Working Paper Series, N. 13, September 2008

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UNIVERSITÀ DEGLI STUDI DI PADOVA
DIPARTIMENTO DI SCIENZE STATISTICHE

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Keywords: bootstrap for time series, long memory, GPH and LW estimator, confidence intervals.

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1 Introduction

Long memory can been observed in natural, economic and financial series and it has been deeply studied in the past years giving raise to a wide and rich literature. We concentrate our attention on AutoRegressive Fractionally Integrated Moving Average (hereafter ARFIMA) processes, a useful and flexible tool to model the aforesaid time series. These processes exhibit long memory when the parameter d assumes any real value in the interval $(0, 0.5)$. Recent and past literature focused on the estimation of the long memory parameter d with parametric and semi-parametric techniques: among the others, we recall the parametric Whittle estimator (Fox and Taqqu, 1986), the semi-parametric GPH (Geweke and Porter-Hudak, 1983) and local Whittle (Robinson, 1995a). Parametric estimators are very efficient when the model is correctly specified but exhibit large biases otherwise, whereas semi-parametric and non-parametric techniques can estimate only the long memory parameter d but they have large standard deviations and slow convergence rates (Bisaglia and Guègan, 1998).

Despite the existence of a wide literature on bootstrap procedures for time series (see, Davison and Hinkley, 1997; Li and Maddala, 1996), at the moment there are no satisfactory bootstrap methods to replicate long memory. On the other hand, short and long memory can confound each other when the parameters, describing the short memory behaviour, are near the boundary of non-stationarity. Bootstrap methods are widely used to improve estimators or to build confidence intervals for the parameters. Usually, they provide estimators with smaller standard errors and confidence intervals with a coverage level closer to the nominal level than confidence intervals obtained by applying asymptotic results.

In this paper we investigate an alternative bootstrap method based on the empirical autocorrelation function and the Durbin-Levinson algorithm that seems to give satisfactory performance especially with Gaussian long memory processes. Even though the method is equivalent to a Cholesky decomposition, its applicability is wider and has some interesting advantages. Cholesky decomposition requires the inversion of a square matrix of the same dimension as the length of the observed series. Increasing the length of the series, also the most powerful calculators can have problems to finish the calculation in a reasonable time and anyway the effect of rounding can lead to huge and unpredictable errors. On the contrary, the method we introduce is iterative and avoids the problem of large matrices. The *ACF bootstrap*, as we call it, is based on a result of Ramsey (1974) (see below) and requires Gaussianity of the observed process X_t. This assumption is quite restrictive, however we will show that some deviations from Normality do not affect substantially the method.

We assess, by mean of a wide Monte Carlo experiment, the validity of ACF bootstrap for ARFIMA(p, d, q) processes in three different scenarios:

- ACF bootstrap improves the performance of semi-parametric estimators of the memory parameter d;
- the proposed method is robust against non-Gaussian innovations, asymmetry and fat tails, and
- it gives better results if applied to build confidence intervals for d in terms of coverage level.

In the first scenario, we perform an extensive Monte Carlo experiment to estimate the memory parameter d when the data generating process (shortly DGP) is a fractionally integrated noise (ARFIMA$(0, d, 0)$). We compare ACF bootstrap performance with the performance of other well known bootstrap procedures like local and sieve bootstraps in terms of reduction of standard error and mean square error of the estimates.

Then, we conduct experiments on processes to test robustness of the ACF bootstrap when the observed series is non-Normal, using Chi-squared and Student t innovations to test against skewness and fat tails, respectively.

In the last scenario, we aim to improve the coverage of confidence intervals for the memory parameter d in two different situations. Firstly, we consider the Whittle estimator. Even though the Whittle estimator is asymptotically normal, if the assumption of correctly specified model is satisfied then confidence intervals based on short series ($n = 128, 300$) have an actual coverage level lower than the nominal coverage level. Secondly, we study the confounding effects when both long and short memory are present in the series. It has already been highlighted by Agiakloglou et al. (1993) that short memory introduces bias in the GPH estimates, and also the coverage of confidence intervals is affected.

The plan of the paper is the following. In Section 2 we briefly introduce long memory $ARFIMA(p, d, q)$ processes and the estimation techniques we use to estimate the long memory parameter d. Section 3 recalls bootstrap for long memory time series. Section 4 is dedicated to introducing the new bootstrap method: the ACF bootstrap. Sections 5 and 6 present the Monte Carlo results on the estimation of the memory parameter d and on the bootstrap confidence intervals. Conclusions are reported in the last section.

2 ARFIMA(p, d, q) processes: an introduction

There exist different definitions of long memory processes. In the time domain, a stationary discrete time series is said to be long memory if its autocorrelation function decays to zero like a power function. This definition implies that the dependence between successive observations decays slowly as the number of lags tends to infinity. On the other hand, in the frequency domain, a stationary discrete time series is said to be long memory if its spectral density is unbounded at low frequencies. Other definitions are equivalent and can be found in Beran (1994). More recently the paper of Boutahar et al. (2007) provides an updated review on the topic.

In this paper we consider one of the most popular long memory processes that takes into account this particular behaviour of the autocorrelation and of the spectral density function, i.e. the ARFIMA(p, d, q), independently introduced by Granger and Joyeux (1980) and Hosking (1981). This process simply generalizes the usual ARIMA(p, d, q) process by allowing d to assume any real value.

Let ε_t be a white noise process having $E[\varepsilon_t^2] = \sigma^2$. The process $\{X_t,\ t \in \mathbf{Z}\}$ is said to be an ARFIMA(p, d, q) process with $d \in (-0.5, 0.5)$, if it is stationary and satisfies the difference equation

$$\Phi(B)\,\Delta(B)\,(X_t - \mu) = \Theta(B)\,\varepsilon_t, \tag{1}$$

where $\Phi(z)$ and $\Theta(z)$ are polynomials of degree p and q, respectively, satisfying $\Phi(B) \neq 0$ and $\Theta(B) \neq 0$ for all z such that $|z| \leq 1$, B is the backward shift operator, $\Delta(B) = (1 - B)^d = \sum_{j=0}^{\infty} \pi_j B^j$ with $\pi_j = \Gamma(j - d)/[\Gamma(j + 1)\Gamma(-d)]$, and $\Gamma(\cdot)$ is the gamma function.

If $p = q = 0$ the process $\{X_t,\ t \in \mathbf{Z}\}$ is called Fractionally Integrated Noise and denoted by $I(d)$. When $d \in (0, 0.5)$ the ARFIMA(p, d, q) process is stationary and the autocorrelation function decays to zero hyperbolically at a rate $O(k^{2d-1})$, where k denotes the lag. In this case we say that the process has a long memory behaviour. When $d \in (-0.5, 0)$ the ARFIMA(p, d, q) process is a stationary process with intermediate memory. In the following we will concentrate on $I(d)$ processes with $d \in (0, 0.5)$: for this range of values the process is stationary, invertible and possesses long range dependence. Without loss of generality, we will assume for convenience and without loss of generality that $\sigma^2 = 1$ and $\mu = 0$.

The estimation of the long memory parameter d has been of interest to many authors. Many estimators are well described in Palma (2007). In this contest, three of the most common will be considered. We will try to improve the semi-parametric estimators local Whittle and GPH, while the parametric Whittle is used as benchmark.

2.1 The Whittle estimator

Fox and Taqqu (1986) introduced a maximum likelihood method based on the frequency domain, i.e. the MLE is found by maximizing the function

$$\frac{1}{\sigma} \exp\left\{-\frac{\mathbf{Z}T_n^{-1}\mathbf{Z}}{2n\sigma^2}\right\},$$

where $\mathbf{Z} = (\mathbf{X}_n - \hat{\mu}\mathbf{1})$, $\vartheta = (\sigma^2, d, \phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)$ is the vector of parameters of dimension $m = p + q + 2$, $\hat{\mu}$ is a consistent estimate of the mean of the process (e.g., the sample mean $\bar{X}$), $\mathbf{1}$ is a column vector of ones and $T_n(\vartheta)$ is the Toeplitz matrix of generic element j, k

$$T_{n;j,k}(\vartheta) = \int_{-\pi}^{\pi} \exp\{i\omega(j-k)\} f(\omega;\vartheta) d\omega \qquad j, k = 1, 2, \ldots, n,$$

with $f(\omega;\vartheta)$ the spectral density of the process X_t where we highlight the dependence on the parameter ϑ. They followed a suggestion of Whittle (1951), who proposed to use an approximation to invert the Toeplitz matrix $T_n(\vartheta)$. By Parseval's relation it is possible to show that a good approximation for T_n^{-1} is given by the matrix $A_n(\vartheta)$ of generic element j, k

$$a_{j,k}(\vartheta) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \exp\{i(j-k)\omega\} \left[f(\omega;\vartheta)\right]^{-1} d\omega.$$

This estimator extends the results of Hannan (1973), who applied Whittle's method to the estimation of the parameters of ARMA models. Fox and Taqqu's result, later generalized by Dahlhaus (1989) to the exact maximum likelihood estimator, is the basis of one of the most used methods for estimating the long (and short, if both are present) memory parameters in Gaussian time series. Giraitis and Surgailis (1990) generalized the result of Fox and Taqqu in order to prove the asymptotic normality of Whittle's estimator relaxing the Gaussianity assumption.

The exact maximum likelihood estimator has the drawback of requiring a large computational burden and it might also cause computational problems when calculating the autocovariances needed to evaluate the likelihood function (Sowell, 1992). These difficulties do not occur when using the Whittle estimator, which has the further advantage of not requiring the estimation of the mean of the series (generally unknown in practice). Besides, under some regularity assumptions (Fox and Taqqu, 1986; Dahlhaus, 1989) fulfilled by ARFIMA(p, d, q) processes, it is possible to prove that the Whittle estimator has the same asymptotic distribution as the exact maximum likelihood estimator and it converges to the true value of the parameter at the usual rate of $n^{-1/2}$. Eventually, for Gaussian processes the Whittle estimator is asymptotically efficient in the sense of Fisher.

If the Whittle approximation to the log-likelihood function is used, the parameter vector ϑ is estimated by minimizing, with respect to ϑ, the estimated variance of the underlying white noise process:

$$\sigma^2(\vartheta) = \frac{1}{2\pi} \sum_{j=1}^{[(n-1)/2]} \frac{I(\omega_j)}{f(\omega_j; \vartheta)},$$

where $f(\omega_j; \vartheta)$ indicates the spectral density of the ARFIMA process at the Fourier frequencies $\omega_j = (2\pi j/n)$, $j = 1, 2, \ldots, n^*$ and n^* is the integer part of $(n-1)/2$.

The drawback of this estimator is that it is necessary to assume the parametric form of the spectral density to be known *a priori*. If the specified spectral density function is not the correct one (as it is often the case), the estimated parameters may be dramatically biased.

2.2 The GPH estimator

This is one of the best known methods to estimate in a semi-parametric way the fractional parameters d of long range dependence behaviour. The advantage of this method is that the specification of the model is not really necessary because the only information we need is the behaviour of the spectral density near the origin. Furthermore, the long memory parameter can be estimated alone.

This method was first introduced by Geweke and Porter-Hudak (1983) for the Gaussian case when d belongs to $(-0.5, 0)$ and then it was developed by Robinson (1995b).

Assume that the process $\{X_t\}$, $t = 1, 2, \ldots, n$, is an ARFIMA(p, d, q) model as defined in equation (1), then we can observe that the spectral density of this model is proportional to $(4\sin^2(\omega/2))^{-d}$ near the origin, i.e.

$$f(\omega) \sim c_f(4\sin^2(\omega/2))^{-d}, \tag{2}$$

when ω tends to 0, c_f being a slowly varying function at zero. Since the periodogram $I(\omega)$ is an asymptotically unbiased estimate of the spectral density, i.e. $\lim_{\omega\to 0} E[I(\omega)] = f(\omega)$, it is possible to estimate d applying the least squares method to the equation

$$\log(I(\omega_j)) = \log c_f - d\log(4\sin^2(\omega_j/2)) + u_j \tag{3}$$

where u_j, $j = 1, 2, \ldots, n^*$ are i.i.d. error terms, ω_j are the Fourier frequencies defined above.

Equation (2) is an asymptotic relation that holds only in a neighbourhood of the origin, thus if we use this relation from all periodogram ordinates $(-\pi < \omega < \pi)$

the estimator of d can be highly biased. Geweke and Porter-Hudak (1983) proposed to consider only the first $\sqrt{n}$ frequencies since the long memory feature influences mostly the lower frequencies. The higher frequencies are influenced only by the short memory ARMA part.

An interesting advantage with respect to the Whittle is that the GPH estimator can be easily applied without specifying the orders p and q of the ARMA part. The main drawback of this estimator is its high standard deviation. Moreover Agiakloglou et al. (1993) showed that it is biased in presence of ARMA parameters near the non-stationary boundary.

2.3 The local Whittle estimator

The local Whittle estimator is another semi-parametric estimator of the memory parameter d developed by Robinson (1995a) following a suggestion of Künsch (1987). Robinson (1995a) demonstrated that the local Whittle estimator is asymptotically more efficient than the GPH in the stationary case, although it is not defined in closed form and numerical optimization methods are needed to calculate it.

It can be found minimizing the following expression:

$$R(d) = \log \left[\frac{1}{m} \sum_{j=1}^{m} \omega_j^d I(\omega_j) \right] - d \frac{1}{m} \sum_{j=1}^{m} \log \omega_j^d, \tag{4}$$

where $I(\omega_j)$ is the periodogram at the Fourier frequencies and m is an integer less than $n/2$.

Under slight conditions Robinson (1995a) showed that this estimator is weakly consistent. Moreover, under stronger conditions, he proved the asymptotic normality even if the convergence rate is slower than in the Whittle case. The rate depends on $m^{1/2}$, the number of frequencies considered in the estimate. Usually it is considered a value of $m = \lfloor \sqrt{n} \rfloor$. Thus, the local Whittle estimate is much less efficient than parametric estimates, like, for example, the Whittle one, when they happen to be based on a correct model, but it is asymptotically more efficient than the GPH estimate.

3 Bootstrap for time series

Bootstrap methods were introduced firstly by Efron (1979) and they have since become a popular statistical tool caused by their easiness of use combined to the advent of strong calculators. For a review of the bootstrap methodology, see Hinkley (1988); monographs on the topic include Efron and Tibshirani (1993) and Davison and Hinkley (1997).

Special care is needed when applying bootstrap techniques to time series analysis, since the correlation structure among the variables is possibly complicated and simple methods designed for independent and identically distributed variables are not appropriate. Li and Maddala (1996) discussed the difficulties found in the use of bootstrap for time series models, and gave some guidelines. More recently, Bühlmann (2002) and Politis (2003) review and compare some bootstrap methods for time series illuminating theoretical aspects of the procedure as well as their performance on finite-sample data. In spite of the great number of papers on bootstrap techniques for time series, the problem is still open since these techniques are not always satisfactory, especially if the time series exhibits long range dependence.

In this section we define the bootstrap methods used when data present long memory behaviour. Parametric bootstrap is not considered since it requires the knowledge of the correct model to work well. We briefly introduce sieve bootstrap, local bootstrap and log-periodogram regression.

3.1 Sieve bootstrap

The sieve bootstrap was first introduced by Kreiss (1992) and then developed by Bühlmann (1997). This method is based on the idea of sieve approximation: it approximates a general linear, invertible process by a finite autoregressive model with order increasing with the series length, and resampling from the approximated autoregressions. By viewing such autoregressive approximations as a sieve for the underlying infinite-order process, the bootstrap procedure may still be regarded as a non-parametric one. This method is computationally simple and yields a (conditionally) stationary bootstrap sample that does not exhibit artefacts in the dependence structure. In a very recent paper, Kapetanios and Psaradakis (2006) study the properties of the sieve bootstrap for a class of linear processes with long range dependence. The authors established the first order asymptotic validity of the sieve bootstrap in the case of the sample mean and sample autocovariances, but the results of a Monte Carlo experiment are disappointing. Poskitt (2008) proves some theoretical results on the validity of the sieve bootstrap to replicate Gaussian long memory.

Given the time series $X_1, X_2, \ldots, X_n$, the scheme for the sieve bootstrap is as follows. Fit an AR($p(n)$) model to the data choosing the optimal p using the AIC criterion. It is important to note that we fit the autoregressive process with increasing order $p(n)$ as the sample size n increases. Estimate the residuals:

$$\hat{\varepsilon}_{t,n} = \sum_{j=0}^{p(n)} \hat{\phi}_{j,n}(X_{t-j} - \bar{X}), \quad \hat{\phi}_{0,n} = 1 \quad (t = p+1, \ldots, n), \tag{5}$$

where $\bar{x}$ is the sample mean and $\hat{\phi}_{j,n}$ are the autoregressive coefficients' estimates. Before bootstrapping the residuals, they have to be centred. At last each bootstrap replicate can be calculated using the following recursion:

$$\sum_{j=0}^{p(n)} \hat{\phi}_{j,n}(X^*_{t-j} - \bar{X}) = \hat{\varepsilon}^*_{t,n} \tag{6}$$

where $\hat{\varepsilon}^*_{t,n}$ are the centred bootstrapped residuals.

3.2 The local bootstrap

Paparoditis and Politis (1999) have proposed the non-parametric local bootstrap for weakly dependent stationary processes. It produces surrogate versions of the periodogram $I(\omega_j)$ of the observed process $\{X_t\}$ so that it is useful when the aim is to make inference through the spectrum (e.g. confidence interval for the memory parameter d in case of long memory).

Silva et al. (2006) apply the local bootstrap to the estimation of the long memory parameter d and, by means of simulations, compare its performance with that of other bootstrap approaches. The authors established the efficacy of the local bootstrap in terms of low bias, short confidence intervals and low CPU times.

Given the data $X_1, \cdots, X_n$, the local bootstrap algorithm that generates bootstrap replicates $I_x^*(\omega_j)$, $j = 0, 1, \cdots, n^*$ of the periodogram can then be described as follows.

1. Select a resampling width k_n where $k_n = k(n) \in \mathbb{N}$ and $k_n \leq [n/2]$.

2. Define i.i.d discrete random variables $J_1, J_2, \cdots, J_N$ taking values in the set $\{-k_n, -k_n+1, \cdots, k_n\}$ with probability $p_{k_n,s}$, i.e. $P(J_i = s) = p_{k_n,s}$ for $s = 0, \pm 1, \cdots, \pm k_n$ such that $p_{k_n,s} = p_{k_n,-s}$.

3. The bootstrap periodogram is then defined by $I_X^*(\omega_j) = I_X(\omega_{J_j+j})$ for $j = 1, 2, \cdots, n/2$, $I_X^*(\omega_j) = I_X^*(-\omega_j)$ for $\omega_j \leq 0$ and for $\omega_j = 0$ we set $I_X^*(0) = 0$.

Paparoditis and Politis (1999) have showed that the local bootstrap is asymptotically valid but some care should be taken for the choice of the resampling widths k_n, in the case of a finite sample size n. Following a suggestion of Silva et al. (2006), in this paper we will consider $k_n = 1$ with uniform sample probability since the results are very similar when $k_n = 2$.

3.3 The log-periodogram regression

This method has been introduced by Arteche and Orbe (2005) to improve the efficiency of the GPH estimator in presence of short memory behaviour. At the moment the applicability of the method is specific only to the GPH estimator of the memory parameter.

It assumes the residuals of the regression model given in Equation (3) to be independent and identically distributed. The three steps to obtain the bootstrap distribution of $\hat{d}$ are quite straightforward.

1. Calculate the least-squared estimates of $a = \log\{\sigma^2 f_\varepsilon(0)2\pi\}$ and d to estimate the residuals $\hat{u}_j = \log(I(\omega_j)) - \hat{a} + \hat{d}\log(4\sin^2(\omega_j/2))$;

2. resample B bootstrap samples from the residuals $\hat{u}_j$. Using the empirical distribution function of the residuals we obtain the corresponding bootstrap dependent variable $\log(I(\omega_{j,N})) = \hat{a} - \hat{d}\log(4\sin^2(\omega_j/2)) + \hat{u}_j^*$;

3. estimate d from the new models and compute its bootstrap distribution.

Even if the method is very specific, it gave nice results on building confidence intervals for d and we will compare its performance with the new bootstrap introduced in a later section of this paper.

3.4 Confidence intervals

We dedicate this section to study the performance of the ACF bootstrap to build confidence intervals for the memory parameter in two different scenarios. This is an interesting problem and is directly related with the problem of testing the hypothesis of existence of long memory.

We analyse the finite sample performance of the Whittle estimator in building confidence intervals for the memory parameter and test whether or not they can be improved by building confidence intervals based on bootstrap replications.

As we already pointed out, it is useful to estimate and study long range behaviour separately from short range memory, because the two behaviours tend to confound each other and it can be difficult to distinguish between them. This is not possible with parametric estimators since we need to specify the whole model *a priori*. On the other hand, Agiakloglou et al. (1993) showed that the GPH estimator is influenced by the short memory part and its bias increases when the parameters of the short memory part approach the non-stationarity boundary. In their paper Arteche and Orbe (2005) reduced the coverage error of confidence intervals for the memory parameter built with the GPH estimator. Their method is specifically designed for, and limited to, the GPH estimator. The methodology proposed in this paper, the ACF bootstrap, has a more general applicability and can be used not only with the GPH estimator but also with other estimators of d, such as the local Whittle and the Whittle estimators.

In the following we introduce briefly four of the most common confidence intervals for the memory parameter d, describing their principal advantages and drawbacks. We consider the methods that gave best results in the paper of Arteche and Orbe (2005): they performed simulations with many methods, but not all improved the actual coverage level and it is not worth to consider them.

1. *Asymptotic distribution of* $\hat{d}$: this interval is based on the asymptotic distribution of $\hat{d}$ and is symmetric by construction. It is given by

$$\mathrm{CI}_{se}(1-\alpha) = \hat{d} \pm z_{\alpha/2}\mathrm{se}(\hat{d}),$$

where z_α is the 100α percentile of the standard normal distribution.

2. *Percentile confidence intervals*:

$$\mathrm{CI}_{pc}(1-\alpha) = \left(\hat{d}^*_{\alpha/2}, \hat{d}^*_{1-\alpha/2}\right),$$

where d^*_α is the 100α percentile of the bootstrap distribution of $\hat{d}^*$. This interval can be asymmetric but it is equal-tailed.

3. *Percentile-t confidence intervals* (Hall, 1988, 1992)

$$\mathrm{CI}_{pt}(1-\alpha) = \left(\hat{d} - se(\hat{d})\hat{t}_{1-\alpha/2}, \hat{d} - se(\hat{d})\hat{t}_{\alpha/2}\right),$$

where t_α is the 100α percentile of $t = (\hat{d}^* - \hat{d})/\hat{se}(\hat{d}^*)$. Percentile-$t$ has been criticized because it produces bad results if the estimate of the variance is poor and because it is not invariant to transformations.

4. *Bootstrap standard error confidence intervals*

$$\mathrm{CI}_{se^*}(1-\alpha) = \hat{d} \pm z_{\alpha/2}\mathrm{se}^*(\hat{d}^*).$$

4 ACF bootstrap

In this section we introduce an alternative bootstrap method based on a theorem of Ramsey (1974) that derives the distribution of X_t conditionally on the past values $X_0, \ldots, X_{t-1}$ of the process. Its distribution is Normal with mean and variance given in the theorem below if the observed process is Gaussian itself.

Theorem 4.1: *Let X_t be a Gaussian, wide-sense stationary time series with mean μ and variance γ_0. Then the conditional distribution of X_t given $X_0, \cdots, X_{t-1}$ is Gaussian with mean and variance given by*

$$m_t = \mathrm{E}(X_t | X_0, \cdots, X_{t-1}) = \sum_{j=1}^{t} \phi_{tj} X_{t-j}, \tag{7}$$

$$v_t = \mathrm{Var}(X_t | X_0, \cdots, X_{t-1}) = \gamma_0 \prod_{j=1}^{t} (1 - \phi_{jj}), \tag{8}$$

where ϕ_{jj} is the jth partial autocorrelation and ϕ_{tj} is the jth autoregressive coefficient in an autoregressive fit of order t.

Proof. See Ramsey (1974).

The coefficients ϕ_{tj} and ϕ_{jj} can easily be obtained through the Durbin-Levinson (see, e.g., Brockwell and Davis, 1991) recursion:

$$\phi_{tt} = N_t / D_t \tag{9}$$

$$\phi_{tj} = \phi_{t-1,j} - \phi_{tt}\phi_{t-1,t-j}, \quad j = 1, \cdots, n-1, \tag{10}$$

where

$$\begin{aligned} N_0 &= 0 \\ D_0 &= 1 \\ N_t &= \rho_t - \sum_{j=1}^{t-1} \phi_{t-1,j}\rho_{t-j} \\ D_t &= D_{t-1} - N_{t-1}^2 / D_{t-1} \end{aligned}$$

and ρ_t is the autocorrelation function of X_t at lag t.

The hypotheses of Theorem 4.1 admit all processes with an MA-infinite representation, e.g., stationary ARMA processes, ARFIMA processes with $0 \le d < 1/2$.

Instead of using a theoretical autocovariance function, the idea is to use the empirical autocorrelation function of an observed time series to generate bootstrap copies through the conditional mean and the conditional variance given in Equations (7)-(8). The steps to generate a bootstrap series X_t^* are:

1. compute the empirical autocorrelation function, $\hat{\rho}_k$, from the observed time series X_t;

2. perform the Durbin-Levinson recursion, given in Equations (9) and (10), for $\hat{\phi}_{tt}$ and $\hat{\phi}_{tj}$ based on the empirical autocorrelation function;

3. generate a starting value of X_0^* from an $N(0, v_0)$ distribution where v_0 is the sample variance of X_t;

4. calculate v_t based on Equation (8) and m_t^* as follows

$$m_t^* = E(X_t^* | X_0^*, \cdots, X_{t-1}^*) = \sum_{j=1}^{t} \hat{\phi}_{tj} X_{t-j}^*,$$

   and thus m_t^* depends on the past values of the bootstrap series and the observed autocorrelation function of the original one; and

5. generate the bootstrap replicate of X_t^* from $N(m_t^*, v_t)$; and

6. repeat steps 4 and 5 until $t = n$, where n is the series length.

It is interesting to notice that the conditional means m_t^* depend on each bootstrap time series, whereas the conditional variances v_t are determined from the observed process and do not change for each bootstrap replication. The steps 3 to 6 have to be repeated for $b = 1, \ldots, B$, where B is the total number of bootstrap replicates. We omit the b subscript in the following when it is clear that we are referring to bootstrap replicates and the double subscript (e.g. $X_{t,b}^*$) would be redundant.

5 Monte Carlo and bootstrap estimates of the memory parameter

In this section we conduct experiments with simulated data to assess the validity of the ACF bootstrap method with respect to the existing methods in the literature. In particular, we apply the proposed bootstrap method to long memory time series. We use the ACF, the sieve and the local bootstrap methods to replicate the observed series, and GPH and local Whittle estimators to estimate the long memory parameter d.

As we said in the introduction, we are interested in fractionally integrated processes, and especially in improving the performance of two semi-parametric estimators for the memory parameter d: the GPH and the local Whittle. Both of them estimate the parameter d through the periodogram of the observed series. We use the Whittle estimator as a benchmark since it is a parametric estimator used without the risk of misspecification, given the parametric assumptions in our simulation, thus it is the most efficient.

In the simulation study we generated series by $I(d)$ models for different values of the long memory parameter, $d = 0.1, 0.2, 0.3, 0.4, 0.45$, and increasing sample sizes, $n = 200, 500, 1000$.

The series are generated using the recursive Durbin-Levinson algorithm (see Brockwell and Davis, 1991). For each model we consider $S = 1000$ realizations and $B = 1000$ bootstrap replications. For each estimation method we calculate the Monte Carlo estimate, i.e.,

$$\hat{d} = \frac{1}{S}\sum_{j=1}^{S}\hat{d}_j,$$

where $\hat{d}_j$ is the estimated value for a single realization obtained with one of the estimators (Whittle, GPH, local Whittle) or the average of the bootstrap estimates. To compare the performance of estimators and bootstrap methods, we compute standard errors

$$\hat{\text{se}}(\hat{d}) = \sqrt{\frac{1}{S-1}\sum_{j=1}^{S}(\hat{d}_j - d)^2}$$

and mean squared errors

$$\hat{\text{MSE}}(\hat{d}) = \text{Var}(\hat{d}) + \text{Bias}(\hat{d})^2 = \frac{1}{S-1}\sum_{j=1}^{S}(\hat{d}_j - d)^2 + (\hat{d} - d)^2$$

The results are presented in Tables 1-2 where also the Whittle estimator is included as a benchmark. The tables report results on $\hat{d}$ (in boldface), standard error of $\hat{d}$ (italic font) and MSE of $\hat{d}$ (normal font) for the two estimators treated and for the three bootstrap methods.

The Monte Carlo estimates are in accordance with known results (see, for example, Bisaglia and Guègan, 1998). As we expected, the Whittle estimator largely outperforms all the others, since it is a parametric estimator in the best conditions, i.e., the estimates are based on the correctly specified parametric model.

With regard to the other two methods, both gave satisfactory results compared with the same estimators in the Monte Carlo simulations: the ACF is a slightly more biased but its standard deviation and the mean squared error are always smaller than using the local bootstrap.

Table 2 reports the gain, namely

$$\text{GAIN\%} = \frac{\hat{\text{se}}(\hat{d}_i) - \hat{\text{se}}(\hat{d}_i^*)}{\hat{\text{se}}(\hat{d}_i)} \times 100 \tag{11}$$

(where i = GPH, lW) calculated as a percentage, when using the ACF and local bootstrap with respect to the Monte Carlo estimates, for the GPH and local Whittle bootstrap estimators. The results confirm that the gain is always greater for the ACF bootstrap even if it decreases with increasing the series length. It is interesting that for the local bootstrap the gain is almost irrelevant for $n = 500$ and negative for $n = 1000$.

5.1 Non-Gaussian innovations

The assumption of Gaussianity is very restrictive and it would be interesting to see how much deviations from Normality affect the performance of the proposed bootstrap methods even though Gaussianity is one of the assumptions of Theorem 4.1. To this end we perform some simulations to compare the estimators when the observed process is non-Gaussian. We consider two different deviations from Normality. To test robustness against asymmetry, we generate long memory processes with Chi-squared innovations with one degree of freedom, giving skewness $\gamma_1 = 2\sqrt{2}$. To test robustness against fat tails, we use the Student t distributions with four and six degrees of freedom: the former does not have the fourth moment finite, the latter has excess of kurtosis $\gamma_2 = 3$.

In Tables 3-5 we report the results. All the estimates, Monte Carlo and bootstrap, are very similar to the results obtained with Gaussian innovations, in terms of both standard error and mean squared error. This suggests that ACF bootstrap can be useful also relaxing the Gaussianity assumption. There is not the danger of obtaining bad results when there is the suspect of non-Gaussian innovations and it is not necessary to correct or exclude extreme values. Also in this case the standard error and the mean squared error are smaller for the bootstrap estimates. These results are very important in view of applying the method to replicate the dependence structure of heteroskedastic data, such as data with GARCH effects or stochastic volatility processes.

6 Monte Carlo and bootstrap confidence intervals for the memory parameter

In this section we describe two Monte Carlo experiment, which aim is to show the improvements, given by bootstrap methods, in estimating confidence intervals for the memory parameter d.

6.1 Finite sample performance of ACF bootstrap

Even if the Whittle estimator is very efficient in the case of correct specification of the model, its performance in estimating long memory for small samples is not very good. Also the nominal level of confidence intervals for the memory parameter d is usually far from the actual level. Especially when detecting long range behaviour, it is necessary to have quite long series. We deem that bootstrap methods can improve the coverage level and give satisfactory results with finite sample sizes. We compare Monte Carlo results of the Whittle estimator with the results given applying the ACF and local bootstrap.

Following the simulation plan of Arteche and Orbe (2005), we run simulations with $n = 200$, 500, 1000 and $d = 0, 0.2, 0.45, -0.45$. For each model we consider $S = 1000$ realizations and $B = 1000$ bootstrap replications. The results are given in Table 6. For small values of the parameter, $d = 0, 0.2$, the confidence intervals based on ACF bootstrap are all better than the Monte Carlo intervals. The best results are given by the percentile method. For larger values of the parameter in absolute value, $d = |0.45|$, it appears to be more difficult to have actual coverage close to the nominal, but the intervals obtained with the standard deviation and with percentile of ACF bootstrap outperform the asymptotic results especially for small sample sizes. Overall the best intervals are built with the standard deviation estimated using the ACF bootstrap.

The confidence intervals built with the local bootstrap give very poor results. Only the t percentile method gives reasonable results for $d = 0, 0.2$ but these intervals are very similar to the Monte Carlo, and it is not worth using a bootstrap method if it does not lead to any improvement.

6.2 The influence of the short memory part

It is known that the semi-parametric estimators, which we introduced above, are biased in the presence of short memory behaviour affecting also the coverage level of confidence intervals. In this work we aim to improve confidence intervals for the memory parameter when the data generating process is a simple ARFIMA$(1, d, 0)$, given by

$$(1 - B)^d(1 - \phi)X_t = \varepsilon_t,$$

where $|\phi| < 1$ to assure stationarity. The problem is when ϕ gets close to unity: short memory and long memory confound each other and it is really difficult to distinguish the effects of the two parameters and consequently to build reliable confidence intervals for d. Arteche and Orbe (2005) proposed a solution to the problem of the bias introduced by short memory behaviour in the estimation of d, bootstrapping the log-periodogram, however the method can be applied only to the GPH estimator.

Since in the first part of the article we showed that the ACF bootstrap can replicate long memory behaviour, it is rational to think that it could also improve the coverage level of confidence intervals.

The Monte Carlo experiments were run for all combinations of n = 200, 300 1000, d = 0, 0.2, 0.45, -0.45 and $\phi = 0, 0.2, 0.4, 0.6, 0.8$. The number of bootstrap replicates was $B = 1000$ and each experiment was repeated $S = 2000$ times.

All the results are reported in Tables 7, 8 and 9 and may be summarized as follows:

- the log-periodogram regression is still the best method to build confidence intervals through the GPH for the memory parameter d;
- the local bootstrap gives results quite similar to the Monte Carlo, thus it is not useful for this problem; and
- the ACF bootstrap gives satisfactory results applied to the GPH estimator but not as good as the log-periodogram; however it is the best method to build confidence intervals when estimating d with the local Whittle.

7 Conclusions

In this paper, we proposed a new bootstrap method for time series, the ACF bootstrap, which seems to be promising for long memory Gaussian processes. The Monte Carlo experiments showed that the ACF bootstrap is better than the existing methods, local and sieve bootstrap. It outperformed both of them in terms of reduction of standard error and mean squared error of the estimates of d. The method is robust against deviations from Normality, asymmetry (Chi-squared distribution with one degree of freedom) and fat tails (Student t distribution with four and six degrees of freedom), suggesting the possibility of wider application.

Nominal coverage for confidence intervals for d based on the asymptotic distribution of the Whittle estimator are improved by using ACF bootstrap especially when the sample size is small, such as $n = 128$; also in this case the method we proposed outperformed the local bootstrap. In the presence of short memory the ACF bootstrap gave some improvements to the local Whittle. As regard as the GPH estimator, the log-periodogram regression by Arteche and Orbe (2005) is still the best solution to build confidence interval but also the .

The result given in this paper are illustrative, more exhaustive tables are available, upon request, from the corresponding author.

	Monte Carlo			ACF		Local B.		SIEVE	
d	Whittle	GPH	LW	GPH	LW	GPH	LW	GPH	LW
	0.0817	**0.0961**	**0.1232**	**0.0664**	**0.1337**	**0.0897**	**0.1322**	**0.0253**	**0.0831**
0.1	*0.0553*	*0.2318*	*0.1346*	*0.1714*	*0.1012*	*0.2175*	*0.1165*	*0.0706*	*0.0502*
	0.0034	0.0537	0.0186	0.0305	0.0114	0.0474	0.0146	0.0106	0.0028
	0.1802	**0.2074**	**0.2065**	**0.1672**	**0.1996**	**0.1946**	**0.2052**	**0.0789**	**0.1197**
0.2	*0.0613*	*0.2305*	*0.1620*	*0.1773*	*0.1249*	*0.2162*	*0.1417*	*0.1072*	*0.0842*
	0.0041	0.0532	0.0263	0.0325	0.0156	0.0468	0.0201	0.0262	0.0135
	0.2786	**0.3070**	**0.2916**	**0.2625**	**0.2700**	**0.2853**	**0.2825**	**0.1497**	**0.1742**
0.3	*0.0618*	*0.2270*	*0.1767*	*0.1752*	*0.1401*	*0.2148*	*0.1581*	*0.1382*	*0.1159*
	0.0043	0.0516	0.0313	0.0321	0.0205	0.0464	0.0253	0.0417	0.0293
	0.3817	**0.4107**	**0.3871**	**0.3584**	**0.3533**	**0.3846**	**0.3736**	**0.2481**	**0.2577**
0.4	*0.0629*	*0.2316*	*0.1890*	*0.1820*	*0.1564*	*0.2189*	*0.1736*	*0.1652*	*0.1497*
	0.0043	0.0538	0.0359	0.0348	0.0266	0.0481	0.0308	0.0504	0.0427
	0.4366	**0.4604**	**0.4365**	**0.4073**	**0.3955**	**0.4284**	**0.4181**	**0.3067**	**0.3110**
0.45	*0.0621*	*0.2306*	*0.1875*	*0.1784*	*0.1578*	*0.2214*	*0.1773*	*0.1752*	*0.1628*
	0.0040	0.0533	0.0353	0.0337	0.0279	0.0495	0.0325	0.0513	0.0458
	0.0945	**0.0981**	**0.1024**	**0.0807**	**0.1074**	**0.0950**	**0.1069**	**0.0153**	**0.0486**
0.1	*0.0256*	*0.1393*	*0.0908*	*0.1089*	*0.0730*	*0.1361*	*0.0823*	*0.0354*	*0.0272*
	0.0007	0.0194	0.0083	0.0122	0.0054	0.0185	0.0068	0.0084	0.0034
	0.1954	**0.1955**	**0.1907**	**0.1775**	**0.1812**	**0.1889**	**0.1873**	**0.0645**	**0.0844**
0.2	*0.0258*	*0.1394*	*0.1078*	*0.1135*	*0.0919*	*0.1375*	*0.1004*	*0.0718*	*0.0623*
	0.0007	0.0194	0.0117	0.0134	0.0088	0.0190	0.0102	0.0235	0.0173
	0.2972	**0.3059**	**0.2930**	**0.2813**	**0.2733**	**0.2970**	**0.2857**	**0.1574**	**0.1658**
0.3	*0.0258*	*0.1353*	*0.1105*	*0.1105*	*0.1019*	*0.1338*	*0.1078*	*0.1110*	*0.1075*
	0.0007	0.0183	0.0123	0.0126	0.0111	0.0179	0.0118	0.0326	0.0296
	0.3984	**0.4031**	**0.3925**	**0.3801**	**0.3697**	**0.3919**	**0.3841**	**0.2755**	**0.2826**
0.4	*0.0257*	*0.1381*	*0.1120*	*0.1126*	*0.1062*	*0.1355*	*0.1115*	*0.1325*	*0.1363*
	0.0007	0.0191	0.0126	0.0131	0.0122	0.0184	0.0127	0.0330	0.0324
	0.4488	**0.4657**	**0.4535**	**0.4383**	**0.4276**	**0.4522**	**0.4445**	**0.3489**	**0.3586**
0.45	*0.0264*	*0.1389*	*0.1163*	*0.1145*	*0.1111*	*0.1370*	*0.1170*	*0.1461*	*0.1516*
	0.0007	0.0195	0.0135	0.0132	0.0128	0.0188	0.0137	0.0316	0.0313

Table 1: Results of the estimators (Whittle, GPH and local Whittle, LW) analysed with parameter values $d = 0.1, 0.2, 0.3, 0.4, 0.45$, bootstrap replications $B = 1000$ and replications $S = 2000$: average value (boldface), standard error (italic), mean squared error (normal font): top of the table series length $n = 200$, bottom part $n = 1000$.

d	$n=200$ GPH	$n=200$ LW	$n=500$ GPH	$n=500$ LW	$n=1000$ GPH	$n=1000$ LW
0.1	*26.05*	*24.82*	*19.67*	*21.25*	*21.79*	*19.66*
	43.23	39.03	32.96	36.31	36.93	34.84
0.2	*23.11*	*22.91*	*20.38*	*18.60*	*18.56*	*14.75*
	38.92	40.66	33.92	32.50	31.13	24.86
0.3	*22.82*	*20.71*	*19.99*	*13.45*	*18.31*	*7.75*
	37.75	34.40	33.55	20.58	31.48	9.41
0.4	*21.43*	*17.28*	*20.37*	*9.91*	*18.49*	*5.25*
	35.17	25.83	34.01	11.61	31.51	3.35
0.45	*22.61*	*15.83*	*18.11*	*6.74*	*17.60*	*4.49*
	36.81	21.12	31.08	5.96	32.25	5.16

d	$n=200$ GPH	$n=200$ LW	$n=500$ GPH	$n=500$ LW	$n=1000$ GPH	$n=1000$ LW
0.1	*6.17*	*13.39*	*3.26*	*10.31*	*2.31*	*9.34*
	11.80	21.58	6.33	18.24	4.45	17.30
0.2	*6.21*	*12.53*	*3.90*	*8.84*	*1.34*	*6.88*
	12.06	23.50	7.59	16.69	2.12	12.55
0.3	*5.37*	*10.53*	*3.02*	*5.60*	*1.10*	*2.41*
	10.12	19.14	5.99	9.62	2.32	3.47
0.4	*5.49*	*8.17*	*3.69*	*3.40*	*1.94*	*0.45*
	10.42	14.12	7.23	5.14	3.55	-0.68
0.45	*3.98*	*5.43*	*2.77*	*0.94*	*1.41*	*-0.61*
	7.11	8.14	5.66	0.40	4.00	-1.36

Table 2: Percentage of gain (see Equation (11))comparing the Monte Carlo results of estimators GPH and local Whittle (LW) with the bootstrap results in terms of standard deviation (italic) and mean squared error (normal font): in the first part of the table there is the gain using the ACF bootstrap, whereas in the second part there is the gain using the local bootstrap.

	Monte Carlo			ACF		Local B.	
d	Whittle	GPH	LW	GPH	LW	GPH	LW
	0.0816	**0.0978**	**0.1158**	**0.0630**	**0.1286**	**0.0906**	**0.1264**
0.1	*0.0558*	*0.2230*	*0.1303*	*0.1655*	*0.0977*	*0.2111*	*0.1146*
	0.0035	0.0498	0.0172	0.0288	0.0104	0.0446	0.0138
	0.1818	**0.1974**	**0.1940**	**0.1577**	**0.1897**	**0.1837**	**0.1949**
0.2	*0.0608*	*0.2294*	*0.1571*	*0.1705*	*0.1192*	*0.2145*	*0.1367*
	0.0040	0.0526	0.0247	0.0309	0.0143	0.0463	0.0187
	0.2804	**0.3051**	**0.2908**	**0.2607**	**0.2696**	**0.2859**	**0.2830**
0.3	*0.0606*	*0.2299*	*0.1760*	*0.1763*	*0.1413*	*0.2174*	*0.1592*
	0.0041	0.0529	0.0311	0.0326	0.0209	0.0475	0.0256
	0.3856	**0.4091**	**0.3896**	**0.3605**	**0.3549**	**0.3832**	**0.3745**
0.4	*0.0621*	*0.2274*	*0.1824*	*0.1772*	*0.1523*	*0.2123*	*0.1680*
	0.0041	0.0518	0.0334	0.0329	0.0252	0.0453	0.0289
	0.4360	**0.4628**	**0.4387**	**0.4088**	**0.3972**	**0.4319**	**0.4207**
0.45	*0.0627*	*0.2266*	*0.1861*	*0.1769*	*0.1568*	*0.2156*	*0.1745*
	0.0041	0.0515	0.0348	0.0330	0.0274	0.0468	0.0313
	0.0913	**0.0995**	**0.1096**	**0.0787**	**0.1170**	**0.0948**	**0.1155**
0.1	*0.0361*	*0.1753*	*0.1066*	*0.1324*	*0.0837*	*0.1699*	*0.0960*
	0.0014	0.0307	0.0115	0.0180	0.0073	0.0289	0.0095
	0.1911	**0.1997**	**0.1935**	**0.1743**	**0.1865**	**0.1921**	**0.1927**
0.2	*0.0361*	*0.1755*	*0.1279*	*0.1396*	*0.1058*	*0.1696*	*0.1177*
	0.0014	0.0308	0.0164	0.0201	0.0114	0.0288	0.0139
	0.2932	**0.3072**	**0.2940**	**0.2764**	**0.2727**	**0.2946**	**0.2852**
0.3	*0.0370*	*0.1678*	*0.1353*	*0.1352*	*0.1186*	*0.1618*	*0.1271*
	0.0014	0.0282	0.0183	0.0188	0.0148	0.0262	0.0164
	0.3949	**0.4092**	**0.3950**	**0.3770**	**0.3661**	**0.3926**	**0.3834**
0.4	*0.0367*	*0.1705*	*0.1399*	*0.1379*	*0.1276*	*0.1667*	*0.1365*
	0.0014	0.0292	0.0196	0.0195	0.0174	0.0278	0.0189
	0.4448	**0.4633**	**0.4447**	**0.4266**	**0.4128**	**0.4460**	**0.4335**
0.45	*0.0382*	*0.1675*	*0.1389*	*0.1348*	*0.1284*	*0.1621*	*0.1377*
	0.0015	0.0282	0.0193	0.0187	0.0179	0.0263	0.0192

Table 3: Simulated series with Chi-squared 1 d.f.innovations: results of the estimators (Whittle, GPH and local Whittle, LW) analysed with memory parameter values $d = 0.1, 0.2, 0.3, 0.4, 0.45$, bootstrap replications $B = 1000$ and simulation replications $S = 2000$: average value (boldface), standard error (italic), mean squared error (normal font). In the upper part $n = 200$, whereas in the lower part $n = 500$.

	Monte Carlo			ACF		Local B.	
d	Whittle	GPH	LW	GPH	LW	GPH	LW
	0.0822	**0.0995**	**0.1232**	**0.0686**	**0.1344**	**0.0915**	**0.1325**
0.1	*0.0554*	*0.2334*	*0.1328*	*0.1744*	*0.1005*	*0.2189*	*0.1168*
	0.0034	0.0545	0.0182	0.0314	0.0113	0.0480	0.0147
	0.1779	**0.2050**	**0.2014**	**0.1632**	**0.1968**	**0.1924**	**0.2026**
0.2	*0.0614*	*0.2304*	*0.1623*	*0.1791*	*0.1265*	*0.2174*	*0.1437*
	0.0043	0.0531	0.0263	0.0334	0.0160	0.0473	0.0206
	0.2796	**0.2936**	**0.2869**	**0.2538**	**0.2681**	**0.2771**	**0.2805**
0.3	*0.0625*	*0.2347*	*0.1816*	*0.1861*	*0.1462*	*0.2231*	*0.1641*
	0.0043	0.0551	0.0332	0.0368	0.0224	0.0503	0.0273
	0.3832	**0.4092**	**0.3890**	**0.3593**	**0.3544**	**0.3819**	**0.3729**
0.4	*0.0644*	*0.2310*	*0.1852*	*0.1788*	*0.1530*	*0.2157*	*0.1691*
	0.0044	0.0534	0.0344	0.0336	0.0255	0.0469	0.0293
	0.4365	**0.4650**	**0.4407**	**0.4094**	**0.3995**	**0.4368**	**0.4237**
0.45	*0.0623*	*0.2382*	*0.1941*	*0.1864*	*0.1629*	*0.2219*	*0.1804*
	0.0041	0.0570	0.0378	0.0364	0.0291	0.0494	0.0332
	0.0899	**0.1008**	**0.1076**	**0.0747**	**0.1150**	**0.0960**	**0.1140**
0.1	*0.0376*	*0.1698*	*0.1035*	*0.1316*	*0.0814*	*0.1645*	*0.0921*
	0.0015	0.0288	0.0108	0.0180	0.0068	0.0271	0.0087
	0.1907	**0.1980**	**0.1913**	**0.1710**	**0.1842**	**0.1898**	**0.1899**
0.2	*0.0374*	*0.1759*	*0.1259*	*0.1386*	*0.1027*	*0.1691*	*0.1137*
	0.0015	0.0309	0.0159	0.0201	0.0108	0.0287	0.0130
	0.2933	**0.3023**	**0.2902**	**0.2728**	**0.2701**	**0.2915**	**0.2830**
0.3	*0.0365*	*0.1720*	*0.1362*	*0.1359*	*0.1184*	*0.1667*	*0.1289*
	0.0014	0.0296	0.0186	0.0192	0.0149	0.0279	0.0169
	0.3950	**0.4104**	**0.3924**	**0.3767**	**0.3642**	**0.3948**	**0.3819**
0.4	*0.0372*	*0.1716*	*0.1413*	*0.1386*	*0.1282*	*0.1638*	*0.1371*
	0.0014	0.0295	0.0200	0.0198	0.0177	0.0268	0.0191
	0.4448	**0.4645**	**0.4474**	**0.4288**	**0.4154**	**0.4448**	**0.4344**
0.45	*0.0375*	*0.1678*	*0.1401*	*0.1366*	*0.1306*	*0.1623*	*0.1374*
	0.0014	0.0284	0.0196	0.0191	0.0183	0.0264	0.0191

Table 4: Simulated series with Student t 4 d.f. innovations: results of the estimators (Whittle, GPH and local Whittle, LW) analysed with memory parameter values $d = 0.1, 0.2, 0.3, 0.4, 0.45$, bootstrap replications $B = 1000$ and simulation replications $S = 2000$: average value (boldface), standard error (italic), mean squared error (normal font). In the upper part $n = 200$, whereas in the lower part $n = 500$.

	Monte Carlo			ACF		Local B.	
d	Whittle	GPH	LW	GPH	LW	GPH	LW
	0.0808	**0.0987**	**0.1227**	**0.0662**	**0.1330**	**0.0909**	**0.1317**
0.1	*0.0544*	*0.2303*	*0.1336*	*0.1735*	*0.1009*	*0.2167*	*0.1166*
	0.0033	0.0530	0.0184	0.0312	0.0113	0.0470	0.0146
	0.1793	**0.1930**	**0.1930**	**0.1551**	**0.1897**	**0.1820**	**0.1940**
0.2	*0.0626*	*0.2296*	*0.1602*	*0.1758*	*0.1234*	*0.2138*	*0.1407*
	0.0044	0.0528	0.0257	0.0329	0.0153	0.0461	0.0198
	0.2799	**0.3082**	**0.2902**	**0.2598**	**0.2685**	**0.2896**	**0.2825**
0.3	*0.0625*	*0.2254*	*0.1724*	*0.1754*	*0.1376*	*0.2116*	*0.1548*
	0.0043	0.0509	0.0298	0.0324	0.0199	0.0449	0.0243
	0.3834	**0.4084**	**0.3865**	**0.3580**	**0.3519**	**0.3845**	**0.3722**
0.4	*0.0622*	*0.2307*	*0.1871*	*0.1804*	*0.1556*	*0.2155*	*0.1725*
	0.0041	0.0533	0.0352	0.0343	0.0265	0.0467	0.0305
	0.4372	**0.4719**	**0.4462**	**0.4138**	**0.4034**	**0.4402**	**0.4275**
0.45	*0.0626*	*0.2349*	*0.1924*	*0.1834*	*0.1620*	*0.2214*	*0.1791*
	0.0041	0.0557	0.0370	0.0350	0.0284	0.0491	0.0326
	0.0908	**0.1006**	**0.1099**	**0.0778**	**0.1175**	**0.0961**	**0.1167**
0.1	*0.0369*	*0.1740*	*0.1075*	*0.1336*	*0.0848*	*0.1699*	*0.0970*
	0.0014	0.0303	0.0117	0.0183	0.0075	0.0289	0.0097
	0.1913	**0.2021**	**0.1938**	**0.1740**	**0.1858**	**0.1946**	**0.1927**
0.2	*0.0369*	*0.1700*	*0.1254*	*0.1332*	*0.1022*	*0.1647*	*0.1150*
	0.0014	0.0289	0.0158	0.0184	0.0106	0.0271	0.0133
	0.2918	**0.3029**	**0.2891**	**0.2740**	**0.2690**	**0.2910**	**0.2818**
0.3	*0.0363*	*0.1714*	*0.1353*	*0.1366*	*0.1174*	*0.1669*	*0.1288*
	0.0014	0.0294	0.0184	0.0193	0.0147	0.0279	0.0169
	0.3943	**0.4065**	**0.3909**	**0.3733**	**0.3626**	**0.3913**	**0.3798**
0.4	*0.0382*	*0.1731*	*0.1435*	*0.1407*	*0.1307*	*0.1656*	*0.1396*
	0.0015	0.0300	0.0207	0.0205	0.0185	0.0275	0.0199
	0.4455	**0.4635**	**0.4470**	**0.4288**	**0.4153**	**0.4442**	**0.4345**
0.45	*0.0388*	*0.1756*	*0.1432*	*0.1410*	*0.1320*	*0.1705*	*0.1415*
	0.0015	0.0310	0.0205	0.0203	0.0186	0.0291	0.0203

Table 5: Simulated series with Student t 6 d.f. innovations: results of the estimators (Whittle, GPH and local Whittle, LW) analysed with memory parameter values $d = 0.1, 0.2, 0.3, 0.4, 0.45$, bootstrap replications $B = 1000$ and simulation replications $S = 2000$: average value (boldface), standard error (italic), mean squared error (normal font). In the upper part $n = 200$, whereas in the lower part $n = 500$.

		ACF bootstrap			local bootstrap			
d	n	MC	SD	P	PT	SD	P	PT
	200	0.9025	0.9705	0.9635	0.9065	0.8450	0.8190	0.9055
0	500	0.9200	0.9765	0.9705	0.9290	0.8520	0.8465	0.9230
	1000	0.9360	0.9880	0.9830	0.9405	0.8690	0.8615	0.9365
	200	0.9000	0.9770	0.9570	0.9090	0.8280	0.8010	0.9070
0.2	500	0.9350	0.9830	0.9770	0.9350	0.8765	0.8690	0.9340
	1000	0.9470	0.9870	0.9820	0.9500	0.8825	0.8735	0.9520
	200	0.9240	0.9755	0.9680	0.6975	0.8080	0.8355	0.7140
0.45	500	0.9405	0.9390	0.9665	0.7160	0.7925	0.8445	0.7565
	1000	0.9330	0.9515	0.9810	0.7995	0.8450	0.8730	0.8455
	200	0.9670	0.9670	0.9905	0.5810	0.7340	0.8425	0.6075
-0.45	500	0.9635	0.9510	0.9905	0.6635	0.7665	0.8490	0.7140
	1000	0.9235	0.9490	0.9880	0.7685	0.8160	0.8575	0.8175

Table 6: Observed coverage (%) of the Whittle estimator relative to a confidence interval with nominal level $1 - \alpha = 0.95$ for different values of the parameter $d = 0, 0.2, 0.45, -0.45$ and different sample sizes $n = 200, 500, 1000$ computed using Monte Carlo (MC), bootstrap standard deviation (SD), percentile (P), percentile t (PT) methods and two different bootstrap procedures (ACF and local bootstrap).

			ACF bootstrap			local bootstrap			log-periodogram		
n	ϕ	MC	SD	P	PT	SD	P	PT	SD	P	PT
	0	92.45	97.75	99.05	82.45	88.35	82.75	87.35	95.80	90.40	92.25
	0.2	93.70	98.55	99.45	83.55	88.45	81.45	88.00	95.90	91.55	93.00
200	0.4	93.25	97.65	99.30	82.05	86.40	81.70	87.15	95.30	91.25	92.60
	0.6	91.35	95.10	99.60	78.30	79.80	76.50	86.45	91.15	89.35	90.60
	0.8	69.15	76.15	91.95	53.35	48.45	47.20	63.95	63.80	65.75	66.10
	0	93.55	98.30	99.25	83.85	89.65	82.50	89.95	97.05	92.25	93.90
	0.2	93.65	98.85	98.60	83.20	90.15	82.85	90.35	97.10	91.80	93.70
500	0.4	93.55	97.95	99.15	83.15	88.35	81.40	90.60	96.40	92.05	93.25
	0.6	93.50	97.75	99.05	82.70	86.15	81.70	90.60	95.20	92.80	92.90
	0.8	80.35	86.20	96.10	64.85	62.55	61.10	77.20	78.50	79.10	78.30
	0	93.70	98.75	98.70	83.75	90.60	83.35	90.80	97.35	92.25	93.65
	0.2	93.05	97.50	98.45	84.55	89.85	83.30	91.05	96.05	92.15	92.35
1000	0.4	93.25	97.80	98.85	84.05	88.80	81.95	91.15	96.40	92.00	93.45
	0.6	93.10	97.25	99.20	82.60	86.55	81.95	90.45	95.15	92.20	93.00
	0.8	88.60	91.65	98.10	76.05	75.95	74.10	86.40	88.65	88.75	86.90
	0	92.75	98.40	98.95	83.35	90.30	83.55	86.95	96.25	91.00	92.90
	0.2	92.30	98.70	98.40	82.60	90.80	83.25	87.40	96.00	89.85	92.10
200	0.4	92.65	97.75	98.70	81.70	86.60	82.05	87.10	95.00	90.45	92.60
	0.6	91.00	96.10	99.05	78.15	80.30	77.75	84.60	90.35	89.15	89.90
	0.8	68.05	74.50	91.65	50.65	48.05	48.30	61.05	63.30	64.70	65.65
	0	93.45	98.65	98.25	83.95	90.80	83.75	90.30	97.30	92.30	93.55
	0.2	94.05	98.45	98.55	84.90	90.90	84.35	91.05	96.55	92.45	93.80
500	0.4	93.40	97.80	98.50	81.70	87.60	81.80	89.30	95.70	91.60	92.85
	0.6	93.15	96.70	99.05	81.30	85.65	81.60	89.50	94.10	91.95	92.35
	0.8	81.80	86.00	96.90	66.45	64.80	64.55	76.15	79.85	81.00	79.90
	0	93.35	98.70	98.35	85.20	91.40	83.90	91.30	97.15	92.80	93.40
	0.2	94.90	98.50	98.05	85.30	91.55	85.15	92.50	97.35	93.55	94.80
1000	0.4	94.20	97.65	97.90	83.75	89.65	83.10	92.05	97.00	93.20	94.15
	0.6	94.20	97.15	99.05	83.65	87.00	83.10	91.25	95.55	93.15	93.65
	0.8	89.15	92.25	98.30	74.70	74.85	73.90	85.50	88.70	88.30	87.80

Table 7: Observed coverage (%) of the GPH estimator relative to a confidence interval with $1-\alpha = 0.95$ for $d = 0, 0.2$ (top and bottom part, respectively), series length $n = 200, 500, 1000$, different values of $\phi = 0, 0.2, 0.4, 0.6, 0.8$ computed using Monte Carlo (MC), bootstrap standard deviation (SD), percentile (P), percentile t (PT) methods and three different bootstrap procedures (ACF, local bootstrap and log-periodogram regression).

		ACF bootstrap			local bootstrap			log-periodogram			
ϕ	MC	SD	P	PT	SD	P	PT	SD	P	PT	
	0	91.90	98.10	98.05	81.25	89.90	83.75	85.95	95.10	89.65	91.45
	0.2	91.65	97.55	98.10	80.80	87.65	82.80	85.10	93.95	89.75	91.30
200	0.4	92.20	97.15	98.90	79.70	85.35	82.00	86.45	93.05	90.65	91.85
	0.6	88.40	94.15	98.95	73.85	78.75	79.30	80.60	87.80	86.30	87.30
	0.8	66.65	73.80	94.05	48.70	49.45	52.55	55.65	60.80	63.35	63.65
	0	93.60	98.05	97.65	83.80	89.35	83.30	89.55	96.15	92.25	93.25
	0.2	93.40	98.25	98.15	82.35	88.25	82.95	88.60	95.50	92.20	93.15
500	0.4	94.25	98.15	98.35	82.80	89.25	85.70	89.65	95.70	93.15	93.90
	0.6	91.90	96.15	98.20	79.70	85.45	82.50	86.85	93.60	91.15	91.70
	0.8	80.40	84.95	96.95	64.55	66.25	67.60	72.90	78.60	79.95	77.95
	0	93.35	98.00	97.85	84.15	90.20	83.95	90.85	96.65	92.70	92.85
	0.2	94.10	97.95	98.15	83.85	89.05	84.95	91.20	96.65	93.30	93.70
1000	0.4	93.55	97.15	98.05	81.95	87.40	83.40	89.25	95.50	92.20	93.15
	0.6	93.80	97.30	98.50	82.70	87.75	85.00	90.10	95.40	93.30	93.15
	0.8	87.15	90.95	98.00	72.50	74.40	74.35	81.45	86.45	86.80	85.10
	0	91.65	97.40	99.70	77.75	85.30	79.75	87.20	94.35	89.35	91.45
	0.2	91.50	97.65	99.75	76.85	85.00	78.70	87.10	93.40	89.30	90.65
200	0.4	91.35	96.60	99.60	78.80	82.90	76.70	88.10	92.60	88.90	90.75
	0.6	90.35	94.45	99.35	76.90	76.20	71.00	86.70	88.90	88.15	88.75
	0.8	65.25	72.05	85.05	55.00	46.60	42.00	64.15	60.00	62.65	62.35
	0	91.75	97.00	99.65	80.90	85.55	79.70	89.85	94.55	90.90	91.35
	0.2	93.85	97.60	99.70	81.55	86.35	80.40	91.00	95.95	93.10	93.45
500	0.4	92.70	97.25	99.80	79.40	84.20	78.85	90.70	94.05	91.15	92.25
	0.6	91.65	95.95	99.30	79.40	82.70	77.70	89.40	93.15	90.70	90.80
	0.8	80.10	85.85	93.25	68.90	61.40	57.95	79.70	78.25	79.40	77.80
	0	92.15	97.15	99.60	79.65	85.10	80.70	90.80	94.55	91.75	91.80
	0.2	93.10	96.80	99.45	80.35	86.05	80.75	91.60	94.95	92.50	92.85
1000	0.4	93.15	96.85	99.25	79.45	85.35	80.60	90.90	95.45	92.35	92.80
	0.6	92.55	96.20	98.95	80.80	82.20	77.25	90.85	94.10	92.50	91.65
	0.8	85.85	90.25	95.90	74.75	71.80	67.90	85.85	85.35	86.40	84.10

Table 8: Observed coverage (%) of the GPH estimator relative to a confidence interval with $1-\alpha = 0.95$ for $d = 0.45, -0.45$ (top and bottom part, respectively), series length $n = 200, 500, 1000$, different values of $\phi = 0, 0.2, 0.4, 0.6, 0.8$ computed using Monte Carlo (MC), bootstrap standard deviation (SD), percentile (P), percentile t (PT) methods and three different bootstrap procedures (ACF, local bootstrap and log-periodogram regression).

			lW	ACF bootstrap			local bootstrap						lW	ACF bootstrap			local bootstrap		
d	n	ϕ	MC	SD	P	PT	SD	P	PT	d	n	ϕ	MC	SD	P	PT	SD	P	PT
		0	81.65	97.85	97.20	69.15	91.25	82.25	71.40			0	82.30	96.75	97.05	65.90	90.10	82.00	69.35
		0.2	83.45	98.40	98.05	70.40	91.70	81.45	73.80			0.2	82.95	96.15	96.60	66.35	90.35	81.90	68.75
	200	0.4	83.05	97.65	98.60	70.55	89.40	82.80	74.55		200	0.4	82.20	94.85	97.20	60.95	86.40	82.30	65.25
		0.6	80.05	93.75	98.30	64.00	82.65	79.75	69.50			0.6	77.50	88.70	98.30	52.40	77.95	80.70	58.55
		0.8	40.75	63.55	87.05	28.35	44.35	50.45	35.45			0.8	41.85	51.90	87.10	17.70	38.90	53.00	22.25
		0	87.10	99.00	97.75	82.65	93.50	81.75	82.70			0	86.70	97.15	96.65	77.50	92.80	82.55	79.25
		0.2	86.15	99.10	97.35	82.30	93.25	81.95	81.10			0.2	87.15	97.25	97.65	78.40	92.20	83.60	80.60
0	500	0.4	84.70	98.90	97.40	80.75	92.55	82.25	80.60	0.2	500	0.4	87.15	96.55	97.20	75.20	90.60	82.50	79.05
		0.6	85.30	98.50	98.55	80.05	89.05	82.95	80.10			0.6	85.60	94.95	98.40	70.60	86.95	82.80	74.70
		0.8	63.10	87.00	94.70	56.40	64.40	64.80	58.75			0.8	63.50	74.65	94.85	39.75	60.10	68.55	47.85
		0	88.10	99.30	97.50	87.25	94.35	82.50	85.25			0	88.85	98.50	97.40	83.15	93.75	84.00	85.05
		0.2	87.70	99.20	97.05	86.25	93.15	83.00	85.10			0.2	87.45	98.80	96.80	82.10	93.65	83.65	84.35
	1000	0.4	88.30	99.30	97.55	86.15	92.55	83.65	84.40		1000	0.4	88.30	97.95	97.00	81.00	92.50	83.20	83.85
		0.6	88.70	98.85	97.95	86.00	91.20	82.95	85.50			0.6	89.40	97.15	98.10	81.20	90.55	85.50	84.60
		0.8	79.05	95.00	97.45	75.35	77.50	75.95	74.80			0.8	77.20	89.50	97.30	60.80	75.50	76.95	68.50
		0	89.25	99.90	96.80	48.60	89.90	83.05	46.90			0	90.65	98.25	99.60	49.05	83.90	81.10	49.85
		0.2	91.40	99.95	97.40	45.65	90.30	83.60	44.55			0.2	89.90	98.15	99.65	48.00	88.10	81.10	48.20
	200	0.4	93.40	99.50	98.00	42.45	86.05	82.85	41.75		200	0.4	86.95	97.55	99.25	52.30	86.30	78.85	52.80
		0.6	96.10	99.05	97.80	33.40	81.50	82.70	33.55			0.6	79.50	95.90	98.20	55.80	79.45	72.95	56.75
		0.8	99.45	90.35	88.75	7.15	48.85	55.55	7.50			0.8	39.65	70.95	82.25	36.55	45.90	44.45	37.90
		0	92.00	99.30	96.85	50.40	89.30	83.90	50.15			0	92.25	98.05	99.00	52.45	84.05	80.65	54.65
		0.2	91.80	99.60	97.30	48.00	88.55	85.10	47.45			0.2	92.90	98.55	99.45	56.05	88.95	83.30	57.20
0.45	500	0.4	94.00	99.70	97.95	48.65	89.25	86.35	49.45	-0.45	500	0.4	91.10	97.90	99.50	54.90	87.75	81.65	56.30
		0.6	94.55	99.00	97.90	41.50	85.10	84.10	42.55			0.6	88.35	97.50	98.70	56.70	86.20	79.90	57.40
		0.8	98.90	92.40	94.45	16.15	61.60	69.45	16.80			0.8	61.75	87.50	91.05	54.25	62.30	60.45	54.55
		0	93.50	99.35	97.25	53.85	89.15	84.95	54.90			0	92.30	98.65	99.45	56.90	84.25	81.15	58.25
		0.2	94.00	99.55	97.60	52.45	87.80	85.30	52.75			0.2	92.55	98.40	99.45	55.70	89.20	82.45	57.55
	1000	0.4	94.05	98.25	97.45	49.40	85.85	83.35	50.00		1000	0.4	93.10	98.00	99.15	59.35	89.35	83.75	60.15
		0.6	95.40	99.20	98.20	44.10	86.25	86.45	45.55			0.6	89.45	97.40	98.50	58.35	86.05	80.45	59.55
		0.8	98.75	94.10	96.70	26.80	68.45	77.05	28.00			0.8	76.45	92.55	95.35	62.00	72.30	69.25	62.70

Table 9: Observed coverage (%) of the local Whittle estimator relative to a confidence interval with $1-\alpha = 0.95$ for $d = 0, 0.2, 0.45, -0.45$, series length $n = 200, 500, 1000$, different values of $\phi = 0, 0.2, 0.4, 0.6, 0.8$ computed using Monte Carlo (MC), bootstrap standard deviation (SD), percentile (P), percentile t (PT) methods and three different bootstrap procedures (ACF and local bootstrap).

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