of tables spanning on one or several time periods, have passed through a reconciliation process.

The accounting restrictions can be of two types: the *contemporaneous* constraints, assuming the form of linear combinations of the variables which should be fulfilled in every observed period, and the *temporal* aggregation constraints, which require that the high-frequency adjusted series be in line with known (e.g., more reliable) low-frequency aggregates (say, the annual series of the variables of interest). The former set is problem dependent, inasmuch as the set of linear restrictions is directly identified from the system of accounts to be reconciled. Instead, the restrictions in the time dimension are generally limited to few cases, being related to the nature of the data: preliminary high-frequency variables are usually adjusted such that linear combinations[1] of their values equal known low-frequency series.

The statistical procedures to restore consistencies between variables and within variables are very similar, but the two problems are often treated separately in literature. The former type of reconciliation is generally known as the balancing problem, while the process of adjustment in the time dimension is called benchmarking (or temporal disaggregation) of time series (for references, see Di Fonzo, 2003).

It is common opinion that the main difficulty of solving a reconciliation problem with both types of constraints "once for all", i.e. by simultaneously considering all the time periods covered by the series and the whole set of aggregation constraints, is related to the features (dimensions and rank) of the matrices involved in the calculation (Chen and Dagum, 1997; Di Fonzo and Marini, 2005b). In fact, as we shall see later, when the number of variables, the length of series and/or the number of constraints increase, the required memory space becomes huge and the computational burden may be significant. Recently, however, an interesting two-

[1] Appropriately defined either for flow or stock variables.

step reconciliation strategy has been proposed (Quenneville and Rancourt, 2005; see Dagum and Cholette, 2006, for a generalization), which "splits" the reconciliation in two steps: in the former the preliminary series are benchmarked to their known temporal aggregates in such a way as their temporal profiles remain untouched as much as possible, in the latter the fulfilment of the contemporaneous constraints is performed for each low-frequency time period (e.g., year-by-year) according to a well-established statistical procedure[2] of estimation of the cells of a two-way classified table given the marginal totals (Deming and Stephan, 1940; Friedlander, 1961).

Besides the advantage of making simpler the reconciliation problem, by reducing it in smaller 'pieces', it has been stressed (see, for example, Quenneville and Rancourt, 2005, and Dagum and Cholette, 2006) that following a two-step approach in order to reconcile a system of time series[3] there is no need to 'preserve the movement' (e.g., month-to-month growth rates) in the second step, because this is done in the first step. We think that this is an important point, which merits to be discussed and verified in practice.

In this paper we present both simultaneous and two-step reconciliation procedures. A simultaneous procedure is developed according to a quadratic constrained minimization approach (Di Fonzo and Marini, 2005a), discussing both the features of the linear system from which the solution can be recovered, and feasible ways for solving it. Then we present the two-step procedure worked out by Quenneville and Rancourt (2005), and discuss two sensible alternatives for the second step of the procedure. In any case, we look at all procedures from both a technical (mathematical) and practical (feasibility) point of view. In fact, working with time series of accounting relationships, at least three issues must be considered when a table of preliminarly estimated time series must be reconciled in order to be in line with pre-specified accounting constraints: (i) the dimension of the problem, which is basically proportional to the number of variables involved in the reconciliation process and to the length of the time span considered, (ii) the number and the nature of the constraints imposed to the series, and (iii) the preservation at the best of the original dynamic profile (in time) of the preliminary series.

A further, not minor issue should be the capability of the procedure of taking into account reliability measures for the variables, so that 'good' preliminary estimates are touched less than 'bad' ones. This can be done - in the same quadratic constrained minimization framework - by means of a straightforward extension of the least squares adjustment procedure by Stone *et al.* (1942), the main problem rather being how to recover sensible reliability indicators. Even if this is not the main focus of the paper, we do believe that this is a major point to be considered when one wishes to adjust a system of time series.

The paper is organized as follows. In section 2 we describe the linear constraints usually faced in a reconciliation problem, and some ways of writing them in matrix form, which is very useful in order to provide a general formulation of the reconciliation problem when temporal and contemporaneous aggregation constraints are

[2] Often referred to as *raking* (Fagan and Greenberg, 1988; Evans, 2004).

[3] As far as we know, statistical reconciliation according to a two-step approach has been used to restore additivity in tables of SA time series and of time series recalculated according to a new classification scheme.

jointly considered. In section 3 we discuss the movement preservation principle on which the reconciliation procedures we consider are grounded, and a multivariate extension of the modified Denton *Proportional First Differences* (PFD) benchmarking procedure is presented. Then, three two-step reconciliation procedures sharing the same first step are presented in section 4, while section 5 presents summary indices to assess the quality of the reconciled estimates (also) in terms of capability of preservation of the movement of the preliminary series. In section 6 the proposed reconciliation procedures are applied to two real life datasets. First a medium-sized problem is considered, namely the reconciliation of the European Union Quarterly Sector Accounts (EU-QSA), where a system of 183 quarterly series has to be reconciled with respect to the annual sector accounts and 30 accounting relationships along a 7 years time span. Then we consider a large-sized reconciliation problem, coming from the Canadian Monthly Retail Trade Survey (MRTS), where the component SA monthly time series, classified by 13 regions (provinces and territories) and 19 trading groups, are reconciled according to the marginal totals of both classifications along a 13 years time span. Finally, section 7 offers some conclusive remarks.

2 Aggregation constraints in a system of time series

In several practical situations a system of sub-annual time series (say, monthly or quarterly) may be required to be coherent with known aggregated information (e.g., annual and/or marginal totals). In practice, we wish to estimate M unknown $(n \times 1)$ vectors of high-frequency (say, monthly or quarterly) data, $\mathbf{Y}_j$, $j = 1, \ldots, M$, each pertaining to M basic (i.e., component, disaggregate) variables Y_j, which have to satisfy both known contemporaneous and temporal aggregation constraints. The available information to be exploited is given by M high-frequency preliminary series, M temporally aggregated (say, annual) series and possibly a number of high-frequency contemporaneously aggregated series.

In this section we consider a matrix formulation of the constraints, which may reveal itself useful to deal with reconciliation procedures of systems of time series.

2.1 Contemporaneous aggregation constraints

As regards the contemporaneous constraints of the system, let $\mathbf{G}$ be a $(k \times M)$ matrix of known constants (usually 0, 1 and -1) defining the (contemporaneous) accounting relationships between $\mathbf{Y}_j$, k being the number of linear relationships to be fulfilled. Let $\mathbf{Z}_h$, $h = 1, \ldots, k$, be the $(n \times 1)$ vectors of high-frequency known quantities associated to the k accounting constraints in $\mathbf{G}$. Denoting by $\mathbf{Y} = [\mathbf{Y}_1' \mathbf{Y}_2' \ldots \mathbf{Y}_M']'$ the $(Mn \times 1)$ vector of all the unknown component series, the contemporaneous aggregation constraints can be written in compact form as[4]

$$(\mathbf{G} \otimes \mathbf{I}_n) \mathbf{Y} = \mathbf{Z} \tag{1}$$

[4] When $\mathbf{G}$ is a $(1 \times M)$ row vector of constant values we have a single contemporaneous constraint. In addition, $\mathbf{G} = \mathbf{1}_M'$ when a simple summation constraint links the variables, $\mathbf{1}_M$ being a $(M \times 1)$ vector of one.

where $\otimes$ is the Kronecker product and $\mathbf{Z} = [\mathbf{Z}_1'\mathbf{Z}_2' \dots \mathbf{Z}_k']'$ has dimension $(kn \times 1)$. We try to clarify how matrix $\mathbf{G}$ may be designed by some examples.

A (very) simple system of National Accounts variables. Consider the stylized accounting relationship defining the GDP from the expenditure side, $y = c+i+g+x-m$, where y, c, i, g, x and m denote GDP, private consumption, investment, government expenditures, exports, and imports, respectively. Consider also a (even more stylized) definition of GDP from the production side (excluding VAT): $y = v - n$, where v and n denote total production and intermediate consumption, respectively. If we define the (8×1) vector $Y = (y \quad c \quad i \quad g \quad x \quad m \quad v \quad n)'$, we have $k = 2$, $\mathbf{G} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$, and $Z = [0 \quad 0]'$.

One-way classified systems of time series (Dagum and Cholette, 2006, chapter 12). Consider M component variables, Y_j, $j = 1, \dots, M$, linked by a summation relationship to a grand-total series. We can write $\sum_{j=1}^{M} Y_j = Z$ if the grand-total series is given (i.e., it is exogenous). In this case $k = 1$ and $\mathbf{G} = \mathbf{1}_M'$. If also the grand-total has to be reconciled (i.e., it is endogenous), we have $\sum_{j=1}^{M} Y_j - Y_{M+1} = 0$, where Y_{M+1} denotes the grand-total, $Z \equiv 0$ and $\mathbf{G} = [\mathbf{1}_M' \quad -1]$.

Marginal totals of two-way classified systems of time series (Dagum and Cholette, 2006, chapter 13). Let $Y_{i.}$, $i = 1, \dots, R$, and $Y_{.j}$, $j = 1, \dots, C$, respectively, R and C marginal totals series deriving from a two-way classified system of variables (table 1). The cases of interest are basically two: either the reconciled grand-total series is implicitly (indirectly) derived by summation of the reconciled marginal totals, or the grand-total series has to be reconciled along with the $M = R + C$ marginal totals. In the former case the contemporaneous constraint can be expressed as $\sum_{i=1}^{R} Y_{i.} - \sum_{j=1}^{C} Y_{.j} = 0$, so $k = 1$ and $\mathbf{G} = [\mathbf{1}_R' \quad -\mathbf{1}_C']$. In the latter case we have $\sum_{i=1}^{R} Y_{i.} - Y_{..} = 0$ and $\sum_{j=1}^{C} Y_{.j} - Y_{..} = 0$. Thus, $k = 2$, $\mathbf{G} = \begin{bmatrix} \mathbf{1}_R' & \mathbf{0} & -1 \\ \mathbf{0} & \mathbf{1}_C' & -1 \end{bmatrix}$ and $Z = [0 \quad 0]'$.

Two-way classified systems of time series (Dagum and Cholette, 2006, chapter 14). Let Y_{ij}, $i = 1, \dots, R$, $j = 1, \dots, C$ be the $R \cdot C$ component series of the system. There are R constraints linking the component series to the 'row' marginal totals $Y_{i.}$,

$$\sum_{j=1}^{C} Y_{ij} = Y_{i.}, \quad i = 1, \dots, R, \tag{2}$$

and C constraints linking the component series to the 'column' marginal totals $Y_{.j}$,

$$\sum_{i=1}^{R} Y_{ij} = Y_{.j}, \quad j = 1, \dots, C. \tag{3}$$

Table 1: A two-way classified system of variables.

	1	$\cdots$	j	$\cdots$	C	
1	Y_{11}	$\cdots$	Y_{1j}	$\cdots$	Y_{1C}	$Y_{1.}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$
i	Y_{i1}	$\cdots$	Y_{ij}	$\cdots$	Y_{iC}	$Y_{i.}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$
R	Y_{R1}	$\cdots$	Y_{Rj}	$\cdots$	Y_{RC}	$Y_{R.}$
	$Y_{.1}$	$\cdots$	$Y_{.j}$	$\cdots$	$Y_{.C}$	$Y_{..}$

Depending on the 'hierarchy' of the reconciliation (Dagum and Cholette, 2006), several different situations can be considered (see also Di Fonzo and Marini, 2005b). For example, if we assume that the marginal totals according to both classifications are exogenously given (and thus the grand-total series too is exogenous), the constraints (2) and (3) can be expressed by considering the $((R \cdot C) \times 1)$ vector $Y = (Y_{11} \dots Y_{R1} \dots Y_{1C} \dots Y_{RC})'$, the $((R{+}C){\times}1)$ vector $Z = (Y_{1.} \dots Y_{R.} Y_{.1} \dots Y_{.C})'$, $k = R + C$ and the $(k \times (R \cdot C))$ contemporaneous aggregation matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{1}'_C \otimes \mathbf{I}_R \\ \mathbf{I}_C \otimes \mathbf{1}'_R \end{bmatrix}.$$

2.2 Temporal aggregation and the complete set of constraints

We assume that low-frequency counterparts of the vectors $\mathbf{Y}_j$, denoted by $\mathbf{Y}_{0j}$, are available. Each $\mathbf{Y}_{0j}$ can be viewed as a non overlapping $(N \times 1)$ linear combination of $\mathbf{Y}_j$, with coefficients given by the $(s \times 1)$ vector $\mathbf{c}$, s being the temporal aggregation order[5]. Thus, we define the $(N \times n)$ aggregation matrix converting high-frequency into low-frequency series as $\mathbf{C} = \left[\mathbf{I}_N \otimes \mathbf{c}' \vdots \mathbf{0}\right]$, where $\mathbf{0}$ is a null $(N \times (n - Ns))$ matrix which permits to deal, for example, with high-frequency preliminary series pertaining to the unknown most recent (e.g., current) low-frequency period[6]. The temporal constraints linking the high-frequency component series to their temporal aggregated counterparts can be expressed as $\mathbf{C}\mathbf{Y}_j = \mathbf{Y}_{0j}$, $j = 1, \dots, M$, that is

$$(\mathbf{I}_M \otimes \mathbf{C})\,\mathbf{Y} = \mathbf{Y}_0, \tag{4}$$

where $\mathbf{Y}_0 = [\mathbf{Y}'_{01}\mathbf{Y}'_{02} \dots \mathbf{Y}'_{0M}]'$.

[5] $s = 3$ for monthly-to-quarterly aggregation, $s = 4$ for quarterly-to-annual and $s = 12$ for monthly-to-annual. In addition, we assume $n \geq Ns$.

[6] For simplicity, here and in the rest of the paper we are assuming that all the series (i) cover the same time span, (ii) start on the first high-frequency period of the first low-frequency period, and (iii) there are no 'holes' either in the preliminary or in the known aggregates to be filled within the considered time span. The generalization of the results to such situations involves more sophisticated and cumbersome notation and algebra (see, for example, Dagum and Cholette, 2006).

Let $\mathbf{H}$ be the $((kn + NM) \times nM)$ aggregation matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{G} \otimes \mathbf{I}_n \\ \mathbf{I}_M \otimes \mathbf{C} \end{bmatrix}$$

and $\mathbf{Y}_a = [\mathbf{Z}'\mathbf{Y}_0']'$ the $((kn + NM) \times 1)$ vector containing both contemporaneous and temporal aggregates. The complete set of constraints between the unknown high-frequency component series and the available aggregated information can be expressed in compact form as

$$\mathbf{HY} = \mathbf{Y}_a. \tag{5}$$

Notice that the contemporaneous aggregation of temporally aggregated series implies $(\mathbf{G}_h \otimes \mathbf{I}_N)\,\mathbf{Y}_0 = \mathbf{C}\mathbf{Z}_h$, $h = 1, \ldots k$, where $\mathbf{G}_h$ is the h-th $(1 \times M)$ row-vector of matrix $\mathbf{G}$. Considering the whole set of constraints, we have

$$(\mathbf{G} \otimes \mathbf{I}_N)\,\mathbf{Y}_0 = (\mathbf{I}_k \otimes \mathbf{C})\,\mathbf{Z}. \tag{6}$$

Relationship (6) reflects the fact that the exogenous information have to be consistent with the system constraints. Thus, the low-frequency component series, when 'longitudinally' aggregated through matrix $\mathbf{G}$, must be equal to the series obtained by temporal aggregation of the high-frequency series in $\mathbf{Z}$. In other words, we are assuming that $\mathbf{Y}_0$ and $\mathbf{Z}$ fulfill, respectively, all contemporaneous and temporal aggregation constraints[7]. This point must be stressed, because it is a strong pre-requisite in order the reconciliation procedure may work.

Hereafter we denote by $\mathbf{P}_j$ the $(n \times 1)$ vectors of preliminary series to be adjusted, by $\mathbf{R}_j$ the corresponding $(n \times 1)$ estimated data after the reconciliation process, $j = 1, \ldots, M$, and by $\mathbf{P}$ and $\mathbf{R}$ the $(Mn \times 1)$ vectors $\mathbf{P} = [\mathbf{P}_1'\mathbf{P}_2' \ldots \mathbf{P}_M']'$ and $\mathbf{R} = [\mathbf{R}_1'\mathbf{R}_2' \ldots \mathbf{R}_M']'$, respectively. Provided that $\mathbf{HP} \neq \mathbf{Y}_a$, we look for reconciled estimates of the high-frequency component series for which, while the temporal profile of the original preliminary series is preserved "at the best" (*movement preservation principle*), the same linear relationships valid for the M unknown high-frequency component series must hold, $(\mathbf{G} \otimes \mathbf{I}_n)\,\mathbf{R} = \mathbf{Z}$ and $(\mathbf{I}_M \otimes \mathbf{C})\,\mathbf{R} = \mathbf{Y}_0$:

$$\mathbf{HR} = \mathbf{Y}_a. \tag{7}$$

3 Movement preservation principle and simultaneous reconciliation

Let us start by considering the reconciliation of a single preliminary series with respect to its low-frequency counterpart. In this case we have a classical *benchmarking* problem (Denton, 1971, Cholette, 1984), no contemporaneous constraint being in order. A widely used solution to this problem is given by the modified Denton PFD procedure worked out by Cholette (1984)[8].

[7]For example, for a one-way classified table of time series linked by an additivity constraint, relationship (6) simply becomes $\sum_{j=1}^{M} Y_{0j,T} = \sum_{t=(T-1)s+1}^{Ts} Z_t$, $T = 1, \ldots, N$.

[8]For generalizations and extensions of Denton's approach to benchmarking, see Cholette and Dagum (1994), and Quenneville *et al.* (2004).

This procedure performs the constrained minimization of an objective function which is generally seen as a good approximation[9] of the 'ideal' (Bloem *et al.*, 2001) movement preservation objective function. For, a rather natural measure of the movement preservation is founded on the distance – suitably defined – between the rates of change of the preliminary and target series. Denoting by $\mathbf{B} = \{B_t\}_{t=1}^n$ the $(n \times 1)$ benchmarked values, and taking the squared differences, one can consider the following objective function (Helfand *et al.*, 1977):

$$F^{MPP} = \sum_{t=2}^{n} \left(\frac{B_t - B_{t-1}}{B_{t-1}} - \frac{P_t - P_{t-1}}{P_{t-1}} \right)^2 . \tag{8}$$

Unfortunately, the constrained optimization of expression (8) has not an explicit solution, and thus requires the use of numerical optimization techniques in order to find the benchmarked figures (Bozik and Otto, 1988). The modified Denton PFD procedure (Cholette, 1984) considers instead an objective function according to which the proportionate period-to-period changes of the benchmarked series should be as close as possible to those of the preliminary figures:

$$F^{MD} = \sum_{t=2}^{n} \left(\frac{B_t - P_t}{P_t} - \frac{B_{t-1} - P_{t-1}}{P_{t-1}} \right)^2 . \tag{9}$$

In this case the solution to the constrained optimization problem can be explicitly expressed by means of standard linear algebra operations.

3.1 The modified Denton PFD procedure

An extension of the modified Denton's criterion (9) to a system of $M > 1$ time series[10] is the function:

$$F^{MMD} = \sum_{j=1}^{M} \sum_{t=2}^{n} \left(\frac{R_{j,t} - P_{j,t}}{P_{j,t}} - \frac{R_{j,t-1} - P_{j,t-1}}{P_{j,t-1}} \right)^2 , \tag{10}$$

where we use symbol R to denote the reconciled series, which are wished to fulfill the constraint (7).

In matrix notation, the simultaneously reconciled series can be obtained by solving the following quadratic constrained minimization problem:

$$\min_{\mathbf{R}} (\mathbf{R} - \mathbf{P})' \boldsymbol{\Omega} (\mathbf{R} - \mathbf{P}) \quad \text{subject to} \quad \mathbf{H}\mathbf{R} = \mathbf{Y}_a \tag{11}$$

where $\boldsymbol{\Omega} = \hat{\mathbf{P}}^{-1} (\mathbf{I}_M \otimes \boldsymbol{\Delta}_n' \boldsymbol{\Delta}_n) \hat{\mathbf{P}}^{-1}$, $\hat{\mathbf{P}} = \text{diag}(\mathbf{P})$ and $\boldsymbol{\Delta}_n$ is the $((n-1) \times n)$ first

[9] See Cholette (1987, p. 23), for a justification of this "approximation" assumption, and U.S. Census Bureau (2007, pp. 100–101) for a critical assessment of it.

[10] A reconciliation procedure according to the movement preservation principle (8) extended to a system of time series *via* a suitable log–transformation, has been discussed in Di Fonzo and Marini (2005b).

differences matrix[11]:

$$\mathbf{\Delta}_n = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

The first order minimization conditions are

$$\begin{bmatrix} \mathbf{\Omega} & \mathbf{H}' \\ \mathbf{H} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{\Omega P} \\ \mathbf{Y}_a \end{bmatrix} \tag{12}$$

where λ is a $((kn + Mn) \times 1)$ vector of Lagrange multipliers. In addition, given that $\hat{\mathbf{P}}^{-1}\mathbf{P} = \mathbf{1}_{Mn} = \mathbf{1}_M \otimes \mathbf{1}_n$, and $\mathbf{\Delta}_n \mathbf{1}_n = \mathbf{0}$, it is $\mathbf{\Omega P} = \mathbf{0}$, so the rhs of system (12) simplifies to $[\mathbf{0}'\mathbf{Y}_a']'$. In summary, the reconciled estimates are (part of) the solution of the linear system

$$\mathbf{Ax} = \mathbf{b} \tag{13}$$

where $\mathbf{A} = \begin{bmatrix} \mathbf{\Omega} & \mathbf{H}' \\ \mathbf{H} & \mathbf{0} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \mathbf{R} \\ \lambda \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \mathbf{0} \\ \mathbf{Y}_a \end{bmatrix}$.

3.2 Direct solution of system (13)

The square coefficient matrix $\mathbf{A}$ of the linear system (13) has some noticeable characteristics[12]. It is symmetric, indefinite, and, in most practical situations, large and sparse. In addition, matrix $\mathbf{A}$ is singular, that is it cannot be directly inverted, and usually its dimensions make the adoption of the Moore-Penrose generalized inverse, which gives a formal solution of the system, generally unfeasible on a practical ground.

Table 2 shows the dimensions (number of rows) of the coefficient matrix $\mathbf{A}$ of the linear system (13) for both $s = 4$ (quarterly series) and $s = 12$ (monthly series), and growing values of N (number of low-frequency time periods) and M (number of component series to be reconciled). Table 3 reports instead the *sparsity* of such a matrix, that is the ratio between the number of non-zero entries and the total number of its cells, when a simultaneous modified Denton PFD reconciliation procedure is applied[13].

From table 2 it clearly appears that the dimensions involved in calculations dramatically increase as M and/or n grow up. For example, if we wish to reconcile according to the simultaneous modified Denton PFD procedure a system of 250 monthly time series over 15 years while respecting 30 contemporaneous accounting

[11] In agreement with Cholette and Dagum (2006, p. 139), it is generally advisable to avoid using the approximated (not singular) $(n \times n)$ first differences matrix originally proposed by Denton (1971). This holds mostly when the time series in the system do not start in the same period.

[12] For a thorough analysis of the 'fundamental bordered matrix of linear estimation', as a matrix like $\mathbf{A}$ is known in the literature, see Magnus (1990).

[13] For simplicity, we assume $n = Ns$. Notice that the values reported in table 2 are calculated by assuming the maximum number of possible non-zero entries in $\mathbf{A}$, which means we consider (i) that no entry of vector $\mathbf{c}$ is zero (e.g., no interpolation of stock variables), and (ii) that all the kM entries of matrix $\mathbf{G}$ are non-zero, which is a very safe assumption (see the examples in section 2.1).

Table 2: Dimensions of the linear system (13) for different number of variables (M), low-frequency time periods (N), temporal aggregation order (s), and contemporaneous constraints (k).

	N				N			
M	1	5	10	15	1	5	10	15
	$s = 4,\ k = 1$				$s = 12,\ k = 1$			
10	54	270	540	810	142	710	1420	2130
50	254	1270	2540	3810	662	3310	6620	9930
100	504	2520	5040	7560	1312	6560	13120	19680
200	1004	5020	10040	15060	2612	13060	26120	39180
250	1254	6270	12540	18810	3262	16310	32620	48930
	$s = 4,\ k = 30$				$s = 12,\ k = 30$			
50	370	1850	3700	5550	1010	5050	10100	15150
100	620	3100	6200	9300	1660	8300	16600	24900
200	1120	5600	11200	16800	2960	14800	29600	44400
250	1370	6850	13700	20550	3610	18050	36100	54150

Table 3: Sparsity of matrix $\mathbf{A}$ for the simultaneous modified Denton PFD reconciliation procedure (matrix $\mathbf{G}$ assumed full).

	N				N			
M	1	5	10	15	1	5	10	15
	$s = 4,\ k = 1$				$s = 12,\ k = 1$			
10	.05761	.01174	.00588	.00393	.02539	.00511	.00256	.00171
50	.01252	.00251	.00126	.00084	.00555	.00111	.00056	.00037
100	.00633	.00127	.00063	.00042	.00281	.00056	.00028	.00019
200	.00318	.00064	.00032	.00021	.00141	.00028	.00014	.00009
250	.00255	.00051	.00026	.00017	.00113	.00023	.00011	.00008
	$s = 4,\ k = 30$				$s = 12,\ k = 30$			
50	.09064	.01813	.00907	.00604	.03650	.00730	.00365	.00243
100	.06454	.01291	.00645	.00430	.02701	.00540	.00270	.00180
200	.03955	.00791	.00396	.00264	.01699	.00340	.00170	.00113
250	.03304	.00661	.00330	.00220	.01427	.00286	.00143	.00095

constraints, the coefficient matrix of the linear system has dimension $(54,150 \times 54,150)$, which means about 2,932 millions entries.

However, less than 0.1% of these entries are non-zero (sparsity 0.00095 in table 3). Indeed, the sparsity ratio is smaller and smaller as the dimensions of the reconciliation problem grow up. On the other hand, for $N = 1$, that is when only one low-frequency period (e.g., one year) is considered for the reconciliation, the dimensions are such that any standard statistical/mathematical package may solve the linear system without any particular trouble.

It is important to notice that if $\mathbf{A}$ were (sparse and) non singular, $\mathbf{A}^{-1}$ usually would be dense, in fact preventing the solution of the system be expressed in terms of explicit inverse of $\mathbf{A}$ when large systems are involved.

In some instances, iterative methods (Saad, 2003) can be used, but these are not guaranteed to converge on general systems and usually require very sophisticated preconditioning[14]. Instead, we look at sparse direct methods (Davis, 2006) that involve some matrix factorization representation of the inverse. The method that we consider here is based on Gaussian Elimination, that generates the factorization $\mathbf{\Pi A Q} = \mathbf{LU}$, where $\mathbf{\Pi}$ and $\mathbf{Q}$ are permutation matrices chosen to preserve sparsity and maintain stability, and $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices, respectively. Thanks to the symmetry of $\mathbf{A}$, the factorization is of the form

$$\mathbf{\Pi A \Pi}' = \mathbf{LDL}', \tag{14}$$

where $\mathbf{D}$ is a diagonal matrix. The solution to system (13) is then easily obtained by solving the lower triangular system $\mathbf{Ly} = \mathbf{\Pi b}$ followed by the upper triangular system $\mathbf{DL}'\mathbf{\Pi x} = \mathbf{y}$.

The rank-deficiency of $\mathbf{A}$ is due to the fact that the aggregation matrix $\mathbf{H}$ has not full row-rank. For, from relationship (6) it follows that only $r = kn + N(M-k)$ aggregated observations are 'free', while kN aggregated observations are redundant, thus matrix $\mathbf{H}$ has rank r. This fact can be dealt with either by reformulating the constraints[15], which is obviously a problem-dependent solution and sometimes may reveal itself tedious and prone to error, or numerically, by a (possibly sparse and rank-revealing, see Davis, 2008) QR factorization of matrix $\mathbf{H}$. For the constrained minimization problem in hand, however, a very interesting result is that, due to the peculiar pattern of the system coefficient matrix $\mathbf{A}$, and provided the rhs vector of system (13) fulfills the fundamental relationship (6), a very efficient algorithm performing direct solution of sparse and large systems can be adopted, regardless of the rank deficiency of $\mathbf{A}$ (Duff, 2004)[16].

4 Two-step reconciliation procedures

As we have seen in the previous section, when the system of time series is very large, a simultaneous solution can be computationally burdensome, mostly if the practitioner either does not intend to or cannot use sparse matrices computation facilities. Simplified solutions are however possible, based on a generalization of the two-step approach proposed by Quenneville and Rancourt (2005) for restoring the additivity of a system of SA time series such that their sum is in line with directly-derived SA totals: firstly, a univariate benchmarking procedure (e.g., the modified Denton

[14] An iterative procedure to balance National Accounts aggregates, based on the conjugate gradient algorithm, has been proposed by Byron (1978). Danilov and Magnus (2008) stress that the application of iterative procedures has to be somewhat customized to the problem and sometimes poses problems of convergence. Besides, the singularity of matrix $\mathbf{\Omega}$ results in solving the complete system, not the smaller sub-system valid for the Lagrange multipliers. A similar situation was faced by Byron (1993).

[15] Constraints matrices having full row-rank for systems of time series linked by contemporaneous summation constraints have been shown by Di Fonzo (1990) and Di Fonzo and Marini (2005b).

[16] In Matlab©, starting from the release R2007b, a symmetric indefinite sparse system is efficiently solved by the routine MA57 of the Harwell Subroutine Library (HSL) for real sparse matrices. As Duff (2004, p. 122) claims, the solution procedure therein is able to flush any singularities of matrix $\mathbf{A}$ to the end of the factorization, and performs an LDL' factorization on the nonsingular block.

PFD benchmarking procedure or the more general regression based benchmarking procedure by Cholette and Dagum, 1994) is used to restore the temporal additivity of every SA series; in the second step, the SA component series are reconciled *one year at a time* using a least squares balancing procedure.

Denoting by $\mathbf{B} = \{B_{j,t}\}$ the $(Mn \times 1)$ vector containing all the temporally benchmarked series[17], the second step of this procedure is a quadratic constrained matrix minimization problem, where the constraints are a 'reduced version' of (7), valid for a single year. In order to restore additivity in a table of time series, Quenneville and Rancourt (2005) consider the objective function

$$F_T^{QR} = \sum_{j=1}^{M} \sum_{t=(T-1)s+1}^{Ts} \frac{(R_{j,t} - B_{j,t})^2}{B_{j,t}}, \tag{15}$$

where suffix T, $T = 1, \ldots, N$, denotes that the optimization is performed for each low-frequency period separately. According to the results of the previous section, the proposed two-step approach can be promptly extended to reconcile complex systems of accounts while dealing with different types of aggregation between (not only summation) and within (interpolation and extrapolation) variables.

Quenneville and Rancourt (2005) give an interpretation of their procedure in terms of weighted least squares regression, with weights given by the reciprocal of $\sqrt{B_{j,t}}$. In other terms, this step can be interpreted as the least squares adjustment of the temporally benchmarked estimates obtained in the first step according to the least squares procedure by Stone *et al.* (1942), with variances given by $B_{j,t}$ (thus admitting heteroskedasticity), and assuming autocorrelation neither between nor within the variables of the system. This requires the positivity of $B_{j,t}$, an assumption that may create some practical problems when we want to reconcile aggregates of a system of accounts, e.g. accounting balances, which may well not be positive. In this case, $|B_{j,t}|$ should be considered instead of $B_{j,t}$ as denominator into the expression (15)[18]:

$$F_T^{BB} = \sum_{j=1}^{M} \sum_{t=(T-1)s+1}^{Ts} \frac{(R_{j,t} - B_{j,t})^2}{|B_{j,t}|}. \tag{16}$$

For space reason here we do not consider the certainly very important issue related to the use of some estimates of data reliability in the least squares adjustment (van der Ploeg, 1982). Nevertheless we wish to stress that, assuming that $B_{j,t}$ is a (positive) unbiased[19] estimate of $R_{j,t}$ (i.e., $E(B_{j,t} - R_{j,t}) = 0$, $B_{j,t} > 0$), and if we agree on

[17] The series $\mathbf{B}_j$ are in line with the known temporal aggregates, i.e. $\mathbf{C}\mathbf{B}_j = \mathbf{Y}_{0j}$, but they do not fulfill the contemporaneous aggregation constraints, i.e. $(\mathbf{G} \otimes \mathbf{I}_n)\,\mathbf{B} \neq \mathbf{Z}$, so that, in general, $\mathbf{H}\mathbf{B} \neq \mathbf{Y}_a$.

[18] This choice has been made by Beaulieu and Bartelsman (2004) and Chen (2006) in balancing a system of tables of National Accounts for a given year. See also Dagum and Cholette (2006, p. 274).

[19] As far as the balancing of NA data is concerned, Byron (1978, p. 361) writes: "The argument for the importance of unbiasedness, while formally correct, is unconvincing because most statisticians will have not sufficient confidence in their initial estimates. An alternative, perhaps more satisfactory, approach is to set up a loss function, and to interpret the new estimates in relation to it".

using the coefficient of variability[20] (CV, the ratio between standard deviation and mean) as reliability indicator (higher CV's correspond to variables of comparatively worse quality, see, for example, Chen, 2006, Danilov and Magnus, 2008), the choice of $B_{j,t}$ as denominator in expression (15) implicitly involves $CV_{j,t} = \frac{1}{\sqrt{B_{j,t}}}$, that is: (i) different reliabilities for all variables are considered in the least-squares adjustment, and (ii) large variables are considered relatively more reliable, and thus they are touched relatively less by the reconciliation procedure than small variables[21].

Conversely, in absence of any information about data reliabilities, one can adopt the idea that the relative reliabilities of all variables are equal. This corresponds to the situation in which $CV_{j,t} = \kappa$ is constant for any j and t, which means $\sigma^2_{j,t} = \kappa^2 B^2_{j,t}$. As in least squares reconciliation only relative variances play a role, in this case there is no loss of generality in assuming $\sigma^2_{j,t} = B^2_{j,t}$[22]. As a consequence, it seems sensible to consider the following objective function (Round, 2003; Stuckey *et al.*, 2004) in order to reconcile the temporally benchmarked series in the system at the second step[23]:

$$F_T^{ST} = \sum_{j=1}^{M} \sum_{t=(T-1)s+1}^{Ts} \left(\frac{R_{j,t} - B_{j,t}}{B_{j,t}} \right)^2 . \tag{17}$$

Alternatively, we may consider to adopt the simultaneous modified Denton PFD criterion referred to a single low-frequency period:

$$F_T^{MD} = \sum_{j=1}^{M} \sum_{t=(T-1)s+2}^{Ts} \left(\frac{R_{j,t} - B_{j,t}}{B_{j,t}} - \frac{R_{j,t-1} - B_{j,t-1}}{B_{j,t-1}} \right)^2 . \tag{18}$$

In line of principle, nothing prevents us to consider a reconciliation procedure grounded on the analogous of criterion (18), in which for each low-frequency period the preliminary series P_{jt} are considered instead of the temporally benchmarked series B_{jt}. In this case, one might be concerned about a possible 'step' problem[24] in the series, which in turn would have been limited if temporal benchmarking was performed at the first step[25].

Finally, notice that the objective function (18) involves the squared levels of the series to be reconciled at denominator, like criterion (17). In addition, it can

[20]Quenneville and Rancourt (2005), Dagum and Cholette (2006), Bikker and Buijtenhek (2006), Buijtenhek (2006), and Daalmans and Mushkudiani (2009), discuss about using 'reliability weights/alterability coefficients', generally based on *a priori* and/or subjective information on some variables, both in order to take into account different reliabilities of the preliminary data and to deal with exogenous and endogenous variables in the system. This extension can be easily incorporated into the procedures considered in the paper.

[21]From a purely statistical point of view, we have thus some doubt in considering "neutral" the choice of $|B_{j,t}|$ (Chen, 2006).

[22]See Round, 2003, p. 178. Notice that even in this case eteroskedasticity is assumed, and still neither contemporaneous nor temporal correlation is allowed.

[23]Although they do consider SA monthly time series, annual temporal aggregation constraints are not dealt with by Stuckey *et al.* (2004).

[24]We mean the introduction of irregularities in the temporal dynamics of the series between, e.g., the last month of one year and the first month of the following one (Bloem *et al.*, 2001).

[25]In practical applications the differences registered for the two procedures (either using or not B_{jt}) are irrelevant.

be shown (Appendix 1) that in the simultaneous modified Denton PFD reconciliation procedure both eteroskedasticity and temporal autocorrelation in the data are assumed, while cross-correlation is still not allowed.

4.1 Some remarks

In the following (sec. 6) we will show that it is interesting to evaluate the corrections produced by the procedures considered so far, in connection with the dimension of the variables to be reconciled. For, the nature of the various adjustments is always proportional, in the sense that - if a direct relationship between the level and the variance of a variable is assumed - the amount of correction is strictly related to the size of the variable: the larger (smaller) is a variable, the larger (smaller) will be its adjustment.

In one dimension, that is if one considers only contemporaneous (accounting) relationships linking the variables of the system, the following considerations of Kim *et al.* (1994, p. 5) hold[26]: "at equal levels of accuracy (as quantified by relative standard errors) and under equivalent accounting constraints, if one element is k times larger than the others in the unbalanced accounts, its level of adjustment will be k^2 times larger than the rest and its proportional adjustment will be k times larger. If this element is k times larger than the rest and m times more inaccurate than others, the level of adjustment will be k^2m^2 the level of others while the proportional adjustment will be km^2 that of the others. If this element is no larger than others but its relative standard error is m times greater, both its level of adjustment and its proportional adjustment will be m^2 greater. It is important to note that the adjustments also depend upon the accounting constraints (...) on the elements. This explains why data items of the same magnitude and comparable levels of accuracy may be revised differently".

Instead, if a *pro-rata* criterion of reconciliation is used, the level of the adjustment will be k times larger than the rest, while the proportional adjustment will be the same for all variables, regardless their relative dimensions.

It is clear that extending these findings in more dimensions (number of accounting relationships and/or time periods greater than 1) does not result in known-in-advance amounts of adjustment. Nevertheless, one may guess that the nature and the 'scale' of the corrections done to the levels by the various procedures should be somewhat similar to what happens in one dimension.

We have shown (section 4) that both criterion F_T^{ST} and F_T^{MD} are consistent with the assumption of equal reliability of the variables in the system, while criterion F_T^{BB} (equivalent to F_T^{QR} when no negative preliminary values are involved)[27], implicitly postulates a quality ranking dictated by the dimensions of the variables. It follows that the adjustment of larger variables will be relatively smaller in the latter than in the former case, while smaller variables will be touched relatively more.

[26]We leave untouched the notation used by Kim *et al.* (1994), so in this context k does not mean the number of contemporaneous constraints, as it is in the rest of the paper.

[27]If all the preliminary values are positive, it can be shown that the well-known RAS bi-proportional adjustment procedure is a first order approximation of the adjustment procedure which follows the criterion F_T^{QR} (Bacharach, 1970; Rampa, 2008).

This should hold for the levels of the variables. However, given the relevance of the time dimension in the problem under study, a question naturally rises: how large is, and what (if any) are the peculiarities of the impact of the various reconciliation procedures on the temporal profile (e.g., the movement) of the original preliminary series? From the results we have found (sec. 6), it appears that the distinctive characteristics we have discussed so far have a notably different impact, at least when evaluated in terms of corrections done to the rates of change of the series.

5 The assessment of reconciliation

The result of a reconciliation procedure may be assessed using summary indices of the corrections (adjustments) made to the original preliminary figures.

A range of quality measures may be used in order to assess the reconciliation results. For example, in their work on post-adjustment of a system of SA series (which play the role of preliminary estimates) Stuckey *et al.* (2004) consider

- that the reconciled estimates result in small corrections to the level of the preliminary series;
- that the reconciliation result in a small correction to the period to period movement of the preliminary series;
- that highly volatile series are altered more than less volatile series.

Another important point to be evaluated if the reconciliation of SA series is in order, is that there is no introduction of residual seasonality into the reconciled SA series[28].

In this paper, we limit ourselves in considering simple indices which summarize the size of the adjustments to both levels and percentage rates of change. More precisely, for each series we calculate the Mean Squared Percentage Adjustment ($MSPA$) to the levels, and the Mean Squared Adjustment (MSA) to the percentage growth rates, that is:

$$\begin{aligned} MSPA(R_j, P_j) &= 100 \times \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\frac{R_{j,t}-P_{j,t}}{P_{j,t}}\right)^2} \\ MSA(r_j, p_j) &= 100 \times \sqrt{\frac{1}{n-1}\sum_{t=2}^{n}(r_{j,t}-p_{j,t})^2} \end{aligned}$$

where $r_{j,t} = \left(\frac{R_{j,t}-R_{j,t-1}}{R_{j,t-1}}\right)$ and $p_{j,t} = \left(\frac{P_{j,t}-P_{j,t-1}}{P_{j,t-1}}\right)$ are the growth rates of the reconciled and preliminary series, respectively.

In addition, the standard deviation of the percentage change in the proportional movement (Dagum and Cholette, 2006, p. 294) is considered, that is:

$$SDPA_j = 100 \times \sqrt{\frac{1}{n-1}\sum_{t=2}^{n}\left[\left(\frac{R_{j,t}-P_{j,t}}{P_{j,t}} - \frac{R_{j,t-1}-P_{j,t-1}}{P_{j,t-1}}\right) - \overline{CM}_j\right]^2}$$

[28] On this issue, see Evans (2004).

where $\overline{CM}_j = \dfrac{1}{n-1}\sum_{t=2}^{n}\left(\dfrac{R_{j,t}-P_{j,t}}{P_{j,t}} - \dfrac{R_{j,t-1}-P_{j,t-1}}{P_{j,t-1}}\right)$.

Accordingly, overall indices for the whole system of time series are considered:

$$MSPA(R,P) = 100 \times \sqrt{\frac{1}{Mn}\sum_{j=1}^{M}\sum_{t=1}^{n}\left(\frac{R_{j,t}-P_{j,t}}{P_{j,t}}\right)^2}$$

$$MSA(r,p) = 100 \times \sqrt{\frac{1}{M(n-1)}\sum_{j=1}^{M}\sum_{t=2}^{n}(r_{j,t}-p_{j,t})^2}$$

$$SDPA = 100 \times \sqrt{\frac{1}{M(n-1)}\sum_{j=1}^{M}\sum_{t=2}^{n}\left[\left(\frac{R_{j,t}-P_{j,t}}{P_{j,t}} - \frac{R_{j,t-1}-P_{j,t-1}}{P_{j,t-1}}\right) - \overline{CM}\right]^2}$$

where $\overline{CM} = \dfrac{1}{M(n-1)}\sum_{j=1}^{M}\sum_{t=2}^{n}\left(\dfrac{R_{j,t}-P_{j,t}}{P_{j,t}} - \dfrac{R_{j,t-1}-P_{j,t-1}}{P_{j,t-1}}\right)$.

6 Two applications

In the following we apply simultaneous and two-step procedures for the reconciliation of two systems of time series. We denote by Sim MD the simultaneous modified Denton PFD procedure described in section 3.1, while the two-step procedures described in section 4 are labelled as 2-S QR/2-S BB, provided either objective function (15) or (16) is considered, 2-S ST for criterion (17), and 2-S MD for criterion (18).

First a medium-sized problem is considered, namely the reconciliation of the European Union Quarterly Sector Accounts (EU-QSA). Then we consider a large-sized reconciliation problem, coming from the Canadian Monthly Retail Trade Survey (MRTS), where we want to restore additivity in a cross-classified system of component SA monthly time series[29].

Albeit the two systems (and the nature of their discrepancies) are rather different, both cases can be seen as typical situations encountered in the current activity of a data producer agency.

6.1 The EU Quarterly Sector Accounts

The compilation of the EU-QSA is done jointly by Eurostat (the European statistical office) and the European Central Bank. The aggregates of the accounts are derived from the data collected by member States, but the compilation process is such that they are not calculated simply as the sum of the national components. For this reason, many inconsistencies arise after this process, so that the accounting relationships between the variables are not fulfilled. In addition, in some cases the quarterly series are not in line with the corresponding annual figures. A reconciliation of the system is thus necessary to restore the required consistency.

[29] Restoring additivity in systems of SA series could be problematic from a logical point of view. For reasons of space, here we do not address this issue (see, for example, U.S. Census Bureau, 2007, p. 102).

The EU-QSA dataset at disposal nominally comprises 183 series. In fact, in 8 out of 183 cases the aggregates are always zero, and need not be adjusted[30]. So they are excluded from the system, leaving 175 series, which are linked by 30 accounting constraints. Figure 1 shows the non-zero pattern of the (30×175) matrix of contemporaneous constraints $\mathbf{G}$.

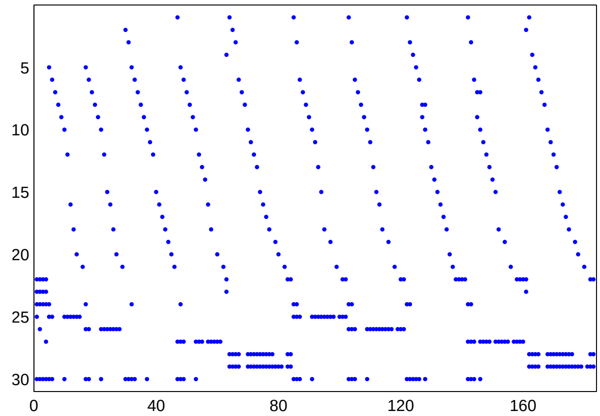

Figure 1: EU-QSA system. Non-zero pattern of the accounting constraints.

The first 22 equations of the system establish the consistency for a set of variables between institutional sectors and total economy (the known quantities are zero vectors). For example, in the first equation the sum of the compensation of employees by institutional sector is equal to that of the total economy. The last 8 equations 'close' the system with respect to important balances of the accounts (e.g., value added, net lending/net borrowing, operating surplus, etc.), for both the total economy and some institutional sectors[31].

The quarterly data span the period from 1999-Q1 to 2005-Q4 (28 quarters). Annual benchmarks are available for each variable, along the 7 years. The dimensions of this reconciliation problem are summarized in Table 4.

In 113 out of 175 cases, the quarterly data are perfectly in line with the annual counterparts. The remaining 62 variables present temporal discrepancies which are in some cases very small (less than 0.5% of the original level) and in other cases rather large (up to 50% of the original level). The temporal discrepancies for a single variable are often either all negative or all positive, which is a clear indication that the quarterly series are biased with respect to the annual benchmarks[32].

[30]In addition, 11 series present negative values, so the two-step reconciliation procedure by Quenneville and Rancourt (2005) has been applied according to the objective function (16) rather than (15). For the same reason, in the first step the temporal benchmarking was performed according to a variant of the objective function (9), where the absolute values of the preliminary series have been considered at denominator. A similar device was adopted for the objective functions (10) and (18) too.

[31]It should be noted that quarterly preliminary series for these balances are available, which are not aligned to the corresponding annual series. Given that we choose to treat these balances as exogenous contemporaneous constraints of the system, we have preliminarly benchmarked them *via* the (univariate) modified Denton PFD benchmarking procedure in order to fulfill the fundamental relationship (6).

[32]This is rather usual in National Accounts, where annual data are generally more reliable than

Table 4: Dimensions of the EU-QSA reconciliation problem.

n. of series (M)	175
n. of quarters (n)	28
n. of complete years (N)	7
n. of contemporaneous constraints (k)	30
n. of preliminary values to be reconciled	4,900
n. of constraints	2,065
n. of non-redundant constraints	1,855
n. of rows of $\mathbf{A}$ (simultaneous modified Denton PFD)	6,965
sparsity of $\mathbf{A}$	0.00086

Most of the series are non-stationary, with a clear seasonal component (although sometimes really unstable). We stress that here we do not deal with any issue of indirect *vs.* direct seasonal adjustment, such as the treatment of discrepancies between components and aggregates. Rather, the focus is on the performance of reconciliation procedures used to achieve consistency of a system of raw quarterly NA time series.

As regards the contemporaneous discrepancies, in one out of the 30 cases the accounting constraint is always fulfilled by the original data. In the other 29 cases the ranges of the discrepancies (levels) hover between (in millions of €) 300 (min. -113, max 187) and 185,973 (min -87,561, max 98,412). In addition, in 6 accounts the preliminary data are systematically biased (all discrepancies have the same sign).

The EU-QSA have been reconciled by means of the four reconciliation procedures described so far. As regards the two-step procedures, the first step is common to them all: the quarterly series have been individually aligned to the annual series using the modified Denton PFD benchmarking procedure. The second step is the year-by-year reconciliation of the benchmarked series obtained at the first step, performed by constrained optimization of the objective functions F_T^{BB}, F_T^{ST}, and F_T^{MD}, as expressed by (16), (17), and (18), respectively. The system has been reconciled also by means of the simultaneous procedure presented in section 3.1.

Firstly, the results are evaluated in terms of the distance between reconciled and preliminary series, using the summary indices illustrated in section 5. Table 5 shows the $MSPA$ to the levels, the MSA to the percentage growth rates and the $SDPA$ in the proportional movement, calculated on all series and all quarters.

The reconciled series from 2-S BB present always higher statistics than those from the other three procedures, which instead show very similar performances. In particular, the impact of 2-S BB procedure on the rates of change seems to be significantly higher (11.99% *vs.* 5.63% of the simultaneous procedure). The practical equivalence of 2-S ST and Sim MD is a very interesting result: the two-step approach may be implemented quite easily with any statistical package, and it requires less computational time with respect to a simultaneous approach that considers all years

quarterly data because their calculation relies on more solid and comprehensive sources. The preliminary series are thus standardized to the overall level of the annual series, according to the bias correction procedure described in Quenneville *et al.* (2009).

Table 5: EU-QSA system. Assessment of results from different reconciliation procedures.

Procedure	$MSPA(R,P)$	$MSA(r,p)$	$SDPA$
2-S BB	4.1286	11.9946	4.1551
2-S ST	3.7809	5.6096	3.2455
2-S MD	3.7853	5.6518	3.2460
Sim MD	3.7883	5.6318	3.2400

in a single step[33]. Another interesting aspect of the exercise can be drawn from the slight superiority of 2-S ST relative to 2-S MD (5.61% against 5.63%); this implies that with this dataset, dealing with the movement preservation at the second step through function (18) does not improve on the results obtained when using criterion (17).

To further investigate the quality of the results, there are two interesting aspects to assess:

- the step problem: the procedure should not introduce breaks in the movement from the fourth quarter of a year to the first quarter of the next;
- the preservation of the signs: the adjustment should not alter the signs of the original estimates. Also, it is highly desirable that the signs of growth rates do not change that much after the reconciliation.

Table 6 shows the MSA to growth rates considering only the rates of the first quarter. The statistic worsens only for 2-S BB (19.28% *vs* 14.50%), while it improves in the other cases. Considering the preservation of the signs, there are no change from positive to negative values (and *vice versa*) caused by the reconciliation procedures. The same does not hold for growth rates. Again, 2-S BB is the least performing procedure: it changes sign to the 12.12% of the rates, against less than 9% of the other procedures.

Table 6: EU-QSA system. Some measures on problematic aspects of reconciliation: step problem and signs preservation.

	$MSA(r,p)$		*% signs pres.*	
Procedure	*all quarters*	*only Q1*	*levels*	*rates*
2-S BB	11.9946	19.2805	100.0	87.88
2-S ST	5.6096	3.3738	100.0	91.30
2-S MD	5.6518	3.4817	100.0	91.34
Sim MD	5.6318	3.3881	100.0	91.34

Figure 2 compares the $MSA(r,p)$ statistics of procedures 2-S BB and Sim MD calculated on different groups of variables, grouped according to their dimension[34].

[33] However, our Matlab© script performing the simultaneous reconciliation of EU-QSA takes about 0.25 seconds on an Intel T2050 (1.60Ghz and 1Gb of memory Ram) notebook working under Windows XP.

[34] The groups are determined as follows: the large series (10) are those with an annual average level greater than €1,000,000 mln; the group of medium-sized series (52) between €100,000 and €1,000,000; the group of small series (113) up to €100,000.

As expected, the procedure 2-S BB shows a smaller distance than Sim MD for the 10 largest series (1.25% *vs* 2.05%). For the medium-sized group the MSA statistic is slightly in favor of Sim MD (3.31% compared to 3.51% of 2-S BB). But the performances of the two procedures are really different for the small-sized group, the largest one with 113 variables: the adjustment produced by the 2-S BB procedure is such that the original temporal profiles of these variables are drastically changed. The MSA statistic is indeed 16.62%, compared to 6.61% of Sim MD. From this analysis it is confirmed that the 2-S BB procedure tends to preserve better the movement of the larger variables at the expense of the smaller ones, whose movements after reconciliation might be very far from those of the preliminary series.

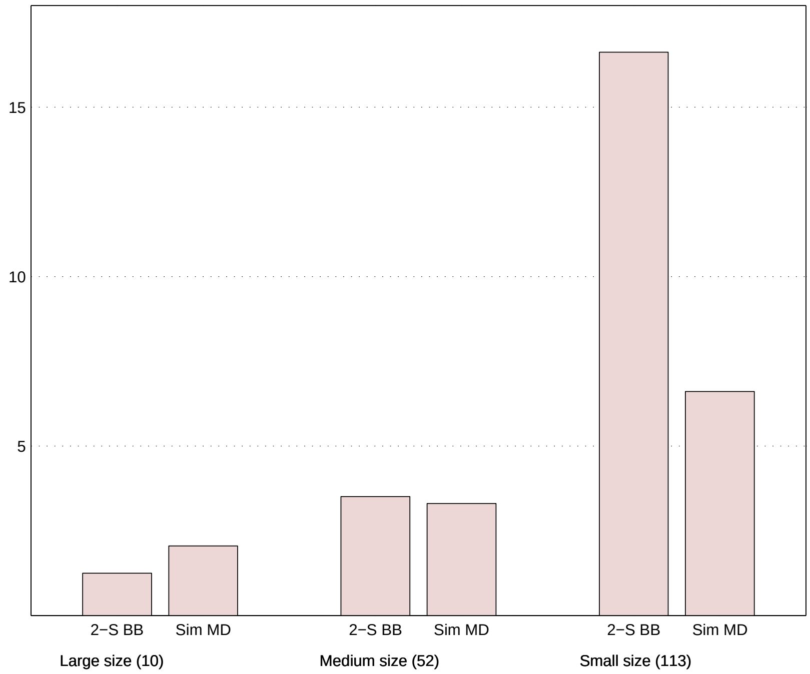

Figure 2: EU-QSA system. $MSA(r, p)$ by dimension of the variables.

Figure 3 and 4 show preliminary and reconciled series of a large variable (B-S11-B1G, gross value added of non-financial corporations) and a small variable of the EU-QSA (B-S13-D71, net premium (non-life insurance) of general government), in order to visually detect the different adjustment carried out by the 2-S BB and Sim MD procedures. The adjustments done on the two series are typical of their group. The top graph contains the original series (straight line) along with the two reconciled series (dashed line for 2-S BB, dot-dashed line for Sim MD); the bottom graph shows the differences from the original series. The two procedures adjust the large variable in a similar way, although the simultaneous procedure seems to have a more marked impact in comparison with the two-step approach. As far as the small variable is concerned, the reconciliation with 2-S BB radically changes the short-term movements of the series; on the contrary, the Sim MD procedure leaves them practically unchanged.

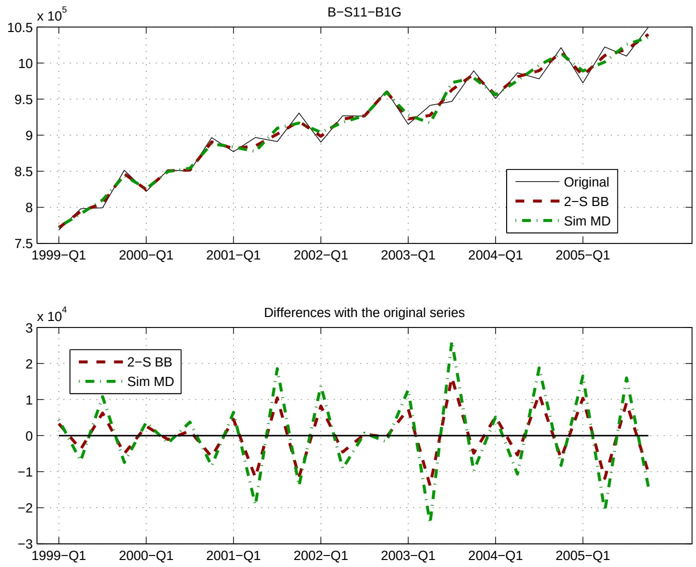

Figure 3: EU-QSA system. Original and reconciled estimates for series B-S11-B1G (Gross value added of non-financial corporations).

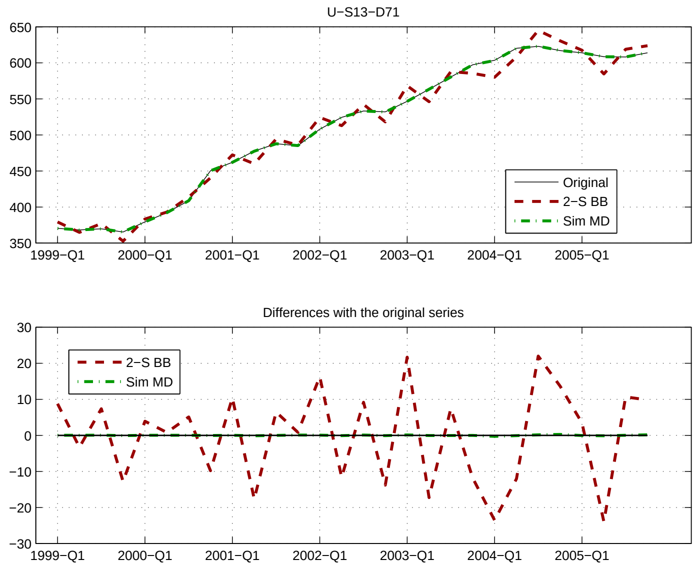

Figure 4: EU-QSA system. Original and reconciled estimates for series B-S13-D71 (Net Premium (non-life insurance) of General Government).

6.2 The Canadian Monthly Retail Trade Survey

Typically, seasonal adjustment introduces discrepancies in a system of time series linked by aggregation constraints. Firstly, the annual sum of SA series might show differences with the annual totals from the raw series, due to the fact that a non-deterministic seasonal component is normally assumed. Secondly, the direct SA aggregate does not necessarily equal the sum of its SA components series, due to the presence of non-linearities in the SA procedures.

This kind of problem is faced by Statistics Canada with the Monthly Retail

Trade Survey (MRTS). The data are collected according to a two-way classification: by industry (19 trade groups, the TG system) and by region (10 provinces and 3 territories, the PR system). The cross-classified MRTS system is thus nominally composed of 247 component series (the TGPR system). SA time series are only published for the national total, the 19 trade groups and the 13 regions (the marginal systems). The SA national total is derived indirectly as the sum of the 19 SA components by industry. Then, the 13 SA regional components are aligned to the SA national total (Statistics Canada, 2009).

In this section we present a comparison of simultaneous and two-step techniques for reconciling the MRTS data[35]. We use a dataset covering the period from January 1991 to December 2003, except for the Nunavut territory, whose series begin on January 1999. We consider 236 raw series out of the 247 components of the TGPR system as the starting point of the exercise. In fact, 2 component series are always zero, while other 9 series do not show any visible seasonal pattern and contain some observed zero values[36]. Thus we decided not to seasonally adjust them. The marginal series by region (13) and by industry (19), the national total, and the corresponding annual totals are derived as direct aggregation of the 236 component series. The X12-ARIMA procedure was applied to the 269 (236+13+19+1) monthly series with automatic options[37], using the interface program Demetra (version 2.2, see Eurostat, 2007). The SA series resulting from X12-ARIMA are called hereafter preliminary SA series.

6.2.1 Reconciliation of the PR system

As a first step, the 19 marginal SA totals by industry are benchmarked according to the univariate modified Denton PFD procedure[38], so that their annual totals are equal to those of the corresponding raw series. These series will be referred to as benchmarked SA series. The SA national total is calculated (indirectly) as the sum of the 19 benchmarked SA series by industry. This practice forces the national total to be coherent with the SA data derived from the TG system: as Quenneville and Rancourt (2005) stress, this is justified by the fact that it is easier to identify breaks, outliers, calendar effects, and the seasonal component in general, at the industry level. The annual constraints for the national total are automatically satisfied.

Such a procedure makes the 13 SA regional series not in line with the SA national total. Then, the 13 SA series by region need to be adjusted such that their sum

[35] This example has been already considered by Di Fonzo and Marini (2005b), Quenneville and Rancourt (2005), and Dagum and Cholette (2006).

[36] The series we have not considered in the exercise pertain to 10 trading groups of the Nunavut territory and 1 trading group of the Yukon territory. The raw series account for a negligible share of the Canadian total (at most 0.02% in the time span under study).

[37] We did not use the optional spec `FORCE` (U.S. Census Bureau, 2007), so the yearly sums of the SA series are in general different from those of the original series. Besides, we stress that the quality of seasonal adjustment is not a primary concern of the paper. We have replicated the exercise performing the seasonal adjustment by TRAMO-SEATS, and the results we found as regards the different impact of the various reconciliation procedures on the temporal profiles of the directly SA series, were not significantly affected.

[38] This can be accomplished with X-12 ARIMA by setting the argument `type=denton` in the `FORCE` spec.

equal the SA national total derived from the TG system and, also, their yearly sums match the annual totals of the corresponding raw series. The dimension of this reconciliation problem is relatively small, as shown in table 7.

Table 7: Dimensions of the PR system reconciliation problem.

n. of series (M)	13
n. of months (n)*	156
n. of complete years (N)*	13
n. of contemporaneous constraints (k)	1
n. of preliminary values to be reconciled	1,932
n. of constraints	317
n. of non-redundant constraints	304
n. of rows of $\mathbf{A}$ (simultaneous modified Denton PFD)	2,249
sparsity of $\mathbf{A}$	0.00267

* For Nunavut $n = 60$ (1999.01–2003.12) and $N = 5$ (1999–2003).

The temporal discrepancies generated by the seasonal adjustment are rather small, being in a range from -0.7% to 0.3% of the original annual levels. The contemporaneous discrepancies (the percentage differences between the SA national total and the sum of the preliminary SA series by region) are shown in figure 5. They are always less than 2% in absolute values (from a minimum of -1.55% to a maximum of 1.73%), with a mean absolute percentage discrepancy equal to 0.3%.

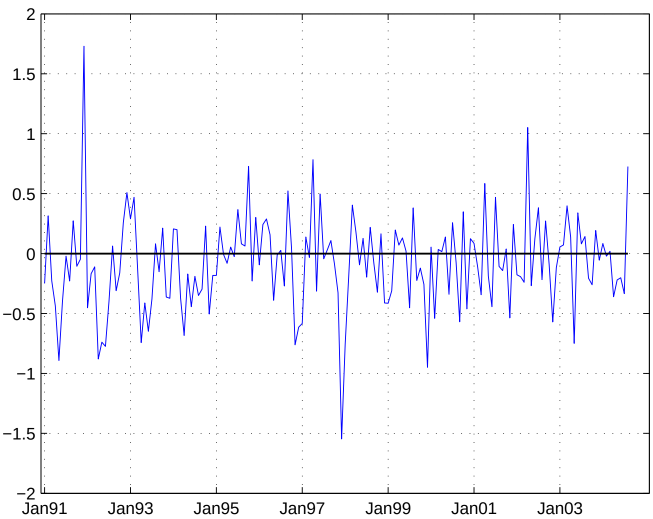

Figure 5: PR system. Contemporaneous discrepancies (%) between the SA Canada total from the TG system and the sum of the 13 SA regional series.

The reconciliation is performed with the four procedures described at the beginning of this section (the 2-S QR procedure can be used because preliminary series are always positive). Table 8 shows the statistics for assessing their relative performances. The reconciled estimates from the simultaneous solution Sim MD and those from the two step procedures 2-S ST and 2-S MD are very close to each other: the statistics are practically identical. Instead the 2-S QR displays higher values, both in terms of levels and growth rates. In particular, the MSA to growth rates between preliminary and reconciled SA series is 0.54% for 2-S QR, quite higher than 0.32% achieved by Sim MD.

Table 8: PR system. Assessment of results from different reconciliation procedures.

Procedure	$MSPA(R,P)$	$MSA(r,p)$	$SDPA$
2-S QR	0.4209	0.5408	0.5377
2-S ST	0.2893	0.3196	0.3182
2-S MD	0.2900	0.3196	0.3183
Sim MD	0.2900	0.3196	0.3181

Like the EU-QSA system, it is interesting to look at the $MSA(r_j,p_j)$ on the basis of the relative dimension of the series in the system. Figure 6 shows graphically the relationship between the $MSA(r_j,p_j)$ statistics and the regional sales in percentage of the national total (the top graph refers to Sim MD, the bottom graph to 2-S QR). Both relationships are very peculiar: the adjustment is directly proportional to the dimension of the regional sales for Sim MD, while the 2-S QR has always the same impact on the rates of change, irrespective of their relative size. The $MSA(r_j,p_j)$ for the four largest regions (in terms of retail sales' share) are identified in both graphs. The $MSA(r_j,p_j)$ statistic is lower for the 2-S QR procedure only for Ontario, which is the largest region (37.9% share): 0.53% *vs.* 0.87% of Sim MD. The adjustment for Quebec is almost the same, while for the remaining 11 regions a lower MSA statistic is always achieved by Sim MD. It clearly appears that the 2-S QR procedure has a deep impact on the smallest series: for example, the $MSA(r_j,p_j)$ for Newfoundland and Labrador (about 1.68% share of the Total Canada amount) is about ten times higher than the one produced by Sim MD (0.54% *vs.* 0.05%). This is an empirical confirm that the 2-S QR procedure tends to 'touch' less the larger variables, at the expense of the smaller ones, implicitly assuming a greater reliability for the former.

6.2.2 Reconciliation of the cross-classified TGPR system

A second exercise is performed with the MRTS data in the industry-by-region classification (the TGPR system). The 19 benchmarked SA series by industry, the indirectly-derived SA series for the Canada total, and the 13 reconciled SA series by region (with the Sim MD procedure) are now considered as given constraints for the TGPR system. The 236 preliminary SA series must be adjusted such that:

- for every month, the sum over the 13 regions for an industry matches the given industry total (19 constraints);
- for every month, the sum over the 19 industries of a region matches the given region total (13 constraints);
- for every complete year, and for every industry-region, the sum over the 12 months complies with the annual total from the corresponding raw series (236

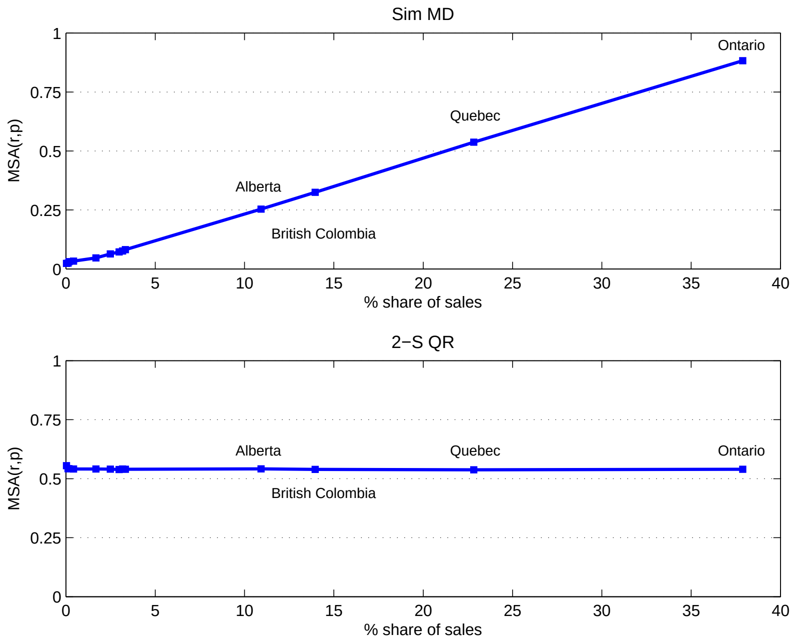

Figure 6: PR system. $MSA(r_j, p_j)$ by percentage share of sales for Sim MD and 2-S QR.

constraints).

This application can be seen as a reconciliation problem of a two-way table of time series subject to marginal totals and temporal constraints. The problem can be formulated in different ways, according to the role played by the marginal totals, which can be treated as either endogenous or exogenous (Di Fonzo and Marini, 2005b). In this exercise we consider the 32 marginal totals by province and trading group as exogenous constraints of the system.

Table 9 shows the (quite large) dimensions of the problem. The system matrix $\mathbf{A}$ for the Sim MD reconciliation procedure, whose sparsity pattern is shown in figure 7, can be managed in a computing software only using sparse matrices facilities, and system (13) can be directly solved only with algorithms that exploit the sparsity of $\mathbf{A}$ (see section 3).

The discrepancies, both temporal and contemporaneous, are quite variable: the mean absolute percentage temporal discrepancies of the 236 component series range from 0.02% (Pharmacies and Personal Care Stores Sales of Yukon Territory) to 4.52% (Computer and Software Stores Sales of Newfound and Labrador), whereas the same indices for the 32 monthly additivity constraints vary between 0.29% (Pharmacies and Personal Care Stores) and 1.65% (Used and Recreational Motor Vehicle and Parts Dealers).

Table 10 presents the statistics to assess the four procedures of reconciliation. Once again, 2-S QR shows higher distances from the preliminary series than the other procedures, both in terms of levels and rates of change. The MSA to the growth rates is almost double the one of the simultaneous solution (2.52% compared

Table 9: Dimensions of the TGPR system reconciliation problem.

n. of series (M)	236
n. of months (n)*	156
n. of complete years (N)*	13
n. of contemporaneous constraints (k)	32
n. of preliminary values to be reconciled	35,952
n. of constraints	7,988
n. of non-redundant constraints	7,572
n. of rows of **A** (simultaneous modified Denton PFD)	43,940
sparsity of **A**	0.00017

* In 9 out of 236 series $n = 60$ (1999.01–2003.12) and $N = 5$ (1999–2003).

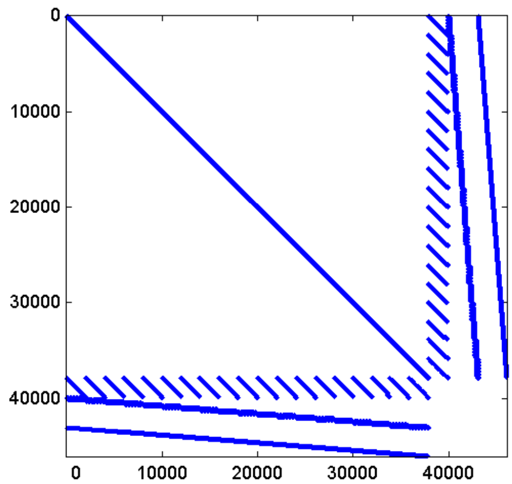

Figure 7: TGPR system. Sparsity pattern of matrix **A**.

to 1.26%). The 2-S ST procedure represents a good alternative, again better than 2-S MD, to the simultaneous solution in regard to movement preservation. Besides, it is also less demanding in terms of computational resources[39].

It is interesting to inspect the distribution of the MSA statistics over the 236 series of the system. Figure 8 displays the boxplots of $MSA(r_j, p_j)$ for 2-S QR (left) and Sim MD (right). Differently from the PR system, 2-S QR does not alter the temporal profiles of the variables with the same intensity[40]: for 6 series the $MSA(r_j, p_j)$ statistic is higher than 5%, in 2 cases even higher than 10%. Sim MD

[39] It should be noted that, for each two-step procedure, the reconciled estimates are obtained by solving 8 linear systems of 3,310 equations for each year in 1991-1998, when Nunavut territory was not present, and 5 linear systems of 3,452 equations for the remaining years. On the other hand, the Matlab© script performing the simultaneous reconciliation of the TGPR system takes about 13 seconds. In any case the linear systems are solved by using sparse matrices facilities and algorithms made available by the software.

[40] We guess that this fact is due to the need of fulfilling 32 contemporaneous constraints instead of only 1, as it was in the reconciliation of the marginal PR system.

Table 10: TGPR system. Assessment of results from different reconciliation procedures.

Procedure	$MSPA(R,P)$	$MSA(r,p)$	$SDPA$
2-S QR	1.7137	2.5158	2.0328
2-S ST	1.3308	1.2668	1.2410
2-S MD	1.3637	1.2880	1.2473
Sim MD	1.3599	1.2635	1.2364

seems instead more robust, the largest $MSA(r,p)$ being equal to 5.6%.

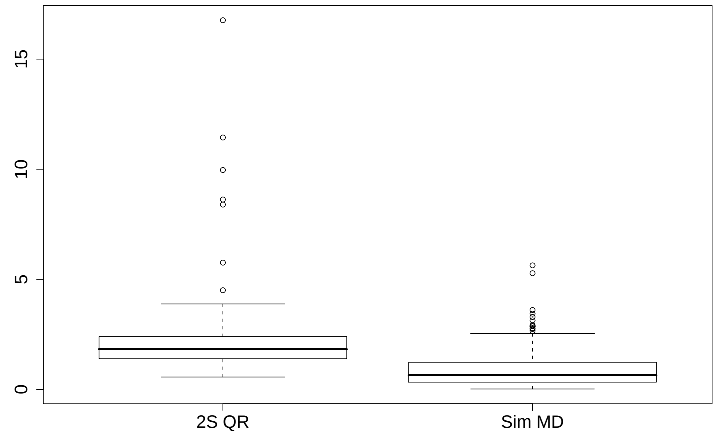

Figure 8: TGPR system. Boxplots of $MSA(r_j, p_j)$ for 2-S QR and Sim MD.

This picture is somewhat enriched by figure 9, which displays the scatterplot of the $MSA(r_j, p_j)$ statistics *vs.* the relevant percentage shares of sales. It clearly appears that the series whose dynamic profile is strongly modified by 2-S QR are relatively very small. On the other hand, the two series much adjusted by Sim MD as compared to the rest of the system, are not the largest ones[41].

In order to have a better look at the relationship between the corrections and the dimension of the variables, we classify the 236 component series in groups of regions. The first group is composed of the 38 series of Ontario and Quebec, having these provinces the largest share of sales (60.7% of total Canada). The other 8 provinces form the second group, with a share of 39.0%. The last group includes the 46 series of the 3 territories (Yukon, Northwest and Nunavut), which amounts to the remaining 0.3% of Canada retail sales. Table 11 presents the number of times the procedures 2-S QR and Sim MD produce the smallest adjustment to the preliminary series' growth rates. The two-step approach prevails to the simultaneous solution in 47 cases, more than half concentrated on the series of Ontario and Quebec (25). For the other regions (8 provinces and 3 territories) Sim MD achieves a lower

[41]The two series of sales refer to Supermarkets of Yukon Territory ($MSA(r_j, p_j) = 5.63$, % share=0.03), and Used and Recreational Motor Vehicle and Parts Dealers of Ontario ($MSA(r_j, p_j) = 5.28$, % share=1.64), respectively.

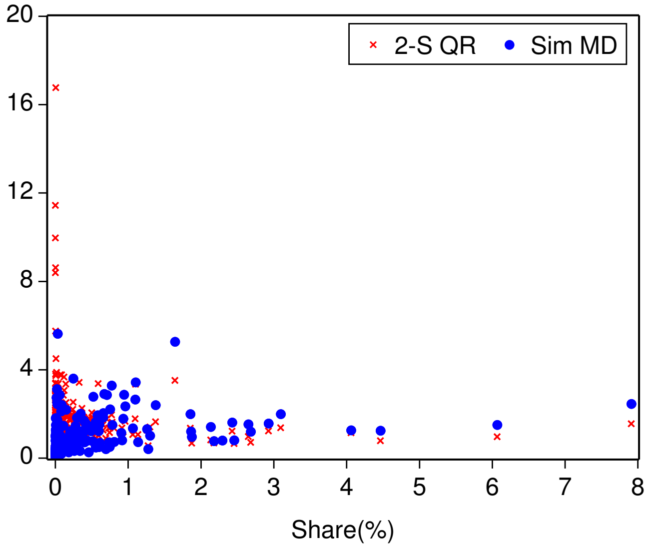

Figure 9: TGPR system. $MSA(r_j, p_j)$ by share (%) of sales for Sim MD and 2-S QR. $MSA(r_j, p_j)$ in 176 out of 198 cases.

Table 11: TGPR system. Number of series with minimum $MSA(r_j, p_j)$ by group of regions and reconciliation procedure

	Group of regions			
Procedure	*Ontario-Quebec*	*Other provinces*	*Territories*	*Total*
2-S QR	25	17	5	*47*
Sim MD	13	135	41	*189*
Total	*38*	*152*	*46*	*236*

We calculate the $MSA(r, p)$ statistic for the three groups; figure 10 compares those referred to 2-S QR and Sim MD. In the first group (Ontario and Quebec), the statistic is lower for 2-S QR, 1.6% against 2.0% for Sim MD. The results are opposite for the other two groups. For the other 8 provinces, the $MSA(r, p)$ statistic is equal to 0.9% for Sim MD and 2.5% for 2-S QR; for the three territories, we obtain 1.5% for Sim MD and 3.2% for 2-S QR.

The step problem and the sign preservation (discussed in the previous section) can be evaluated from table 12. Differently from the EU-QSA exercise, the statistics computed considering only the first months of the year increase for all procedures. As expected, Sim MD provides the smoothest transition from one year to the next. The step problem is more pronounced if the system is adjusted by the 2-S QR procedure, which also gives rise to a larger number of changes of sign of the preliminary growth rates (about 15.6% *vs.* 10.2%).

Finally, figure 11 shows the percentage differences between preliminary and reconciled estimates for two component series of sales: New car dealers in Ontario and Computer software stores in Saskatchewan.

The first series is the largest of the system, with a share of 7.9%: the adjustment through 2-S QR both in levels and growth rates is consequently smaller, with

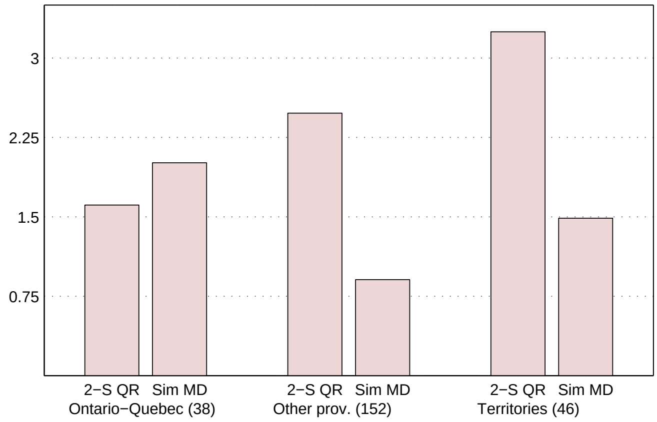

Figure 10: TGPR system. $MSA(r,p)$ for three groups of regions.

Table 12: TGPR system. Some measures on problematic aspects of reconciliation: step problem and signs preservation.

	$MSA(r,p)$		*% signs pres.*	
Procedure	*all months*	*only Jan*	*levels*	*rates*
2-S QR	2.5158	3.0178	100.0	84.41
2-S ST	1.2668	1.8549	100.0	89.82
2-S MD	1.2880	2.0701	100.0	89.78
Sim MD	1.2635	1.8282	100.0	89.82

$MSPA(R_j, P_j)$=1.1% and $MSA(r_j, p_j)$=1.6%, *vs.* 1.8% and 2.5%, respectively, of Sim MD. It can be seen from the graph that the corrections made by Sim MD are more pronounced (max correction 6.3% in absolute value) but share the same pattern as those made by 2-S QR (max correction 4.1% in absolute value).

The second series, a very small one (0.01% of share), when reconciled *via* 2-S QR registers a marked adjustment both in levels ($MSPA(R_j, P_j)$=1.8%) and in growth rates ($MSA(r_j, p_j)$=16.8%, the largest one of the system), while Sim MD leaves this series practically unchanged ($MSPA(R_j, P_j)$=0.2% and $MSA(r_j, p_j)$=0.2%).

7 Conclusions

In this paper we have discussed and applied to two real-life economic datasets a number of procedures for reconciling systems of time series in such a way that all the *a priori* (contemporaneous and/or temporal) constraints be fulfilled while preserving “at the best” the temporal dynamics of the original series. Many aspects of the problem have not been dealt with, like the treatment of non-linear constraints, an evaluation of the procedures’ performance when extrapolation is involved (e.g., when reconciling systems of monthly or quarterly time series when the most recent year is not yet concluded), possible adoption of temporal benchmarking procedures other than modified Denton PFD procedure in two-step procedures, the adoption of

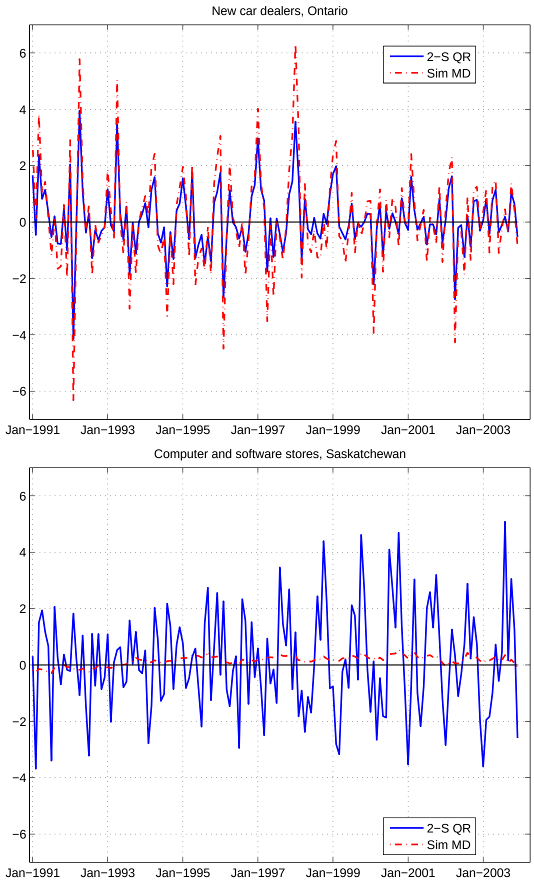

Figure 11: TGPR system. Relative differences (%) between preliminary and reconciled values for two series.

(simple) parameterizations for matrix $\mathbf{\Omega}$ in the objective function of the simultaneous reconciliation, and other issues concerning this important and awkward phase of the data production process.

Nevertheless, we think that on the basis of the results we have found, some remarks are worthy of being stressed.

First, we think that, as a kind of prerequisite when one wishes to reconcile a system of time series, if 'genuine' variability measures (i.e., reliability indicators, either 'true' or coming from evaluations of experts) for one or more variables of interest are available, they should be used in a statistical procedure which explicitly takes care about them, like the least squares adjustment procedure by Stone *et al.* (1942). When (if) no such measures are available, a sensible approach consists in considering all the series to be reconciled of equal reliability, and in looking for reconciliation procedures which pay attention to the *movement preservation principle*.

At this regard, we showed that a straightforward generalization to systems of $M > 1$ time series of the well known Denton's PFD benchmarking criterion is feasible even for large-sized problems, because the possible large matrices involved in the simultaneous reconciliation procedure can be managed without any particular trouble if their pattern, and mostly their sparsity, are conveniently taken into account.

Simple two-step procedures might however be a valid alternative to the simultaneous approach. We considered the raking technique by Quenneville and Rancourt (2005), an extension of the proposal by Stuckey *et al.* (2004) in order to encompass temporal aggregation constraints in the reconciliation, and finally the simultaneous extension of the modified Denton PFD procedure working on a single low-frequency period. All the above two-step procedures share the same temporal benchmarking for each component series (*via* modified Denton PFD) in the first step, the main difference being the way in which the quadratic additive terms in the objective function considered in the second step are normalized.

The results found for a medium-sized and a large-sized system of time series suggest that the simultaneous reconciliation procedure seems to work well in practice. In both cases this procedure demonstrated itself as feasible, provided the practitioner has at disposal a software capable of dealing with linear systems with sparse and large coefficient matrices in efficient way, as most of the modern statistical/mathematical packages are. In addition, and most important, in the two real-life applications we considered, we never found results worse than those obtained by the most performing two-step reconciliation procedures. On the other hand, very good performances have been registered for two-step procedures in which the squared temporal benchmarked series is considered as normalizing factor rather than the (absolute) level. At least for the datasets we used in the paper, these procedures may be considered as valid alternatives to the simultaneous reconciliation procedure.

Finally, we want to stress that, as far as two-step procedures are concerned, contrary to what it has been claimed in the past, the size of the impact on the original (preliminary) dynamics of the series due to the second step is not at all irrelevant, and depends on the reconciliation procedure one chooses. The practitioner should thus control the performance offered by different techniques, for example looking for the one producing, on the whole, the smallest adjustment to the rates of change, or to the levels, or both.

Appendix 1: Matrix formulation of the objective functions (16), (17) and (18)

Function F_T^{BB}, F_T^{ST} and F_T^{MD} can be written in matrix form as

$$F_T^{\nu} = (\mathbf{R}_T - \mathbf{B}_T)' \, \mathbf{\Omega}_T^{\nu} \, (\mathbf{R}_T - \mathbf{B}_T)\,, \quad \nu = BB, ST, QR,$$

where $\mathbf{R}_T$ and $\mathbf{B}_T$ are $(Ms \times 1)$ vectors of, respectively, high-frequency reconciled and temporally benchmarked values for a single low-frequency period T, $T = 1, \ldots, N$. In other words, $\mathbf{R}_T = (\mathbf{R}'_{1T} \ldots \mathbf{R}'_{MT})'$, $\mathbf{B}_T = (\mathbf{B}'_{1T} \ldots \mathbf{B}'_{MT})'$, where $\mathbf{R}_{jT} = \{R_{jt}\}_{t=(T-1)s+1}^{Ts}$, $\mathbf{B}_{jT} = \{B_{jt}\}_{t=(T-1)s+1}^{Ts}$. The $(Ms \times Ms)$ matrices $\mathbf{\Omega}_T^{BB}$ and $\mathbf{\Omega}_T^{ST}$ are both diagonal, with nonzero entries given by $\dfrac{1}{|B_{j,t}|}$ and $\dfrac{1}{B_{j,t}^2}$, respectively. Matrix $\mathbf{\Omega}_T^{MD}$ is in turn given by:

$$\mathbf{\Omega}_T^{MD} = \hat{\mathbf{B}}_T^{-1} \left(\mathbf{I}_M \otimes \mathbf{\Delta}'_s \mathbf{\Delta}_s\right) \hat{\mathbf{B}}_T^{-1},$$

where $\hat{\mathbf{B}}_T = \mathrm{diag}(\mathbf{B}_T)$ and $\boldsymbol{\Delta}_s$ is the $((s-1)\times s)$ first differences matrix. So, for example, for $s=4$ and dropping index j (i.e., $M=1$), we find

$$\boldsymbol{\Omega}_T^{MD} = \begin{bmatrix} \frac{1}{B_1^2} & -\frac{1}{B_1B_2} & 0 & 0 \\ -\frac{1}{B_1B_2} & \frac{2}{B_2^2} & -\frac{1}{B_2B_3} & 0 \\ 0 & -\frac{1}{B_2B_3} & \frac{2}{B_3^2} & -\frac{1}{B_3B_4} \\ 0 & 0 & -\frac{1}{B_3B_4} & \frac{1}{B_4^2} \end{bmatrix}.$$

References

[1] Aceituno Puga, G. (2008), “Consistencia Transversal en Cuentas Nacionales: Métodos de Reconciliación a través de Técnicas de Optimización,” Central Bank of Chile, Studies in Economic Statistics, 66.

[2] Bacharach, M. (1970), *Biproportional matrices and input-output change*, Cambridge University Press, Cambridge.

[3] Beaulieu, J.J., and Bartelsman, E.J. (2004), “Integrating Expenditure and Income Data: What to Do with the Statistical Discrepancy?,” Tinbergen Institute Discussion Paper TI 2004-078/3 (http://www.tinbergen.nl/discussionpapers/04078.pdf).

[4] Bikker, R., and Buijtenhek, S. (2006), “Alignment of Quarterly Sector Accounts to annual data,” CBS Voorburg (http://www.cbs.nl/NR/rdonlyres/D918B487-45C7-4C3C-ACD0-0E1C86E6CAFA/0/Benchmarking_QSA.pdf).

[5] Bloem, A., Dippelsman, R., and Mæhle, N. (2001), *Quarterly National Accounts Manual. Concepts, Data Sources, and Compilation*, International Monetary Fund, Washington DC.

[6] Bozik, J., and Otto, M. (1988), “Benchmarking: Evaluating methods that preserve month-to-month changes,” Bureau of the Census - Statistical Research Division, CENSUS/SRD/RR-88/07.

[7] Buijtenhek, S. (2006), “A model based approach for benchmarking seasonally adjusted time series,” paper presented at the Conference on Seasonality, Seasonal Adjustment and their Implications for Short-Term Analysis and Forecasting, Luxembourg, 10-12 May 2006.

[8] Byron, R. (1978), “The estimation of large social accounts matrices,” *Journal of the Royal Statistical Society A*, **141**, 359–367.

[9] Byron, R. (1993), “A re-estimation of Australia’s quarterly national accounts and balance of payments tables,” Bond University, School of Business Discussion Paper n. 44, August 1993 (http://epublications.bond.edu.au/cgi/viewcontent.cgi?article=1046&context=discussion_papers).

[10] Chen, B. (2006), “A Balanced System of Industry Accounts for the U.S. and Structural Distribution of Statistical Discrepancy,” BEA Working Papers WP2006-8 (http://www.bea.gov/papers/pdf/reconciliation_wp.pdf).

[11] Chen, Z., and Dagum, E. (1997), “A recursive method for predicting variables with temporal and contemporaneous constraints,” American Statistical Association, Proceedings of the Business and Economic Statistics Section, 229-233.

[12] Cholette, P. (1984), “Adjusting sub-annual series to yearly benchmarks,” *Survey Methodology*, **10**, 35–49.

[13] Cholette, P. (1987), "Concepts, definitions and principles of benchmarking and interpolation of time series," Statistics Canada, Time Series Research and Analysis Division, Working Paper No TSRA-87-014E.

[14] Cholette, P. (1988), "Benchmarking systems of socio-economic time series," Statistics Canada, Time Series Research and Analysis Division, Working Paper No TSRA-88-017E.

[15] Cholette, P., and Dagum, E.B. (1994), "Benchmarking time series with autocorrelated survey errors," *International Statistical Review*, **62**, 365–377.

[16] Daalmans, J., and Mushkudiani, M. (2009), "Towards the implementation of a multivariate Benchmark method on Dutch National Accounts," CBS Statistics Nederland, Discussion paper 09012 (www.cbs.nl/NR/rdonlyres/3B11CEEE-28FB-4FF1-A8B4-0A9DD04A1EAF/0/200912x10pub.pdf).

[17] Dagum, E.B., and Cholette, P. A. (2006), *Benchmarking, Temporal Distribution, and Reconciliation Methods for Time Series*, Springer, New York.

[18] Danilov, D., and Magnus, J.R. (2008), "On the estimation of a large sparse Bayesian system: The Snaer program," *Computational Statistics and Data Analysis*, **52**, 4203–4224.

[19] Davis, T. (2006), *Direct Methods for Sparse Linear Systems*, SIAM, Philadelphia.

[20] Davis, T. (2008), "Multifrontal multithreaded rank-revealing sparse QR factorization", (http://www.cise.ufl.edu/ davis/techreports/SPQR/spqr.pdf).

[21] Deming, W., and Stephan, F. (1940), "On a least-squares adjustment of a sampled frequency table when the expected marginal totals are known," *Annals of Mathematical Statistics*, **11**, 427–444.

[22] Denton, F. (1971), "Adjustment of monthly or quarterly series to annual totals: An approach based on quadratic minimization," *Journal of the American Statistical Association*, **66**, 99–102.

[23] Di Fonzo, T. (1990), "The estimation of M disaggregated time series when contemporaneous and temporal aggregates are known," *The Review of Economics and Statistics*, **72**, 178–182.

[24] Di Fonzo, T. (2003), "Temporal disaggregation using related series: log-transformation and dynamic extensions," *Rivista Internazionale di Scienze Economiche e Commerciali*, **50**, 371–400.

[25] Di Fonzo, T., and Marini, M. (2005a), "Benchmarking a system of time series: Denton's movement preservation principle *vs.* a data based procedure," paper presented at the workshop on 'Frontiers in benchmarking techniques and their application to official statistics', Luxembourg, 7-8 April 2005.

[26] Di Fonzo, T., and Marini, M. (2005b), "Benchmarking systems of seasonally adjusted time series," *Journal of Business Cycle Measurement and Analysis*, **2**, 89–123.

[27] Duff, I.S. (2004), "MA57 - a code for the solution of sparse symmetric definite and indefinite systems," *ACM Trans. Math. Software*, **30**, 118–144.

[28] Eurostat (2007), "Demetra. Advanced Seasonal Adjustment Interface," (http://circa.europa.eu/irc/dsis/eurosam/info/data/demetra.htm).

[29] Evans, T.D. (2004), "Analysis of Raking on Seasonally Adjusted Household Gross Flows Data," American Statistical Association, Proceedings of the Business & Economic Statistics Section, 1166–1173.

[30] Fagan, J.T., and Greenberg, B.T. (1988), "Algorithms for making tables additive: raking, maximum likelihood and minimum chi-square," American Statistical Association, Proceedings of the Section on Survey Research Methods, 467–472.

[31] Friedlander, D. (1961), "A technique for estimating a contingency table, given the marginal totals and some supplementary data," *Journal of the Royal Statistical Society A*, **124**, 412–420.

[32] Helfand, S., Monsour, N., and Trager, M. (1977), "Historical revision of current business survey estimates," American Statistical Association, Proceedings of the Business and Economic Statistics Section: 246-250.

[33] Kim, C., Salou, G., and Rossiter, P. (1994), "Balanced Australian National Accounts," Australian Bureau of Statistics, Working Papers in Econometrics and Applied Statistics, 94/2.

[34] Magnus, J.R. (1990), "On the fundamental bordered matrix of linear estimation," in van der Ploeg, F. (ed.), *Advanced Lectures in Quantitative Economics*, Academic Press, London, 583–604.

[35] Quenneville, B., Cholette, P., Huot, G., Chiu, K., and Di Fonzo, T. (2004), "Adjustment of Seasonally Adjusted Series to Annual Totals," American Statistical Association, Proceedings of the Business & Economic Statistics Section, 1294–1301.

[36] Quenneville, B., and Fortier, S. (2006), "Balancing seasonally adjusted series as a complement to the direct and indirect approaches to seasonal adjustment," American Statistical Association, Proceedings of the Business & Economic Statistics Section, 1118–1125.

[37] Quenneville, B., Fortier, S., and Gagnè, C. (2009), "A non-parametric iterative smoothing method for benchmarking and temporal distribution," *Computational Statistics & Data Analysis*, **53**, 3386–3396.

[38] Quenneville, B., and Rancourt, E. (2005), "Simple methods to restore the additivity of a system of time series," paper presented at the workshop on 'Frontiers in benchmarking techniques and their application to official statistics', Luxembourg, 7-8 April 2005.

[39] Rampa, G. (2008), "Using Weighted Least Squares to Deflate Input-Output Tables," *Economic Systems Research*, **20**, 259–276.

[40] Round, J.I. (2003), "Constructing SAMs for Development Policy Analysis: Lessons Learned and Challenges Ahead," *Economic Systems Research*, **15**, 161–183.

[41] Saad, Y. (2003), *Iterative Methods for Sparse Linear Systems*, SIAM, Philadelphia.

[42] Statistics Canada (2009), "Retail Trade. February 2009," (http://www.statcan.gc.ca/pub/63-005-x/63-005-x2009002-eng.pdf).

[43] Stone, R., Champernowne, D., and Meade, J. (1942), "The precision of national income estimates," *Review of Economic Studies*, **9**, 111–125.

[44] Stuckey, A., Zhang, X., and McLaren, C.H. (2004), "Aggregation of Seasonally Adjusted Estimates by a Post-Adjustment," (www.uow.edu.au/ craigmc/abs_agg_2004.pdf).

[45] Taillon, J. (1988), "Dessaisonalisation d'un tableau avec contraintes linéaires: Application aux flux financiers canadiens," *The Canadian Journal of Statistics*, **16**, 97–107.

[46] U.S. Census Bureau (2007), *X-12-ARIMA Reference manual, Version 0.3*, U.S. Census Bureau, U.S. Department of Commerce.

[47] van der Ploeg, F. (1982), "Reliability and the adjustment of sequences of large economic accounting matrices," *Journal of the Royal Statistical Society A*, **145**, 169–194.

Acknowledgements

Many thanks are due to Benoit Quenneville, who made the raw data of the Canadian Monthly Retail Trade Survey available to us, for his support, advices and critical assessment of the early stages of our work. Obviously, we alone are responsible for the errors of the paper. The views contained here are those of the authors and do not necessarily reflect those of the institutions for which they work. Financing from MIUR is also gratefully acknowledged by T. Di Fonzo.

Working Paper Series
Department of Statistical Sciences, University of Padua

You may order paper copies of the working papers by emailing wp@stat.unipd.it

Most of the working papers can also be found at the following url: http://wp.stat.unipd.it

UNIVERSITA DEGLI STUDI DI PADOVA
DIPARTIMENTO DI SCIENZE STATISTICHE

Department of Statistical Sciences
University of Padua
Italy