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ON TWO APPROACHES FOR PARTIAL DIFFERENTIAL EQUATIONS IN SEVERAL COMPLEX VARIABLES

Direttore della Scuola: Ch.mo Prof. Paolo Dai Pra
Coordinatore dindirizzo: Ch.mo Prof. Franco Cardin
Supervisore: Ch.mo Prof. Giuseppe Zampieri

Dottorando : LY KIM HA
(firma del dottorando)

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RIASSUNTO

Lo scopo di questa tesi è quello di presentare l'influenza di notazioni di " tipo" su equazioni differenziali alle derivate parziali in più variabili complesse. Le notazioni di "tipo" qui includono il finito e il tipo di infinito, nel senso di Hörmander ", e D'Angelo. In particolare, nella prima parte, a condizione tipo finito, prenderemo in considerazione l'esistenza e l'unicità delle soluzioni per il problema del valore iniziale associato ai operatore calore $\partial_s + \Box_b$ su varietà CR. Il tipo finito m è la condizione fondamentale per fornire stime puntuali del nucleo del calore attraverso la teoria degli operatori integrali singolari sviluppate da E. Stein e A. Nagel, D.H. Phong e E. Stein. Prossimo, nella seconda parte, introdurremo un nuovo metodo per indagare la equazioni Cauchy-Riemann $\bar{\partial} u = \phi$. Le soluzioni sono costruite con via metodo rappresentazione integrale. Inoltre, mostreremo che il nuovo metodo qui viene applicato anche ben al complesso operatore Monge-Ampère ere $(dd^c)^n$ in $\mathbb{C}^n$. Il punto principale è che il nostro metodo può passare alcuni risultati noti dal caso di tipo finito al tipo di infinito.

The shortest and best way between two truths of the real domain often passes through the imaginary one - *"J. Hadamard"*

A mia moglie

ABSTRACT

The aim of this thesis is to present influence of notations of "type" on partial differential equations in several complex variables. The notations of "type" here include the finite and the infinite type in the sense of Hörmander, and D'Angelo. In particular, in the first part, under the finite type condition, we will consider the existence and uniqueness of solutions for the initial value problem associated to the heat operator $\partial_s + \Box_b$ on CR manifolds. The finite type m is the critical condition to provide pointwise estimates of the heat kernel via theory of singular integral operators developed by E. Stein and A. Nagel, D.H. Phong and E. Stein. Next, in the second part, we will introduce a new method to investigate the Cauchy-Riemann equations $\bar{\partial} u = \phi$. The solutions are constructed via the integral representation method. Moreover, we will show that the new method here is also applied well to the complex Monge-Ampère operator $(dd^c)^n$ in $\mathbb{C}^n$. The main point is that our method can pass some well-known results from the case of finite type to infinite type.

The shortest and best way between two truths of the real domain often passes through the imaginary one - *"J. Hadamard"*

To my wife

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Part I

SINGULAR INTEGRAL METHODS VIA NAGEL-STEIN THEORY

Chapter 1

Preliminaries

In Riemannian geometry, the Laplace-Beltrami operator acting on functions on a Riemannian manifold M is $\Delta = d^*d$. In order to study the relation between geometrical objects and analytic ones on M, one of well-known methods is to consider the heat equation associated to the Laplace-Beltrami operator. Let u defined on $\mathbb{R}^+ \times M$. We say that u solves the heat equation if

$$\frac{\partial u}{\partial s} + \Delta u = 0.$$

Moreover, we also have the initial value problem for the heat equation. That is to find a function $u(s,x)$ solving the heat equation on M with

$$\lim_{s \to 0^+} u(s,x) = f(x).$$

The well-known fact is that there is a unique fundamental solution $H(s,x,y)$ of the initial value problem, a distribution on $\mathbb{R}^+ \times M \times M$ such that

$$u(s,x) = \int_M H(s,x,y) f(y) dV(y).$$

The kernel $H(s,x,y)$ is smooth.
Now, in the first part of this thesis, we will consider one analogue of the heat equation in Cauchy-Riemann geometry. That is an equation associated to the $\Box_b$-heat operator. Here, $\Box_b$ is a second-order system of partial differential operators associated to the tangential Cauchy-Riemann operator $\bar{\partial}_b$. Unfortunately, both of these are non elliptic on Cauchy-Riemann manifolds without boundary. Hence, the classical approach in Riemannian geometry is not able to proceed the $\Box_b$-heat equation.
The main purpose in this part is to introduce the singular integral operators approach in Nagel-Stein sense to investigate kernels of solutions solving to heat $\Box_b$- initial value problems. This work is motivated to the fourth level in Fefferman's hierarchy, deriving estimates directly from the singularities of the integral kernels.

1.1 CR manifolds and Kohn-Laplacian Operator

We summarize the material background we will need. First of all, we begin with the basic notations of CR manifolds, pseudo-convexity, Hermitian metrics, and operators $\bar{\partial}_b$, $\Box_b$. For more discussions, we refer the books by [FoKo72] and [ChSh01], or [Za08].
Let M be a C^∞ compact, oriented manifold of real dimension $2n-1$, $n \geq 3$. Let $T(M)$, $T^*(M)$ be tangent bundle and cotangent bundle respectively associated with M . Let $\mathbb{C}T(M) = T(M) \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{C}T^*(M) = T^*(M) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle and complexified cotangent bundle respectively over M .

Definition 1.1.1. An integrable Cauchy-Riemann (CR) structure on M is an $(n-1)$-dimensional complex subbundle $T^{1,0}$ of the complexified tangent bundle $\mathbb{C}T(M)$ such that

1. $T^{1,0} \cap T^{0,1} = \{0\}$ where $T^{0,1} = \overline{T^{1,0}}$.
2. If Z and W are smooth sections of $T^{1,0}$, then $[Z, W]$ is as well.

Such a manifold M endowed with an integrable CR structure is called a CR manifold.

In particular, if M is an oriented real hypersurface in $\mathbb{C}^n$, M inherits a CR structure, with $T^{1,0}(M) = T^{1,0}(\mathbb{C}^n) \cap \mathbb{C}T(M)$. Indeed, since $T^{1,0}(\mathbb{C}^n) \cap T^{0,1}(\mathbb{C}^n)$, (1) holds. Now, take any vector fields Z, W in $T^{1,0}(M)$ defined on some open subset U of M. By definition, $Z = \sum_{j=1}^{n-1} a_j(z) \dfrac{\partial}{\partial z_j}$ and $W = \sum_{j=1}^{n-1} b_j(z) \dfrac{\partial}{\partial z_j}$, so $[Z, W] = \sum_{j=1}^{n-1} c_j(z) \dfrac{\partial}{\partial z_j}$, that means $[Z, W]$ is a section of $T^{1,0}(M)$.
Let θ be a real, non-vanishing one form which annihilates $T^{1,0}$ (and thus $T^{1,0} \oplus T^{0,1}$). It determines a Hermitian form L_θ, the Levi form, on $T^{1,0}$ by

$$L_\theta(Z, W) = -id\theta(Z, W),$$

for $Z, W \in T^{1,0}$ and where $i = \sqrt{-1}$.

The conformal class of the Levi form does not depend of the choice of θ. It is an intrinsic invariant of the CR structure, because any choice θ' is of the form $f\theta$ and $L_{f\theta} = fL_\theta$. Thus, this is also true for the following definition.

Definition 1.1.2. The CR manifold M is called weakly pseudoconvex if there is a form θ such that the Levi form L_θ is positive semi-definite. And M is called strongly pseudoconvex if there is a form θ such that the Levi form is positive definite.

Example 1.1.1. *If M is the boundary of a sphere in $\mathbb{C}^n$, M is strongly pseudoconvex. And if M is the boundary of the complex ellipsoid $\Omega = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1|^{2m_1} + |z_2|^{2m_2} + ... + |z_n|^{2m_n} - 1 < 0\}$, for $m_1, ..., m_n > 1$, then M is weakly pseudoconvex manifold.*

Definition 1.1.3. A Hermitian metric on the CR manifold M is a Riemannian metric, extended to be Hermitian on $\mathbb{C}T(M)$, such that $T^{1,0} \perp T^{0,1}$.

In order to study the operator $\bar{\partial}_b$ and its adjoint operator with Hilbert space techniques, we must equip M with a such Hermitian metric, this induces an inner product of (p, q)-forms in Hilbert space theory.

Now, denote by $T^{*1,0}(M)$ and $T^{*0,1}(M)$ the dual spaces of $T^{1,0}(M)$ and $T^{1,0}(M)$ respectively and $T^*(M)$ be the dual bundle of $T(M)$. Let

$$E = (T^{1,0} \oplus T^{0,1})^{\perp},$$

and, we define $\Lambda^{p,0}$ to be the p-forms in $\mathbb{C}T^*(M)$ which annihilate $E \oplus T^{1,0}$. $\Lambda^{p,q}$ is the subbundle of $\Lambda^{p+q}(\mathbb{C}T^*(M))$ generated by $\Lambda^{p,0} \wedge \Lambda^{0,q}$, where $\Lambda^{0,q} = \overline{\Lambda^{q,0}}$.

The **tangential Cauchy-Riemann** operator $\bar{\partial}_b : C^{\infty}(\Lambda^{p,q}) \to \mathcal{C}^{\infty}(\Lambda^{p,q+1})$ is defined as follows

$$\bar{\partial}_b := \pi_{p,q+1} d,$$

where $\pi_{p,q+1}$ is the orthogonal projection of Λ^{p+q+1} onto $\Lambda^{p,q+1}$ and d is exterior differentiation. This operator forms the tangential Cauchy-Riemann complex

$$0 \longrightarrow C^{\infty}(\Lambda^{p,0}(M)) \xrightarrow{\bar{\partial}_b} C^{\infty}(\Lambda^{p,1}(M)) \xrightarrow{\bar{\partial}_b} \dots \xrightarrow{\bar{\partial}_b} C^{\infty}(\Lambda^{p,n-1}(M)) \longrightarrow 0.$$

The operator $\bar{\partial}_b$ is a derivation, i.e, if $\phi \in C^{\infty}(\Lambda^{p,q})$, and $\psi \in C^{\infty}(\Lambda^{r,s})$, then

$$\bar{\partial}_b(\phi \wedge \psi) = (\bar{\partial}_b \phi) \wedge \psi + (-1)^{p+q} \phi \wedge \bar{\partial}_b \psi.$$

Moreover, from (2) in Definition 1.1.1, we imply that $\bar{\partial}_b^2 = 0$.
Notice that p plays no role in the formulation of the tangential Cauchy-Riemann operators. Thus, in this thesis, it suffices to consider the action of $\bar{\partial}_b$ on type $(0, q)$-forms, for $0 \leq q \leq n-1$.

To describe $\bar{\partial}_b$ more explicitly, let U be an open set such that $\Lambda^1(U)$ is trivial. We pick an orthogonal basis $\{\omega_1, \omega_1, ..., \omega_{n-1}, \overline{\omega}_1, \overline{\omega}_2, ..., \overline{\omega}_{n-1}, \omega_0\}$ of $\Lambda^1(U)$ such that $\{\omega_1, ..., \omega_{n-1}\}$ is a basis of $\Lambda^{1,0}(U)$, and ω_0 is a real annihilator of $T^{1,0}$. Next, let $\{L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}, T\}$ be the (local) basis dual to $\{\omega_1, \omega_1, ..., \omega_{n-1}, \overline{\omega}_1, \overline{\omega}_2, ..., \overline{\omega}_{n-1}, \omega_0\}$. We may assume T is the real vector field.

Definition 1.1.4. The Hermitian matrix $\left(c_{kj}\right)_{k,j=1,...,n-1}$ defined by

$$[L_k, \overline{L}_j] = i c_{kj} T, \quad \operatorname{mod}(L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1})$$

is called the Levi form associated with the given CR structure.

The Levi matrix (c_{ij}) clearly depends on the choices of $L_1, ..., L_{n-1}$ and T. However, the number of nonzero eigenvalues and the absolute value of the signature of (c_{kj}) at each point are independent of the choice of $L_1, ..., L_{n-1}$ and T. Hence, after changing T to $-T$, it makes sense

to consider positive definiteness of the matrix (c_{ij}).
It turns out that if the Levi form is semi-definite positive, we say that M is a weakly pseudoconvex manifold, but in this thesis, we will only require the weaker condition on the Levi form (in the next section), that means the condition of pseudoconvexity does not necessarily hold.

We fix a Hermitian metric on $\mathbb{C}T(M)$ such that $L_1, ..., L_{n-1}$ is an orthonormal basis and $T^{1,0}(M) \perp T^{0,1}(M)$, the adjoint operator of $\bar{\partial}_b$ is defined relative to this metric.
We extend $\bar{\partial}_b$ to $L^2_{(0,q)}(M)$ in the sense of distribution, where $L^2_{(0,q)}(M)$ is the space of $(0,q)$-forms on M whose coefficients belong to $L^2(M)$. In particular, we can define the domain of $\bar{\partial}_b$.

Definition 1.1.5. $\mathrm{Dom}(\bar{\partial}_b)$ is the subset of $L^2_{(0,q)}(M)$ composed of all forms ϕ for which there exists a sequence of $\{\phi_m\}$ in $C^\infty(\Lambda^{0,q}(M))$ satisfying:

1. $\phi = \lim_{m\to\infty} \phi_m$ in L^2,
2. $\{\bar{\partial}_b \phi_m\}$ is a Cauchy sequence in $L^2_{(0,q+1)}(M)$

For all $\phi \in \mathrm{Dom}(\bar{\partial}_b)$, let $\lim\limits_{m\to\infty} \bar{\partial}_b \phi_m = \bar{\partial}_b \phi$ which is thus well-defined. We have the complex

$$0 \longrightarrow L^2_{(0,0)}(M) \xrightarrow{\bar{\partial}_b} L^2_{(0,1)}(M) \xrightarrow{\bar{\partial}_b} ... \xrightarrow{\bar{\partial}_b} L^2_{(0,n-1)}(M) \longrightarrow 0.$$

We define the domain of the adjoint $\bar{\partial}_b^*$ as follows

$$\begin{aligned} \mathrm{Dom}(\bar{\partial}_b^*) = \{\phi \in L^2_{0,q}(M) : \; & \text{there exixts a unique } (0,q-1)\text{-form } g \in L^2_{(0,q-1)}(M) \text{ such that} \\ & (\phi, \bar{\partial}_b \psi) = (g, \psi) \text{ for every } (0,q-1)\text{-form } \psi \in L^2_{(0,q-1)}(M)\}. \end{aligned}$$

In this case, we define $\bar{\partial}_b^* \phi = g$. Next, we denote the domain of $\Box_b$ by

$$\begin{aligned} \mathrm{Dom}(\Box_b) = \{\phi \in L^2_{(0,q)}(M) : \; & \phi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*), \\ & \bar{\partial}_b \phi \in \mathrm{Dom}(\bar{\partial}_b^*) \text{ and } \bar{\partial}_b^* \phi \in \mathrm{Dom}(\bar{\partial}_b)\}, \end{aligned}$$

where, the Kohn-Laplacian operator

$$\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b.$$

Notice that $\Box_b$ is a linear, closed, densely defined self-adjoint operator from $L^2_{(0,q)}(M)$ into itself.
On U, we can express a smooth $(0,q)$-form ϕ as

$$\phi = {\sum_{|J|=q}}' \phi_J \, \overline{\omega}_J,$$

where $J = (j_1, ..., j_q)$ are multi-indices, and the prime means that we take the sum over only increasing multi-indeces. In fact, any $(0,q)$-form can be expressed as this way. Then, the operator $\bar{\partial}_b$ is

$$\bar{\partial}_b \phi = {\sum_{|J|=q}}' \sum_{j=1}^{n-1} \overline{L}_j(\phi_J) \, \overline{\omega}_j \wedge \overline{\omega}_J + \text{terms of order zero}, \tag{1.1.1}$$

and integration by parts yields

$$\bar{\partial}_b^* \phi = -\sum_{|K|=q-1}{}' \sum_{j=1}^{n-1} L_j(\phi_{jK})\,\overline{\omega}_K + \text{terms of order zero.} \tag{1.1.2}$$

Then, a straightforward calculation shows that

$$\begin{aligned} \Box_b \phi = -\,[&\textstyle\sum'_{\substack{j\notin J,\\ |J|=q}} (L_j \overline{L}_j \phi_J)\,\overline{\omega}_J + \sum'_{\substack{j\in J,\\ |J|=q}} (\overline{L_j} L_j \phi_J)\,\overline{\omega}_J \\ &+ \textstyle\sum'_{\substack{j\neq k,\\ |J|=q}} [\overline{L_j}, L_k]\phi_J\,\overline{\omega}_j \wedge (\overline{\omega}_k \rfloor \overline{\omega}_J)] + L\phi \end{aligned}\ , \tag{1.1.3}$$

where $\overline{\omega}_k \rfloor \overline{\omega}_J = 0$ if $k \notin \{J\}$, otherwise $\overline{\omega}_k \rfloor \overline{\omega}_L = (-1)^{l-1}\overline{\omega}_{j_1} \wedge ... \wedge \overline{\omega}_{j_{l-1}} \wedge \overline{\omega}_{j_{l+1}} \wedge ... \wedge \overline{\omega}_{j_q}$ if $k = j_l$, and L is a first order differential operator involving only differentiations in the directions $L_1, ..., L_{n-1}$ and $\overline{L}_1, ..., \overline{L}_{n-1}$.

Definition 1.1.6. The operator $\bar{\partial}_b$ is said to have the closed range in L^2 if $\overline{\text{range}(\bar{\partial}_b)} = \text{range}(\bar{\partial}_b)$. The closed range hypothesis implies that $(\ker\bar{\partial}_b^{(0,q)})^{\perp} = \text{range}(\bar{\partial}_b^{*(0,q+1)})$ and $(\ker\bar{\partial}_b^{*(0,q+1)})^{\perp} = \text{range}(\bar{\partial}_b^{(0,q)})$. Moreover, for given $f \in \text{range}(\bar{\partial}_b^{(0,q)})$, there exists $u \perp \ker\bar{\partial}_b^{(0,q)}$ such that $\bar{\partial}_b^{(0,q)} u = f$ and $||u||_{L^2_{0,q}(M)} \lesssim ||f||_{L^2_{0,q+1}(M)}$. In particular, this hypothesis is always satisfied when $M = b\Omega \subset \mathbb{C}^n$ is the boundary of a (smoothly bounded) pseudoconvex domain. If M is a compact, oriented, weakly pseudoconvex manifold of dimension $(2n-1), n \geq 3$, embedded in $\mathbb{C}^N, (n \leq N)$, of codimension one or above, and endowed with the induced CR structure, then $\bar{\partial}_b$ has closed range (see [Ni06]). Recently, in [Ba11], the hypothesis of closed range holds when M is a smooth, compact, connected, CR manifold of hypersurface type, pseudoconvex-oriented.

In this thesis, the considered CR-manifold M is an abstract one, so it is assumed to have the closed range. Next, we will introduce two critical geometrical conditions in our approach.

1.2 The condition $D^{\epsilon}(q)$

This condition was introduced first by K. Koenig [Koe02].
Let M be any CR-manifold of dimension $2n-1$ with $n \geq 3$. For $1 \leq q \leq n-1$, let σ_q denote any of the $\binom{n-1}{q}$ sums of q eigenvalues λ_j of the Levi matrix (c_{kj}) and $\tau = \sum_{j=1}^{n-1} \lambda_j$ be the trace

Definition 1.2.1. Let $U \subset M$ be an open subset. We say that the $D^{\epsilon}(q)$ condition holds in U if for every compact set $K \subset U$, there exists $\epsilon_K > 0$ such that $\epsilon_K \tau \leq \sigma_q \leq (1-\epsilon_K)\tau$, for all possible sums σ_q. However, since this is a local condition, we can always assume that

$$\epsilon_K \tau \leq \sigma_q \leq (1-\epsilon_K)\tau \quad \forall \text{ possible } \sigma_q, \tag{1.2.1}$$

in U, for some sufficiently small $\epsilon > 0$.

In particular, the condition $D^\epsilon(1)$ implies that the Levi matrix (c_{kj}) has comparable eigenvalues. And $D^\epsilon(2)$ holds when at most one eigenvalue is degenerate with $n \geq 5$.

Example 1.2.1. *Let $M = b\Omega$ be the boundary of the following domain*

$$\Omega = \{(z_1, ..., z_n) \in \mathbb{C}^n : \operatorname{Im} z_n > P(z_1) + (|z_2|^2 + ... + |z_{n-1}|^2)^k\}$$

in a neighborhood of the origin, for $n \geq 5$ and $k \geq 1$. Here P is a smooth, subharmonic function such that $P(0) = 0$. We can see that the hypersurface M can be identified with $\mathbb{C}^{n-1} \times \mathbb{R}$ via the following map

$$(z_1, ..., z_{n-1}, t) \mapsto (z_1, ..., z_{n-1}, t + i(P(z_1) + (|z_2|^2 + ... + |z_{n-1}|^2)^k)).$$

We define

$$\begin{aligned} \frac{1}{2}L_1 &= \frac{\partial}{\partial z_1} + i\frac{\partial P}{\partial z_1}(z_1)\frac{\partial}{\partial t}; \\ \frac{1}{2}L_j &= \frac{\partial}{\partial z_j} + i\bar{z}_j\frac{\partial}{\partial t} \quad \textit{for } j = 2, ..., n-1; \\ T &= \frac{\partial}{\partial t}. \end{aligned}$$

Then, M is pseudoconvex and satisfies the condition $D^\epsilon(2)$ near the origin.

Definition 1.2.2. We say that M is *weakly q-convex* near a point $x_0 \in M$ if every possible σ_q is non-negative on some neighborhoods of x_0.

Proposition 1.2.3. *If M satisfies the $D^\epsilon(q)$ condition near a point $x_0 \in M$, then M is weakly q-convex near x_0.*

Proof. Since $\epsilon > 0$ is small, so from $D^\epsilon(q)$ condition, we have $\tau \geq 0$. Hence, M is weakly q-convex. □

Next, we will consider the range of the $D^\epsilon(q)$ condition, for some $1 \leq q \leq n-1$.

Lemma 1.2.4. *Fix $\epsilon > 0$.*

1. $\sigma_q \geq \epsilon\tau, \quad \forall\sigma_q \quad \Leftrightarrow \quad \sigma_{n-1-q} \leq (1-\epsilon)\tau, \quad \forall\sigma_{n-1-q}$.
2. $\sigma_q \leq (1-\epsilon)\tau, \quad \forall\sigma_q \quad \Leftrightarrow \quad \sigma_{n-1-q} \geq \epsilon\tau, \quad \forall\sigma_{n-1-q}$.
3. *$D^\epsilon(q)$ holds $\quad \Leftrightarrow \quad D^\epsilon(n-1-q)$ holds.*

Proof. (1)

$$\begin{aligned} \sigma_q \geq \epsilon\tau, \quad \forall\sigma_q \quad &\Leftrightarrow \quad -\sigma_q \leq -\epsilon\tau, \quad \forall\sigma_q \\ &\Leftrightarrow \quad \tau - \sigma_q \leq \tau - \epsilon\tau, \quad \forall\sigma_q \\ &\Leftrightarrow \quad \sigma_{n-1-q} \leq (1-\epsilon)\tau, \quad \forall\sigma_{n-1-q}. \end{aligned}$$

Part (2) is obtained similarly to the proof of Part (1). Then, (1) and (2) immediately imply Part (3). □

Proposition 1.2.5. *1. Assume that $q \le n-1-q$*

(a) If $n = 2q+1$, then $\sigma_q \ge \epsilon\tau, \quad \forall\sigma_q \quad \Leftrightarrow \quad \sigma_q \le (1-\epsilon)\tau, \quad \forall\sigma_q$.

(b) If $n \ge 2q+2$, then $\sigma_q \ge \epsilon\tau, \quad \forall\sigma_q \Rightarrow \quad \sigma_q \le (1-\epsilon')\tau, \quad \forall\sigma_q$, for some $\epsilon' > 0$. Similarly, $\sigma_{n-1-q} \ge \epsilon\tau, \quad \forall\sigma_{n-1-q} \Rightarrow \quad \sigma_{n-1-q} \le (1-\epsilon')\tau, \quad \forall\sigma_{n-1-q}$, for some $\epsilon' > 0$. However, the converse statements are not true in general.

2. Let $1 \le q_0 \le n-2$, then the $D^\epsilon(q_0)$ condition implies $D^\epsilon(q)$ condition for $\min(q_0, n-1-q_0) \le q \le \min(q_0, n-1-q_0)$.

Proof. **Part (1)**
(a). Since $q = n-1-q$, this is the part (1) of the above lemma.
(b). Let $q < n-1-q$, so, given $n-1-q$ eigenvalues, choose q at a time and apply the assumption $\sigma_q \ge \epsilon\tau$, then taking the sum of all possible q in those $n-1-q$ eigenvalues, we get

$$\frac{q\binom{n-1-q}{q}}{n-1-q}\sigma_{n-1-q} \ge \binom{n-1-q}{q}\epsilon\tau.$$

This implies that

$$\sigma_{n-1-q} \ge \frac{n-1-q}{q}\epsilon\tau.$$

Therefore, replacing ϵ to $\frac{n-1-q}{q}\epsilon$, the part (2) of the previous lemma implies $\sigma_q \le (1-\epsilon')\tau$, where $\epsilon' = \frac{n-1-q}{q}\epsilon$. The converse statement is not true. Indeed, the strictly inequality $q < n-1-q$ means we could have q zero eigenvalues and $q+1$ strictly positive eigenvalues in $\lambda_j's$, $j = 1, ..., n-1$. Final, also applying the part (1) of the previous lemma, we yield the assertion (b).
Part (2).
We already know that $D^\epsilon(q)$ is equivalent to $D^\epsilon(n-1-q)$. This means we have $D^\epsilon(n-2-q_0)$ if $D^\epsilon(q_0+1)$ holds. We can assume that $n \ge 2(q_0+1)+1$, and it is sufficent to prove that $D^\epsilon(q_0)$ implies $D^\epsilon(q_0+1)$. This is a consequence of Part (1). Since $\epsilon'\tau \le \sigma_{q_0+1} \le (1-\epsilon")\tau$, then we choose $\epsilon = min\{\epsilon', \epsilon"\}$. The proof of the proposition is completed.

□

The last statement in the above proposition says that : for $1 \le q_0 \le n-1$, if the $D^\epsilon(q_0)$ holds, then $D^\epsilon(q)$ conditions holds as well, for all $\min(q_0, n-1-q_0) \le q \le \min(q_0, n-1-q_0)$. And we will always assume that $q_0 \le n-1-q_0$ for convenience.

Next, we will show that the condition $D^\epsilon(q)$ implies the following *maximal L^2 estimates* for the operator $\Box_b$ on $(0,q)$-forms.

Theorem 1.2.6. *[Koe02] Let M be a $(2n-1)$-dimensional CR-manifold with $n \ge 3$ with hypothesis of closed-range of $\bar\partial_b$. Assume that the condition $D^\epsilon(q)$ holds near x_0. Then, in neighborhood U of x_0,*

$$\sum_{|J|=q}{}' \sum_{j=1}^{n-1} \left(||L_j u_J||^2 + ||\overline{L}_j u_J||^2\right) \lesssim ||\bar\partial_b u||^2 + ||\bar\partial_b^* u||^2 + ||u||^2 \tag{1.2.2}$$

for a smooth $(0,q)$*-form* $u = \sum'_{|J|=q} u_J \overline{\omega}_J$ *with compact support in* U.
Moreover, as Proposition 1.2.5, the estimate (1.2.2) *holds as well for all smooth* $(0,q')$*-forms, with* $q \le q' \le n-1-q$.

Here and in what follows, $\lesssim$ and $\gtrsim$ denote inequality up to a positive constant. Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

Proof. For $u = \sum'_{|J|=q} u_J \overline{w}_J$, with $1 \le q \le n-2$, a direct computation shows that

$$||\bar{\partial}_b u||^2 = \sum_{|J|=q}{}' \sum_{\substack{j=1,\\ j\notin J}}^{n-1} ||\overline{L}_j u_J||^2 + \sum{}'_{\substack{|I|=q,\\ |J|=q}} \sum_{\substack{i,j=1,\\ i\neq j}}^{n-1} \epsilon_{iI}^{jJ} \big(\overline{L}_j u_J, \overline{L}_i u_I\big) + O\big(||\overline{L}u||.||u|| + ||u||^2\big),$$

where $\epsilon_{iI}^{jJ} = 0$, unless $i \notin I$, $j \notin J$, and $\{i\} \cup I = \{j\} \cup J$, in which case ϵ_{iI}^{jJ} is the sign of permutation $\begin{pmatrix} iI \\ jJ \end{pmatrix}$. Similarly,

$$||\bar{\partial}_b^* u||^2 = \sum_{|K|=q-1}{}' \sum_{\substack{|I|=q\\ |J|=q}} \sum_{i,j=1}^{n-1} \epsilon_J^{jK} \epsilon_I^{iK} \big(L_j u_J, L_i u_I\big) + O\big(||\overline{L}u||.||u|| + ||u||^2\big),$$

where $\epsilon_{J(I)}^{jK}$ is the sign of the permutation $\begin{pmatrix} jK \\ J(I) \end{pmatrix}$ if $\{j\} \cup K = J(I)$, and is zero otherwise. Notice that, when $i = j$, $\epsilon_J^{jK} \epsilon_I^{iK} = 1$ if $I = J$ (and is zero otherwise), when $i \neq j$, $\epsilon_J^{jK} \epsilon_I^{iK} = -\epsilon_{iJ}^{jI}$. Thereupon,

$$||\bar{\partial}_b^* u||^2 = \sum_{|J|=q}{}' \sum_{\substack{j=1\\ j\in J}}^{n-1} \big(L_j u_J, L_j u_J\big) - \sum{}'_{\substack{|I|=q\\ |J|=q}} \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \epsilon_{jJ}^{iI} \big(L_i u_J, L_j u_J\big) + O\big(||\overline{L}u||.||u|| + ||u||^2\big).$$

In particular, if $i \neq j$, $\epsilon_J^{jK} = \epsilon_{iJ}^{ijK}$ and $\epsilon_I^{iK} = \epsilon_{jI}^{jiK} = -\epsilon_{ijK}^{jI}$, both of them are not zero when $i \notin J, j \notin I$. And so $\epsilon_J^{jK} \epsilon_I^{iK} = -\epsilon_{iJ}^{jI}$.
If $i = j$, $\epsilon_J^{jK} \epsilon_I^{iK} = 1$ when $I = J$, if $I \neq J$, this equals to 0. Accordingly, we obtain

$$||\bar{\partial}_b^* u||^2 = \sum_{|J|=q}{}' \sum_{\substack{j=1\\ j\in J}}^{n-1} \big(L_j u_J, L_j u_J\big) - \sum{}'_{\substack{|I|=q\\ |J|=q}} \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \epsilon_{jI}^{iJ} \big(L_i u_J, L_j u_I\big) + O\big(||\overline{L}u||.||u|| + ||u||^2\big).$$

We know that

$$L_j^* = -\overline{L_j} + \text{term of order } 0, \quad \overline{L_j}^* = -L_j + \text{term of order } 0,$$

and, $[L_i, \overline{L}_j] = ic_{ij}T$, $\big(c_{ij}Tu_J, u_I\big) = \big(L_iu_J, L_ju_I\big) - \big(\overline{L}_ju_J, \overline{L}_iu_I\big) +$ terms of order 0. Hence, integration by parts, we obtain

$$||\bar\partial_b^* u||^2 = \sum_{|J|=q}{}' \sum_{\substack{j=1\\ j\in J}}^{n-1} ||\overline{L}_j u_J||^2 + \sum_{|J|=q}{}' \sum_{\substack{j=1\\ j\in J}}^{n-1} \big(c_{jj}Tu_J, u_J\big) - \sum{}'_{\substack{|I|=q\\ |J|=q}} \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \epsilon^{jJ}_{iI}\big(\overline{L}_ju_J, \overline{L}_iu_I\big)$$
$$- \sum{}'_{\substack{|I|=q\\ |J|=q}} \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \epsilon^{jI}_{iJ}\big(c_{ij}Tu_J, u_I\big) + O\big(||\overline{L}u||.||u|| + ||u||^2\big).$$

So,

$$\begin{aligned} ||\bar\partial_b u||^2 + ||\bar\partial_b^* u||^2 &= ||\overline{L}u||^2 + \sum_{|J|=q}{}' \sum_{\substack{j=1\\ j\in J}}^{n-1} \big(c_{jj}Tu_J, u_J\big) - \sum{}'_{\substack{|I|=q\\ |J|=q}} \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \epsilon^{iJ}_{jI}\big(c_{ij}Tu_J, u_J\big) \\ &\quad + O\big(||\overline{L}u||.||u|| + ||u||^2\big) \\ &= ||\overline{L}u||^2 + \sum{}'_{\substack{|I|=q\\ |J|=q}} \big(d^q_{IJ}Tu_I, u_J\big) + O\big(||\overline{L}u||.||u|| + ||u||^2\big), \end{aligned} \tag{1.2.3}$$

where,the $\binom{n-1}{q} \times \binom{n-1}{q}$ matrix $\big(d^q_{IJ}\big)_{\substack{|I|=q\\ |J|=q}}$ is denoted by

$$\begin{aligned} d^q_{II} &= \sum_{i\in I} c_{ii} \\ d^q_{IJ} &= -\sum_{\substack{i\in I\\ j\in J}} \epsilon^{iJ}_{jI} c_{ij}, \quad \text{for} \quad I \neq J \quad \big(\text{in particular} \quad \big(d^1_{IJ}\big) = (c_{ij})\big). \end{aligned}$$

To proceed the matrix $\big(d^q_{IJ}\big)_{\substack{|I|=q\\ |J|=q}}$, we need the following lemma (proved by induction in q)

Lemma 1.2.7. *For fixed $1 \le q \le n-2$, the eigenvalues of the matrix $\big(d^q_{IJ}\big)_{\substack{|I|=q\\ |J|=q}}$ are the $\binom{n-1}{q}$ sums $\lambda_{j_1} + ... + \lambda_{j_q}$. Therefore, if $D^\epsilon(q)$ condition holds, the matrix $(d^q_{IJ})_{I,J}$ is semi-definite positive .*

From this lemma, it is arised that $||\overline{L}u||^2 \lesssim ||\bar\partial_b u||^2 + ||\bar\partial_b^* u||^2 + ||u||^2$, but we also have better estimates by the method of microlocalization introduced in [De91] when $q = 1$. In our case, when $q \geq 1$, we will replace the $(n-1)\times(n-1)$- matrix with indexes $j, k = 1$ for $(0,1)$-forms in Derridj's paper by the $\binom{n-1}{q} \times \binom{n-1}{q}$-matrix with indexes I, J, where $|I| = |J| = q$. In U, we choose the local coordinates $(x,t) = (x_1, ..., x_{2n-2}, t)$, and let (ξ, τ) be the dual coordinates to $\mathbb{R}^{2n-1} - (x,t)$. We define the following non-negative , C^∞-functions whose ranges in

$[0,1]$:

$$\begin{aligned}
\psi^+(\xi,\tau) \quad \text{such that} \quad & \psi^+(\xi,\tau) = 0 \,\text{when}\, \tau < 0\\
& \psi^+(\xi,\tau) = 1 \,\text{when}\, \tau > 1,\\
\psi^-(\xi,\tau) \quad \text{such that} \quad & \psi^-(\xi,\tau) = 0 \,\text{when}\, \tau > 0\\
& \psi^-(\xi,\tau) = 1 \,\text{when}\, \tau \le -1,\\
\psi^0(\xi,\tau) = 1 - \psi^+(\xi,\tau) - \psi^-(\xi,\tau),&
\end{aligned}$$

so supp $\psi^0 \subset \{|\tau| \le 1\}$. Let ϕ be a distribution on U, $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^{2n-1}$, we define

$$\begin{aligned}
\phi^+ &:= \mathcal{F}^{-1}[\psi^+\widehat{\phi}]\\
\phi^- &:= \mathcal{F}^{-1}[\psi^-\widehat{\phi}]\\
\phi^0 &:= \mathcal{F}^{-1}[\psi^0\widehat{\phi}].
\end{aligned}$$

The microlocal decomposition is interpreted as follows

$$\phi = \mathcal{P}^+\phi + \mathcal{P}^-\phi + \mathcal{P}^0\phi,$$

where $\mathcal{P}^+, \mathcal{P}^-, \mathcal{P}^0$ are the pseudodifferential operators of order 0 defined by

$$\begin{aligned}
\mathcal{P}^+\phi &= \zeta\mathcal{F}^{-1}[\psi^+\widehat{\phi}]\\
\mathcal{P}^-\phi &= \zeta\mathcal{F}^{-1}[\psi^-\widehat{\phi}]\\
\mathcal{P}^0\phi &= \zeta(\phi - \mathcal{F}^{-1}[(\psi^+ + \psi^-)\widehat{\phi}]),
\end{aligned}$$

for all distribution ϕ, where $\zeta \in C_0^\infty(U')$, $\bar{U} \subset U'$ and $\zeta = 1$ on U.
Let $\delta > 0$ be determined later. It is sufficiently small since we can shrink U if necessary. We will estimate $(1-\delta)||\bar{L}u||^2$ from lower in (1.2.3). For any $(0,q)$-form $u = \sum'_{|J|=q} u_J\bar{\omega}_J$, by construction, we imply that

$$||\bar{L}_j u_J||^2 \ge ||\bar{L}_j(\mathcal{P}^+u_J + \mathcal{P}^-u_J)||^2 - ||\bar{L}_j\mathcal{P}^0u||^2.$$

By the ellipticity of $||\bar{\partial}_b(.)||^2 + ||\bar{\partial}_b^*(.)||^2$ in complex directions, we also obtain (see [ChSh01])

$$||\bar{L}_j\mathcal{P}^0u||^2 \le C_0(||\bar{\partial}_b u||^2 + ||\bar{\partial}_b^* u||^2 + ||u||^2).$$

As a consequence,

$$||\bar{L}_j u_J||^2 \ge ||\bar{L}_j(\mathcal{P}^+u_J + \mathcal{P}^-u_J)||^2 - C_0(||\bar{\partial}_b u||^2 + ||\bar{\partial}_b^* u||^2 + ||u||^2), \tag{1.2.4}$$

for every $(0,q)$-form u. On the other hand, $||\bar{L}_j(\mathcal{P}^+u_J + \mathcal{P}^-u_J)||^2 = ||\bar{L}_j\mathcal{P}^+u_J)||^2 + ||\bar{L}_j\mathcal{P}^+ - u_J||^2 + 2\,\mathrm{Re}(\bar{L}_j\mathcal{P}^+u_J, \bar{L}_j\mathcal{P}^-u_J)$. And also, the last term can be written as

$$(\bar{L}_j\mathcal{P}^+u_J, \bar{L}_j\mathcal{P}^-u_J) = (\mathcal{P}^-\mathcal{P}^+\bar{L}_j u_J, \bar{L}_j u_J) + O(||u||.||\bar{L}u|| + ||u||^2),$$

Notice that $\mathcal{P}^-\mathcal{P}^+$ is the pseudodifferential operator of order zero since $[\mathcal{P}^+, \mathcal{P}^-]$ is of order zero, so, $\mathcal{P}^-\mathcal{P}^-\bar{L}_j$ is of order zero. Then, $(\mathcal{P}^-\mathcal{P}^+\bar{L}_j u_J) = O(||u||)$. Thus, combine all of the above estimates, we obtain

$$\begin{aligned}(1-\delta)||\bar{L}_j u_J||^2 = (1-\delta)||\bar{L}_j\mathcal{P}^+u_J)||^2 + (1-\delta)||\bar{L}_j\mathcal{P}^-u_J)||^2 \\ + O(||u||.||\bar{L}u|| + ||u||^2) - C_0(||\bar{\partial}_b u||^2 + ||\bar{\partial}_b^* u||^2 + ||u||^2).\end{aligned} \tag{1.2.5}$$

Now, let α, β be positive number, less than $(1-\delta)$, be chosen later. The same calculation yields that

$$\begin{aligned}\alpha||\bar{L}_j\mathcal{P}^+u_J||^2 = \alpha||L_j\mathcal{P}^+u_J||^2 - \alpha(\lambda_{jj}T\mathcal{P}^+u_J, \mathcal{P}^+u_J) + O(||u||.||\bar{L}u|| + ||u||^2); \\ \beta||\bar{L}_j\mathcal{P}^-u_J||^2 = \beta||L_j\mathcal{P}^-u_J||^2 - \beta(\lambda_{jj}T\mathcal{P}^-u_J, \mathcal{P}^-u_J) + O(||u||.||\bar{L}u|| + ||u||^2).\end{aligned} \tag{1.2.6}$$

Hence, from (1.2.3) and rewriting $||\bar{L}u|| = \delta||\bar{L}u|| + (1-\delta)||\bar{L}u||$, the following holds for $(0,q)$-form $u = \sum'_{|J|=q} u_J\bar{\omega}_J$

$$\begin{aligned}||\bar{\partial}_b u||^2 + ||\bar{\partial}_b^* u||^2 \geq \delta||\bar{L}u||^2 + \alpha\sum_{|J|=q}{}' \sum_{j=1}^{n-1}||L_j\mathcal{P}^+u_J||^2 + \beta\sum_{|J|=q}{}' \sum_{j=1}^{n-1}||L_j\mathcal{P}^-u_J||^2 \\ + \sum{}'_{\substack{|I|=q\\|J|=q}}\big(d^q_{IJ}Tu_I, u_J\big) - \alpha\sum_{|J|=q}{}'\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J, \mathcal{P}^+u_J) \\ - \beta\sum_{|J|=q}{}'\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^-u_J, \mathcal{P}^-u_J) + O(||u||.||\bar{L}u|| + ||u||^2) \\ - C_0(||\bar{\partial}_b u||^2 + ||\bar{\partial}_b^* u||^2 + ||u||^2),\end{aligned} \tag{1.2.7}$$

when we choose $0 < \alpha, \beta < (1-\delta)$.
On the other hand,

$$\begin{aligned}\big(d^q_{IJ}Tu_I, u_J\big) = (d^q_{IJ}T\mathcal{P}^+u_I, \mathcal{P}^+u_J) + (d^q_{IJ}T\mathcal{P}^-u_I, \mathcal{P}^-u_J) + (d^q_{IJ}T\mathcal{P}^0u_I, \mathcal{P}^0u_J) \\ + (d^q_{IJ}T\mathcal{P}^+u_I, \mathcal{P}^-u_J) + (d^q_{IJ}T\mathcal{P}^+u_I, \mathcal{P}^0u_J) + (d^q_{IJ}T\mathcal{P}^-u_I, \mathcal{P}^+u_J) \\ + (d^q_{IJ}T\mathcal{P}^-u_I, \mathcal{P}^0u_J) + (d^q_{IJ}T\mathcal{P}^0u_I, \mathcal{P}^+u_J) + (d^q_{IJ}T\mathcal{P}^0u_I, \mathcal{P}^-u_J).\end{aligned} \tag{1.2.8}$$

We have the symbols of $\mathcal{P}^0, \mathcal{P}^0\mathcal{P}^+, \mathcal{P}^0\mathcal{P}^-, \mathcal{P}^+\mathcal{P}^-, \mathcal{P}^-\mathcal{P}^+, \mathcal{P}^+\mathcal{P}^0$ and $\mathcal{P}^-\mathcal{P}^0$ supported in $|\tau| \leq 1$. Since the matrix $(d^q_{IJ})_{I,J}$ is positive semi-definite, the seven last terms in (1.2.8) equal to

$O(||u||^2)$. It follows

$$
\begin{aligned}
(C_0+1)(||\bar\partial_b u||^2+||\bar\partial_b^* u||^2) \geq \delta||\bar L u||^2 &+ \alpha\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^+u_J||^2+\beta\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^-u_J||^2 \\
&+ \sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}}(d^q_{IJ}T\mathcal{P}^+u_I,\mathcal{P}^+u_J)+\sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}}(d^q_{IJ}T\mathcal{P}^-u_I,\mathcal{P}^-u_J) \\
&- \alpha\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J,\mathcal{P}^+u_J)-\beta\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^-u_J,\mathcal{P}^-u_J) \\
&+ O(||u||.||\bar L u||+||u||^2).
\end{aligned}
\tag{1.2.9}
$$

Now,we will apply the lemma to $\sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}}(d^q_{IJ}T\mathcal{P}^+u_I,\mathcal{P}^+u_J)-\alpha\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J,\mathcal{P}^+u_J)$,

$$
\begin{aligned}
&\sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}}(d^q_{IJ}T\mathcal{P}^+u_I,\mathcal{P}^+u_J)-\alpha\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J,\mathcal{P}^+u_J) \\
&=\sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}}(d^q_{IJ}T\mathcal{P}^+u_I,\mathcal{P}^+u_J)-\epsilon\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J,\mathcal{P}^+u_J) \\
&+(\epsilon-\alpha)\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J,\mathcal{P}^+u_J).
\end{aligned}
\tag{1.2.10}
$$

We define the following $\binom{n-1}{q}\times\binom{n-1}{q}$ matrix

$$
\begin{aligned}
D^q_{IJ}&=d^q_{IJ},\quad I\neq J\\
D^q_{II}&=d^q_{II}-\epsilon\sum_{j=1}^{n-1}c_{jj}.
\end{aligned}
$$

By Lemma (1.2.7) and $\sigma_q\geq\epsilon\tau$, $(D^q_{IJ})_{I,J}$ is a positive semi-definite matrix. On the other hand $\mathcal{P}^+T\mathcal{P}^+$ has non-negative symbol (since ψ^+ is supported in $\tau\geq 0$), and is of degree 1. Thereupon, from the fact of pseudodifferential operators in [LaNi66] (Section 3), we imply that

$$\sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}}(D^q_{IJ}\mathcal{P}^+T\mathcal{P}^+u_I,u_J)\geq -C||u||^2.$$

So,

$$
\begin{aligned}
&\sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}}(d^q_{IJ}T\mathcal{P}^+u_I,\mathcal{P}^+u_J)-\alpha\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J,\mathcal{P}^+u_J) \\
&\geq(\epsilon-\alpha)\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+u_J,\mathcal{P}^+u_J)+O(||u||^2).
\end{aligned}
\tag{1.2.11}
$$

Now, we put

$$E^q_{IJ} = d^q_{IJ}, \quad I \neq J$$
$$E^q_{II} = d^q_{II} - (1-\epsilon)\sum_{j=1}^{n-1} c_{jj}.$$

This matrix is negative semi-definite since $\sigma_q \leq (1-\epsilon)\tau$. However, $\mathcal{P}^- T \mathcal{P}^-$ has non-positive symbol (since ψ^- is supported in $\tau \leq 0$), so we also have

$$\begin{aligned} &\sideset{}{'}\sum_{\substack{|I|=q\\|J|=q}} (d^q_{IJ} T\mathcal{P}^- u_I, \mathcal{P}^- u_J) - \beta \sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^- u_J, \mathcal{P}^- u_J) \\ &\geq (1-\epsilon-\beta)\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^- u_J, \mathcal{P}^- u_J) + O(||u||^2). \end{aligned} \tag{1.2.12}$$

Hence, combine (1.2.9, 1.2.11, 1.2.12),

$$\begin{aligned} (C_0+1)(||\bar\partial_b u||^2 + ||\bar\partial_b^* u||^2) \geq \delta||\bar L u||^2 &+ \alpha \sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^+ u_J||^2 + \beta\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^- u_J||^2 \\ &+ (\epsilon-\alpha)\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^+ u_J, \mathcal{P}^+ u_J) \\ &+ (1-\epsilon-\beta)\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}(c_{jj}T\mathcal{P}^- u_J, \mathcal{P}^- u_J) \\ &+ O(||u||.||\bar L u|| + ||u||^2). \end{aligned} \tag{1.2.13}$$

On the orther hand,

$$\begin{aligned} \alpha\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^+ u_J||^2 + \beta\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^- u_J||^2 &\geq \inf(\alpha,\beta)(\sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^+ u_J||^2 + \sideset{}{'}\sum_{|J|=q}\sum_{j=1}^{n-1}||L_j\mathcal{P}^- u_J||^2) \\ &\geq \inf(\alpha,\beta)||Lu||^2 - C||L\mathcal{P}^0 u||^2 \\ &\geq \inf(\alpha,\beta)||Lu||^2 - C(||\bar\partial_b u||^2 + ||\bar\partial_b^* u||^2 + ||u||^2). \end{aligned} \tag{1.2.14}$$

Next, we choose $\delta > 0$ is small enough such that

$$\alpha, \beta < 1-\delta,\ \epsilon - \alpha \geq \gamma > 0,\ 1-\epsilon-\beta \leq -\gamma.$$

Immediately, we obtain

$$\begin{aligned}\sum_{|J|=q}{}' \sum_{j=1}^{n-1} ||L_j \mathcal{P}^+ u_J||^2 + \sum_{|J|=q}{}' \sum_{j=1}^{n-1} ||\bar{L}_j \mathcal{P}^+ u_J||^2 + & \\ + \sum_{|J|=q}{}' (\tau T \mathcal{P}^+ u_J, \mathcal{P}^+ u_J) - \sum_{|J|=q}{}' (\tau T \mathcal{P}^- u_J, \mathcal{P}^- u_J) & \\ \lesssim ||\bar{\partial}_b u||^2 + ||\bar{\partial}_b^* u||^2 + ||u||^2. &\end{aligned} \tag{1.2.15}$$

The property that $\sum_{|J|=q}' (\tau T \mathcal{P}^+ u_J, \mathcal{P}^+ u_J) - \sum_{|J|=q}' (\tau T \mathcal{P}^- u_J, \mathcal{P}^- u_J) \geq O(||u||^2)$ implies the maximal L^2 estimate. $\square$

Remark 1.2.8. The estimate (1.2.2) has showed that how we control the L^2-norm of the derivatives Lu and $\overline{L}u$. And applying this observation, combining a compactness estimate, the operator $\Box_b$ is globally (real) analytic hypoelliptic. (see Appendix for more details).

1.3 Condition of finite commutator type (Hömander's condition)

First, we recall the length of a commutator of vector fields.

$$\begin{aligned} \{L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}\} & \quad \text{: are commutators of length 1;} \\ \{[L_1, L_2], [L_1, \overline{L}_1], ...\} & \quad \text{: are commutators of length 2;} \\ \{[L_{i_k}, [..., [L_{i_2}, L_{i_1}]...]], ...\} & \quad \text{for } L_{i_j} \in \{L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}\}\text{: are commutators of length } k. \end{aligned}$$

Let $U \subset M$ be an open subset. We say that U is of *finite commutator type* (shortly, finite-type) m if m is the least strictly positive integer for which the vector fields $L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}$ and their commutators of length $\leq m$ span the tangent space at each point of U.
Now, we put $X_j = \mathrm{Re}(L_j)$ and $X_{n+j-1} = \mathrm{Im}(L_j)$, for $j = 1, ..., n-1$. The finite type associated from the complex vector fields $\{L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}\}$ is equivalent to the finite-type associated from the real vector fields $\{X_1, X_2, ..., X_{2n-2}\}$

It is well-known that the condition of finite-type implies the *subelliptic estimate* for the system of vector fields $\{X_1, X_2, ..., X_{2n-2}\}$ (eg, see [ChSh01], Theorem 8.2.5), that means

Theorem 1.3.1. *Assume $U \subset M$ is of finite-type m , then there exists $\epsilon > 0$ depending on m such that*

$$||u||_\epsilon^2 \lesssim \sum_{|J|=q}{}' \sum_{j=1}^{2n-2} ||X_j u_J||^2 + ||u||^2,$$

for any smooth $(0,q)$-form $u = \sum_{|J|=q}' u_J \overline{\omega}_J$ with compact support in U. In particular, we can choose $\epsilon \leq \frac{1}{2^m}$ (ϵ here is different from the similar symbol in Definition of $D(q)$ condition). Where, we denote $||.||_s$ is the usual Sobolev norm, for any $s \in \mathbb{R}$.

1.4 The Szegö projection operators

Let $\mathcal{S}_q$ and $\mathcal{S}'_q$ denote the orthogonal projections in $L^2_{0,q}(M)$ onto $\ker(\bar{\partial}_b^{0,q})$ and $\ker(\bar{\partial}_b^{*0,q})$, respectively, where $\bar{\partial}_b^{0,q}$ and $\bar{\partial}_b^{*0,q}$ mean $\bar{\partial}_b$, $\bar{\partial}_b^*$ acting on $(0,q)$-forms. We also define $\mathcal{H}_q$ is the orthogonal projection in $L^2_{0,q}(M)$ onto the space of *CR harmonic* $(0,q)$*-forms* $\ker(\Box_b)$, where $\Box_b$ acting on $(0,q)$-forms. We also notice that $\ker(\Box_b) = \{\phi \in L^2_{0,q}(M) : \bar{\partial}_b\phi = \bar{\partial}_b^*\phi = 0\}$

In this section, we assume that $\bar{\partial}_b$ has the closed range in $L^2_{0.q}(M)$ if M is a general CR-manifold, then

$$L^2_{0.q}(M) = \text{range}(\Box_b) \oplus \ker(\Box_b),$$

and

$$\text{range}(\bar{\partial}_b\bar{\partial}_b^*) \perp \text{range}(\bar{\partial}_b^*\bar{\partial}_b),$$

so, we have the strong Hodge type decomposition on $L^2_{0,q}(M)$

$$L^2_{0,q}(M) = \bar{\partial}_b\bar{\partial}_b^*(\text{Dom}(\Box_b)) \oplus \bar{\partial}_b^*\bar{\partial}_b(\text{Dom}(\Box_b)) \oplus \ker(\Box_b).$$

Now, we could define the *relative* inverse of $\Box_b$ in $L^2_{0,q}(M)$. Let any $\alpha \in L^2_{0,q}(M)$, if $\alpha \in \ker(\Box_b)$, we set $\mathcal{K}_q\alpha = 0$. If $\alpha \in \text{range}(\Box_b)$, we set $\mathcal{K}_q\alpha = \phi$, where ϕ is the unique solution of $\Box_b\phi = \alpha$ with $\phi \perp \ker(\Box_b)$. This definition implies that $\mathcal{K}_q\mathcal{H}_q = \mathcal{H}_q\mathcal{K}_q = 0$.
Therefore, for any $\alpha \in L^2_{0,q}(M)$,

$$\alpha = \Box_b\mathcal{K}_q\alpha + \mathcal{H}_q\alpha = \bar{\partial}_b\bar{\partial}_b^*\mathcal{K}_q\alpha + \bar{\partial}_b^*\bar{\partial}_b\mathcal{K}_q\alpha + \mathcal{H}_q\alpha,$$

and we say that $\mathcal{K}_q$ is the relative inverse operator of $\Box_b$ in the sense that

$$\Box_b\mathcal{K}_q = \mathcal{K}_q\Box_b = I - \mathcal{H}_q.$$

Three these operators have the following relations.

Properties 1.4.1. *[Koe02]*

1. *$\mathcal{K}_q$ is self-adjoint.*
2. *For $1 \leq q \leq n-2$,*
$$I = (I - \mathcal{S}_q) + (I - \mathcal{S}'_q) + \mathcal{H}_q = \mathcal{S}_q + \mathcal{S}'_q - \mathcal{H}_q.$$
3. *For $1 \leq q \leq n-2$,*
$$\bar{\partial}_b^{0,q-1}\bar{\partial}_b^{*0,q}\mathcal{K}_q = I - \mathcal{S}'_q, \text{ and } \bar{\partial}_b^{*0,q+1}\bar{\partial}_b^{0,q}\mathcal{K}_q = I - \mathcal{S}_q$$

4. For $1 \leq q \leq n-1$, $\bar{\partial}_b^{0,q-1}\mathcal{K}_{q-1} = \mathcal{K}_q\bar{\partial}_b^{0,q-1}$ and $\bar{\partial}_b^{*0,q}\mathcal{K}_q = \mathcal{K}_{q-1}\bar{\partial}_b^{*0,q}$.

Remark 1.4.2. We have some remarks for these properties.

1. We suppose that $\bar{\partial}_b$ has closed range in $L^2_{0,q}(M)$. Since $\Box_b\mathcal{H}_q u = 0$, for every $u \in L^2_{0,q}(M)$,

$$\big(I - \mathcal{S}_q, \mathcal{H}_q\big)u = \big(I - \mathcal{S}'_q, \mathcal{H}_q\big)u = 0.$$

Then, by Cauchy's Inequality and the hypothesis of closed-range of $\bar{\partial}_b$, we imply

$$||u||^2_{L^2_{0,q}(M)} \lesssim ||\bar{\partial}_b u||^2_{L^2_{0,q+1}(M)} + ||\bar{\partial}_b^* u||^2_{L^2_{0,q-1}(M)} + ||\mathcal{H}_q u||^2_{L^2_{0,q}(M)}.$$

2. If $q = 0$, $\Box_b = \bar{\partial}_b^{*0,1}\bar{\partial}_b^{0,0}$ operates on functions in $L^2(M)$. In this case, $\mathcal{K}_0$ is the unique relative inverse of $\Box_b$ in the sense $\Box_b\mathcal{K}_0 = I - \mathcal{S}_0$, where $\mathcal{S}_0$ is the usual Szegö projection on space of holomorphic functions in $L^2(M)$. And from the property (4), we have $\mathcal{S}_0 = I - \bar{\partial}_b^{*0,1}\mathcal{K}_1\bar{\partial}_b^{0,0}$. Similarly, if $q = n-1$, $\Box_b = \bar{\partial}_b^{0,n-2}\bar{\partial}_b^{*0,n-1}$ operates on members of $L^2_{0,n-1}(M)$. In this case, $\mathcal{K}_{n-1}$ is the unique relative inverse of $\Box_b$ in the sense $\Box_b\mathcal{K}_{n-1} = I - \mathcal{S}'_{n-1}$.

Definition 1.4.3. We say that a *subelliptic* estimate of order $\epsilon > 0$ holds at $x_0 \in M$ for $(0,q)$-forms for $\bar{\partial}_b$-Neumann problem if there is a neighborhood U of x_0, $C > 0$, $0 < \epsilon < 1$ such that

$$||u||^2_\epsilon \leq C\left(||\bar{\partial}_b u||^2_{L^2_{0,q+1}(M)} + ||\bar{\partial}_b^* u||^2_{L^2_{0,q-1}(M)} + ||u||^2_{L^2_{0,q}(M)}\right),$$

where $u(x) = \sum'_{|I|=q} u_I(x)\bar{\omega}_I(x)$ defined on U.

Now, we assume that $U \subset M$ satisfies the condition of $D^\epsilon(q)$ and the condition of finite commutator type of m. Then, from Theorem 1.3.1, we get the following subelliptic estimate

$$\begin{aligned}
||u||^2_\epsilon &\lesssim ||\bar{\partial}_b u||^2_{L^2_{0,q+1}(M)} + ||\bar{\partial}_b^* u||^2_{L^2_{0,q-1}(M)} + ||u||^2_{L^2_{0,q}(M)} \\
&\lesssim ||\bar{\partial}_b u||^2_{L^2_{0,q+1}(M)} + ||\bar{\partial}_b^* u||^2_{L^2_{0,q-1}(M)} + ||\mathcal{H}_q u||^2_{L^2_{0,q}(M)} \\
&= (\Box_u, u)_{L^2_{0,q}(M)} + ||\mathcal{H}_q u||^2_{L^2_{0,q}(M)}.
\end{aligned} \tag{1.4.1}$$

On the other hand, from the hypothesis of closed range of $\bar{\partial}_b$, $||u||_{L^2_{0,q}(M)} \lesssim ||\Box_b u||_{L^2_{0,q}(M)}$, accordingly, we have

$$||u||^2_\epsilon \lesssim ||\Box_b u||^2 + ||\mathcal{H}_q u||^2 \lesssim ||\Box_b u||^2 + ||u||^2. \tag{1.4.2}$$

As a consequence of Theorem 1.2.6, the following *a priori estimate* holds

$$||u||^2_{\delta+\epsilon} + ||Lu||^2 + ||\bar{L}u||^2 \lesssim ||\Box_b u||^2_\delta + ||u||^2, \tag{1.4.3}$$

for any $(0,q)$-form $u = \sum'_{|J|=q} u_J\bar{\omega}_J$ for each $s \in \mathbb{R}$. It turns out that the inverse operator $\Box_b^{-1}$: $H^0_{0,q}(M) \to H^{\epsilon'}_{0,q}(M)$ is compact, for $0 < \epsilon' < \epsilon$. By positivity of $\Box_b$, the spectrum of $\Box_b$ is

contained in $[0,\infty)$, see [FoKo72], Proposition (3.1.11).
Following the technique in [FoKo72], we also obtain the corresponding localized estimate

$$||\zeta u||^2_{\delta+\epsilon} \lesssim ||\zeta_1 \bar{\partial}_b u||^2_\delta + ||\zeta_1 \bar{\partial}^*_b u||^2_s + ||\zeta_1 u||^2, \tag{1.4.4}$$

where $\zeta \prec \zeta_1$ (i.e., $\zeta = 1$ on $\text{supp}(\zeta_1)$), $\zeta, \zeta_1 \in C^\infty_0(U)$. This property says that the operator $\Box_b$ is hypoelliptic on $(0,q)$-forms.

In the end, we summarize this section: The condition of finite commutator type and closed range of $\bar{\partial}_b$ provide the subelliptic estimate for the operator $\Box_b$ 1.4.2 on $(0,q)$-from. We can insert $||Lu||_{L^2}, ||\overline{L}u|||_{L^2}$ into (1.4.2) by the condition of $D^\epsilon(q)$ to get (1.4.3). And from the range of $D^\epsilon(q)$ and the following theorem, (1.4.3) also holds for all smooth $(0,q')$ forms, with $q' \in [q, n-1-q]$. In [Ho91], we have

Theorem 1.4.4. *If M satisfies condition of q-convexity near $x_0 \in M$, and if a subelliptic estimate holds for $(0,q)$-forms at x_0, then a subelliptic estimate also holds for $(0,r)$-forms, $q \le r \le n-1$.*

1.5 The heat equation

Let M be the boundary of a (smooth) domain in $\mathbb{C}^n$, for $n \geq 3$, or more generally any compact CR-manifold of dimension $2n-1$ for which the range of $\bar{\partial}_b$ is closed in L^2. Assume that M satisfies the condition of $D^\epsilon(q)$ and the condition of finite commutator type. In this part, we study the *initial value problem* and the regularity properties of the *heat operator* $\mathfrak{H}$ on $(0,q)$-forms defined on $\mathbb{R}^+ \times M$

$$\begin{aligned} &\mathfrak{H}[u](s,x) := \frac{\partial u}{\partial s}(s,x) + \Box_b u(s,x) = 0 \quad \text{for } s > 0 \text{ and } x \in M, \text{ and} \\ &\lim_{s \to 0^+} u(s,.) = \phi(.) \quad \text{with convergence in appropriate norm,} \end{aligned} \tag{1.5.1}$$

where $u(s,x) = \sum'_{|I|=q} u_I(s,x)\bar{\omega}_I(x)$.
The problem is to find a smooth $(0,q')$-form u on $\mathbb{R} \times M$ such that u solves the heat equation (1.5.1) with the given $(0,q')$-form $\phi(z)$ defined on M, for all $q \le q' \le n-1-q$.
The sub-elliptic estimate for $\Box_b$ (1.4.2) on M also implies the hypoellipticity for the heat operator on $(0,\infty) \times M$.

Theorem 1.5.1. *Assume $U \subset M$ is of finite commutator type and satisfies a $D^\epsilon(q)$ condition defined as above. Then, the heat operator $\mathfrak{H}$ is hypoelliptic in $(0,\infty) \times U$ on $(0,q')$-forms, $q \le q' \le n-1-q$.*

Proof. It suffices to prove the theorem on $(0,q)$-forms.
Since the conditions of finite type and $D^\epsilon(q)$, the a priori estimate (1.4.2) holds. We have

$$\begin{aligned} ||\mathfrak{H}[u]||^2_{L^2_{0,q}((0,\infty)\times M)} &= ||\partial_s u||^2_{L^2_{0,q}((0,\infty)\times M)} + ||\Box_b u||^2_{L^2_{0,q}((0,\infty)\times M)} \\ &+ < \partial_s u, \Box_b u >_{L^2_{0,q}((0,\infty)\times M)} + < \Box_b u, \partial_s u >_{L^2_{0,q}((0,\infty)\times M)} . \end{aligned} \tag{1.5.2}$$

but $\Box_b$ is self-adjoint and $\partial_s^* = -\partial_s$, then

$$||\mathfrak{H}[u]||^2_{L^2_{0,q}((0,\infty)\times M)} = ||\partial_s u||^2_{L^2_{0,q}((0,\infty)\times M)} + ||\Box_b u||^2_{L^2_{0,q}((0,\infty)\times M)}.$$

Let $u = u(s,x)$, for $s > 0$ and $x \in U$. The condition of commutator finite type for $X_1, ..., X_{2n-2}$ in U also implies the commutator finite type for $X_1, ..., X_{2n-2}, \partial_s$ in $(0,\infty) \times U$. The well-known subelliptic estimate ([Hor67]) implies

$$||u||^2_\epsilon \lesssim ||\partial_s u||^2_{L^2_{0,q}((0,\infty)\times M)} + ||\Box_b||^2_{L^2_{0,q}((0,\infty)\times M)} + ||u||^2_{L^2_{0,q}((0,\infty)\times M)},$$

or

$$||u||^2_\epsilon \lesssim ||\mathfrak{H}[u]||^2_{L^2_{0,q}((0,\infty)\times M)} + ||u||^2_{L^2_{0,q}((0,\infty)\times M)}.$$

Let ζ, ζ_1 be two smooth real-valued cut-off functions supported in U, with $\zeta \prec \zeta_1$. For any $\delta \in \mathbb{R}$ and $N > 0$, by the same method to prove Theorem 8.2.9 in [ChSh01], the following estimate holds

$$||\zeta u||_{\delta+\epsilon} \leq C_{\delta,N}(||\zeta_1 \mathfrak{H}[u]||_\delta + ||\zeta_1 u||_{-N}).$$

Therefore, $\mathfrak{H}$ is hypoelliptic on all $(0,q)$-forms defined on $(0,\infty) \times U$. Then, from the property of q-convexity, the hypoellipticity of $\mathfrak{H}$ holds as well on $(0,q')$-forms, with $q \leq q' \leq n-1-q$. $\Box$

Moreover, as an immediate consequence of Appendix B, $\mathfrak{H}$ is globally analytic hypoelliptic. This means if $\mathfrak{H}[u](s,x) = f(s,x)$, where f is globally analytic hypoelliptic on $\mathbb{R} \times M$, i.e,

$$\Big|(\partial_s)^k D^\alpha f(s,x)\Big| \leq C_f C_f^{2k+|\alpha|}(2k+|\alpha|)!,$$

for any $s > 0$, $x \in U$, and $\alpha = (\alpha_1, ..., \alpha_{2n-1})$, $D^\alpha = X_1^{\alpha_1}...X_{2n-2}^{\alpha_{2n-2}} T^{\alpha_{2n-1}}$. Then, u is globally analytic as well.

Definition 1.5.2. *A fundamental solution* of the $\Box_b$-heat equation (1.5.1) is a one parameter family of bounded operators $\mathbb{H}_s$, $s > 0$, acting on $L^2_{0,q}(M)$ such that for $\phi \in \Lambda_0^{0,q}(M)$ (space of $(0,q)$-forms with compact support in M):

1. For fixed $s > 0$, $\mathbb{H}_s[\phi](x)$ satisfies the $\bar\partial_b$-Neumann boundary conditions

$$\begin{aligned} \mathbb{H}_s[\phi](x) &\in \mathrm{Dom}(\bar\partial_b^*), \\ \bar\partial_b(\mathbb{H}_s[\phi](x)) &\in \mathrm{Dom}(\bar\partial_b^*), \end{aligned} \tag{1.5.3}$$

   so, $\mathbb{H}_s[\phi](x) \in \mathrm{Dom}(\Box_b)$.

2. $\mathbb{H}_s[\phi](x)$ solves the initial value problem for the heat equation, i.e.,

$$\mathfrak{H}[[\mathbb{H}_s\phi](x)] = 0 \quad \text{for } s > 0 \text{ and } x \in M;$$
$$\lim_{s\to 0^+} \mathbb{H}_s[\phi](.) = \phi(.) \quad \text{in } L^2_{0,q}(M).$$

3. For $s \in [0,T]$ and $(0,q)$-form ϕ fixed, $||\mathbb{H}_s[\phi]|| \leq C$, C maybe only depends on T and ϕ,

4. $\mathbb{H}_s[\phi]$ is differentiable in s,

5. Commutative with $\Box_b$, i.e, $\Box_b \mathbb{H}_s[\phi] = \mathbb{H}_s \Box_b[\phi]$.

Now, we will construct a fundamental solution of the initial value problem (1.5.1) via spectral theory.

Theorem 1.5.3. *The fundamental solution $\mathbb{H}_s$ of the heat equation* (1.5.1) *exists and it is unique. Furthermore, $\mathbb{H}_s$ is the semi-group generated by $-\Box_b$. Hence, for each $s > 0$, $\mathbb{H}_s$ is self-adjoint and positive.*

Proof. (*Existence*). We know that $\Box_b$ is a closed-range, densely defined, self-adjoint operator on $L^2_{0,q}(M)$. Then, Spectral Theorem (see Appendix A, the *continuous functional calculus* version) guarantees the existence of a unique algebra homomorphism Φ from the algebra $\mathcal{B} = \mathcal{B}[0,\infty)$ of bounded Borel (complex valued) functions on the non-negative real axis to the algebra $\mathcal{L}(L^2_{0,q}(M))$ of bounded linear operators on $L^2_{0,q}(M)$. In particular, for each $s \geq 0$, the function $e_s(a) = e^{-sa}, a \geq 0$ is bounded, continuous on $[0,\infty)$. Then, we denote $\Phi[e_s] = \mathbb{H}_s[.] = e^{-s\Box_b}[.]$ be the corresponding bounded linear operator. Again, from the spectral theorem, the family of bounded linear operators $\{e^{-s\Box_b}[.]\}_{s\geq 0}$ is a strongly continuous semi-group generated by $-\Box_b$, called the *heat semi-group* for $\Box_b$.
From properties of strongly continuous semi-group, Spectral Theorem and the observation that $\frac{d}{ds}(e^{-sa}) = -ae^{-sa}$, (1), (2), (3) and (4) in the definition 1.5.2 are satisfied . Now, we prove (5) as following. (see [Na]).
Let $f \in \mathrm{Dom}(\Box_b)$, $f = (\Box_b + iI)^{-1}[g]$ with $g \in L^2_{0,q}(M)$. Putting $F_1(x) = x(x+i)^{-1}$, then $F_1(\Box_b)[g] = \Box_b[g]$. But,

$$(e^{-sa})(\frac{a}{a+i}) = (ae^{-sa})(\frac{1}{a+i}),$$

it now implies $e^{-s\Box_b}F_1(\Box_b) = \Box_b e^{-s\Box_b}(\Box_b + iI)^{-1}$. Hence

$$e^{-s\Box_b}[\Box_b[f]] = e^{-s\Box_b}F_1(\Box_b)[g] = \Box_b e^{-s\Box_b}(\Box_b + iI)^{-1}[g] = \Box_b e^{-s\Box_b}[f].$$

(*Uniqueness*)
Let $\mathbb{H}'_s$ be another fundamental solution of the heat equation (1.5.1). We define

$$G_s[f] = (\mathbb{H}_s - \mathbb{H}'_s)[f],$$

for $f \in \Lambda^{0,q}(M)$, so $G_s[f]$ also solves (1.5.1). Let $T > 0$ and $s \in [0,T]$, since $\mathbb{H}_s$ and $\mathbb{H}'_s$ satisfy (3) and (5) in Definition 1.5.2, we obtain

$$(\Box_b G[f], G_s[f]) = (G_s \Box_b[f], G_s[f]) \in L^1(M \times [0,T]).$$

Similarly, $(G_s[f], \Box_b G_s[f]) \in L^1(M \times [0,T])$. Let

$$g(t) = -\int_0^t \int_M [(\Box_b G_s[f], G_s[f]) + (G_s[f], \Box_b G_s[f])] dV ds$$
$$= -\int_M \int_0^t [(\Box_b G_s[f], G_s[f]) + (G_s[f], \Box_b G_s[f])] ds dV.$$

By (2) in Definition 1.5.2,

$$g(t) = \int_M \int_0^t [(\frac{\partial}{\partial s} G_s[f], G_s[f]) + (G_s[f], \frac{\partial}{\partial s} G_s[f])] ds dV$$
$$= \int_M \int_0^t \frac{\partial}{\partial s} |G_s[f]|^2 ds dV.$$

By the initial value condition,

$$g(t) = \int_M |G_t[f]|^2 dV = ||G_t[f]||^2_{L^2_{0,q}(M)} \geq 0.$$

In the other hand, the positivity of $\Box_b$ implies

$$g'(t) = -- \int_M [(\Box_b G_t[f], G_t[f]) + (G_t[f], \Box_b G_t[f])] dV \leq 0.$$

Therefore, $g(t)$ is non-negative, decreasing of t, $g(0) = 0$. Immediately, $g(t) = 0$, for any $t \in [0, T]$. Therefore, $G_s[f] = 0$ for any $f \in \Lambda^{0,q}(M)$. By density of $\Lambda^{0,q}(M)$, we have $\mathbb{H}_s[f] = \mathbb{H}'_s[f]$, for every $f \in L^2_{0,q}(M)$. $\Box$

Now, we will see that the heat semi-group gives us a solution to the initial value problem posed in (1.5.1).

Theorem 1.5.4. *For each $f \in L^2_{0,q}(M)$, let $u(s,x) = e^{-s\Box_b}[f](x)$. Then $u \in C^\infty_{0,q}((0,\infty) \times M)$ satisfies the following properties*

1. $\left[\frac{\partial}{\partial s} + \Box_b\right][u](s,x) = 0$ *in the sense of distributions for $s > 0$ and $x \in M$;*

2. $\lim_{s \to 0^+} \int_M |u(s,x) - f(x)|^2 dV(x) = 0,$

   where $\Box_b$ operating on x-variables.

Proof. Define a $(0,q)$-form Ψ_f on $(0,\infty) \times M$ with distribution coefficients as follows,

$$< \Psi_f, \psi >= \int_0^\infty < e^{-s\Box_b}[f], \bar{\psi}_s >_{L^2_{0,q}(M)} ds,$$

for any $(0,q)$-form ψ whose coefficients in $C_0^\infty((0,\infty)\times M)$, and $\psi_s(x)=\psi(s,x)$.
Note that $e^{-s\Box_b}[f]$ and ψ_s are differentiable in s-variables with values in $L^2_{0,q}$-norm, hence the function $s\mapsto <e^{-s\Box_b}[f],\bar{\psi}_s>_{L^2_{0,q}(M)}$ is in $C_0^\infty(0,\infty)$. Then, let the support of ψ be contained in the set $\{(t,x)\in(0,\infty)\times M: 0<a\le s\le b<\infty\}$, we obtain

$$|<\Psi_f,\psi>|\le(b-a)||f||_{L^2_{0,q}(M)\sup_{a\le s\le b}}||\psi_s||_{L^2_{0,q}(M)}.$$

This implies that the distribution Ψ_f is continuous.
Let $\psi\in C_0^\infty((0,\infty)\times M)$,

$$\int_0^\infty\frac{d}{ds}[<e^{-s\Box_b}[f],\bar{\psi}_s>_{L^2_{0,q}(M)}]ds=0$$

$$\text{or}\quad\int_0^\infty<\Box_b e^{-s\Box_b}[f],\bar{\psi}_s>_{L^2_{0,q}(M)}-\int_0^\infty<e^{-s\Box_b}[f],\partial_s\bar{\psi}_s>_{L^2_{0,q}(M)}=0.$$

Since $\Box_b$ is self-adjoint,

$$\int_0^\infty<e^{-s\Box_b}[f],\Box_b\bar{\psi}_s>_{L^2_{0,q}(M)}-\int_0^\infty<e^{-s\Box_b}[f],\partial_s\bar{\psi}_s>_{L^2_{0,q}(M)}=0,$$

and by the definition

$$<\Psi_f,[-\partial_s+\Box_b]\psi>=0.$$

That means $[\partial_s+\Box_b][\Psi_f]=0$ in the sense of distributions. We know that $\mathfrak{H}$ is hypoelliptic, so $u(s,x)=e^{-s\Box_b}[f](x)$ is $C_0^\infty((0,\infty)\times M)$. And then, $\left[\dfrac{\partial}{\partial s}+\Box_b\right][u](s,x)=0$ in the classical sense. $\Box$

Remark 1.5.5. Theory of the heat semigroup $e^{-s\Box_b}$ also provides an argument to study the relative inverse of the $\Box_b$ operator under the view point of the spectral theory for unbounded, self-adjoint operators. In particular, recall that $\mathcal{H}_q$ be the orthogonal projection from $L^2_{0,q}(M)$ onto null space of $\Box_b$, then there exists the unique relative inverse of $\Box_b$ by $\mathcal{K}_q$ in the sense that $\Box_b\mathcal{K}_q=\mathcal{K}_q\Box_b=I-\mathcal{H}_q$. Moreover, $\lim_{s\to\infty}e^{-s\Box_b}[f]=\mathcal{H}_q[f]$ and

$$\int_0^\infty\left(e^{-s\Box_b}[f]-\mathcal{H}_q\right)[f]ds=\mathcal{K}_q[f].$$

In [NaSt06], from this observation, Alex Nagel and Eli Stein firstly applied pointwise estimates of the heat kernel in $\mathbb{C}^2$ to investigate the operator $\mathcal{K}_q$ when M is the model of a decouple boundary in $\mathbb{C}^n$.

1.6 Some models of the $\Box_b$-Heat equations in Several Complex Varibles.

1.6.1 Strong pseudoconvexity case.

Let M be a compact strongly pseudoconvex CR manifold of dimension $2n-1, n\ge 3$. In [St78], the author proved that the fundamental solution $p(s,x,y)$ of the heat equation for $\Box_b$ is a smooth $(0,q)\otimes(0,q)$-form on $\mathbb{R}^+\times M\times M$. And then, in [StTa84], the authors also obtained the smoothness of $p(s,x,y)$ on $\overline{\mathbb{R}^+}\times((M\times M)\backslash\Delta)$, where Δ is the diagonal on $M\times M$, $\Delta=\{(x,y)\in M\times M : x=y\}$. In particular, $p(s,x,y)$ has singularities on $\{0\}\times\Delta$. Let $\delta(x,y)$ be a pseudo-distance on M, the following estimate holds

$$||X_X^I X_Y^J p(s,.,.)||_{L^\infty(M)}\le C\epsilon^{-m}s^{-5n+\frac{m-3l}{2}},$$

where the given $\epsilon>0$ such that $\delta(x,y)>\epsilon$, $m>10n-3(|I|+|J|)$, C only depends on $|I|,|J|$.

1.6.2 Weakly Pseudoconvexity case.

In this case, the problem is more difficult.
Let M be the boundary of a weakly pseudoconvex domain of finite type in $\mathbb{C}^2$. In [NaSt01], the authors showed that the singularities of the heat kernel $H(s,x,y)$ of the solution are exactly the same as those of kernel of Szegö projection. Moreover, the rapid decay estimate for $H(s,x,y)$ holds

$$|\partial_s^j X_z^I X_w^J H(s,x,y)|\le C_{\{N,|I|,|J|,j\}}\frac{d_M(x,y)^{-2j-|I|-|J|}}{V_M(x,y)}\Big[\frac{s^N}{s^N+d_M(x,y)^{2N}}\Big],$$

where $d_M(z,w)$ is the non-isotropic distance and $V_M(z,w)$ is correspondingly the volume of a non-isotropic ball. It turns out that in the case of 3-dimension, we do not have any result for hypoellipticity of $\Box_b$ in general. Hence, the heat operator is not hypoelliptic. However, the authors proved that it is relative-hypoelliptic. The smoothness of the heat $\Box_b$-operator is depend on the smoothness of Szegö projection.
Second, when M is an unbounded polynomial model of finite type, i.e,

$$M=\{(z,w)\in\mathbb{C}^2 : \mathrm{Im}(w)=h(z)\},$$

where h is a sub-harmonic, non-harmonic polynomial of degree m. In [BoRa10], the authors provided the exponential decay estimates to the heat kernel via partial Fourier transform. More general, the authors also investigated the case that M is a decoupled polynomial model. Here, the considered metrics are non-isotropic and controlled by the tangent vectors fields. However, higher regularity estimates for the heat kernel are not established.

Chapter 2

Geometry and Analysis on Carnot-Carathéodory Spaces

In this chapter, integral representations of the Szegö projections as well as their regularity will be introduced. The basic ideals come from the earliest one by D.H. Phong and E. Stein [PS83, PS86a, PS86b], in which a key connection with singular Radon transforms was first identified there. Moreover, the notation of finite commutator type implies some important properties in Carnot-Carathéodory geometry, for more discussions, see [FP83, NSW85], [Na86],and also [Koe02], [Na]. As claims later, we always assume that M is the boundary of a smoothly bounded domain in $\mathbb{C}^n$ ($n \geq 3$), or more generally any CR-manifold of dimension $2n-1$ for which the range of $\bar{\partial}_b$ is closed in L^2. We also assume that the holomorphic vectors fields $L_1, ..., L_{n-1}, \bar{L}_1, ..., \bar{L}_{n-1}$ defined on $U \subset M$ - a neighborhood of a base fixed point $x_0 \in M$ - satisfy the condition of commutator finite-type of m. To obtain subelliptic estimates for higher order forms, the $D^\epsilon(q)$ is also assumed. The real vector fields $X_1, ..., X_{2n-2}$ are defined by $X_j = \operatorname{Re} L_j, X_{n+j-1} = \operatorname{Im} L_j, j = 1, ..., n-1$. Finally, we choose a real vector field T such that $L_1, ..., L_{n-1}, T$ is a local basic of the tangent space at each point of M.

2.1 Geometry on Carnot-Carathéodory Spaces

For each finite sequence $i_1, ..., i_k$ of integers with $1 \leq i_j \leq 2n-2$, setting $I = (i_1, ..., i_k)$ and the length $|I| = k$. We can write the commutator

$$[X_{i_k}, [X_{i_{k-1}}, ..., [X_{i_2}, X_{i_1}]...]] = \lambda_{i_1 \dots i_k} T \quad (\mathrm{mod} X_1, ..., X_{2n-2}),$$

where $\lambda_{i_1 \dots i_k} \in C^\infty(U)$.

Definition 2.1.1. For $x \in U$ and $r > 0$, set

$$\Lambda_l(x) = \Big(\sum_{2 \leq |I| \leq l} |\lambda_{i_1 \dots i_k}(x)|^2 \Big)^{\frac{1}{2}}, \quad l \geq 2,$$

and

$$\Lambda(x,r)=\sum_{l=2}^{m}\Lambda_l(x)r^l.$$

Note that $\lambda_l(x)$ is a function whose size measures how much T component the commutators of $X_1,...,X_{2n-2}$ of length $\le k$ can have. Since M is of commutator finite type of m, Λ_m never vanishes, so there are positive constants C_1, C_2 such that for $0<r\le 1$, we have

$$C_1 r^m \le \Lambda(x,r)\le C_2 r^2.$$

Definition 2.1.2. For each $x,y\in U$, the natural non-isotropic distance $\rho_M(x,y)$ corresponding to the vector fields $X_1,...,X_{2n-2}$ is given by

$$\begin{aligned}\rho_M(x,y)=\inf\{\delta>0 :&\text{there exists a continuous piecewise smooth}\\ &\text{map } \phi:[0,1]\to U \text{ such that } \phi(0)=x,\phi(1)=y,\\ &\text{and almost everywhere } \phi'(t)=\sum_{j=1}^{2n-2}\alpha_j(t)X_j,\\ &\text{with } |\alpha_j(t)|\ <\delta, \text{ for } j=1,...,2n-2\}.\end{aligned}$$

The non-isotropic ball centered at $x\in U$, with radius $r>0$ is

$$B_M(x,r)=\{y\in U:\rho_M(x,y)<r\}.$$

Remark 2.1.3. The fact that ρ_M is finite follows because there are commutators of finite length of the vectors fields $X_1,...,X_{2n-2}$ span the tangent space at each $x\in U$. This was first proved by Carathéodory.

For any $x,y\in U$, we also define $V(x,y)=|B_M(x,\rho_M(x,y))|$ be the volume of the non-isotropic ball centered at x with the radius is the non-isotropic distance of x,y.

Next, we define the family of exponential balls generated by exponential mapping corresponding to the vector fields $X_1,...,X_{2n-2}$. Let $\mathbb{B}_0$ denote the unit ball (defined by Euclidean metric) in $\mathbb{R}^{2n-1}$. For $x\in U$ and $r>0$ we set

$$\Phi_{x,r}(u)=\exp(ru_1X_1+...+ru_{2n-2}X_{2n-2}+\Lambda(x,r)u_{2n-1}T)(x),$$

where $u=(u_1,...,u_{2n-1})\in\mathbb{B}_0$.
There is $R_0>0$ depending on the manifold M so that for all $0<r<R_0$, the map $\Phi_{x,r}$ is a diffeomorphism of the unit ball $\mathbb{B}_0$ to its image. Hereafter, $0<r<R_0$ when we have calculations on the exponential map $\Phi_{x,r}$. Now, let

$$\widetilde{B}_M(x,r)=\Phi_{x,r}(\mathbb{B}_0),$$

that is

$$\widetilde{B}_M(x,r)=\{y\in U: y=\exp(a_1X_1+...+a_{2n-2}X_{2n-2}+aT)(x)$$
$$\text{where } |a_j|<r \text{ for } j=1,...,2n-2, \text{ and } |a|<\Lambda(x,r)\}.$$

We have the following facts about the size function Λ and the above families of non-isotropic balls.

Theorem 2.1.4. *Under the assumption of finite commutator type of m, there exists $R_0>0$ so that we have*

1. *There are positive constants C_1, C_2 so that for all $x\in U$ and $0<r<R_0$,*
$$B_M(x,C_1r)\subset \widetilde{B}_M(x,r)\subset B_M(x,C_2r),$$
that means $B_M(x,r)$ is equivalent to $\widetilde{B}_M(x,r)$.
2. *$|B_M(x,r)|\approx|\widetilde{B}_M(x,r)|\approx r^{2n-2}\Lambda(x,r)$ uniformly in x and $0<r<R_0$.*
In particular, there are two constants $C_3, C_4>0$ such that for all $x,y\in U$
$$C_3\leq \frac{V_M(x,y)}{\rho_M(x,y)\Lambda(x,\rho_M(x,y))}\leq C_4.$$
3. *Let $J_{x,r}(u)$ denote the Jacobian matrix of $\Phi_{x,r}(u)$, i.e,*
$$J_{x,r}(u)=(d\Phi_{x,r}(\frac{\partial}{\partial u_1}),...,d\Phi_{x,r}(\frac{\partial}{\partial u_{2n-1}})),$$
then $|det(J_{x,r}(u))|\approx r^{2n-2}\Lambda(x,r)$ uniformly in x and $0<r<R_0$.
4. $\left|\frac{\partial^\alpha}{\partial u^\alpha}det(J_{x,r}(u))\right|\lesssim r^{2n-2}\Lambda(x,r)$ *uniformly in x and $0<r<R_0$, for each multi-index α.*

About the proofs for these results, see [NSW85], Theorem 7.

Next, we will apply the exponential mapping to scaling method which was introduced in [Ch91]. First of all, for any function $f\in C^1(\mathbb{B}_0)$, the scaled pullbacks to $\mathbb{B}_0$ of the vector fields X_j on U are given by

$$(\widehat{X}_jf)(u)=(\widehat{X}_jf)_{x,r}(u)=r(X_j\check{f})(\Phi_{x,r}(u)),$$

where $\check{f}(y)=f\circ\Phi_{x,r}^{-1}(y)$, for $y\in\widetilde{B}_M(x,r)$. Therefore, $\widehat{X}_1,...,\widehat{X}_{2n-2}$ may be written in the u-coordinates as linear combinations of the vector fields $\frac{\partial}{\partial u_1},...,\frac{\partial}{\partial u_{2n-1}}$ on $\mathbb{B}_0$. Also, we define the scaled pullback to $\mathbb{B}_0$ of the function ϕ on $\widetilde{B}_M(x,r)$ by

$$\widehat{\phi}(u)=\phi(\Phi_{x,r}(u)),$$

for $u\in\mathbb{B}_0$.
The following facts are also from [NSW85].

Theorem 2.1.5. *1. The vector fields $\widehat{X_1}, ..., \widehat{X}_{2n-2}$ are of finite type on $\mathbb{B}_0$, uniformly in x and r. And $|det(\widehat{X_1}, ..., \widehat{X}_{2n-2}, Z)| > C$ for a commutators Z (of the $\widehat{X}_j$) of length $\leq m$ such that $\widehat{X_1}, ..., \widehat{X}_{2n-2}, Z$ span the tangent space. ($C > 0$ is independent of x and r).*

2. The coefficients of the $\widehat{X}_j$ (expressed in u-coordinates), together with their derivatives, are bounded above uniformly in x and r, i.e, if

$$\frac{\partial}{\partial u_j} = \sum_{l=1}^{2n-2} b_{jl}\widehat{X}_l + b_{j,2n-1}Z,$$

then $b_{l,j}$ and its derivatives are bounded above uniformly in x and r, for $j, l = 1, ..., 2n-1$.

Now, let $\widehat{L}_j$ and $\widehat{\bar{L}}_j$ denote the scaled pullback of L_j and $\bar{L}_j$ by

$$\widehat{L}_j = \widehat{X}_j + i\widehat{X}_{n+j-1}, \widehat{\bar{L}}_j = \widehat{X}_j - i\widehat{X}_{n+j-1},$$

and the basis of $(0,1)$-forms dual to $\widehat{\bar{L}}_1, ..., \widehat{\bar{L}}_{n-1}$ by $\widehat{\bar{\omega}}_1, ..., \widehat{\bar{\omega}}_{n-1}$.
We consider the equation on $\widetilde{B}(x,r)$

$$\bar{L}_j\phi = f,$$

from the definition,

$$\widehat{\bar{L}}_j\widehat{\phi} = r(\bar{L}_j\phi)(\Phi_{x,r}(u)) = rf(\Phi_{x,r}(u)) = r\widehat{f} = r\widehat{\bar{L}_j\phi}$$

the equation $\widehat{\bar{L}_j\phi} = r^{-1}\widehat{\bar{L}}_j\widehat{\phi}$ is the scaled pullback of the equation $\bar{L}_j\phi = f$. Now, $\bar{L}_j^* = -L_j + a_j$, for some $a_j \in C^\infty(U)$, we also define

$$\widehat{\bar{L}_j^*} = -\widehat{L}_j + r\widehat{a}_j.$$

Similarly we obtain $\widehat{\bar{L}_j^*\phi} = r^{-1}\widehat{\bar{L}}_j^*\widehat{\phi}$. Therefore, the scaled pullback of $\bar{\partial}_b$-equation is

$$\widehat{(\bar{\partial}_b\phi)} = r^{-1}\widehat{\bar{\partial}}_b\widehat{\phi},$$

and also

$$\widehat{(\bar{\partial}_b^*\phi)} = r^{-1}\widehat{\bar{\partial}_b^*}\widehat{\phi}.$$

Finally, we also extend the map $\Phi_{x,r}$ on $\mathbb{B}_0$ to the map $\Phi_{(s,x),r}$ on $\mathbb{R} \times \mathbb{B}_0$ by

$$\Phi_{(s,x),r}(s,u) = (s^2, \Phi_{x,r}(u)),$$

with $0 < r < R_0$.
The scaled pullback of the heat equation on $\mathbb{R} \times M$ to $\mathbb{R} \times \mathbb{B}_0$ is

$$((\frac{\partial}{\partial s} + \Box_b)\phi(s,x))\widehat{\ } = r^{-2}\frac{\partial}{\partial s}\widehat{\phi}(s,u) + r^{-2}\widehat{\Box}_b\widehat{\phi}(s,u),$$

where $\widehat{\Box_b} = \widehat{\bar\partial_b}\widehat{\bar\partial^*_b} + \widehat{\bar\partial^*_b}\widehat{\bar\partial_b}$ is the scaled pullback of $\Box_b$.

In [Koe02], the author applied this method to show that: the Szegö projections are smooth away from the diagonal, and their kernels satisfy size estimates. Now, we again apply this method to estimate the Szegö kernel in T-direction when x closed y.

Theorem 2.1.6. *Suppose that $U \subset M$ satisfies the condition of $D^\epsilon(q)$ and finite commutator type of m as above. Let $\mathcal{S}_{q'}$ be the Szegö projection on $(0, q')$-forms, for $q \le q' \le n-1$. Then, the components $S^{IJ}_{q'}$ ($|I| = |J| = q'$) of the kernel of $\mathcal{S}_{q'}$ satisfies*

$$|T^k_x T^l_y S^{IJ}_{q'}(x,y)| \le C_{k,l} r^{-m(k+l)} |B(x,r)|^{-1},$$

when x sufficiently closed y and $r = \rho_M(x,y) > 0$.
An analogous statement holds for the kernel of the projection $\mathcal{S}'_{q'}$.

Proof. This theorem can be proved by applying the argument in [Koe02]. Let x, y be two different points in U, then there is a constant $C > 0$ independent to x, y such that $\widetilde{B}(x,r) \cap \widetilde{B}(y,r) = \emptyset$, where $r = C^{-1}\rho(x,y)$.
Let f be a $(0,q')$-form with coefficients in $C^\infty_0(\widetilde{B}(y,r))$. Hence, for some $(0, q'+1)$-form u which is orthogonal to $\mathrm{Ker}(\bar\partial^*_b)$, we have $(I - \mathcal{S}_{q'})[f] = \bar\partial^*_b u$, so that $\bar\partial_b\bar\partial^*_b u = \bar\partial_b f = 0$ on $\widetilde{B}(x,r)$. Moreover,

$$T^k_z(\bar\partial^*_b u)_J(z) = -\sum_{|I|=q'} \int_M T^k_z S^{IJ}_{q'}(z,y') f_I(y') dV(y'), \tag{2.1.1}$$

for $z \in \widetilde{B}(x,r)$.
We will begin the proof by showing the following estimate

$$|T^k_z(\bar\partial^*_b u)_J(z)| \le C_k r^{-km} |B(x,r)|^{-\frac{1}{2}} ||f||_{L^2_{0,q'}(M)}. \tag{2.1.2}$$

On $\widetilde{B}(x,r)$, $\bar\partial_b\bar\partial^*_b u = \bar\partial_b f = 0$, so with pullbacks given by the map $g_{x,r}$, we have $\widehat{\bar\partial_b}\widehat{\bar\partial^*_b}\widehat{u} = 0$ on the Euclidean unit ball $\mathbb{B}_0$. Now, applying the same argument in [Koe02], we have

$$|\widehat{D}^k(\widehat{\bar\partial^*_b}\widehat{u})_J(v)| \le C_k |B(x,r)|^{-\frac{1}{2}} r ||f||_{L^2_{0,q'}(M)},$$

or

$$|\widehat{D}^k(\widehat{\bar\partial^*_b u})_J(v)| \le C_k |B(x,r)|^{-\frac{1}{2}} ||f||_{L^2_{0,q'}(M)},$$

where $v \in g^{-1}_{x,r}(\widetilde{B}(x, \frac{r}{2}))$ and $\widehat{D}^k$ be the composition of k factors of $\{\widehat{X}_j, j = 1, ..., 2n-2\}$. Since the finite type condition, for some multi-indics $\alpha = (i_1, ..., i_m)$, there is a smooth function $\lambda_\alpha \ne 0$ on U such that

$$[X_{i_m}, [..., [X_{i_2}, X_{i_1}]...] = \lambda_\alpha T + \sum_{j=1}^{2n-2} \beta_j X_j,$$

for some functions β_j on M. Scaling this equality to $\mathbb{B}_0$, we have

$$\widehat{\lambda_\alpha}\widehat{T}\phi + \sum_{j=1}^{2n-2}\widehat{\beta}_j\widehat{X_j}\phi = [X_{i_m}, \widehat{[..., [X_{i_2}}, X_{i_1}]...]\phi,$$

where $\phi \in C_0^\infty(\mathbb{B}_0)$. This yields the formulae to the pullback of T-direction

$$\widehat{T}\phi(v') = \frac{1}{\widehat{\lambda_\alpha}}\big(r^m[X_{i_m}, [..., [X_{i_2}, X_{i_1}]...]\check{\phi}(g_{x,r}(v')) + r\sum_{j=1}^{2n-2}\widehat{\beta}_j X_j\check{\phi}(g_{x,r}(v'))\big).$$

We apply $\widehat{T}$ as the sum of derivatives

$$|\widehat{T}_z^k(\widehat{\bar{\partial}_b^* u})_J(v)| \le C_k|B(x,r)|^{-\frac{1}{2}}||f||_{L^2_{0,q'}(M)},$$

for $v \in g_{x,r}^{-1}(\widetilde{B}(x,\frac{r}{2}))$. By rescaling, and then induction for k, it is not to hard to imply that

$$|T_z^k(\bar{\partial}_b^* u)_J(z)| \le C_k r^{-km}|B(x,r)|^{-\frac{1}{2}}||f||_{L^2_{0,q'}(M)},$$

for $z \in \widetilde{B}(x,\frac{r}{2})$ with r is sufficiently small. This is also true in place of $z \in \widetilde{B}(x,r)$.
So, by L^2-duality, we yield

$$||T_z^k S_{q'}^{IJ}(z,.)||_{L^2_{0,q'}(\widetilde{B}(y,r))} \le C_k r^{-km}|B(x,r)|^{-\frac{1}{2}},$$

and

$$||T_w^l S_{q'}^{IJ}(.,w)||_{L^2_{0,q'}(\widetilde{B}(x,r))} \le C_l r^{-lm}|B(y,r)|^{-\frac{1}{2}},$$

since $\overline{S_{q'}^{IJ}(z,w)} = S_{q'}^{IJ}(w,z)$. Again, by scaling and induction in k, we can show that

$$||\widehat{T}_{v'}^k\widehat{S}_{q'}^{IJ}(v',.)||_{L^2_{0,q'}(\mathbb{B}_0)} \le C_k|B(x,r)|^{-1},$$

and

$$||\widehat{T}_v^l\widehat{S}_{q'}^{IJ}(.,v)||_{L^2_{0,q'}(\mathbb{B}_0)} \le C_l|B(x,r)|^{-1},$$

since $r = C^{-1}\rho_M(x,y)$ and so $|B(x,r)|^{-\frac{1}{2}} \approx |B(y,r)|^{-\frac{1}{2}}$.
From [Koe02], we already know that

$$\left|\left|\frac{\partial^\alpha}{\partial v'^\alpha}\frac{\partial^\beta}{\partial v^\beta}(|B(x,r)|\widehat{S}(v',v))\right|\right|_{L^2_{0,q'}(\mathbb{B}_0\times\mathbb{B}_0)} \le C_{\alpha,\beta}.$$

By Embedding Soblev Theorem,

$$\big|\widehat{T}_{v'}^k\widehat{T}_v^l(|B(x,r)|\widehat{S}(v',v))\big| \le C_{k,l} \quad \text{uniformly},$$

and hence, by the same previous argument,

$$|T_x^k T_y^l S_{q'}^{IJ}(x,y)| \le C_{k,l} r^{-m(k+l)}|B(x,r)|^{-1},$$

when x closed y as well. □

Next, under condition of commutators finite type on M, we also define the *parabolic* non-isotropic metric on $\mathbb{R} \times M$. Recall that the *family of dilation* δ_λ on $\mathbb{R} \times M$ is

$$\delta_\lambda(s, x) = (\lambda^2 s, \lambda x),$$

for $s \in \mathbb{R}, x \in M$ and the parameter $\lambda > 0$. This implies the following.

Definition 2.1.7. Denote by Y the vector fields $\dfrac{\partial}{\partial s}$ on $\mathbb{R}$, then the family of the vector fields $\{Y, X_1, ..., X_{2n-2}\}$ also satisfies the condition of finite commutator type. For every $p = (s, x), q = (t, y) \in \mathbb{R} \times U$, the following function is finite

$$\rho_{\mathbb{R}\times M}(p, q) = \rho_M(x, y) + \sqrt{|s - t|},$$

then

$$\rho_{\mathbb{R}\times M}(\delta_\lambda(s, x), \delta_\lambda(t, y)) = \lambda \rho_{\mathbb{R}\times M}((s, x), (t, y)),$$

and $\rho_{\mathbb{R}\times M}$ is called non-isotropic parabolic distance. This metric associates to the corresponding balls $B_{\mathbb{R}\times M}((s, x), r)$ on $\mathbb{R} \times M$. Note that

$$|B_{\mathbb{R}\times M}((s, x), |t - s|)| \approx (t - s)^2 |B_M(x, |t - s|)|.$$

Let $\widetilde{\mathbb{B}}_0$ denote the unit ball in $\mathbb{R}^{2n-1}$. For each $(u, u_0) \in \widetilde{\mathbb{B}}_0$ the exponential mapping on $\mathbb{R} \times M$ is

$$\widetilde{\Phi}_{(s,x),r}(u, u_0) = \exp(r^2 u_0 Y + r u_1 X_1 + ... + r u_{2n-2} X_{2n-2} + \Lambda(x, r) u_{2n-1} T)(s, x),$$

and

$$\widetilde{B}_{\mathbb{R}\times M}((s, x), r) = \widetilde{\Phi}_{(s,x),r}(\widetilde{\mathbb{B}}_0).$$

2.2 Analysis on Carnot-Carath$\acute{e}$odory Spaces

In this section, we briefly introduce the definition of the class of *non-isotropic smoothing* (NIS) operators on the manifold M. These operators generalize the classical Calderón-Zygmund operators to do the standard coordinate vector fields $\left\{\dfrac{\partial}{\partial x_1}, ..., \dfrac{\partial}{\partial x_{2n-1}}\right\}$. For more discussions, see [NaSt01], [Na], [Koe02].

Let $\rho_M(x, y)$ be a non-isotropic metric on M defined as before, where M satisfies the condition of finite commutator type. Let $\mathbb{I}_k$ denote the set of ordered k-tuples I of integers $I = (i_1, ..., i_k), 1 \leq i_j \leq 2n - 2$, and denote by $X^I = X_{i_1} ... X_{i_k}$. The diagonal of $M \times M$ is Δ_M and $\mathcal{D}'(M)$ denote the space of distributions on M.

We begin the definition of the class of NIS operators acting on functions on M.

Definition 2.2.1. An operator $\mathcal{T} : C_0^\infty(M) \to \mathcal{D}'(M)$ is called a non-isotropic smoothing operator of order r if the following conditions hold:

1. There is a function $T_0(x,y) \in C^\infty(M \times M \setminus \Delta_M)$ so that if $\phi, \psi \in C_0^\infty(M)$ have disjoint supports,
$$\langle \mathcal{T}[\phi], \psi \rangle = \int_M \int_M \phi(y)\psi(x) T_0(x,y) dV(x) dV(y). \tag{2.2.1}$$

2. For any $s \geq 0$, there exist parameters $\alpha(s) < \infty, \beta < \infty$ such that if $\zeta, \zeta' \in C^\infty(M), \zeta \prec \zeta'$, then there is a constant C_s so that
$$||\zeta \mathcal{T}[f]||_s \leq C_s(||\zeta' f||_{\alpha(s)} + ||f||_\beta) \tag{2.2.2}$$
for all $f \in C^\infty(M)$.

3. For any $I \in \mathbb{I}_k, J \in \mathbb{I}_l$, there exists a constant $C_{k,l}$ so that
$$|X^I X^J T_0(x,y)| \leq C_{k,l} \rho_M(x,y)^{r-k-l} V_M(x,y)^{-1}. \tag{2.2.3}$$

4. For any ball $B_M(x_0, r) \subset U$, for each integer $k \geq 0$, there is a positive integer N_k and a constant C_k so that if $\phi \in C_0^\infty(B_M(x_0,r))$ and $I \in \mathbb{I}_k$, we have
$$\sup_{x \in B_M(x_0,r)} |X^I \mathcal{T}[\phi](x)| \leq C_k r^{r-k} \sup_{y \in M} \sum_{|J| \leq N_k} \delta^{|J|} |X^J[\phi](y)|. \tag{2.2.4}$$

5. The above conditions also hold for the adjoint operator T^* with kernel $T_0(y,x)$.

We have some remarks to this definition.

Remark 2.2.2. 1. The condition (1) says that the operator $\mathcal{T}$ has the distribution kernel which is given by integration againts the function T_0 is smooth away from the diagonal Δ_M on M. Thus, in addition to the condition (2) and (5), the operator $\mathcal{T}$ is *pseudo-local*, means, if $\phi \in C_0^\infty(M)$, then away from the support of ϕ, $\mathcal{T}[\phi]$ is given by the infinitely differentiable function $\mathcal{T}[\phi](x) = \int_M T_0(x,y)\phi(y)dV(y)$. Moreover, the condition (2) was imposed in order to prove that the class of NIS operators forming an algebra under composition.

2. The condition (3) clearly shows how large the sizes of singularities of the distribution kernel $T_0(x,y)$ at diagonal on M are. And from the generalized theory of singular integral operator developed by A. Nagel and E. Stein in [NaSt04], this estimate is critical to show the L^p-boundedness for the operator $\mathcal{T}$. In particular, in [Koe02], Koenig showed that if $\mathcal{T}$ is a NIS operator of order zero on M, then $\mathcal{T}$ is bounded from $L^p(M)$ to itself. The analogue in $\mathbb{C}^2$ is also true in [NaSt04].

3. The condition (4) encodes the basic cancellation hypothesis needed to show that NIS operators of order zero are bounded on $L^2(M)$. In the condition (3), when $m \leq 0$, the integral $\displaystyle\int_M T_0(x,y)\phi(y)dV(y)$ is able not to converge absolutely, even if $\phi \in C_0^\infty(M)$. Thus, the estimate in condition (4) is required for $\mathcal{T}[\phi]$.

Now, we generalize the class of NIS operators acting on $(0,q)$-forms. Let $\mathcal{T}$ be an operator from $C^\infty_{0,q_1}(M)$ into $C^\infty_{0,q_2}(M)$, and $\phi = \sum'_{|I|=q_1} \phi_I \bar{\omega}_I$, then

$$\mathcal{T}[\phi](x) = \sum\nolimits'_{|J|=q_2} (\mathcal{T}[\phi])_J(x)\bar{\omega}_J,$$

where

$$(\mathcal{T}[\phi])_J(x) = \sum\nolimits'_{|I|=q_1} < \mathcal{T}[\phi_I(x)\bar{\omega}_I], \bar{\omega}_J >_{L^2} .$$

And we define $\mathcal{T}^{IJ}[g](x) =< \mathcal{T}[g(x)\bar{\omega}_I], \bar{\omega}_J >_{L^2}$, for $g \in C^\infty_0(M)$. Here, the appearing of the primes in these sums means the follow forms are represented uniquely. Naturally, we say that $\mathcal{T}$ is a NIS operator of order r on $(0,q_1)$-form if and only if $\mathcal{T}$ and $\mathcal{T}^*$ satisfy the estimate in condition (2) of the definition 2.2.1 and each $\mathcal{T}^{IJ}$ is a NIS operator of order r on functions.

We will apply this definition to the Szegö projection $\mathcal{S}_q$ and $\mathcal{S}'_q$. We can rewrite these operators by

$$\mathcal{S}_q[\phi](x) = \sum\nolimits'_{|J|=q_2}\sum\nolimits'_{|I|=q_1} < \mathcal{S}_q[\phi_I(x)\bar{\omega}_I], \bar{\omega}_J >_{L^2} \bar{\omega}_J = \sum\nolimits'_{|J|=q_2}\big(\sum\nolimits'_{|I|=q_1} \mathcal{S}^{IJ}_q[\phi_I](x)\big)\bar{\omega}_J,$$
$$\mathcal{S}'_q[\phi](x) = \sum\nolimits'_{|J|=q_2}\sum\nolimits'_{|I|=q_1} < \mathcal{S}'_q[\phi_I(x)\bar{\omega}_I], \bar{\omega}_J >_{L^2} \bar{\omega}_J = \sum\nolimits'_{|J|=q_2}\big(\sum\nolimits'_{|I|=q_1} (\mathcal{S}^{IJ}_q)'[\phi_I](x)\big)\bar{\omega}_J,$$

for $\phi = \sum_{|I|=q_1} \phi_I\bar{\omega}_I$. Now, by Riesz Representation theorem,

$$\mathcal{S}_q[\phi](x) = \sum\nolimits'_{|J|=q_2}\big(\sum\nolimits'_{|I|=q_1} \int_M S^{IJ}_q(x,y)\phi_I(y)dV(y)\big)\bar{\omega}_J$$
$$\mathcal{S}'_q[\phi](x) = \sum\nolimits'_{|J|=q_2}\big(\sum\nolimits'_{|I|=q_1} \int_M (S^{IJ}_q)'(x,y)\phi_I(y)dV(y))\big)\bar{\omega}_J.$$

With these integral representations and the fact that the identity operator is non-isotropic smoothing of order zero, the operators $\mathcal{S}_{q'}, \mathcal{S}'_{q'}$ and $\mathcal{K}_{q'}$ are the NIS operators, with $q \leq q' \leq n-1-q$.

Theorem 2.2.3. *[Koe02] Assume the condition $D^\epsilon(q)$ holds for $q_0 \leq q \leq n-1-q_0$ for some fixed $q_0 \geq 1$ near a point of finite commutator type $x_0 \in M$. Then, there is a neighborhood U of x_0 such that*

1. *$\mathcal{S}_q$ and $\mathcal{S}'_q$ are NIS operators of order zero in U.*
2. *$\mathcal{S}_{q_0-1}$ and $\mathcal{S}'_{n-q_0}$ also are NIS operators of order zero in U.*
3. *$\mathcal{K}_q$ is a NIS operator of order 2 in U*

As a consequence, we have

Proposition 2.2.4. *Let α be a muilti-index with $|\alpha| = k \geq 1$. For $0 \leq j \leq \left[\dfrac{k}{2}\right]$, there are NIS operators $A_{j,1}, A_{j,2}, A_{j,3}$ smoothing of order zero such that*

$$X^\alpha (I - \mathcal{H}_q) = \sum_{j=0}^{\left[\frac{k}{2}\right]} (A_{j,1} + A_{j,2} + A_{j,3}) \Box_b^j.$$

In particular, if $k = 2l$, $A_{l,1} = A_{l,2} = 0$.

Proof. The proof really follows the lines in Proposition 3.4.7 [NaSt01] with the replacing of the terms $I - \mathcal{S}_q$ by $I - \mathcal{H}_q$. And we can omit here. □

The scaling method also provides following Sobolev Type Theorem.

Theorem 2.2.5. *Let M satisfy the conditions of $D^\epsilon(q)$ and of finite commutator type. There is a constant C , and an even integer L_m so that if $f \in C^\infty(U)$, then for all $x \in U$ and all $r \leq r_0$*

$$\sup_{B_M(x,r)} |f| \leq C |B_M(x,r)|^{-\frac{1}{2}} \sum_{0 \leq |I| \leq L_m, |I| \ even} r^{|I|} ||X^I f||_{L^2(B_M(x,2r))}. \tag{2.2.5}$$

If $f \in \Lambda^{0,q'}(C^\infty(M)) \cap L^2_{0,q'}(M)$, with $q \leq q' \leq n-1-q$. Moreover, if $f \in (ker\Box_b)^\perp$, then

$$\sup_{B_M(x,r)} |f| \leq C |B_M(x,r)|^{-\frac{1}{2}} \sum_{j=0}^{L_m/2} r^{2j} ||\Box_b^j f||_{L^2_{0,q}(B_M(x,2r))}. \tag{2.2.6}$$

Proof. We apply the scaling method introduced before. From the property (1) in Theorem 2.1.4, we can change the ball under supremum

$$\sup_{y \in B_M(x,r)} |f(y)| \leq \sup_{y \in \widetilde{B}_M(x,C_1 r)} |f(y)| \leq \sup_{u \in \mathbb{B}_0} |f(\Phi_{x,C_2 r}(u))|.$$

Setting $F(u) = f(\Phi_{x,C_2 r}(u))$, for $u \in \mathbb{B}_0$. Let $G(u) = F(u)\theta(u)$, where $\theta \in C_0^\infty(\mathbb{R}^{2n-1})$, $\theta = 1$ on $\mathbb{B}_0$, and $\theta = 0$ outside the ball $\mathbb{B}(0,2) \subset \mathbb{R}^{2n-1}$. So,

$$\begin{aligned}
\sup_{u \in \mathbb{B}_0} |F(u)| &\leq \sup_{u \in \mathbb{R}^{2n-1}} |G(u)| \leq \int_{\mathbb{R}^{2n-1}} |\widehat{G}(\xi)| dV(\xi) \\
&\leq ||(1+|\xi|^4)^{1/2} \widehat{G}(\xi)||_{L^2(\mathbb{R}^{2n-1})} ||(1+|\xi|^4)^{-1/2}||_{L^2(\mathbb{R}^{2n-1})} \\
&\leq C(||\widehat{G}||_{L^2(\mathbb{R}^{2n-1})} + |||\xi|^2 \widehat{G}||_{L^2(\mathbb{R}^{2n-1})}) \\
&\leq C \sum_{0 \leq |I| \leq 2, |I| \text{ even}} ||(\frac{\partial}{\partial u})^I \widehat{F}||_{L^2(\mathbb{B}(0,2))}.
\end{aligned}$$

Now, from the statement (2) in Theorem 2.1.5, there is an positive integer number l depending on m such that

$$\sup_{\mathbb{B}_0}|F| \leq C \sum_{0\leq|I|\leq 2l, |I| \text{ even}} ||(\widehat{X})^I\widehat{F}||_{L^2(\mathbb{B}(0,2))}.$$

Then, after rescaling pullback, by Theorem 2.1.4, we have the first statement.
Now, we have the analogue version for $(0,q)$-form. Let $f = \sum_{|J|=q} f_J\overline{\omega}_J \in \Lambda^{0,q}(C^\infty(M))$, we also obtain

$$\begin{aligned}
\sup_{B_M(x,r)}|f| &= \sup_{|J|=q}\sup_{B_M(x,r)}|f_J| \\
&\leq \sup_{|J|=q} C|B_M(x,r)|^{-\frac{1}{2}} \sum_{0\leq|I|\leq L_m, |I|\,\text{even}} r^{|I|}||X^I f_J||_{L^2(B_M(x,2r))} \\
&\leq C|B_M(x,r)|^{-\frac{1}{2}} \sum_{0\leq|I|\leq L_m, |I|\,\text{even}} r^{|I|}\sum_{|J|=q}||X^I f_J||_{L^2(B_M(x,2r))} \\
&\leq C|B_M(x,r)|^{-\frac{1}{2}} \sum_{0\leq|I|\leq L_m, |I|\,\text{even}} r^{|I|}||X^I f||_{L^2(B_M(x,2r))}.
\end{aligned}$$

In order to estimate $X^I f$ in the terms of $\Box_b^j f$, with $f = f - \mathcal{H}_q[f]$, we will apply the basic decomposition in Proposition 2.2.4. Since $|I|$ is even, there exists a NIS operator of smoothing of order zero A_I such that

$$X^I(I - \mathcal{H}_q) = A_I\Box_b^{\frac{|I|}{2}}.$$

Therefore, let f be orthogonal to the null space of $\Box_b$, we obtain the second assertion. $\square$

The main result in this part is following

Theorem 2.2.6. *Let M satisfy the conditions of $D^\epsilon(q)$ and be of finite commutator type. Then, the heat operators $e^{-s\Box_b}$, with $s > 0$, are NIS operators smoothing of order zero on $(0,q')$-forms, with $q \leq q' \leq n-1-q$, and associated estimates are uniform in $s > 0$. As a consequence, $e^{-s\Box_b}$ is bounded from $L^p_{0,q'}(M)$ to itself.*

We will prove this theorem in the next chapter.

Chapter 3

The Initial Value Problem For The Heat Operator $\partial_s + \Box_b$

In this chapter, for a $(0,q)$-form $\phi = \sum'_{|I|=q} \phi_I \bar{\omega}_I$, we will show that the operator $e^{-s\Box_b}[\phi](x)$ has the following form

$$e^{-s\Box_b}[\phi](x) = \int_M H(s,x,y)\phi(y)dV(y),$$

where the integral means that

$$\int_M H(s,x,y)\phi(y)dV(y) = \sum\nolimits'_{|J|=q} \left(\sum\nolimits'_{|I|=q} \int_M H^{IJ}(s,x,y)\phi_I(y)dV(y) \right) \bar{\omega}_J.$$

We require that M satisfies the condition of finite commutator type and $D^\epsilon(q)$.

3.1 The heat kernel

For convenience, we recall some results about the heat semi-group of unbounded operators $e^{-s\Box_b}$ via Hilbert space theory.

Theorem 3.1.1. *Assume the condition of finite commutator type and $D^\epsilon(q)$ hold on M, and also the operator $\bar{\partial}_b$ has its closed-range in L^2. Let $\phi \in L^2_{0,q}(M)$, then :*

1. $\lim\limits_{s\to 0} ||e^{-s\Box_b}[\phi] - \phi||_{L^2_{0,q}(M)} = 0$;
2. *For* $s > 0$, $||e^{-s\Box_b}[\phi]||_{L^2_{0,q}(M)} \leq ||\phi||_{L^2_{0,q}(M)}$;
3. *If* $\phi \in Dom(\Box_b)$, $||e^{-s\Box_b}[\phi] - \phi||_{L^2_{0,q}(M)} \leq s||\Box_b[\phi]||_{L^2_{0,q}(M)}$;
4. *For* $s > 0$ *and* j *is non-negative integer,* $||(\Box_b)^j e^{-s\Box_b}[\phi]||_{L^2_{0,q}(M)} \leq \left(\frac{j}{e}\right)^j s^{-j}||\phi||_{L^2_{0,q}(M)}$;

5. $e^{-s\Box_b}\mathcal{H}_q[\phi] = \mathcal{H}_q e^{-s\Box_b}[\phi] = \mathcal{H}_q[\phi]$;

6. $e^{-s\Box_b}[\phi] = (I - \mathcal{H}_q)e^{-s\Box_b}[\phi] + \mathcal{H}_q[\phi] = e^{-s\Box_b}(I - \mathcal{H}_q)[\phi] + \mathcal{H}_q[\phi]$;

7. *For any $\phi \in L^2_{0,q}(M)$, and any $s > 0$, the Hilbert space valued form $\mathbb{H}_s = e^{-s\Box_b}[\phi]$ satisfies*

$$\begin{aligned} [\partial_s + \Box_b][\mathbb{H}_s] &= 0 \text{ for } s > 0 \\ \lim_{s\to 0} \mathbb{H}_s &= \phi \text{ in } L^2_{0,q}((0,\infty)\times M). \end{aligned}$$

Next, we have the self-adjointness for heat semi-group.

Lemma 3.1.2. *For any $s \geq 0$, $e^{-s\Box_b}$ is a self-adjoint operator on $L^2_{0,q}(M)$.*

Proof. Let $s, t \geq 0$, since the self-adjointness of $\Box_b$, we have

$$\frac{\partial}{\partial s}\left\langle e^{-s\Box_b}\phi, e^{-t\Box_b}\psi\right\rangle_{L^2_{0,q}(M)} = \frac{\partial}{\partial t}\left\langle e^{-s\Box_b}\phi, e^{-t\Box_b}\psi\right\rangle_{L^2_{0,q}(M)}.$$

Then,

$$\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left\langle e^{-s\Box_b}\phi, e^{-t\Box_b}\psi\right\rangle_{L^2_{0,q}(M)} = 0.$$

The transform of variables

$$\begin{cases} u = s + t \\ v = s - t \end{cases} \quad \text{or} \quad \begin{cases} s = \frac{1}{2}(u + v) \\ t = \frac{1}{2}(s - t) \end{cases}$$

arises

$$\begin{cases} \frac{\partial}{\partial u} = \frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right) \\ \frac{\partial}{\partial v} = \frac{1}{2}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right). \end{cases}$$

So,

$$\frac{\partial}{\partial v}a(s,t) = 0,$$

where $a(s,t) = \left\langle e^{-s\Box_b}\phi, e^{-t\Box_b}\psi\right\rangle_{L^2_{0,q}(M)}$. That means $a(s,t)$ is a function of only u-variables, $a(s,t) = F(u) = F(s+t) = F(t+s) = a(t,s)$.
In this case, we have

$$\left\langle e^{-s\Box_b}\phi, e^{-t\Box_b}\psi\right\rangle_{L^2_{0,q}(M)} = \left\langle e^{-t\Box_b}\phi, e^{-s\Box_b}\psi\right\rangle_{L^2_{0,q}(M)},$$

and in particular when $t = 0$,

$$\left\langle e^{-s\Box_b}\phi, \psi\right\rangle_{L^2_{0,q}(M)} = \left\langle \phi, e^{-s\Box_b}\psi\right\rangle_{L^2_{0,q}(M)}.$$

□

The next lemma says that the linear functional $\phi \mapsto X^\alpha e^{-s\Box_b}[\phi](x)$ is bounded. The key fact is that, when $n \geq 3$, the operator $\Box_b$ has subelliptic estimates (1.4.2).

Lemma 3.1.3. *Let $|\alpha| = a \geq 0$, and let $K \subset M$ be a compact set. Choose a integer N so that $N\epsilon > 2n - 1 + a$, where ϵ defined in (1.4.2). Then, there is an allowable constant C such that for each $s > 0$, if $x \in K$, and for all $\phi \in L^2_{0,q}(M)$,*

$$|X^\alpha e^{-s\Box_b}[\phi](x)| \leq C(1 + t^{-N})||\phi||_{L^2_{0,q}(M)}. \tag{3.1.1}$$

As a consequence of the condition of commutators finite type, for any derivative D on M,

$$|D^\alpha e^{-s\Box_b}[\phi](x)| \leq C(1 + t^{-N})||\phi||_{L^2_{0,q}(M)}.$$

Proof. Choose $\zeta \in C_0^\infty(M)$ with $\zeta(x) = 1$, for all $x \in K$. Then, choose cut-off function $\zeta \prec \zeta_1 \prec \ldots \prec \zeta_N = \zeta'$. By Sobolev Imbedding Theorem, we have

$$|X^\alpha e^{s\Box_b}[\phi](x)| = |X^\alpha \zeta(x) e^{s\Box_b}[\phi](x)| \leq C||\zeta e^{s\Box_b}[\phi]||_{2n-1+a}.$$

Applying the basic subelliptic estimate (1.4.2), we have

$$||\zeta e^{-s\Box_b}[\phi]||_{2n-1+a} \leq C. \left[||\zeta_1 \Box_b e^{-s\Box_b}[\phi]||_{n+a-\epsilon} + ||\zeta_1 e^{-s\Box_b}[\phi]||_0\right].$$

If we repeat this argument N times, by (4) in Theorem 3.1.1, we will obtain

$$||\zeta e^{-s\Box_b}[\phi]||_{2n-1+a} \leq C. \sum_{j=0}^{N} ||\zeta' \Box_b^j e^{-s\Box_b}[\phi]||_0 \leq C.(1 + t^{-N})||\phi||_{L^2_{0,q}(M)},$$

this completes the proof. $\square$

As a consequence of Riesz Represenation Theorem, we have integral forms for $D^a e^{-s\Box_b}[\phi]$.

Lemma 3.1.4. *For $s > 0$, a be a non-negative integer, and $x \in M$, and for any derivative D^α, with $|\alpha| = a$, there exist unique functions $H^{IJ}_{s,x,a} \in L^2(M)$, where $|I| = |J| = q$, so that*

$$D^\alpha e^{-s\Box_b}[\phi](x) = {\sum}'_{|J|=q} \Big({\sum}'_{|I|=q} \int_M H^{IJ}_{s,x,a}(y)\phi_I(y)dV(y)\Big)\bar{\omega}_J, \tag{3.1.2}$$

or in short,

$$D^\alpha e^{-s\Box_b}[\phi](x) = \int_M H_{s,x,a}(y)\phi(y)dV(y),$$

where $\phi = \sum'_{|I|=q}\phi_I\bar{\omega}_I$. Moreover, if $K \subset M$ is compact and if C is the corresponding constant in Lemma 3.1.1, then if $x \in K$,

$$\sum_{I,J} \int_M |H^{IJ}_{s,x,a}(y)|^2 dy \leq C^2.(1 + t^{-N})^2.$$

Proof. For each $s > 0$, $x \in M$, we define the mapping $\phi \mapsto D^\alpha e^{-s\Box_b}[\phi](x)$. By Lemma 3.1.1, this functional is bounded. Moreover, since

$$D^\alpha e^{-s\Box_b}[\phi](x) = \sum\nolimits'_{|J|=q} \sum\nolimits'_{|I|=q} \left\langle D^\alpha \mathbb{H}_s[\phi_I \bar{\omega}_I], \bar{\omega}_J \right\rangle_{L^2}(x) \bar{\omega}_J,$$

and so by Riesz Representation Theorem, there exist functions $H^{IJ}_{s,x,a} \in L^2(M)$ so that

$$D^\alpha (\mathbb{H}_s)_{IJ}(\phi_I)(x) = \left\langle D^\alpha \mathbb{H}_s[\phi_I \bar{\omega}_I], \bar{\omega}_J \right\rangle(x) = \int_M H^{IJ}_{s,x,a}(y)\phi_I(y) dV(y).$$

Hence, by duality, we obtain

$$\sum\nolimits'_{I,J} \int_M |H^{IJ}_{s,x,a}(y) dy| \leq C^2.(1 + t^{-N})^2.$$

□

For each $|\alpha| = a$, we would like to regard for each I, J, $H^{IJ}_{s,x,a}$ as a measurable function of three variable (s, x, y). We proceed as follows. Each element $H^{IJ}_{s,x,a}$ is by definition an equivalence class of measurable, square functions on M which differ only on sets of measure zero. For each s, x, α, $|\alpha| = a$, choose one representative of this class, defined for all $y \in M$, which again called $H^{IJ}_{s,x,a}$. By this way, we can define a function $H^{IJ}_a(s, x, y) = H^{IJ}_{s,x,a}(y)$. In particular, we write $H^{IJ}(s, x, y) = H^{IJ}_{s,x,0}$. Moreover, we have

Proposition 3.1.5. *For each $|\alpha| = a$, H^{IJ}_a is measurable on $(0, \infty) \times M \times M$.*

Now, we can state the main result in this chapter.

Theorem 3.1.6. *We define the following $(0, q)$-forms*

$$H^I_y(s, x) = \sum\nolimits'_{|J|=q} H^{IJ}(s, x, y)\bar{\omega}_J(x),$$

$$H^J_x(s, y) = \sum\nolimits'_{|I|=q} H^{IJ}(s, x, y)\bar{\omega}_I(y),$$

and for $s > 0$, we define the double form

$$H_s(x, y) = \sum\nolimits'_{\substack{|I|=q \\ |J|=q}} H^{IJ}(s, x, y)\overline{\omega}_J(x) \otimes \overline{\omega}_I(y).$$

It turns out that $e^{-s\Box_b}[\phi](x) = \sum'_{|J|=q} < H^J_x(s, .), \phi >_{L^2_{0,q}(M)} \bar{\omega}_J$.
For each fixed $s > 0$ and $x \in U$, the function $y \mapsto H^{IJ}(s, x, y)$ belongs to $L^2(M)$, so each integral above converges absolutely. Moreover, each component $H^{IJ}(s, x, y)$ of $H(s, x, y)$ satisfies

1. *For $s > 0$, and $x, y \in U$, $H^{IJ}(s, x, y) = \overline{H^{JI}(s, y, x)}$.*

2. $[\partial_s + (\Box_b)_x][H^I_y](s,x) = [\partial_s + (\Box_b)_y][H^J_x](s,y) = 0.$
   And hence,
$$[\partial_s + (\Box_b)_x][H_s(x,y)] = [\partial_s + (\Box_b)_y][H_s(x,y)] = 0.$$

3. *For any integer* $j, k \geq 0$*,*
$$(\Box_b)^j_x(\Box_b)^k_y H^{IJ}_s(x,y) = (\Box_b)^{j+k}_x H^I_y(s,x) = (\Box_b)^{j+k}_y H^J_x(s,y).$$

4. *For each* $s > 0$ *and* $y \in U$*, for any non-negative integer* j*, each function*
$$x \mapsto (\Box_b)^j_x H^I_y(s,x)$$
   is orthogonal to the null space of $\Box_b$*.*

Proof. For fixed $s > 0$, since $H^{IJ}(s,x,y) = H^{IJ}_{s,x,0}$, for all I, J, then Lemma 3.1.4 says that the maps $y \mapsto H^{IJ}(s,x,y)$ belong to $L^2(M)$ for all I, J. and
$$e^{-s\Box_b}[\phi](x) = \sum\nolimits'_{|J|=q}\left(\sum\nolimits'_{|I|=q}\int_M H^{IJ}(s,x,y)\phi_I(y)dV(y)\right)\bar{\omega}_J.$$
We denote these sums as $\displaystyle\int_M H(s,x,y)\phi(y)dV(y)$.
Now,
$$D^\alpha_x\big(\int_M g(y)H^{IJ}(s,x,y)dV(y)\big) = \int_M g(y)H^{IJ}_a(s,x,y)dV(y)$$
Now, by Schwartz Kernel Theorem, the following holds
$$\begin{aligned} < D^\alpha_x(\mathbb{H}_s)_{IJ}[\psi], \phi >_{\mathbb{C}} &=< (-1)^{|\alpha|}(\mathbb{H}_s)_{IJ}[\psi], D^\alpha_x\phi > \\ &=< (-1)^{|\alpha|}H^{IJ}, \psi \otimes D^\alpha_x\phi >, \end{aligned}$$
for every $\phi, \psi \in C^\infty_0(M)$. On the other hand,
$$\begin{aligned} < D^\alpha(\mathbb{H}_s)_{IJ}[\psi], \phi > &=< (D^K\mathbb{H}_s)_{IJ}[\psi], \phi, > \\ &=< H^{IJ}_a, \psi(y) \otimes \phi(x) >, \end{aligned}$$
for every $\phi, \psi \in C^\infty_0(M)$. Hence,
$$< (-1)^{|\alpha|}H^{IJ}, \psi \otimes D^\alpha_x\phi >=< H^{IJ}_a, \psi(y) \otimes \phi(x) >, \text{ for every } \phi, \psi \in C^\infty_0(M) \ ,$$
so
$$D^\alpha H^{IJ}(s,x,y) = H^{IJ}_a(s,x,y) \tag{3.1.3}$$
in the sense of distributions.

1. The fact that the operator $\mathbb{H}_s = e^{-s\Box_b}$ is self-adjoint, and so for $\phi = \sum'_{|I|=q}\phi_I\bar{\omega}_I, \psi = \sum'_{|J|=q}\psi_J\bar{\omega}_J$ in $C_0^\infty(\Lambda^{0,q}(M))$, we have

$$\sum\nolimits'_{|J|=q}\int_M \big(\sum\nolimits'_{|I|=q}[\mathbb{H}_s^{IJ}[\phi_I]](x)\big).\bar{\psi}_J(x)dV(x) = \sum\nolimits'_{|I|=q}\int_M \overline{\big(\sum\nolimits'_{|J|=q}[\mathbb{H}_s^{JI}[\psi_J]](y)\big)}\phi_I(y)dV(y).$$

Now, substituting integral representation of $\mathbb{G}_s^{IJ}$, we have

$$\begin{aligned}\sum\nolimits'_{|J|=q}\int_M \big(\sum\nolimits'_{|I|=q} H^{IJ}(s,x,y)\phi_I(y)dV(y)\big).\bar{\psi}_J(x)dV(x) \\ = \sum\nolimits'_{|I|=q}\int_M \overline{\big(\sum\nolimits'_{|J|=q} H^{JI}(s,y,x)\psi_J(x)dV(x)\big)}\phi_I(y)dV(y).\end{aligned}$$

As a sequence, $H^{IJ}(s,x,y) = \overline{H^{JI}(s,y,x)}$, this is the first assertion.

2. We know that for fixed $s>0, x\in M$, the component function $y \mapsto H^{IJ}(s,x,y)$ is square integrable on M. So for fixed $y\in M$, the function $x\mapsto H^{IJ}(s,x,y) = \overline{H^{JI}(s,y,x)}$ belongs to $L^2(M)$. Similarly, the functions $y\mapsto H_a^{IJ}(s,x,y)$ and $x\mapsto H_a^{IJ}(s,x,y)$ also belong to $L^2(M)$. According to (3.1.3), every derivative of $H_K^{IJ}(s,x,y)$ in x-variables and y variables belongs to $L^2(M)$. That means all of derivatives of $H^{IJ}(s,x,y)$ in x,y variables have $L^2(M)$ bounds. Therefore, by Sobolev Embedding Theorem, $H^{IJ}(s,.,y)$ and $H^{IJ}(s,x,.)$ are smooth whenever $s>0$. As a consequence, every derivative (in x-variables and y-variables) in (3.1.3) exists in the classical sense and every derivative $D^\alpha_{x \text{ or } y}$ is bounded in $L^2(M)$-norm. Hence, again by Sobolev Embedding Theorem, $H^{IJ}(s,.,.)$ belongs to $C^\infty(M\times M)$.
Now, let $\phi\in L^2_{0,q}(M)$. Since

$$[\partial_s + \Box_b][\mathbb{H}_s[\phi]] = 0 \tag{3.1.4}$$

and then, fixing $x \in U$, we can integrate against test forms on $(0,\infty) \times M$, we have

$$
\begin{aligned}
0 &= \left\langle \big(\frac{\partial}{\partial s} + \Box_b\big)[\mathbb{H}_s[\phi]](x), \psi(.,x) \right\rangle_{L^2_{0,q}((0,\infty))} \\
&= \left\langle \mathbb{H}_s[\phi](x), \big(-\frac{\partial}{\partial s} + \Box_b\big)[\psi](.,x) \right\rangle_{L^2_{0,q}((0,\infty))} \\
&= \left\langle \mathbb{H}_s[\phi](x), -\frac{\partial}{\partial s}[\psi](.,x) \right\rangle_{L^2_{0,q}((0,\infty))} + \langle \mathbb{H}_s[\phi](x), (\Box_b)_x[\psi](.,x) \rangle_{L^2_{0,q}((0,\infty))} \\
&= \left\langle \sum\nolimits'_{|J|=q} \left(\sum\nolimits'_{|I|=q} (\mathbb{H}_s)_{IJ}[\phi_I](x) \right) \bar{\omega}_J, \sum\nolimits'_{|J|=q} \left(-\frac{\partial}{\partial s} \psi_J(.,x) \right) \bar{\omega}_J \right\rangle_{L^2_{0,q}((0,\infty))} \\
&+ \left\langle \sum\nolimits'_{|J|=q} \left(\sum\nolimits'_{|I|=q} (\mathbb{H}_s)_{IJ}[\phi_I](x) \right) \bar{\omega}_J, \sum\nolimits'_{|J|=q} \underbrace{(\Box_b \psi(.,x), \bar{\omega}_J)}_{\Psi_J(.,x)} \bar{\omega}_J \right\rangle_{L^2_{0,q}((0,\infty))} \\
&= \sum\nolimits'_{|J|=q} \int_{(0,\infty)} \left(\sum\nolimits'_{|I|=q} (\mathbb{H}_s)_{IJ}[\phi_I](x) \right) . \left(-\frac{\partial}{\partial s} \psi_J(s,x) \right) ds \\
&+ \sum\nolimits'_{|J|=q} \int_{(0,\infty)} \left(\sum\nolimits'_{|I|=q} (\mathbb{H}_s)_{IJ}[\phi_I](x) \right) .\Psi_J(s,x) ds \\
&= \sum\nolimits'_{|J|=q} \int_{(0,\infty)} \left(\sum\nolimits'_{|I|=q} \int_M H^{IJ}(s,x,y)\phi_I(y) dV(y) \right) . \left(-\frac{\partial}{\partial s} \psi_J(s,x) \right) ds \\
&+ \sum\nolimits'_{|J|=q} \int_{(0,\infty)} \left(\sum_{|I|=q} \int_M H^{IJ}(s,x,y)\phi_I(y) dV(y) \right) .\Psi_J(s,x) ds. \\
&= \sum\nolimits'_{|J|=q} \sum\nolimits'_{|I|=q} \int_{(0,\infty)} \int_M \frac{\partial}{\partial s} H^{IJ}(s,x,y)\phi_I(y) dV(y) .\psi_J(s,x) dV(y) ds \\
&+ \sum\nolimits'_{|J|=q} \sum\nolimits'_{|I|=q} \int_{(0,\infty)} \int_M H^{IJ}(s,x,y)\phi_I(y) dV(y) .\Psi_J(s,x) dV(y) ds.
\end{aligned}
$$

Therefore,

$$\frac{\partial}{\partial s} H^I_y(s,x) = -(\Box_b)_x H^I_y(s,x)$$

in the sense of distributions. Then, we also get

$$\frac{\partial^2}{\partial s^2} H^I_{s,y}(x) = -\frac{\partial}{\partial_s} (\Box_b)_x H^I_{s,y}(x) = (\Box_b)^2_x H^I_{s,y}(x).$$

Iterating this computation, we really show that

$$\frac{\partial^j}{\partial s^j} H^I_y(s,x) = (-1)^j (\Box_b)^j_x H^I_y(s,x)$$

in the distributional sense. In the other hand, the fact that every derivative in x-variables and y-variables of $H^{IJ}(s,x,y)$ belongs to $L^2(M\times M)$ implies $(\Box_b)^j_x H^I_y(s,x)\in L^2_{0,q}((0,\infty)\times M)$ locally. Then, from the above identity, all derivatives in s-variables of $H^I_{s,y}$ are in $L^2_{0,q}((0,\infty)\times M)$ locally. This is enough to show that H^{IJ}_q is indeed in $C^\infty((0,\infty)\times M)$ from the standard elliptic theory. That means

$$\frac{\partial}{\partial s}H^I_y(s,x)=-(\Box_b)_x H^I_y(s,x)$$

in the classical sense.
Iterating the same argument to y-variables, we also obtain

$$\frac{\partial}{\partial s}H^J_x(s,y)=-(\Box_b)_y H^J_x(s,y)$$

in the classical sense.

3. From the proof of a result before, we know that $(\Box_b)\mathbb{H}_s=\mathbb{H}_s(\Box_b)$. Then,

$$<(\Box_b)(\mathbb{H}_s[\phi]),\psi>=<\mathbb{H}_s(\Box_b\phi),\psi>,$$

and so

$$<\mathbb{H}_s[\phi],\Box_b\psi>=<\mathbb{H}_s(\Box_b\phi),\psi>,$$

for any $\phi=\sum'_{|I|=q}\phi_I\bar{\omega}_I,\psi=\sum'_{|K|=q}\psi_K\bar{\omega}_K\in C^\infty_0(\Lambda^{0,q}(M))$. We can rewrite $\Box_b\psi=\sum_{|J|=q}(\Box_b\psi)_J\bar{\omega}_J$. Wrting out what this means in the terms of the each componnent $H^{IJ}_q(s,x,y)$, we have

$$\sum\nolimits'_{|J|=q}\int_M\left(\sum\nolimits'_{|I|=q}\int_M H^{IJ}_q(s,x,y)\phi_I(y)dV(y)\right).(\Box_b\psi)_J(x)dV(x)).$$
$$=\sum\nolimits'_{|J|=q}\int_M\left(\sum\nolimits'_{|L|=q}\int_M H^{LJ}(s,x,y)(\Box_b\phi)_L(y)dV(y)\right).\psi_J(x)dV(x).$$

By The Schwart Kernel Theorem and the self-adjointness of $\Box_b$, this implies that

$$\langle(\Box_b)_x H_s(x,y),\phi\otimes\psi\rangle=\langle(\Box_b)_y H_s(x,y),\phi\otimes\psi\rangle \tag{3.1.5}$$

for every $\phi,\psi\in C^\infty(\Lambda^{0,q}(M))$.
Therefore, for each $s>0$, $(\Box_b)_x H_s(x,y)=(\Box_b)_y H_s(x,y)$ in the sense of distributions, and also this holds in the classical sense as the arguments before. From this identity, it is not so hard to show that for each $s>0$

$$(\Box_b)^j_x(\Box_b)^k_y H_s(x,y)=(\Box_b)^{j+k}_x H_s(x,y)=(\Box_b)^{j+k}_y H_s(x,y).$$

4. Since $\mathcal{H}_q(\Box_b)^j=(\Box_b)^j\mathcal{H}_q=0$, then $\mathcal{H}_q(\Box_b)^j\mathbb{H}_s=0$. And the fact that $\mathcal{H}_q$ is self-adjoint implies the following identity for each $s>0$ and for all test forms $\phi=\sum'_{|I|=q}\phi_I\bar{\omega}_I,\psi=\sum'_{|J|=q}\phi_J\bar{\omega}_J$,

$$\left\langle\sum\nolimits'_{|I|=q}<\mathcal{H}_q(\Box_b)^j H^I_y(s,.),\psi>\bar{\omega}_I,\phi\right\rangle=0.$$

Hence, for fixed $s > 0$ and $y \in U$, $\mathcal{H}_q(\Box_b)^j_x H^I_y(s,.) = 0$, and then the following map

$$x \mapsto (\Box_b)^j_x H^I_y(s,x)$$

belongs to the orthogonal complement of the null space of $\Box_b$. This completes the proof of the theorem. $\Box$

3.2 The heat equation on $\mathbb{R} \times M$

In this section, we will study the operator $\partial_s + \Box_b$ on whole space $\mathbb{R} \times M$. We will show that for fixed $x \in M$, we can define distributions $\mathbb{H}^{IJ}_x$ on $\mathbb{R} \times M$ such that

$$\mathbb{H}^{IJ}_x(s.y) = \begin{cases} H^{IJ}(s,x,y) & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

in the sense of each component.

To do this, we recall some materials from distributions theory on $\mathbb{R} \times M$.

Definition 3.2.1. Let S be a distribution on $\mathbb{R}$ and T be a distribution on M, then from Appendix, Theorem A.2.2, we can define the distribution $S \otimes T$ on $\mathbb{R} \times M$ by

$$\langle S \otimes T, \chi \otimes \psi \rangle = \langle S, \chi \rangle \langle T, \psi \rangle,$$

where $\chi \in C^\infty_0(\mathbb{R})$, and $\psi \in C^\infty_0(M)$.
Now, for each fixed $x \in M$, we define the distribution $S^{\mathbb{R}}$ on $\mathbb{R}$ by follows

$$S^{\mathbb{R}}[\psi_x] = S^{\mathbb{R}}[\psi](x) = \int_0^\infty \psi(s,x)ds,$$

for every $\psi \in C^\infty_0(\mathbb{R} \times M)$, where $\psi_x = \psi(s,x)$ for fixed $x \in M$.
For each distribution T on M, we also have the corresponding one $T^{\mathbb{R}\times M}$ on $\mathbb{R} \times M$ defined by

$$\langle T^{\mathbb{R}\times M}, \psi \rangle = \langle T, \mathcal{S}^{\mathbb{R}}[\psi_x] \rangle,$$

where $\psi \in C^\infty_0(\mathbb{R} \times M)$. Again, by Theorem A.2.2, it can be rewritten is that

$$\langle T^{\mathbb{R}\times M}, \psi \rangle = S^{\mathbb{R}}\big(T[\psi_s]\big) = \int_0^\infty \langle T, \psi_s \rangle ds,$$

where $\psi_s = \psi(s,x)$ for each fixed $s \in \mathbb{R}$.
Let ∂_x be the any derivative on M in x-variables, we have

$$\partial_x S^{\mathbb{R}}[\psi] = S^{\mathbb{R}}[\partial_x \psi],$$

and

$$\partial_x T^{\mathbb{R}\times M}[\psi] = T^{\mathbb{R}\times M}[\partial_x \psi].$$

Now, from the theory above, we begin our approach with the following definition.

Definition 3.2.2. For $\psi \in \Lambda^{0,q}(C_0^\infty(\mathbb{R} \times M))$, set

$$\langle \mathbb{H}_x, \psi \rangle = \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \int_M H(s,x,y)\psi(s,y)dV(y)ds$$

The main object in this chapter is to show that the above limit exists. It follows that the kernel of this distribution has the following components

$$H_x^{IJ}(s.y) = \begin{cases} \mathbb{H}^{IJ}(s,x,y) & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

Lemma 3.2.3. *The limit defining $\mathbb{H}_x$ exists. Moreover,*

$$[\partial_s + (\Box_b)_y][\mathbb{H}_x] = \delta_0 \otimes \delta_x$$

in the sense of distributions (for the components), i.e,

$$\langle \mathbb{H}_x, [-\partial_s + \Box_b]\psi \rangle = \psi(0,x).$$

Proof. Setting $\psi_s(y) = \psi(s,y)$. Then, $\psi_s \in \Lambda^{0,q}(C_0^\infty(M))$. Choose a positive integer N so that $N\epsilon > \frac{2n-1}{2}$. Choose $\zeta \prec \zeta_1 \prec ... \prec \zeta_N = \zeta'$ with $\zeta(x) = 1$. Then, again, by Sobolev Imbedding Theorem and the basic subelliptic estimate applied N times, we obtain

$$\begin{aligned} \left| \int_M H(s,x,y)\psi(y)dV(y) \right| &= \left| \zeta e^{-s\Box_b}[\psi_s](x) \right| \\ &\leq C.||\zeta e^{-s\Box_b}[\psi_s]||_{N\epsilon} \\ &\leq C.[||\zeta_1 \Box_b[e^{-s\Box_b}[\psi_s]]||_{(N-1)\epsilon} + ||\zeta_1 e^{-s\Box_b}[\psi_s]||_0] \\ &\leq \text{...repeating } N\text{-times as above} \\ &\leq C.\sum_{j=0}^N ||\zeta' \Box_b^j[e^{-s\Box_b}[\psi_s]]||_0. \end{aligned} \tag{3.2.1}$$

But since the operators $\Box_b$ and $e^{-s\Box_b}$ are commutative, hence,

$$\left| \int_M H(s,x,y)\psi(y)dV(y) \right| \leq C \sum_{j=0}^N ||\zeta' e^{-s\Box_b}[\Box_b^j \psi_s]||_0 \leq C. \sum_{j=0}^N ||\Box_b^j \psi_s||_0.$$

The right hand side is uniformly bounded in s, and then, taking integral on $[\eta_1, \eta_2]$, we have

$$\left| \int_{\eta_1}^{\eta_1} \int_M H(s,x,y)\psi(y)dV(y)ds \right| \leq C.|\eta_2 - \eta_1| \sup_s \sum_{j=0}^N ||\Box_b \psi_s||_0.$$

We see that the left hand side goes to zero as $\eta_1 \to \eta_1$, so the limit defining $\mathbb{H}_x$ exists. Again, let $\psi \in \Lambda^{0,q}(C_0^\infty(\mathbb{R} \times M)$, then

$$\begin{aligned}\langle \mathbb{H}_x, [-\partial_s + \Box_b]\psi_s\rangle &= \lim_{\epsilon\to 0}\int_\epsilon^\infty e^{-s\Box_b}[[-\partial_s + \Box_b]\psi_s]ds \\ &= -\lim_{\epsilon\to 0}\int_\epsilon^\infty e^{-s\Box_b}[[\partial_s]\psi_s]ds + \lim_{\epsilon\to 0}\int_\epsilon^\infty e^{-s\Box_b}[[\Box_b]\psi_s]ds \\ &= -\lim_{\epsilon\to 0}\int_\epsilon^\infty \int_M H(s,x,y)\partial_s\psi(s,x)dV(y)ds \\ &+ \lim_{\epsilon\to 0}\int_\epsilon^\infty \int_M H(s,x,y)\Box_b\psi(s,y)dV(y)ds.\end{aligned} \tag{3.2.2}$$

Now, for the first term,

$$\begin{aligned}&-\int_\epsilon^\infty \int_M H(s,x,y)\partial_s\psi(s,x)dV(y)ds \\ &= -\int_\epsilon^\infty \partial_s{\sum}'_{|J|=q}\left({\sum}'_{|I|=q}\int_M H^{IJ}(s,x,y)\psi_I(s,y)dV(y)\right)\bar{\omega}_J ds \\ &+ \int_\epsilon^\infty {\sum}'_{|J|=q}\left({\sum}'_{|I|=q}\int_M \partial_s H^{IJ}(s,x,y)\psi_I(s,y)dV(y)\right)\bar{\omega}_J ds \\ &= {\sum}'_{|J|=q}\left({\sum}'_{|I|=q}\int_M H^{IJ}(\epsilon,x,y)\psi_I(\epsilon,y)dV(y)\right)\bar{\omega}_J \\ &+ \int_\epsilon^\infty {\sum}'_{|J|=q}\left({\sum}'_{|I|=q}\int_M \partial_s H^{IJ}(s,x,y)\psi_I(s,y)dV(y)\right)\bar{\omega}_J ds. \\ &= {\sum}'_{|J|=q}\left({\sum}'_{|I|=q}\int_M H^{IJ}(\epsilon,x,y)\psi_I(\epsilon,y)dV(y)\right)\bar{\omega}_J \\ &+ \int_\epsilon^\infty {\sum}'_{|J|=q}\left\langle \partial_s H_x^J(s,.),\psi(s,.)\right\rangle\bar{\omega}_J ds.\end{aligned} \tag{3.2.3}$$

And the second term,

$$\begin{aligned}\int_\epsilon^\infty \int_M H(s,x,y)\Box_b\psi(s,y)dV(y)ds &= \int_\epsilon^\infty {\sum}'_{|J|=q}\left\langle H_x^J(s,.),\Box_b\psi(s,.)\right\rangle\bar{\omega}_J ds \\ &= \int_\epsilon^\infty {\sum}'_{|J|=q}\left\langle (\Box_b)_y H_x^J(s,.),\psi(s,.)\right\rangle\bar{\omega}_J ds.\end{aligned} \tag{3.2.4}$$

Hence, since $[\partial_s + (\Box_b)_y]H_x^J(s,y) = 0$, (3.2.2), (3.2.3), and (3.2.4) imply

$$\begin{aligned}-\int_\epsilon^\infty \int_M H(s,x,y)\partial_s\psi(s,x)dV(y)ds + \int_\epsilon^\infty \int_M H(s,x,y)\Box_b\psi(s,y)dV(y)ds \\ = {\sum}'_{|J|=q}\left({\sum}'_{|I|=q}\int_M H^{IJ}(\epsilon,x,y)\psi_I(\epsilon,y)dV(y)\right)\bar{\omega}_J.\end{aligned} \tag{3.2.5}$$

It follows that

$$\begin{aligned}\langle \mathbb{H}_x, [-\partial_s + \Box_b]\psi_s \rangle &= \lim_{\epsilon\to 0^+} \sum\nolimits'_{|J|=q} \left(\sum\nolimits'_{|I|=q} \int_M H^{IJ}(\epsilon, x, y)\psi_I(\epsilon, y) dV(y) \right) \bar{\omega}_J \\ &= \psi(0,x) = \delta_0 \otimes \delta_x.\end{aligned} \tag{3.2.6}$$

Hence, this completes the proof of the lemma. $\square$

A remark that, by translation, we also have

$$\langle \mathbb{H}_x, [-\partial_s + \Box_b]\psi_{s+t} \rangle = \delta_t \otimes \delta_x.$$

3.3 Pointwise estimates for the heat kernel

We begin by recalling the scaled pullback of heat equation on $\mathbb{R} \times M$ to $\mathbb{R} \times \mathbb{B}_0$ by

$$\left[\partial_s + \widehat{\Box_b}\widehat{\phi}(s,u)\right] = r^2(\widehat{[\partial_s + \Box_b]\,\phi}(s,x)).$$

Now, we define the pullback of the heat kernel $H(s,x,y)$ using the same change of variable $\Phi_{(s,x_0),r}$, with $s > 0$, $x_0 \in M$, $u, v \in \mathbb{B}_0$,

$$W^{IJ}(s,u,v) = W^{IJ}_{x_0,r}(s,u,v) = H^{IJ}(r^2 s, \Phi_{x_0,r}(u), \Phi_{x_0,r}(v)),$$

for each $|I| = |J| = q$, and $0 < r < R_0$. Hence, from the main results in previous chapter, with the changing map $\Phi_{(s,x_0),r}$, we have

$$\begin{aligned}&[\partial_s + (\widehat{\Box_b})_u][W^I_v](s,u) = 0,\\ &[\partial_s + (\widehat{\Box_b})_v][W^J_u](s,v) = 0,\end{aligned} \tag{3.3.1}$$

where $W^I_v(s,u)$, and $W^J_u(s,v)$ defined by the same formulation to $H^I_y(s,x)$, and $H^J_x(s,y)$. By the similar way, for $s > 0$, and $\phi \in \Lambda^{0,q}(C^\infty_0(\mathbb{B}_0))$, we can define

$$\mathbb{W}_s[\phi](u) = \int_{\mathbb{B}_0} W(s,u,v)\phi(v)dv = \int_{\mathbb{B}_0} H(r^2 s, \Phi_{x_0,r}(u), \Phi_{x_0,r}(v))\phi(v)dv.$$

The key point is that we can bound the norm of the operator $\mathbb{W}_s$ on $L^2_{0,q}(\mathbb{B}_0)$.

Lemma 3.3.1. *There is a constant C which is independent of x_0, r and $s > 0$ so that*

$$||\mathbb{W}_s[\phi]||_{L^2_{0,q}(\mathbb{B}_0)} \le C|B(x_0,r)|^{-1}||\phi||_{L^2_{0,q}(\mathbb{B}_0)}.$$

Proof. Let $x \in B(x_0, r)$, by change of variables, we have

$$\begin{aligned}\mathbb{W}_s[\phi](\Phi_{x_0,r}^{-1}(x)) &= \int_{\mathbb{B}_0} H(r^2 s, \Phi_{x_0,r}(\Phi_{x_0,r}^{-1}(x)), \Phi_{x_0,r}(v))\phi(v)dv \\ &= \int_M H(r^2 s, x, y)(\phi \circ \Phi_{x_0,r}^{-1})(y) J\Phi_{x_0,r}^{-1}(y) dV(y) \\ &= e^{-r^2 s \Box_b}[(\phi \circ \Phi_{x_0,r}^{-1}) J\Phi_{x_0,r}^{-1}](x).\end{aligned} \tag{3.3.2}$$

Here and later, every integral representing to the heat operator is understood by the formulations in Lemma 3.1.4.
Since, $||e^{-r^2 s\Box_b}[(\phi \circ \Phi_{x_0,r}^{-1}) J\Phi_{x_0,r}^{-1}]||_{L^2} \le ||(\phi \circ \Phi_{x_0,r}^{-1}) J\Phi_{x_0,r}^{-1}||_{L^2}$, it follows that

$$\begin{aligned}\int_M |\mathbb{W}_s[\phi](\Phi_{x_0,r}^{-1}(x))|^2 dV(x) &\le \int_M |\phi(\Phi_{x_0,r}^{-1}(x))|^2 (J\Phi_{x_0,r}^{-1}(x))^2 dV(x) \\ &= \int_{\mathbb{B}_0} |\phi(u)|^2 (J\Phi_{x_0,r}^{-1}(\Phi_{x_0,r}(u)))^2 J\Phi_{x_0,r}(u) du \\ &\le C.|B(x_0, r)|^{-1} \int_{\mathbb{B}_0} |\phi(u)|^2 du.\end{aligned} \tag{3.3.3}$$

Here, we have used the facts that $J\Phi_{x_0,r}^{-1}(\Phi_{x_0,r}(u)) = J\Phi_{x_0,r}(u)^{-1}$, and $J\Phi_{X_0,R}(u) \ge C^{-1}|B(x_0, r)|$ for $0 < r < R_0$ according to Theorem 2.1.4. On the other hand,

$$\int_M |\mathbb{W}_s[\phi](\Phi_{x_0,r}^{-1}(x))|^2 dV(x) \ge C^{-1}|B(x_0, r)| \int_{\mathbb{B}_0} |\mathbb{W}_s[\phi](u)|^2 du.$$

Hence, we obtain

$$||\mathbb{W}_s[\phi]||_{L^2_{0,q}(\mathbb{B}_0)} \le C|B(x_0, r)|^{-1}||\phi||_{L^2_{0,q}(\mathbb{B}_0)},$$

and this completes the proof. $\square$

Next, we will obtain local estimates for the functions H^{IJ}'s, $|I| = |J| = q$, and certain of its derivatives in terms of s and the control metric ρ.

Theorem 3.3.2. *Let j, k, l be non-negative integers. For every positive integer N, there is a constant $C_N = C_{N,j,k,l}$ so that if $|\alpha| = k$, $|\beta| = l$,*

$$|\partial_s^j X_x^\alpha X_y^\beta H^{IJ}(s, x, y)| \le \begin{cases} C_N \rho(x, y)^{-2j-k-l} |B(x, \rho(x, y))|^{-1} \left(\dfrac{s}{\rho(x, y)}\right)^N & \text{if } s \le \rho(x, y)^2, \\ C_N s^{-j-k/2-l/2} |B(x, \sqrt{s})|^{-1} & \text{if } s \ge \rho(x, y)^2 \end{cases}, \tag{3.3.4}$$

for all (s, x, y) with $\rho_{\mathbb{R}\times M}((s, x), (0, y)) = |s|^{1/2} + \rho(x, y) \le 1$.

The proof is based the scaling method which was introduced M. Christ [Ch88], and then developed in higher dimensions by K. Koenig [Koe02]. We need the following subelliptic estimate for the pullback of $\Box_b$ operator on $\mathbb{B}_1$ which is a consequence of (1.4.2), and Theorem 2.1.4.

Proposition 3.3.3. *Fix $\zeta, \zeta' \in C_0^\infty(\mathbb{B}_0)$, with $\zeta \prec \zeta'$. For smooth $(0,q)$-forms, $\phi = \sum'_{|K|=q}\phi_K \bar{\widehat{\omega}}_K$ on $\mathbb{B}_0$ and $\delta \geq 0$,*

$$||\zeta\phi||^2_{\delta+\epsilon} \leq C_\delta \left(||\zeta'\widehat{\bar{\partial}}_b\phi||^2_\delta + ||\zeta'\widehat{\bar{\partial}^*_b}\phi||^2_\delta + ||\zeta'\phi||^2_0 \right),$$

where C_δ is a positive constant independent of x and $0 < r < R_0$. As a consequence, the heat operator $\partial_s + \widehat{\Box_b}$ also satisfies the subelliptic estimate

$$||\zeta\phi||^2_{\delta+\epsilon} \leq C_\delta \left(||\zeta'[\partial_s + \widehat{\Box_b}]\phi||^2_\delta + ||\zeta'\phi||^2_0 \right).$$

Proof. (*(Proof of Theorem 3.3.2)*) We will prove the theorem with $N = 0$ first. By compactness, if $R_0 \leq |s|^{1/2} + \rho(x,y) \leq 1$, the estimates are trivial. Hence, it suffices to show that that the estimates hold when $|s|^{1/2} + \rho(x,y) \leq R_0$. Now, let fix $(s_0, x_0) \in \mathbb{R} \times M$, and let $(s,x) \in \mathbb{R} \times M$ be another point so that $\rho_{\mathbb{R}\times M}((s_0,x_0),(s,x)) = r \leq R_0$. There exists a unique point $(t_0, v_0) \in (-1,1) \times \mathbb{B}_0$ such that $(s,x) = (s_0 + r^2 t_0, \Phi_{x_0,r}(v_0))$. Let $\tau > 0$ such that $|t_0|^{1/2} + |v_0| \geq \tau$.
For $(t_1,u),(t_2,v) \in (-1,1)\times\mathbb{B}_0$, put

$$W^\#((t_1,u),(t_2,v)) = H(r^2(t_2-t_1), \Phi_{x_0,r}(u), \Phi_{x_0,r}(v)),$$

in the sense that $\left(W^\#\right)^{IJ}((t_1,u),(t_2,v)) = H^{IJ}(r^2(t_2-t_1), \Phi_{x_0,r}(u), \Phi_{x_0,r}(v))$. Then

$$\begin{aligned} [-\partial_{t_1} + (\widehat{\Box_b})_u][(W^\#)^I_v] &= 0, \\ [\partial_{t_2} + (\widehat{\Box_b})_v][(W^\#)^J_u] &= 0, \end{aligned} \tag{3.3.5}$$

and

$$[\partial_s^j X_x^\alpha X_y^\beta H](r^2(t_2-t_1), \Phi_{x_0,r}(u), \Phi_{x_0,r}(v)) = r^{-2j-k-l}[\partial^j_{t_2}\widehat{X}^\alpha_u \widehat{X}^\beta_v]((t_1,u),(t_2,v)). \tag{3.3.6}$$

Now, for $\phi \in C_0^\infty(\Lambda((-1,1)\times\mathbb{B}_0))$, setting

$$\mathcal{T}^\#[\phi](t_1,u) = \int\int_{\mathbb{R}\times\mathbb{B}_0} W^\#((t_1,u).(t_2,v))\phi(t_2,v)dvdt_2,$$

in the sense as above, i.e.,

$$\left(\mathcal{T}^\#[\phi](t_1,u)\right)_J = \left(\sum\nolimits'_{|I|=q}\int\int_{\mathbb{R}\times\mathbb{B}_0}(W^\#)^{IJ}((t_1,u).(t_2,v))\phi_I(t_2,v)dvdt_2\right)_J,$$

then,

$$\mathcal{T}^\#[\phi](t_1,u) = \sum\nolimits'_{|J|=q}\left(\mathcal{T}^\#[\phi](t_1,u)\right)_J \widehat{\bar{\omega}}_J.$$

Put

$$\begin{aligned} B_1 &= \left\{(t_1,u) : |t_1|^{1/2} + |u| < \frac{1}{3}\tau\right\}, \\ B_2 &= \left\{(t_2,v) : |t_2 - t_0|^{1/2} + |v - v_0| < \frac{1}{3}\tau\right\}. \end{aligned} \tag{3.3.7}$$

Then, the non-isotropic balls B_1 and B_2 are disjointed. Choose cut-off functions $\zeta \prec \zeta' \prec \zeta" \in C_0^\infty(B_2)$ with $\zeta(t_0, v_0) = 1$, and $\eta \prec \eta' \in C_0^\infty(B_1)$ with $\eta(0,0) = 1$. Then, by Sobolev Inequality and the basic subelliptic estimate for the operator $\partial_{t_2} + \widehat{\Box_b}$, we have

$$\begin{aligned}
\left|[\partial_{t_2}^j \widehat{X}_v^\beta (W^\#)_u^J]((0,0),(t_0,v_0))\right| &= \left|\zeta(t_0,s_0)[\partial_{t_2}^j \widehat{X}_v^\beta (W^\#)_u^J]((0,0),(t_0,v_0))\right| \\
&\le C. \left|\left|\zeta'(W^\#)_u^J((0,0).(.,.))\right|\right|_{2n+j+k+l} \\
&\le C. \left[\left|\left|\zeta"[\partial_{t_2} + (\widehat{\Box_b})_v](W^\#)_u^J((0,0),(.,.))\right|\right|_{2n+j+k+l-\epsilon}\right. \\
&\left. + \left|\left|\zeta"(W^\#)_u^J((0,0),(.,.))\right|\right|_0\right] \\
&\le C \left|\left|\zeta"(W^\#)_u^J((0,0),(.,.))\right|\right|_0 \\
&\le C \sup_{\substack{\phi \in C^\infty(B_2) \\ ||\phi||=1}} \left|\mathcal{T}^\#[\zeta'\phi](0,0)\right|.
\end{aligned}$$

Where we have used the facts that: $[\partial_{t_2} + (\widehat{\Box_b})_v](W^\#)_u^J((0,0),(.,.)) = 0$ on B_2 which contains $\text{supp}(\zeta')$, and by Hilbert space duality to convert each operator $(W^\#)_u^J$ to $(\mathcal{T}^\#)_u^J$.
Now, to estimate the term with the supremum sign, again, we use the basic subelliptic for $-\partial_{t_1} + (\widehat{\Box_b})_u$,

$$\begin{aligned}
\sup_{\substack{\phi \in C^\infty(B_2) \\ ||\phi||=1}} \left|\mathcal{T}^\#[\zeta'\phi](0,0)\right| &= \sup_{\substack{\phi \in C^\infty(B_2) \\ ||\phi||=1}} \left|\eta(0,0)\mathcal{T}^\#[\zeta'\phi](0,0)\right| \\
&\le C. \left|\left|\eta\mathcal{T}^\#[\zeta'\phi]\right|\right|_{2n} \\
&\le C \sup_{\substack{\phi \in C^\infty(B_2) \\ ||\phi||=1}} \left[\left|\left|\eta' \underbrace{[-\partial_{t_1} + (\widehat{\Box_b})_u]\mathcal{T}^\#[\zeta'\phi]}_{=0 \text{ on } B_1 \text{ containing supp}(\eta')}\right|\right|_{2n-\epsilon} + \left|\left|\eta'\mathcal{T}^\#[\zeta'\phi]\right|\right|_0\right] \\
&= C \sup_{\substack{\phi \in C^\infty(B_2) \\ ||\phi||=1}} \left|\left|\eta'\mathcal{T}^\#[\zeta'\phi]\right|\right|_0 \\
&\le C \sup_{\substack{\phi \in C^\infty(B_2) \\ ||\phi||=1}} \left|\left|\mathcal{T}^\#[\zeta'\phi]\right|\right|_0 \\
&\le C.||\mathcal{T}^\#||.
\end{aligned}$$

Therefore, we have shown that

$$\left|\partial_s^j X_y^\beta H(r^2 t_0, x_0, x)\right| \le C.r^{-2j-l}||\mathcal{T}^\#||.$$

By the same argument, we can prove that

$$\left|\partial_s^j X_x^\alpha H(r^2 t_0, x_0, x)\right| \le C.r^{-2j-k}||\mathcal{T}^\#||,$$

and combing these estimates, we have

$$\left|\partial_s^j X_x^\alpha X_y^\beta H(r^2 t_0, x_0, x)\right| \leq C.r^{-2j-k-l}||\mathcal{T}^\#||.$$

The last problem is to estimate the norm $||\mathcal{T}^\#||$. Let ϕ, ψ be $(0,q)$-forms whose coefficients are $C_0^\infty((-1,1)\times\mathbb{B}_0)$, and let $\phi_s(v) = \phi(s,v)$, $\psi_t(u) = \psi(t,u)$. Then, in the sense as above, we have

$$\begin{aligned}
&\left|\int\int_{\mathbb{R}\times\mathbb{B}_0} \mathcal{T}^\#[\phi](t,u)g(t,u)dudt\right| \\
&= \left|\sum\nolimits'_{|J|=q}\int\int_{\mathbb{R}\times\mathbb{B}_0}\left(\mathcal{T}^\#[\phi](t,u)\right)_J \psi_J(t,u)dudt\right| \\
&= \left|\sum\nolimits'_{|J|=q}\int\int_{\mathbb{R}\times\mathbb{B}_0}\left(\sum\nolimits'_{|I|=q}\int\int_{\mathbb{R}\times\mathbb{B}_0}(W^\#)^{IJ}((t,u).(s,v))\phi_I(s,v)dvds\right)\psi_J(t,u)dudt\right| \\
&= \left|\sum\nolimits'_{|J|=q}\sum\nolimits'_{|I|=q}\int\int_{\mathbb{R}\times\mathbb{B}_0}\int\int_{\mathbb{R}\times\mathbb{B}_0}(W^\#)^{IJ}((t,u).(s,v))\phi_I(s,v)\psi_J(t,u)dsdtdudv\right| \\
&= \left|\sum\nolimits'_{|J|=q}\sum\nolimits'_{|I|=q}\int\int\int\int H^{IJ}(r^2(s-t),\Phi_{x_0,x}(u),\Phi_{x_0,r}(v))\phi_I(s,v)\psi_J(r,u)dsdtdudv\right| \\
&= \left|\sum\nolimits'_{|J|=q}\sum\nolimits'_{|I|=q}\int\int\int\int H^{IJ}(r^2 s,\Phi_{x_0,x}(u),\Phi_{x_0,r}(v))\phi_I(s+t,v)\psi_J(t,u)dsdtdudv\right| \\
&= \left|\sum\nolimits'_{|J|=q}\int\int\int(\mathbb{W}_s[\phi_{s+t}](u))_J\psi_J(t,u)dsdtdu\right| \\
&\leq C\sum\nolimits'_{|J|=q}\int\int_{\mathbb{R}^2}\int_{\mathbb{R}_0}|(\mathbb{W}_s[\phi_{s+t}](u))_J\psi_J(t,u)dsdtdu|dudsdt \\
&\leq C\int\int_{\mathbb{R}^2}||\mathbb{W}_s[\phi_{s+t}]||_{L^2_{0,q}(\mathbb{B}_0)}.||\psi_t||_{L^2_{0,q}(\mathbb{B}_0)}dsdt.
\end{aligned}$$

Now, by Lemma 3.3.1, $||\mathbb{W}_s[\phi_{s+t}]||_{L^2_{0,q}} \leq C.|B(x_0,r)|^{-1}||\phi_{s+t}||_{L^2_{0,q}(\mathbb{B}_0)}$. Then

$$\begin{aligned}
&\left|\int\int_{\mathbb{R}\times\mathbb{B}_0}\mathcal{T}^\#[\phi](t,u)g(t,u)dudt\right| \\
&\leq C|B(x_0,r)|^{-1}\int\int_{\mathbb{R}^2}||\phi_{s+t}||_{L^2_{0,q}(\mathbb{B}_0)}.||\psi_t||_{L^2_{0,q}(\mathbb{B}_0)}dsdt \\
&= C|B(x_0,r)|^{-1}\int_{\mathbb{R}}||\phi_s||_{L^2_{0,q}(\mathbb{B}_0)}ds.\int_{\mathbb{R}}||\psi_t||_{L^2_{0,q}(\mathbb{B}_0)}dt \\
&\leq C|B(x_0,r)|^{-1}||\phi||_{\mathbb{R}\times\mathbb{B}_0}||\psi||_{\mathbb{R}\times\mathbb{B}_0},
\end{aligned}$$

here the last inequality has been verified by Schwarz's Inequality, and the the supports of ϕ, ψ are contained in $(-1,1)\times\mathbb{B}_0$. So, in the case $N=0$,

$$\left|\partial_s^j X_x^\alpha X_y^\beta H(s,x,y)\right| \leq C.(\rho_{\mathbb{R}\times M}(s,x),(0,y))^{-2j-k-l}|B(x,\rho_{\mathbb{R}\times M}(s,x),(0,y))|^{-1}.$$

This is actually the statement of Theorem 3.3.2 when $N = 0$ using the basic doubling property. To deal with the case $N > 0$, using Taylor's Formulae via s-variables, and note that when $x \neq y$, the infinitely differential map $s \mapsto H^{IJ}(s, x, y)$ and its derivatives are zero at $s = 0$. Then

$$\begin{aligned}
|H^{IJ}(s,x,y)| &\leq \frac{1}{(N-1)!}\int_0^s |\partial_t^N H(t,x,y)|(s-t)^{N-1}dt \\
&\leq C_0 \frac{1}{(N-1)!}\rho(x,y)^{-2N}|B(x,\rho(x,y)|^{-1}\int_0^s (s-t)^{N-1}dt \\
&\leq C_0 \frac{1}{N!}\left(\frac{s}{\rho(x,y)}\right)^N |B(x,\rho(x,y)|^{-1},
\end{aligned}$$

when $s \leq \rho(x,y)$, and replace $\rho(x,y)$ by $s^{1/2}$. This argument also provides the same results when $s \geq \rho(x,y)$. Finally, estimates for other derivatives of $H^{IJ}(s,x,y)$ are handle in the same way. Therefore, this completes the proof of the theorem. $\square$

Next, action of the heat operator on bump function will be provided.

Theorem 3.3.4. *Fix $s > 0$, $0 < r < R_0$, for each multi-index α, there is an integer N_α and a constant C_α so that if $\phi \in \Lambda^{0,q}(C_0^\infty(B(x,r)))$, then*

$$|X_x^\alpha e^{-s\Box_b}[\phi](x)| \leq C_\alpha r^{-|\alpha|}\sup_{y\in M}\sum_{|\beta|\leq N_\alpha} r^{|\beta|}|X^\beta\phi(y)|. \tag{3.3.8}$$

Proof. By Sobolev Type Theorem 2.2.5, and the argument before, with $N\epsilon \geq 2n-1+|\alpha|+|L_m|$ we have

$$\begin{aligned}
&r^{|\alpha|}|X^\alpha e^{-s\Box_b}[\phi](x)| \\
&\quad\leq C.|B_M(x,r)|^{\frac{-1}{2}}\sum_{0\leq|\beta|\leq L_m,|\beta|\,even} r^{|\beta|+|\alpha|}||X^{\alpha+\beta}e^{-s\Box_b}[\phi]||_{L^2_{0,q}(B_M(x,2r))} \\
&\quad\leq C.|B_M(x,r)|^{\frac{-1}{2}}\sum_{l=0,\,l\,even}^{L_m} r^{l+|\alpha|}\sum_{j=0}^{N}||e^{-s\Box_b}[(\Box_b)^j\phi||_{L^2_{0,q}(B_M(x,2r))} \\
&\quad\leq C.|B_M(x,r)|^{\frac{-1}{2}}\sum_{l=0,\,l\,even}^{L_m} r^{l+|\alpha|}\sum_{j=0}^{N}||(\Box_b)^j\phi||_{L^2_{0,q}(B_M(x,2r))} \\
&\quad\leq C.|B_M(x,r)|^{\frac{-1}{2}}\sum_{l=0,\,l\,even}^{L_m} r^{l+|\alpha|}\sum_{|\beta|=0}^{2N}||X^\beta\phi||_{L^2_{0,q}(B_M(x,2r))}.
\end{aligned} \tag{3.3.9}$$

This yields the desired estimate. $\square$

Theorem 3.3.2 and Theorem 3.3.4 say that for each $s > 0$, the heat operator $e^{-s\Box_b}$ is a NIS operator smoothing of order zero on $(0,q')$-forms, $q \leq q' \leq n-1-q$, and the associated estimates are uniform in $s > 0$.

Part II

f-APPROACHES

Chapter 4

Cauchy-Riemann Equations in $\mathbb{C}^2$

In this chapter, we will discuss methods of integral representations in several complex variables. These methods are generalizations of the Cauchy integrals in complex analysis from one variable to several variables. In particular, these methods are applied to estimate solutions of the Cauchy-Riemann equations in several complex variables, which were pioneered in 1969 by Grauert and Lieb, and, independently, by Henkin. Thus far, these methods have been a most "beautiful" argument in the case of domains being strongly pseudoconvex. For instance, formulaes for such integral representations of functions holomorphic in strongly pseudoconvex were developed by Henkin in [He70]. Then, there is a long history of proving L^p estimates for the $\bar{\partial}$-equation based on these such formulaes can be referred in [Ker71, Ovr71]. Also, in [Kr76], Krantz proved essentially optimal Lipschitz and L^p estimates on strongly pseudoconvex domains. In the case of weak pseudoconvexity, there are also some results being obtained on convex domains. The well-known papers by Range [Ra78], Diederich et al. [DFW86] show that the success of these methods has depend on the existence of fairly explicit holomorphic support functions at each boundary point of the domain under consideration. However, it is not true that any pseudoconvex domain has a holomorphic support function, even admitting a real analytic boundary. It was discovered by Kohn and Nirenberg [KoNi73]. In some positive cases , in [BdC84, CKM93, DFF99], these methods were applied to provide Hölder estimates and L^p regularity for solutions of $\bar{\partial}$-equations. The domains in these papers have a same property : they satisfy the condition of finite type in D'Angelo sense. And obviously, the analysis in the referenced works depends in an essential fashion on the type. In $\mathbb{C}^2$, Chang et. al. [CNS92] proved L^p estimates for the $\bar{\partial}$-Neumann operator on weakly pseudoconvex domains of finite type. See [CKM93, FLZ11] and the references within for a more complete history.

Naturally, we will ask that what happens if these above domains are not finite D'Angelo type, i.e. infinite type. Recently, the L^2 regularity for solutions of $\bar{\partial}$ Neumann equations has been established by Kohn, Khanh and Zampieri [Ko02, KZ10]. The superlogarithmic estimates hold on the such type in stead of the subelliptic ones. Nevertheless, the sup-norm and also L^p-norm (with $p \neq 2$) have been still unknown on such cases. In [FLZ11], Fornaess et al. provided the sup-norm estimates which are available when the domains are convex and of infinite type. In particular, let $0 < \alpha < 1$, the following domains in $\mathbb{C}^2$ were considered

$$\begin{aligned} \Omega &= \{\rho(z) = \operatorname{Re} z_2 + \exp\left(-1/|z_1|^\alpha\right) < 0\}, \\ &\text{or} \\ \Omega &= \{\rho(z) = \operatorname{Re} z_2 + \exp\left(-1/|\operatorname{Re} z_1|^\alpha\right) < 0\}. \end{aligned} \tag{4.0.1}$$

Their result asserts that on the such domains, there is a solution (in particular, Henkin integral solution) to the $\bar{\partial}$-equation $\bar{\partial} u = \phi$, for $\phi \in C^1_{(0,1)}(\overline{\Omega})$ and $\bar{\partial}\phi = 0$, so that $||u||_{L^\infty} \lesssim ||\phi||_{L^\infty}$. The main purpose in this chapter is to develop this result on general domains in $\mathbb{C}^2$ as well as to give the positive answer to the question that if the Hölder and L^p estimates hold on such domains while it was not accessible by the classical L^2-approach.

4.1 Preliminaries

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space, $\mathbb{C}^n = \{(z_1, z_2, ..., z_n)\}$, $n \geq 1$. Where $z_j = x_j + iy_j$, and $x_j, y_j \in \mathbb{R}$. We identify $\mathbb{C}^n \approx \mathbb{R}^{2n}$ by $(z_1, ..., z_n) \approx (x_1, y_1, ..., x_n, y_n)$. As usual,

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad j = 1, ..., n \\ \frac{\partial}{\partial \overline{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right). \end{aligned} \tag{4.1.1}$$

For a complex-valued functions u, the gradient ∇u is the $2n$-vector

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial x_n}. \frac{\partial u}{\partial y_n}\right).$$

For any multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^+$, we define

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} ... \partial z_n^{\alpha_n}},$$

and

$$\overline{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial \overline{z}_1^{\alpha_1} ... \partial \overline{z}_n^{\alpha_n}}.$$

$B(z, r)$ stands for the ball of center $z \in \mathbb{C}^n$ and radius r, $B(z, r) = \{z' \in \mathbb{C}^n : |z - z'| < r\}$.
The notation $V \Subset W$ means that the closure of V is a compact subset of W (V and W are contained in some topological space X). W^c is the complement in X of W and $A \setminus W = A \cap W^c$.

Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, not necessarily with a smooth boundary. For $k = 0, 1, ...,$, $0 < \epsilon < 1$, $1 \leq p < \infty$ and V is the Lebesgue measure on Ω. We recall some classical functional spaces.

$$C^k(\Omega) = \left\{ u \text{ defined on } \Omega : ||u||_{C^k(\Omega)} := \sup\{|D^\alpha \overline{D}^\beta u(z)|, z \in \Omega, |\alpha| + |\beta| \le k\} < \infty \right\};$$

$$C^k_{(0,1)}(\Omega) = \left\{ u = \sum_{j=1}^n u_j d\overline{z}_j : ||u||_{C^k_{(0,1)}(\Omega)} = \max_{1\le j\le n} ||u_j||_{C^k(\Omega)} < \infty \right\};$$

$$\Lambda^\epsilon(\Omega) = \left\{ u \in C^0(\Omega) : ||u||_{\Lambda^\epsilon(\Omega)} = ||u||_{C^0(\Omega)} + \sup\left\{ \frac{|u(z) - u(w)|}{|z-w|^\epsilon} : z, w \in \Omega, z \ne w \right\} < \infty \right\};$$

$$\Lambda^\epsilon_{(0,1)}(\Omega) = \{u = \sum_{j=1}^n u_j d\overline{z}_j : ||u||_{\Lambda^\epsilon_{(0,1)}(\Omega)} = \max_{1\le j\le n} ||u_j||_{\Lambda^\epsilon(\Omega)} < \infty\}; \tag{4.1.2}$$

$$L^p_{(0,1)}(\Omega) = \left\{ u = \sum_{j=1}^n u_j d\overline{z}_j : ||u||_{L^p_{0,1}(\Omega)} = \sum_{j=1}^n \int_\Omega |u_j(z)|^p dV < \infty \right\};$$

$$L^\infty_{(0,1)(\Omega)} = \left\{ u = \sum_{j=1}^n u_j d\overline{z}_j : ||u||_{L^\infty_{(0,1)}(\Omega)} = \max_{1\le j\le n} \text{ess}\sup_\Omega u_j < \infty \right\}.$$

Since the components of any $\phi \in L^p_{(0,1)}(\Omega)$, $1 \le p \le \infty$, are locally integrable, it makes sense to define $\bar\partial\phi$ in the distribution sense. That mean, $\bar\partial\phi = \sum_{j=1}^n \left(\bar\partial\phi_j\right) \wedge d\overline{z}_j$, where $\bar\partial\phi_j = \sum_{k=1}^n \left(\frac{\partial\phi_j}{\partial\overline{z}_k}\right) d\overline{z}_k$, and $d\overline{z}_k \wedge d\overline{z}_j = -d\overline{z}_j \wedge d\overline{z}_k$. Hence, $\bar\partial\phi = 0$ ($\bar\partial$-closed) means

$$\frac{\partial\phi_j}{\partial\overline{z}_k} = \frac{\partial\phi_k}{\partial\overline{z}_j}, \quad j,k = 1,...,n,$$

where the derivatives being in the distribution sense. This is a necessary condition in order that there exists a function u such that $\bar\partial u = \phi$, more clearly, $\frac{\partial u}{\partial\overline{z}_j} = \phi_j$, $j = 1,...,n$.
Now, the $\bar\partial$ problem on $(0,1)$-forms is to study the existence of solutions u of Cauhy-Riemann equations

$$\bar\partial u = \phi \quad \text{in } \Omega,$$

where ϕ is a $(0,1)$ form satisfying $\bar\partial\phi = 0$. We recall here the definition concerning the differentiability of the boundary of a domain.

Definition 4.1.1. A domain Ω in $\mathbb{R}^m$, $m \ge 2$, is said to have C^k $(1 \le k \le \infty)$ boundary at the boundary point p if there exists a real-valued C^k function ρ defined in some open neighborhood U of p such that

1. $\Omega \cap U = \{x \in U : \rho(x) < 0\}$;

2. $b\Omega \cap U = \{x \in U : \rho(x) = 0\}$;

3. $\nabla \rho(x) \neq 0$ on $b\Omega \cap U$.

The function ρ is called a C^k local defining function for Ω near p. If U is an open neighborhood of $\overline{\Omega}$, then ρ is called a global defining function for Ω, or simply a defining function for Ω. A remark that if ρ' is another C^k defining function of Ω, then $\rho(x) = h(x)\rho'(x)$, and $d\rho(x) = h(x)d\rho'(x)$, for some positive C^{k-1} function h.

Definition 4.1.2. Let Ω be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let ρ be a C^2 defining function for Ω. Then, Ω is called pseudoconvex, or Levi pseudoconvex, at $p \in \Omega$ if the Levi form

$$\mathcal{L}_p(\rho, t) := \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k \geq 0,$$

for all $t \in T^{1,0}(b\Omega)$. The domain Ω is said to be strongly pseudoconvex at p if the above Levi form is strictly positive for all such $t \neq 0$. Ω is called a (Levi) pseudoconvex domain if Ω is (Levi) pseudoconvex at every point of Ω. We also have the similar definition for strong pseudoconvexity to Ω.

As the mention before, we want to solve the $\bar{\partial}$-equations in Hölder and L^p spaces. The well-known facts by Kerman [Ker71], and then improved by Krantz [Kr76] are followings

Theorem 4.1.3. *Let $\Omega \Subset \mathbb{C}^n$, $n \geq 2$ be strongly pseudoconvex with C^5 boundary $b\Omega$. For any $\bar{\partial}$-closed $(0,1)$ form $\phi \in L^p_{(0,1)}(\Omega)$, $1 \leq p \leq \infty$, there exists a function u on Ω such that $\bar{\partial} u = \phi$ (in the distribution sense), and u satisfies the following estimates:*

1. $||u||_{L^q(\Omega)} \leq A_p ||\phi||_{L^p_{(0,1)}(\Omega)}$, *where* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{2(n+1)}$, *if* $1 < p < 2(n+1)$.

2. *For any small* $\epsilon > 0$, $||u||_{L^{\frac{2n+2}{2n+1} - \epsilon}(\Omega)} \leq A_\epsilon ||\phi||_{L^1_{(0,1)}(\Omega)}$, *if* $p = 1$.

3. $||u||_{\Lambda^\epsilon(\Omega)} \leq A_p ||\phi||_{L^p_{(0,1)}(\Omega)}$, *where* $\epsilon = \dfrac{1}{2} - \dfrac{n+1}{p}$, *if* $2n+2 < p \leq \infty$.

In [Kr76], the author also provided an example which was due to Stein to show that the above estimates can not be improved.
Going on domains of weakly pseudoconvex type, we can also seek an analogue version for these estimates. For instance, Cauchy-Riemann equations on ellipsoids in $\mathbb{C}^n$ were considered. In particular, on complex ellipsoids ($E_{\mathbb{C}}$), and also real ellipsoids ($E_{\mathbb{R}}$)

$$E_{\mathbb{C}} = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1|^{m_1} + ... + |z_n|^{m_n} < 1\};$$
$$E_{\mathbb{R}} = \{(z_1, ..., z_n) \in \mathbb{C}^n : |\operatorname{Re} z_1|^{m_1} + |\operatorname{Im} z_1|^{m_1'} + ... + |\operatorname{Re} z_n|^{m_n} + |\operatorname{Im} z_n|^{m_n'} < 1\}.$$

Both of these models are of finite type in D'Angelo sense. More precise, Diederich et. al. [DFW86] and Chen et. al. [CKM93] used the support functions constructed on such domains to obtain the optimal L^p and Hölder estimates for solutions of $\bar{\partial}$-equations. These papers illustrated a clear effect of the type. In particular, on the complex ellipsoids defined as $E_{\mathbb{C}}$, the holomorphic support functions is defined by

$$\Phi(\zeta, z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j), \quad \text{for } \zeta \in b\Omega, z \in \overline{\Omega},$$

and if $\hat{\Phi} := \Phi(\zeta, z) - \rho(\zeta)$, for all $\zeta, z \in \overline{\Omega}$, then we have

$$|\hat{\Phi}(\zeta, z)| \gtrsim \left\{ |\rho(\zeta)| + |\rho(z)| + |\operatorname{Im}\Phi(\zeta, z)| + \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \bar{\zeta}_j}(\zeta)|\zeta_j - z_j|^2 + \sum_{j=1}^{n} |\zeta_j - z_j|^{m_j} \right\}. \tag{4.1.3}$$

This inequality plays a critical role in the boundary behaviour of the solution constructed via Henkin integral. By this, the kernel can be estimated dependently on the type of the domain under integral sign that we will see later. It turns out that when Ω is a sphere, strongly pseudoconvex domain, the inequality goes back the classical one $|\hat{\Phi}(\zeta, z)| \gtrsim |\rho(\zeta)| + |\rho(z)| + |\operatorname{Im}\Phi(\zeta, z)| + |\zeta - z|^2$. In these cases, the null-sets of the corresponding holomorphic support functions are strong barrier for $b\Omega$, that is for an small ball B,

$$\{z : \Phi(\zeta, z) = 0\} \cap B \cap \overline{\Omega} = \{\zeta\}, \quad \text{for any } \zeta \in B \cap b\Omega.$$

However, it is natural to ask whether the existence of Φ as well as the inequality (4.1.3) are shared by general pseudoconvex domains. For weakly pseudoconvex domains (even with real analytic boundary), there are singularities, related to the existence of barrier functions Φ for $b\Omega$. There is an immediate negative answer given by Kohn and Nirenberg :

Proposition 4.1.4. *(Kohn, Nirenberg 1973).*
Let Ω be the following pseudoconvex domain in $\mathbb{C}^2$:

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : Re(z_2) + |z_1.z_2|^2 + |z_1|^8 + \frac{15}{7}|z_1|^2.Re(z_1^6) < 0\}.$$

Let h be a function holomorphic in a neighborhood of the point $(0, 0) \in b\Omega$, and equal to zero at this point. Then, the set $\{(z_1, z_2) : h(z_1, z_2) = 0\}$ necessarily has both some points in the interior as well as in the exteriorof the domain Ω.

We know that both real analytic pseudoconvex domains in $\mathbb{C}^2$ and strictly pseudoconvex domains are of finite type. The precise notion of type is still a topic of research; essential contributions are due to Catlin and D'Angelo. For a finite type domain, holomorphic support functions, if they exist, may still admit non-trivial contact sets A which are, it is true, not too large. Nevertheless, their existence would be an obstruction to good estimates for integral kernels constructed from these support functions. But we might hope to construct better support functions. This problem gives rise to many interesting open questions. We shall now discuss a case where a solution has recently been found.

Now, in order to begin our answer for the above problems, following the setup by Khanh in [Kha13], we introduce some generalized versions for these above models in $\mathbb{C}^2$, which include many convex domains of D'Angelo infinite type. We investigate domains of the following form: $\Omega \subset \mathbb{C}^2$ is a smooth, bounded domain with the origin 0 in the boundary $b\Omega$. Moreover, there exists $\delta > 0$ so that $b\Omega \setminus B(0, \delta/2)$ is strictly convex and there exists a defining function ρ so that

$$\Omega \cap B(0,\delta) = \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = F(|z_1|^2) + r(z) < 0\} \tag{4.1.4}$$

or

$$\Omega \cap B(0,\delta) = \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = F(x_1^2) + r(z) < 0\} \tag{4.1.5}$$

where $z_j = x_j + iy_j$, for $x_j, y_j \in \mathbb{R}$, $j = 1, 2$, and $i = \sqrt{-1}$. We also assume that the functions $F : \mathbb{R} \to \mathbb{R}$ and $r : \mathcal{C}^2 \to \mathbb{R}$ satisfy:

1. $F(0) = 0$;
2. $F'(t), F''(t), F'''(t)$ and $\left(\dfrac{F(t)}{t}\right)'$ are non-negative on $(0, \delta)$;
3. $r(0) = 0$ and $\dfrac{\partial r}{\partial z_2} \neq 0$;
4. r is convex and strictly convex away from 0.

This class of domains includes two well-known examples. If $F(t) = t^m$, with $m \geq 1$, then Ω is of finite type $2m$. On the other hand, if $F(t) = \exp(-1/t^\alpha)$, then Ω is of infinite type, and this is our main case of interest. On these exponential type domains, recently [FLZ11], Fornaess et. al. provided the sup-norm estimates for the Cauchy-Riemann equations. The authors again obtained the solutions via Henkin's integral formula, with support functions discovered in [DFF99] on convex domains.
Associated to these classes of such domains, we also define the f-Hölder spaces.

Definition 4.1.5. Let f be an increasing function such that $\lim_{t \to +\infty} f(t) = +\infty$. For $\Omega \subset \mathcal{C}^n$, define the f-Hölder space on $\bar{\Omega}$ by

$$\Lambda^f(\bar{\Omega}) = \{u : \|u\|_\infty + \sup_{z,w \in \overline{\Omega}} f(|z-w|^{-1}) \cdot |u(z) - u(w)| < \infty\}$$

and set

$$\|u\|_f = \|u\|_\infty + \sup_{z,w \in \overline{\Omega}} f(|z-w|^{-1}) \cdot |u(z) - u(w)|.$$

Note that the f-Hölder space includes the standard Hölder space $\Lambda_\alpha(\bar{\Omega})$ by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha < 1$.
Here, we recall the construction of the Henkin kernel and Henkin solution to $\bar{\partial}$-equations. For complete details, see [He70, Ra86], or for a more modern treatment, see [?], with the support function introduced in [DFF99].

Definition 4.1.6. A $\mathcal{C}^2$-valued C^1 function $G(\zeta, z) = (g_1(\zeta, z), g_2(\zeta, z))$ is called a Leray map for Ω if $g_1(\zeta, z)(\zeta_1 - z_1) + g_2(\zeta, z)(\zeta_2 - z_2) \neq 0$ for every $(\zeta, z) \in b\Omega \times \Omega$. A support function (or Ramírez-Henkin function) $\Phi(\zeta, z)$ for Ω is a smooth function defined near $b\Omega \times \bar{\Omega}$ so that Φ admits a decomposition

$$\Phi(\zeta, z) = 2\sum_{j=1}^{2} \Phi_j(\zeta, z)(\zeta_j - z_j)$$

where $\Phi_j(\zeta, z)$ are smooth near $b\Omega \times \bar{\Omega}$, holomorphic in z, and vanishes only on the diagonal $\{\zeta = z\}$.

For a convex domain, it is well known that $G(\zeta, z) = \dfrac{\partial \rho}{\partial \zeta} = \left(\dfrac{\partial \rho}{\partial \zeta_1}, \dfrac{\partial \rho}{\partial \zeta_2}\right)$ is a Leray map [?, Lemma 11.2.6], and Φ defined by Leray map

$$\Phi_j(\zeta, z) = \frac{\partial \rho(\zeta)}{\partial \zeta_j},\ j = 1, 2,$$

is a support function for Ω.

Taylor's Theorem and the convexity of F implies a lower bound on $b\Omega$, which generalizes the inequality (4.1.3) in the case of finite type

Lemma 4.1.7. *Let $\Omega \subset \mathcal{C}^2$ be as in (4.1.4) or (4.1.5) with Φ as above. Then there exist $\epsilon, c > 0$ so that*

$$\operatorname{Re}\Phi(\zeta, z) \geq -\rho(z) + \begin{cases} c|z-\zeta|^2 & \zeta \in b\Omega \setminus B(0,\delta) \\ P(z_1) - P(\zeta_1) - 2\operatorname{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1 - \zeta_1)\right\} & \zeta \in b\Omega \cap B(0,\delta) \end{cases}. \quad (4.1.6)$$

for all $z \in \bar{\Omega}$ with $|z - \zeta| \leq \epsilon$,where $P(z_1) = F(|z_1|^2)$ or $P(z_1) = F(x_1^2)$.

Proof. Let h be a $\mathbb{R}$-valued smooth function in $\mathbb{C}^2 = \mathbb{R}^4$ and $x, y \in \mathbb{R}^4$. If $\alpha(t) = tx + (1-t)y$, and $\varphi(t) = h(\alpha(t))$, then it follows from Taylor's Theorem applied to $\varphi(t)$ that there exists $\tilde{y} \in \alpha([0,1])$ so that

$$h(x) = h(y) + \sum_{j=1}^{4} \frac{\partial h(y)}{\partial y_j}(x-y) + \frac{1}{2}\sum_{j,k=1}^{4} \frac{\partial^2 h(\tilde{y})}{\partial y_j \partial y_k}(x_j - y_j)(x_k - y_k).$$

Set $z = (z_1, z_2) = (x_1 + ix_2, x_3 + ix_4)$ and $\zeta = (\zeta_1, \zeta_2) = (y_1 + iy_2, y_3 + iy_4)$. Translating the first order component of the Taylor series expansion to complex coordinates, we compute

$$\begin{aligned} 2\,\mathrm{Re}\left\{\frac{\partial h(\zeta)}{\partial \zeta_j}(z_j - \zeta_j)\right\} &= \mathrm{Re}\left\{\left(\frac{\partial h(\zeta)}{\partial y_{2j-1}} - i\frac{\partial h(\zeta)}{\partial y_{2j}}\right)\left((x_{2j-1} - y_{2j-1}) + i(x_{2j} - y_{2j})\right)\right\} \\ &= \frac{\partial h(\zeta)}{\partial y_{2j-1}}(x_{2j-1} - y_{2j-1}) + \frac{\partial h(\zeta)}{\partial y_{2j}}(x_{2j} - y_{2j}), \end{aligned}$$

$j = 1, 2$. Consequently, if $[\zeta, z]$ is the line segment connecting ζ and z, then

$$h(z) \geq h(\zeta) + 2\sum_{j=1}^{2} \mathrm{Re}\left\{\frac{\partial h(\zeta)}{\partial \zeta_j}(z_j - \zeta_j)\right\} + \min_{\tilde{y}\in[\zeta,z]} \frac{1}{2}\sum_{j,k=1}^{4} \frac{\partial^2 h(\tilde{y})}{\partial y_j \partial y_k}(x_j - y_j)(x_k - y_k). \tag{4.1.7}$$

Applying (4.2.1) to the defining function ρ, with $\rho(\zeta) = 0, \zeta \in b\Omega$, yield

$$\rho(z) \geq -\,\mathrm{Re}\,\Phi(\zeta, z) + \min_{\tilde{y}\in[\zeta,z]} \frac{1}{2}\sum_{j,k=1}^{4} \frac{\partial^2 \rho(\tilde{y})}{\partial y_j \partial y_k}(x_j - y_j)(x_k - y_k).$$

Since ρ is strictly convex on $b\Omega\backslash B(0,\delta)$, there exists $c > 0$ so that $\left|\sum_{j,k=1}^{4} \frac{\partial^2 \rho(\tilde{y}}{\partial y_j \partial y_k}(x_j - y_j)(x_k - y_k)\right| \geq c|x - y|^2$ if $y \in b\Omega \setminus B(0, \delta)$ and $\epsilon > 0$ is sufficiently small. The first case of (4.4.1) now follows.

For the remaining case, we use (4.2.1) and the convexity of r to observe that

$$\begin{aligned} &-\rho(z) + P(z_1) - P(\zeta_1) - 2\,\mathrm{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1 - \zeta_1)\right\} \\ &\quad = r(\zeta) - r(z) + 2\,\mathrm{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(\zeta_1 - z_1)\right\} \\ &\quad \leq 2\sum_{j=1}^{2} \mathrm{Re}\left\{\frac{\partial r(\zeta)}{\partial \zeta_j}(\zeta_j - z_j)\right\} + 2\,\mathrm{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(\zeta_1 - z_1)\right\} \\ &\quad = \mathrm{Re}\,\Phi(\zeta, z). \end{aligned}$$

This completes the proof. □

We take the ϵ constructed in Lemma 4.4.2 to be a global constant in the paper, though we reserve the right to decrease it.
The lemma says that when ζ is far away from the origin, one's problem goes back $\bar{\partial}$-equation on strongly pseudoconvex, and this is trivial. The problematic point is that ζ closed to z.
Choose $\chi \in C^\infty(\mathbb{C}^2 \times \mathbb{C}^2)$ such that $0 \leq \chi \leq 1$, $\chi(z, \zeta) = 1$ for $|z - \zeta| \leq \frac{1}{2}\epsilon$ and $\chi(z, \zeta) = 0$ for $|z - \zeta| \geq \epsilon$. And for $j = 1, 2$, we define

$$\Phi_j^{\#}(z, \zeta) = \chi\frac{\partial \rho}{\partial \zeta_j}(\zeta) + (1 - \chi)(\overline{\zeta}_j - \overline{z}_j),$$

and

$$\Phi^{\#}(z,\zeta) = \Phi_1^{\#}(z,\zeta)(\zeta_1 - z_1) + \Phi_2^{\#}(z,\zeta)(\zeta_2 - z_2).$$

The new support function also has the following properties for any $\zeta \in b\Omega$.

1.
$$\operatorname{Re}\Phi^{\#}(z,\zeta) \geq -\rho(z) + \begin{cases} c|z-\zeta|^2 & \zeta \in b\Omega \setminus B(0,\delta) \\ P(z_1) - P(\zeta_1) - 2\operatorname{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1 - \zeta_1)\right\} & \zeta \in b\Omega \cap B(0,\delta) \end{cases}. \tag{4.1.8}$$
for all $|z-\zeta| \leq \dfrac{1}{2}\epsilon$ and $z \in \overline{\Omega}$.

2. $\Phi^{\#}(z,\zeta)$ and $\Phi_j^{\#}$, $j = 1,2$, are holomorphic on $\{z : |z-\zeta| \leq \dfrac{1}{2}\epsilon\}$.

We are now already to represent the integral solution of the $\bar\partial$. Let $\phi = \sum_{j=1}^{2} \phi_j d\bar z_j$ be a bounded, C^1, $\bar\partial$-closed $(0,1)$-form on $\overline{\Omega}$. The solution u of the $\bar\partial$-equation, $\bar\partial u = \phi$, provided by the Henkin kernel is given by

$$u = T\phi(z) = H\phi(z) + K\phi(z). \tag{4.1.9}$$

where

$$\begin{aligned} H\phi(z) =& \frac{1}{4\pi^2}\int_{\zeta \in b\Omega} \frac{\Phi_1^{\#}(\bar\zeta_2 - \bar z_2) - \Phi_2^{\#}(\bar\zeta_1 - \bar z_1)}{\Phi^{\#}(\zeta,z)|\zeta - z|^2}\phi(\zeta)\wedge\omega(\zeta); \\ K\phi(z) =& \frac{1}{4\pi^2}\int_{\Omega} \frac{\phi_1(\zeta)(\bar\zeta_1 - \bar z_1) - \phi_2(\zeta)(\bar\zeta_2 - \bar z_2)}{|\zeta - z|^4}\omega(\bar\zeta)\wedge\omega(\zeta) \end{aligned} \tag{4.1.10}$$

where $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$. This function is called the solution of $\bar\partial$-equation via Ramírez-Henkin kernel, or simply, Ramírez- Henkin solution. So we have solved the Cauchy-Riemann equations on Ω by exhibiting an explicit solution in terms of a linear integral formula.
Through this chapter, we only consider solution in this way.

4.2 Preparatory Lemmas

In this section, we will provide some tools in the proof of our main results later [Kha13, HKR13]

Lemma 4.2.1. *Let F be a convex function on $[0,\delta]$. Then we have*

$$F(p) - F(q) - F'(q)(p-q) \geq 0 \tag{4.2.1}$$

for any $p,q \in [0,\delta]$. Furthermore, with the extra assumptions $F'(0) = 0$ and $F''(t)$ increasing, we have

$$F(p) - F(q) - F'(q)(p-q) \geq F(p-q), \tag{4.2.2}$$

for any $0 \leq q \leq p \leq \delta$.

Proof. The proof of (4.2.1) is simple and is omitted here. For (4.2.2), let $s := p - q \geq 0$ and $g(s) := F(s+q) - F(q) - sF'(q) - F(s)$. Hence, $g'(s) = F'(s+q) - F'(q) - F'(s)$ and $g''(s) = F''(s+q) - F''(s)$. Using the assumption $F''(t)$ increasing, we have $g''(s) \geq 0$, thus $g'(s)$ is increasing. This implies $g'(s) \geq g'(0) = 0$ (since $F'(0) = 0$). This means $g(s)$ is also increasing, so we obtain $g(s) \geq g(0) = 0$ (since $F(0) = 0$). This completes the proof of (4.2.2). $\square$

Lemma 4.2.2. *For $\delta > 0$ small enough, let F be an invertible function on $[0, \delta]$ such that $\dfrac{F(t)}{t}$ is increasing on $[0, \delta]$. Then*

1. $\displaystyle\int_0^\delta \frac{dr}{\rho + F(r^2)} \lesssim \frac{\sqrt{F^*(\rho)}}{\rho}$,

2. $\displaystyle\int_0^\delta \frac{|\ln r|}{\rho + F(r^2)} dr \lesssim \frac{\sqrt{F^*(\rho)}|\ln\sqrt{F^*(\rho)}|}{\rho}$,

for any sufficiently small $\rho > 0$, where F^* is the inverse function of F.

Proof. 1. In order to prove the first assertion, we divide the integration into two terms

$$\int_0^\delta \frac{dr}{\rho + F(r^2)} = \int_0^{\sqrt{F^*(\rho)}} \frac{dr}{\rho + F(r^2)} + \int_{\sqrt{F^*(\rho)}}^\delta \frac{dr}{\rho + F(r^2)}.$$

. For the first term, it is easy to see that

$$\int_0^{\sqrt{F^*(\rho)}} \frac{dr}{\rho + F(r^2)} \lesssim \frac{\sqrt{F^*(\rho)}}{\rho}.$$

Since $\dfrac{F(t)}{t}$ is increasing, then we have

$$\frac{F(r^2)}{r^2} \geq \frac{F(F^*(\rho))}{F^*(\rho)} = \frac{\rho}{F^*(\rho)}, \quad \text{i.e.} \quad \frac{F(r^2)}{\rho} \geq \frac{r^2}{F^*(\rho)},$$

for any $r \geq \sqrt{F^*(\rho)}$. Applying this observation to the second integration, we obtain

$$\int_{\sqrt{F^*(\rho)}}^\delta \frac{dr}{\rho + F(r^2)} \leq \frac{1}{\rho}\int_{\sqrt{F^*(\rho)}}^\delta \frac{dr}{1 + \frac{r^2}{F^*(\rho)}} \leq \frac{\sqrt{F^*(\rho)}}{\rho}\int_1^\infty \frac{dy}{1+y^2} = \frac{\pi}{4}\frac{\sqrt{F^*(\rho)}}{\rho}.$$

2. The second assertion is proved by the same way, in particular, we divide the integration into two terms

$$\int_0^\delta \frac{|\ln r|}{\rho + F(r^2)} dr = \int_0^{\sqrt{F^*(\rho)}} \frac{|\ln r|}{\rho + F(r^2)} dr + \int_{\sqrt{F^*(\rho)}}^\delta \frac{|\ln r|}{\rho + F(r^2)} dr.$$

. For the first term, it is easy to see that

$$\int_0^{\sqrt{F^*(\rho)}} \frac{|\ln r|}{\rho + F(r^2)} dr \lesssim \frac{1}{\rho} \int_0^{\sqrt{F^*(\rho)}} |\ln r| dr \lesssim \frac{\sqrt{F^*(\rho)} |\ln \sqrt{F^*(\rho)}|}{\rho}.$$

For the second term,

$$\int_{\sqrt{F^*(\rho)}}^{\delta} \frac{|\ln r|}{\rho + F(r^2)} dr \lesssim |\ln \sqrt{F^*(\rho)}| \int_{\sqrt{F^*(\rho)}}^{\delta} \frac{dr}{\rho + F(r^2)} dr \lesssim \frac{\sqrt{F^*(\rho)} |\ln \sqrt{F^*(\rho)}|}{\rho},$$

where the last inequality follows from the same tool above. □

The following will generalize Hardy-Littlewood Lemma when $G(t) = t^\alpha$. In that case, we can find a proof in [ChSh01].

Lemma 4.2.3. (General Hardy-Littlewood Lemma). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and let $\delta_{b\Omega}(x)$ denote the distance function from x to the boundary of Ω. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\frac{G(t)}{t}$ is increasing and $\int_0^d \frac{G(t)}{t} dt < \infty$ for $d > 0$ small enough. If $u \in C^1(\Omega)$ such that*

$$|\nabla u(x)| \lesssim \frac{G(\delta_{b\Omega}(x))}{\delta_{b\Omega}(x)} \quad \text{for every } x \in \Omega, \tag{4.2.3}$$

then $|u(x) - u(y)| \lesssim f(|x-y|^{-1})^{-1}$, for $x, y \in \Omega$, $x \neq y$, and where $f(d^{-1}) := \left(\int_0^d \frac{G(t)}{t} \right)^{-1}$

Proof. Since $u \in C^1$ in the interior of Ω, we only need to prove the assertion when z and w are near the boundary $b\Omega$. Using a partition of unity, we can assume that u is supported in $U\overline{\Omega}$, where U is a neighborhood of a boundary point $x_0 \in b\Omega$. After a linear change of coordinates, we may assume $x_0 = 0$, and for some $\delta > 0$,

$$U \cap \Omega = \left\{ x = (x', x_N) : x_N > \varphi(x'), |x'| < \delta, |x_N| \leq \delta \right\},$$

where $\varphi(0) = 0$ and φ is some Lipschitz function with Lipschitz constant M. Let $x = (x', x_N)$, $y = (y', y_N) \in \Omega$, $\widetilde{x}' = \theta x' + (1-\theta) y'$, $\widetilde{x}_N = \theta x_N + (1-\theta) y_N$, and $d = |x - y|$. For $a \geq 0$, we define the line segment L_a by $\theta(x', x_N + a) + (1-\theta)(y', y_N + a)$, $0 \leq \theta \leq 1$, and . Applying the Lipschitz property of φ, we obtain

$$\begin{aligned}
\widetilde{x}_N + Md &= \theta(x_N + Md) + (1-\theta)(y_N + Md) \\
&\geq Md + \theta\varphi(x') + (1-\theta)\varphi(y') \\
&\geq Md + \theta(\varphi(x') - \varphi(\widetilde{x}')) + (1-\theta)(\varphi(y') - \varphi(\widetilde{x}')) + \varphi(\widetilde{x}') \\
&\geq \varphi(\widetilde{x}').
\end{aligned} \tag{4.2.4}$$

This implies that the line segment L_a lies in Ω, for any $a \geq Md$. Since $u \in C^1(\Omega)$, using Mean Value Theorem, there exists some $(\widetilde{x}', \widetilde{x}_N + 2Md) \in L_{2Md}$ such that

$$|u(x', x_N + 2Md) - u(y', y_N + 2Md)| \leq |\nabla u(\widetilde{x}', \widetilde{x}_N + 2Md)|.d.$$

The distance function $\delta_{b\Omega}(x', x_N)$ is comparable to $x_N - \varphi(x')$, i.e., there are positive constants c, C such that

$$c(x_N - \varphi(x')) \leq \delta_{b\Omega}(x', x_N) \leq C(x_N - \varphi(x')). \tag{4.2.5}$$

Using the assumptions on G and combining with (4.2.3) and (4.2.5), it follows that

$$\begin{aligned} |u(x', x_N + 2Md) - u(y', y_N + 2Md)| &\lesssim \frac{G(\delta_{b\Omega}(\widetilde{x}', \widetilde{x}_N + 2Md))}{\delta_{b\Omega}(\widetilde{x}', \widetilde{x}_N + 2Md)} d \\ &\lesssim \frac{G(c.(\widetilde{x}_N + 2Md - \varphi(\widetilde{x}')))}{c.(\widetilde{x}_N + 2Md - \varphi(\widetilde{x}'))} d \\ &\lesssim \frac{G(cMd)}{cMd} d \lesssim G(d). \end{aligned} \tag{4.2.6}$$

Here the last inequality follows by considering two cases: if $cM < 1$, we use that $G(t)$ is increasing; otherwise, we use that $\dfrac{G(t)}{t}$ is decrasing. We also have

$$\begin{aligned} |u(x) - u(x', x_N + 2Md)| &= \left| \int_0^d \frac{\partial u(u', x_N + 2Mt)}{\partial t} dt \right| \\ &\lesssim \int_0^d \frac{G(\delta_{b\Omega}(x', x_N + 2Mt))}{\delta_{b\Omega}(x', x_N + 2Mt)} dt \lesssim \int_0^d \frac{G(t)}{t} dt. \end{aligned} \tag{4.2.7}$$

Therefore, for any $x, y \in \Omega$,

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x', x_N + 2Md)| + |u(y) - u(y', y_N + 2Md)| \\ &\quad + |u(x', x_N + 2Md) - u(y', y_N + 2Md)| \\ &\lesssim G(d) + \int_0^d \frac{G(t)}{t} \lesssim \int_0^d \frac{G(t)}{t}. \end{aligned} \tag{4.2.8}$$

Here, the last inequality holds since

$$G(d) = \int_0^d \frac{G(d)}{d} \leq \int_0^d \frac{G(t)}{t}.$$

Hence, this completes the proof. □

4.3 Sup-norm and Hölder estimates for $\bar{\partial}$-Solutions

The first goal of the current section is to prove sup-norm estimates on domains satisfying (4.1.4) or (4.1.5) which both generalize the class of domains of finite type as well as the exponential type considered in [FLZ11].

Theorem 4.3.1 (Theorem 1.2, [Kha13]). *If*

1. Ω is defined by (4.1.4) and there exists $\delta > 0$ so that $\displaystyle\int_0^\delta |\ln F(t^2)|\,dt < \infty$, *or*

2. Ω is defined by (4.1.5) and there exists $\delta > 0$ so that $\displaystyle\int_0^\delta |\ln(t)\ln F(t^2)|\,dt < \infty$,

then for any bounded, $\bar\partial$-closed $(0,1)$-form ϕ on $\overline{\Omega}$, there exists a function u defined on Ω satisfies $\bar\partial u = \phi$ and

$$||u||_{L^\infty(\Omega)} \leq C\|\phi\|_{L^\infty(\Omega)}.$$

where $C > 0$ is independent of ϕ.

Proof. It is well known that (e.g., [Fol99], Theorem 6.18)

$$||K\phi||_{L^\infty(\Omega)} \lesssim ||\phi||_{L^\infty_{(0,1)}(\Omega)}.$$

Moreover, we have

$$\begin{aligned} H\phi(z) &= \frac{1}{4\pi^2}\int_{\zeta\in b\Omega} \frac{\Phi_1^\#(\bar\zeta_2 - \bar z_2) - \Phi_2^\#(\bar\zeta_1 - \bar z_1)}{\Phi^\#(\zeta,z)|\zeta - z|^2}\phi(\zeta)\wedge\omega(\zeta); \\ &= \frac{1}{4\pi^2}\int_{\substack{\zeta\in b\Omega\\ |z-\zeta|\leq\epsilon}} \frac{\Phi_1^\#(\bar\zeta_2 - \bar z_2) - \Phi_2^\#(\bar\zeta_1 - \bar z_1)}{\Phi^\#(\zeta,z)|\zeta - z|^2}\phi(\zeta)\wedge\omega(\zeta); \\ &+ \frac{1}{4\pi^2}\int_{\substack{\zeta\in b\Omega\\ |z-\zeta|>\epsilon}} \frac{\Phi_1^\#(\bar\zeta_2 - \bar z_2) - \Phi_2^\#(\bar\zeta_1 - \bar z_1)}{\Phi^\#(\zeta,z)|\zeta - z|^2}\phi(\zeta)\wedge\omega(\zeta); \\ &= \frac{1}{4\pi^2}\int_{\substack{\zeta\in b\Omega\\ |z-\zeta|\leq\epsilon}} \frac{\Phi_1^\#(\bar\zeta_2 - \bar z_2) - \Phi_2^\#(\bar\zeta_1 - \bar z_1)}{\Phi^\#(\zeta,z)|\zeta - z|^2}\phi(\zeta)\wedge\omega(\zeta), \end{aligned} \tag{4.3.1}$$

since $\Phi_1^\#(\bar\zeta_2 - \bar z_2) - \Phi_2^\#(\bar\zeta_1 - \bar z_1) = 0$ if $|z-\zeta| > \epsilon$. Hence, let dS be a surface area measure on $b\Omega$, we have

$$|H\phi(z)| \lesssim ||\phi||_{L^\infty_{(0,1)}} \int_{\substack{\zeta\in b\Omega\\ |z-\zeta|\leq\epsilon}} \frac{dS(\zeta)}{|\Phi(z,\zeta)|.|\zeta - z|}.$$

Now, setting $t = \operatorname{Im}\Phi(z,\zeta)$. It is easy to check that $\dfrac{\partial t}{\partial\zeta_2} \neq 0$. Hence, we can change coordinates and obtain

$$\begin{aligned} \int_{\substack{\zeta\in b\Omega\\ |z-\zeta|\leq\epsilon}} \frac{dS(\zeta)}{|\Phi(z,\zeta)|.|\zeta - z|} &\lesssim \int_{\substack{|t|\leq\delta, |\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon}} \frac{dtd(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|t| + |\operatorname{Re}\Phi|)|\zeta_1 - z_1|} \\ &\lesssim \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon}} \frac{|\ln|\operatorname{Re}\Phi||d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\zeta_1 - z_1|}. \end{aligned} \tag{4.3.2}$$

Now, let Ω defined by (4.1.4), and for some $\delta > 0$, $\displaystyle\int_0^\delta |\ln F(t^2)|\,dt < \infty$, then we apply the identity $2\operatorname{Re} a\bar{b} = |a+b|^2 - |a|^2 - |b|^2$ in (4.4.1) to obtain

$$\begin{aligned}\operatorname{Re}\Phi(z,\zeta) &\geq -\rho(z) + F(|z_1|^2) - F(|\zeta_1|^2) + 2F'(|\zeta_1|^2)\operatorname{Re}\left(\bar{\zeta}_1(z_1-\zeta_1)\right)\\ &\geq -\rho(z) + \left(F'(|\zeta_1|^2)|z_1-\zeta_1|^2 + F(|z_1|^2) - F(|\zeta_1|^2) - F'(|\zeta_1|^2)(|z_1|^2 - |\zeta_1|)\right)\\ &\geq -\rho(z) + \left(F'(|\zeta_1|^2)|z_1-\zeta_1|^2\right).\end{aligned} \tag{4.3.3}$$

Therefore, since $|\ln(t)|$ is decreasing when $0 \leq t$ is small, (4.3.2) becomes

$$\begin{aligned}\int_{\substack{\zeta\in b\Omega\\|z-\zeta|\leq\epsilon}} \frac{dS(\zeta)}{|\Phi(z,\zeta)|.|\zeta-z|} &\lesssim \int_{\substack{|\zeta_1|<\delta\\|z_1-\zeta_1|\leq\epsilon}} \frac{|\ln(|F'(|\zeta_1|^2)||z_1-\zeta_1|^2|)|}{|\zeta_1-z_1|} d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)\\ &\lesssim \int_{\substack{|\zeta_1|<\delta\\|z_1-\zeta_1|\leq\epsilon\\|\zeta_1|<|\zeta_1-z_1|}} \frac{|\ln(|F'(|\zeta_1|^2)||z_1-\zeta_1|^2|)|}{|\zeta_1-z_1|} d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)\\ &\quad+ \int_{\substack{|\zeta_1|<\delta\\|z_1-\zeta_1|\leq\epsilon\\|\zeta_1|\geq|\zeta_1-z_1|}} \frac{|\ln(|F'(|\zeta_1|^2)||z_1-\zeta_1|^2|)|}{|\zeta_1-z_1|} d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)\\ &\lesssim \int_{\substack{|\zeta_1|<\delta\\|z_1-\zeta_1|\leq\epsilon\\|\zeta_1|<|\zeta_1-z_1|}} \frac{|\ln(|F'(|\zeta_1|^2)|\zeta_1|^2)|}{|\zeta_1|} d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)\\ &\quad+ \int_{\substack{|\zeta_1|<\delta\\|z_1-\zeta_1|\leq\epsilon\\|\zeta_1|\geq|\zeta_1-z_1|}} \frac{|\ln(|F'(|z_1-\zeta_1|^2)||z_1-\zeta_1|^2|)|}{|\zeta_1-z_1|} d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)\\ &\lesssim \int_0^{\delta'} |\ln F(r^2)|dr < \infty\end{aligned} \tag{4.3.4}$$

where $\delta' = \max\{\delta,\epsilon\}$, and the last inequality follows from the fact $F'(t)t \gtrsim F(t)$ (since F convex). Hence, $||H(\phi)||_{L^\infty(\Omega)} \leq C.||\phi||_{L^\infty_{0,1}(\Omega)}$.

Let Ω defined by (4.1.5), and for some $\delta > 0$, $\displaystyle\int_0^\delta |\ln(t)\ln F(t^2)|\,dt < \infty$, let $z_1 = x_1 + iy_1$, $\zeta_1 = \xi_1 + i\eta_1$, the same argument shows that

$$\begin{aligned}\operatorname{Re}\Phi(z,\zeta) &\geq -\rho(z) + F(x_1^2) - F(\xi_1^2) - 2F'(\xi_1^2)\xi_1(x_1-\xi_1)\\ &\geq -\rho(z) + F'(\xi_1^2)(x_1-\xi_1)^2 + \left(F(x_1^2) - F(\xi_1^2) - F'(\xi_1^2)(x_1^2-\xi_1^2)\right)\\ &\geq -\rho(z) + \left(F'(\xi_1^2)(x_1-\xi_1)^2\right).\end{aligned} \tag{4.3.5}$$

Hence, apply (4.3.5) to (4.4.1), we obtain

$$
\begin{aligned}
\int_{\substack{\zeta\in b\Omega\\ |z-\zeta|\le\epsilon}} \frac{dS(\zeta)}{|\Phi(z,\zeta)|.|\zeta-z|} &\lesssim \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\le\epsilon}} \frac{|\ln|(F'(\xi_1^2)|(x_1-\xi_1)^2)||}{|x_1-\xi_1|+|y_1-\eta_1|} d(\xi_1)d(\eta_1)\\
&\lesssim \int_{\substack{|\xi_1|<\delta\\ |x_1-\xi_1|\le\epsilon}} \left|\ln|x_1-\xi_1|\ln|(F'(\xi_1^2)|(x_1-\xi_1)^2)|\right| d(\xi_1)\\
&\lesssim \int_{\substack{|\xi_1|<\delta\\ |x_1-\xi_1|\le\epsilon\\ |\xi_1|<|x_1-\xi_1|}} \left|\ln|x_1-\xi_1|\ln|(F'(\xi_1^2)|(x_1-\xi_1)^2)|\right| d(\xi_1)\\
&+ \int_{\substack{|\xi_1|<\delta\\ |x_1-\xi_1|\le\epsilon\\ |\xi_1|\ge|x_1-\xi_1|}} \left|\ln|x_1-\xi_1|\ln|(F'(\xi_1^2)|(x_1-\xi_1)^2)|\right| d(\xi_1) \qquad (4.3.6)\\
&\lesssim \int_{\substack{|\xi_1|<\delta\\ |x_1-\xi_1|\le\epsilon\\ |\xi_1|<|x_1-\xi_1|}} \left|\ln|\xi_1|\ln|(F'(\xi_1^2)|(\xi_1)^2)|\right| d(\xi_1)\\
&+ \int_{\substack{|\xi_1|<\delta\\ |x_1-\xi_1|\le\epsilon\\ |\xi_1|\ge|x_1-\xi_1|}} \left|\ln|x_1-\xi_1|\ln|(F'((x_1-\xi_1^2))|(x_1-\xi_1)^2)|\right| d(\xi_1)\\
&\lesssim \int_0^{\delta'} \left|\ln|r|.\ln|F(r^2)|\right| dr<\infty.
\end{aligned}
$$

Hence, $||H(\phi)||_{L^\infty(\Omega)} \le C.||\phi||_{L^\infty_{0,1}(\Omega)}$. Therefore, we have the sup-norm estimates for solutions of $\bar\partial$-equation when Ω defined by (4.1.4) or (4.1.5)

$$||T(\phi)||_{L^\infty(\Omega)} \le C.||\phi||_{L^\infty_{0,1}(\Omega)}.$$

□

Next, we will provide the f-Hölder estimates for Henkin solutions to the equation $\bar\partial u=\varphi$.

Theorem 4.3.2. *1. Let Ω is defined by (4.1.4) and there exists $\delta>0$ so that $\int_0^\delta |\ln F(t^2)|\,dt<\infty$. Then for any bounded, $\bar\partial$-closed $(0,1)$-form ϕ on $\overline{\Omega}$, there exists a solution u defined on Ω satisfies $\bar\partial u=\phi$ and*

$$||u||_f \lesssim \|\phi\|_{L^\infty(\Omega)}.$$

where

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}}{t}dt\right)^{-1}.$$

2. Let Ω is defined by (4.1.5) and there exists $\delta>0$ so that $\int_0^\delta |\ln t\ln F(t^2)|\,dt<\infty$. Then for any bounded, $\bar\partial$-closed $(0,1)$-form ϕ on $\overline{\Omega}$, there exists a solution u defined on Ω satisfies $\bar\partial u=\phi$

and

$$||u||_f \lesssim \|\phi\|_{L^\infty(\Omega)}.$$

where

$$f(d^{-1}) := \left(\int_0^d \frac{\sqrt{F^*(t)}\ln\sqrt{F^*(t)}}{t}dt\right)^{-1}$$

Proof. It is well-known that

$$||K\phi||_f \lesssim ||\phi||_{L^\infty}$$

for any f with $0 < f(d^{-1}) < d^{-1}$ (see Lemma 1.15, p. 157 in [Ra86]). Hence, it is sufficient to estimate $H(\phi)$. Now, we will control the gradient of $H(\phi)$ by following

$$\begin{aligned}
|\nabla H(\phi)(z)| &\lesssim ||\phi||_{L^\infty_{0,1}(\Omega)} \int_{\substack{\zeta\in b\Omega\\ |z-\zeta|<\epsilon}} \left(\frac{1}{|\Phi(z,\zeta)|.|\zeta-z|^2} + \frac{1}{|\Phi(z,\zeta)|^2.|\zeta-z|}\right) dS(\zeta)\\
&\lesssim ||\phi||_{L^\infty_{0,1}(\Omega)} \left\{\int_{\substack{\zeta\in b\Omega\setminus B(0,\delta)\\ |z-\zeta|<\epsilon}} \left(\frac{1}{|\Phi(z,\zeta)|.|\zeta-z|^2} + \frac{1}{|\Phi(z,\zeta)|^2.|\zeta-z|}\right) dS(\zeta)\right.\\
&\quad \left. + \int_{\substack{\zeta\in b\Omega\cap B(0,\delta)\\ |z-\zeta|\le\epsilon}} \left(\frac{1}{|\Phi(z,\zeta)|.|\zeta-z|^2} + \frac{1}{|\Phi(z,\zeta)|^2.|\zeta-z|}\right) dS(\zeta)\right\}\\
&\lesssim ||\phi||_{L^\infty_{0,1}(\Omega)} \left(|\rho(z)|^{-1/2} + L(z)\right),
\end{aligned} \tag{4.3.7}$$

where the last inequality follows from (4.4.1), and

$$L(z) = \int_{\substack{\zeta\in b\Omega\cap B(0,\delta)\\ |z-\zeta|<\epsilon}} \left(\frac{1}{|\Phi(z,\zeta)|.|\zeta-z|^2} + \frac{1}{|\Phi(z,\zeta)|^2.|\zeta-z|}\right) dS(\zeta).$$

Again, by setting $t = \operatorname{Im}\Phi(z,\zeta)$, and we can change coordinates as before and obtain

$$\begin{aligned}
L(z) &\lesssim \int_{\substack{|t|\le\delta\\ |\zeta_1|<\delta\\ |z_1-\zeta_1|\le\epsilon}} \frac{dtd(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|t| + |\operatorname{Re}\Phi(z,\zeta)|)(|\rho(z)|^2 + |\zeta_1-z_1|^2)}\\
&\quad + \int_{\substack{|t|\le\delta\\ |\zeta_1|<\delta\\ |z_1-\zeta_1|\le\epsilon}} \frac{dtd(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|t|^2 + |\operatorname{Re}\Phi(z,\zeta)|^2)(|\rho(z)| + |\zeta_1-z_1|)}\\
&\lesssim |\ln(|\operatorname{Re}\Phi(z,\zeta|).\ln(-\rho(z))| + \int_{|\zeta_1|<\delta, |z_1-\zeta_1|\le\epsilon} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\operatorname{Re}\Phi(z,\zeta)|.|\zeta_1-z_1|}\\
&\le |\ln(|\rho(z)|)|^2 + \int_{|\zeta_1|<\delta, |z_1-\zeta_1|\le\epsilon} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\operatorname{Re}\Phi(z,\zeta)|.|\zeta_1-z_1|}.
\end{aligned} \tag{4.3.8}$$

Hence, combining with these above equalities, we have showed that

$$|\nabla H(\phi)(z)| \lesssim ||\phi||_{L^\infty_{0,1}(\Omega)} \left(|\rho(z)|^{-1/2} + + \int_{|\zeta_1|<\delta, |z_1-\zeta_1|\le\epsilon} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\operatorname{Re}\Phi(z,\zeta)|.|\zeta_1-z_1|}\right). \tag{4.3.9}$$

We will estimate the last integral on the right hand side separately when Ω defined by (4.1.4) and (4.1.5), for convenience, set $M(z) = \displaystyle\int_{|\zeta_1|<\delta, |z_1-\zeta_1|\leq\epsilon} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{|\operatorname{Re}\Phi(z,\zeta)|.|\zeta_1 - z_1|}$.

Now, let Ω defined by (4.1.4) and F satisfies $\displaystyle\int_0^\delta |\ln F(r^2)|dr < \infty$ for some positive δ. Applying the inequality (4.3.3),

$$\begin{aligned}
M(z) &\leq \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon}} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|\rho(z)| + F'(|\zeta_1|^2)|z_1-\zeta_1|^2)|\zeta_1 - z_1|} \\
&\leq \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon\\ |\zeta_1|\leq|z_1-\zeta_1|}} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|\rho(z)| + F'(|\zeta_1|^2)|z_1-\zeta_1|^2)|\zeta_1 - z_1|} \\
&+ \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon\\ |\zeta_1|\geq|z_1-\zeta_1|}} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|\rho(z)| + F'(|\zeta_1|^2)|z_1-\zeta_1|^2)|\zeta_1 - z_1|} \\
&\leq \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon}} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|\rho(z)| + F'(|\zeta_1|^2)|\zeta_1|^2)|\zeta_1|} \\
&+ \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon}} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|\rho(z)| + F'(|z_1-\zeta_1|^2)|z_1-\zeta_1|^2)|\zeta_1 - z_1|} \\
&\leq \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon}} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|\rho(z)| + F(|\zeta_1|^2))|\zeta_1|} + \int_{\substack{|\zeta_1|<\delta\\ |z_1-\zeta_1|\leq\epsilon}} \frac{d(\operatorname{Re}\zeta_1)d(\operatorname{Im}\zeta_1)}{(|\rho(z)| + F(|z_1-\zeta_1|^2))|\zeta_1 - z_1|}.
\end{aligned} \tag{4.3.10}$$

Choosing $\delta' = \max\{\delta, \epsilon\}$, the previous inequality reads

$$M(z) \lesssim \int_0^{\delta'} \frac{dr}{|\rho(z)| + F(r^2)} \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}, \tag{4.3.11}$$

here the last inequality follows from (i) in Lemma 4.2.2.
We have showed that

$$|\nabla u(z)| = |\nabla T(\phi)(z)| \lesssim \frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}.$$

Now, in order to apply the general Hardy-Littlewood Lemma, we must prove that $\dfrac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|}$ satisfies the hypothesis of Lemma 4.2.3. Since $F(t)$ and $\dfrac{F(t)}{t}$ are increasing, non-negative functions, it follows that $F^*(t)$ ia also increasing but $\dfrac{\sqrt{F^*(t)}}{t}$ is decreasing. Moreover, for some small $\delta > 0$, the function $|\ln(F(t^2))|$ is decreasing when $0 \leq t \leq \delta$, so we can estimate

$$|\ln F(\epsilon^2)|\epsilon \leq \int_0^\epsilon |\ln F(t^2)|dt \leq \int_0^\delta |\ln F(t^2)|dt < \infty$$

for any $0 \leq \epsilon \leq \delta$. Consequently, $\sqrt{F^*(t)}|\ln t| < \infty$ for any $0 \leq t \leq \sqrt{F^*(t)}$ and $\lim_{t\to 0} t|\ln F(t^2)| = 0$. By change of variables via $y = \sqrt{F^*(t)}$, this implies

$$\begin{aligned} \int_0^d \frac{\sqrt{F^*(t)}}{t} dt &= \int_0^{\sqrt{F^*(d)}} y \left(\ln F(y^2)\right)' dy \\ &= \sqrt{F^*(d)} \ln d - \int_0^{\sqrt{F^*(d)}} \left(\ln F(y^2)\right) dy < \infty \end{aligned} \tag{4.3.12}$$

for d sufficiently small, The boundedness in the above inequalities are finite by the hypothesis on F. Therefore, by setting

$$f(d^{-1}) = \left(\int_0^d \frac{\sqrt{F^*(t)}}{t} dt \right)^{-1},$$

we have proved that $||u||_f \lesssim ||\phi||_{L^\infty_{0,1}(\Omega)}$ to the first case of the theorem.

Now, let Ω defined by (4.1.5) and F satisfies $\int_0^\delta |\ln t \ln F(t^2)|\, dt < \infty$ for some $\delta > 0$. Now, since the previous observations, we only need to estimate of the integral term $M(z)$ on $\zeta \in b\Omega \cap B(0, \delta)$. Applying the inequality (4.3.5), and repeating the comparing as before to $|\xi_1|$ and $|x_1 - \xi_1|$

$$\begin{aligned} M(z) &\leq \int_{\substack{|\zeta_1|<\delta \\ |\zeta_1 - z_1| \leq \epsilon}} \frac{d\xi_1 d\eta_1}{(|\rho(z)| + F'(\xi_1^2)(x_1 - \xi_1)^2)(|x_1 - \xi_1| + |y_1 - \eta_1|)} \\ &\lesssim \int_{\substack{|\xi_1|<\delta \\ |x_1 - \xi_1| \leq \epsilon}} \frac{|\ln |x_1 - \xi_1|| d\xi_1}{|\rho(z)| + F'(\xi_1^2)(x_1 - \xi_1)^2} \\ &\lesssim \int_{|r|<\max\{\delta,\epsilon\}} \frac{|\ln r|}{|\rho(z)| + F(r^2)} dr \\ &\lesssim \frac{\sqrt{F^*(|\rho(z)|)} |\ln \sqrt{F^*(|\rho(z)|)}|}{|\rho(z)|}, \end{aligned} \tag{4.3.13}$$

where the last inequality follows from (ii), Lemma 4.2.2. Then, similarly to the setup in the proof of (1), we obtain

$$\int_0^d \frac{\sqrt{F^*(t)} |\ln \sqrt{F^*(t)}|}{t} dt < \infty,$$

for some $d, \delta > 0$ sufficiently small, under the hypothesis $\int_0^\delta |\ln t \ln F(t^2)|\, dt < \infty$. Hence, by setting

$$f(d^{-1}) = \left(\int_0^d \frac{\sqrt{F^*(t)} |\ln \sqrt{F^*(t)}|}{t} dt < \infty \right)^{-1},$$

we have $||u||_f \lesssim ||\phi||_{L^\infty_{0,1}(\Omega)}$.
This completes the proof of the theorem. $\square$

4.3.1 Examples

To illustrate these theorems we will give some examples in which the functions F and f are computed more explicitly.

Example 4.3.1. *Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : F(|z_1|^2) + |z_2 - 1|^2 < 1\}$. Then, the sup-norm and f -Hölder estimates hold for the integral representation solution of $\bar{\partial}$-equation in the following examples:*

1. *If $F(t^2) = t^{2m}$, then $f(d^{-1}) = d^{-1/2m}$, this case essentially is of D'Angelo finite type of $2m$.*
2. *If $F(t^2) = 2\exp\left(\frac{-1}{t^\alpha}\right)$, with $0 < \alpha < 1$, then $f(d^{-1}) = (-\ln d)^{\frac{1}{\alpha}-1}$. This domain is of D'Angelo infinity type.*
3. *If $F(t^2) = 2\exp\left(\frac{-1}{t|\ln t|^\alpha}\right)$, with $2 < \alpha$, then $f(d^{-1}) = (\ln(-\ln d))^{\alpha-1}$. This domain is also of D'Angelo infinity type.*

Example 4.3.2. *Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : F(|\operatorname{Re} z_1|^2) + G(|\operatorname{Im} z_1|^2) + |z_2 - 1|^2 < 1\}$, where $G(t) = 0$ in a neighborhood of 0 and there is a positive constant c such that $t \geq c$ if $G(t) \geq 1$. Then, the sup-norm and f -Hölder estimates hold for the integral representation solution of $\bar{\partial}$-equation in the following examples:*

1. *If $F(t^2) = t^{2m}$, then $f(d^{-1}) = d^{-1/2m}|\ln d|^{-1}$.*
2. *If $F(t^2) = 2\exp\left(\frac{-1}{t^\alpha}\right)$, with $0 < \alpha < 1$, then $f(d^{-1}) = (-\ln d)^{\frac{1}{\alpha}-1}(\ln(-\ln d))^{-1}$.*
3. *If $F(t^2) = 2\exp\left(\frac{-1}{t|\ln t|^\alpha}\right)$, with $2 < \alpha$, then $f(d^{-1}) = (\ln(-\ln d))^{\alpha-2}$.*

Next, we will provide an example to show that the index f in Theorem 4.3.2 can not be improved. The idea behinds the following example is due to E. M. Stein with a modern setup by M. C. Shaw.
Let

$$\Omega = \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = F(|z_1|^2) + |z_2 - 1|^2 < 1\}, \tag{4.3.14}$$

where F satisfies the conditions in Theorem 4.3.2. Setting $\phi(z) = \dfrac{d\bar{z}_1}{\ln(z_2)}$ and $v(z) = \dfrac{\bar{z}_1}{\ln(z_2)}$. Hence, one can check that ϕ is a C^1, $\bar{\partial}$-closed $(0,1)$-form , and $v(z)$ is a solution of the equation $\bar{\partial}u = \phi$. One already know that the solution of $\bar{\partial}$-equation via Henkin kernel belongs to $\Lambda^f(\Omega)$, then, the following lemma says that it can not be in $\Lambda^g(\Omega)$.

Lemma 4.3.3. *Let ϕ and v defined as above. Then, $\phi \in C^\infty_{(0,1)}(\Omega)$, $v \in \Lambda^f(\Omega)$ where f defined as in Theorem 4.3.2. Moreover , let g satisfy $\lim\limits_{t\to\infty} \dfrac{g(t)}{f(t)} = \infty$. If u is a solution of $\bar{\partial}u = \phi$ in the weak sense, then $u \notin \Lambda^g(\Omega)$.*

Proof. Since v gets its singularities on $\mathrm{Sing} = \{z = (z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$, hence it is sufficiently to show that v admits a Hölder estimate in a small neighborhood of each singularity. The fact that $v \in C^\infty(\overline{\Omega} \setminus \mathrm{Sing})$ guarantees that v is Hölder regularity in standard sense outside these neighborhoods.
Following the example in [HKR13], one can choose a small neighborhood U around 0 such that $|\rho(z)| \leq c|z_2|$, with c should be big enough. Then, on $U \setminus \{0\}$,

$$\nabla v(z) = \left(0, \frac{1}{\ln(z_2)}, \frac{-\overline{z}_1}{[\ln(z_2)]^2 z_2}, 0\right).$$

Now, on $\Omega \cap U$, since $|\rho(z)|$, $|z_2| < 1$

$$\left|\frac{1}{\ln(z_2)}\right| \leq \frac{1}{|\ln|\rho(z)||} < C,$$

where the appearance of C follows from the fact that $\lim\limits_{z \to 0} |\ln|\rho(z)|| = +\infty$.
Next, from the property that $F(|z_1|^2) \leq 1 - |z_2 - 1|^2 \leq 2(1 - |z_2 - 1|)$,

$$\begin{aligned}
\frac{|\overline{z_1}|}{|z_2|.|\ln(z_2)|^2} \leq c\frac{\sqrt{F^*(1 - |z_2 - 1|^2)}}{|z_2|.|\ln|\rho(z)||^2} &\leq c\frac{\sqrt{F^*(1 - |z_2 - 1|)}}{|\rho(z)|} \\
&\leq c\frac{\sqrt{F^*(|z_2|)}}{|z_2|} \\
&\leq C_0\frac{\sqrt{F^*(|\rho(z)|)}}{|\rho(z)|},
\end{aligned} \tag{4.3.15}$$

where the last inequality holds since the function $\dfrac{\sqrt{F^*(t)}}{t}$ is decreasing. Therefore, $v \in \Lambda^f(\Omega)$.
In order to prove the non-existence part, by contradiction, one assume that some function u satisfies $\bar{\partial} u = \phi$ and $u \in \Lambda^g(\Omega)$. Since $\bar{\partial}(u - v) = 0$, by Cauchy's Theorem on $F(r^2) + \xi^2 < 1$

$$\int_{\substack{|z_1|=r \\ z_2=\xi}} u(z_1, z_2) dz_1 = \int_{\substack{|z_1|=r \\ z_2=\xi}} v(z_1, z_2) dz_1 = \frac{Kr^2}{\ln \xi}. \tag{4.3.16}$$

By assumption $u \in \Lambda^g(\Omega)$, one obtain

$$\left|\int_{\substack{|z_1|=r, \\ z_2=\xi}} u(z_1, z_2 + h) - u(z_1, z_2) dz_1\right| \leq C.r.g(|h|^{-1})^{-1}. \tag{4.3.17}$$

Now, let $\xi = 2\delta$, $h = \delta$, $r = f(\delta^{-1})^{-1}$, with $\delta > 0$ is small so that (z_1, ξ), $(z_1, \xi + h)$ belong to Ω if $|z_1| = r$. Hence, applying these terms to (5.4.1), (4.3.17) , one have

$$\left|\frac{Kf(\delta^{-1})^{-2}}{\ln(2\delta)} - \frac{Kf(\delta^{-1})^{-2}}{\ln \delta}\right| \leq Cf(\delta^{-1})^{-1} g(\delta^{-1})^{-1}.$$

That is $Cf(\delta^{-1})^{-1} \leq Cg(\delta^{-1})^{-1}$. Therefore, since $\lim_{t\to\infty} \dfrac{g(t)}{f(t)} = \infty$, taking $\delta \to 0$, one yield a contradiction. $\square$

A remark that, in this section, to obtain the sup-norm estimates, we do not need to use the extra assumptions that $F'''(t)$ are non-negative on $(0, \delta)$. They will be applied in the proof of L^p-estimates.

4.4 L^p-Estimates for $\bar{\partial}$-Solutions

In this section, we will study the L^p boundedness for integral representation solutions of $\bar{\partial}$-equations on the classes of convex domains defined by (4.1.4) and (4.1.5), with $1 \leq p \leq \infty$.

Theorem 4.4.1. *(joint work with Khanh, T. V., Raich, A. [HKR13]) If either of the following conditions hold:*

1. *Ω is defined by (4.1.4) and there exists $\delta > 0$ so that $\displaystyle\int_0^\delta |\ln F(t^2)|\, dt < \infty$,*
2. *Ω is defined by (4.1.5) and there exists $\delta > 0$ so that $\displaystyle\int_0^\delta |\ln(t) \ln F(t^2)|\, dt < \infty$,*

then for any $\bar{\partial}$-closed $(0,1)$-form ϕ in $L^p(\Omega)$ with $1 \leq p \leq \infty$, the solution via Henkin kernel u on Ω satisfies $\bar{\partial} u = \phi$ and

$$\|u\|_{L^p(\Omega)} \leq C\|\phi\|_{L^p(\Omega)}.$$

where $C > 0$ is independent of ϕ.

In fact, the sup-norm estimate has been showed in the previous section. Hence, by Riesz-Thorin Interpolation Theorem, it is sufficiently to prove the L^1 boundedness for Henkin solutions. Before the proof, we will re-call the estimate for the support function Φ with a little modification .

Lemma 4.4.2. *Let $\Omega \subset \mathcal{C}^2$ and Φ defined as above. Then there exist $\epsilon, c > 0$ so that*

$$\operatorname{Re}\Phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + \begin{cases} c|z-\zeta|^2 & \zeta \in S_{0,2\delta} \setminus B(0,\delta) \\ P(z_1) - P(\zeta_1) - 2\operatorname{Re}\left\{\frac{\partial P}{\partial \zeta_1}(\zeta_1)(z_1 - \zeta_1)\right\} & \zeta \in S_{0,2\delta} \cap B(0,\delta) \end{cases}. \tag{4.4.1}$$

for all $z \in \bar{\Omega}$ with $|z - \zeta| \leq \epsilon$,where $P(z_1) = F(|z_1|^2)$ or $P(z_1) = F(x_1^2)$, and $S_{0,2\delta} := \{z \in \bar{\Omega} : \rho(z) \geq -2\delta\}$.

4.4.1 Proof of Theorem 4.4.1

As a consequence of the Riesz-Thorin Interpolation Theorem and Theorem 4.3.1, proving that T is a bounded, linear operator on $L^1(\Omega)$ suffices to establish that T is a bounded linear operator on $L^p(\Omega)$, $1 \leq p \leq \infty$.

The L^1-estimate of $|K\phi(z)|$ is standard and does not require interpolation. Indeed, since $|\zeta - z|^{-3} \in L^1(\Omega)$ in both ζ and z (separately), L^p boundedness of K, $1 \leq p \leq \infty$, follows from [Fol99, Theorem 6.18].

For the boundedness of H, we first begin the analysis of $H\phi(z)$ by using Stokes' Theorem. Using the assumption that ϕ is $\bar{\partial}$-closed, we observe

$$H\phi(z) = \frac{1}{2\pi^2}\int_\Omega \bar{\partial}_\zeta\left(\frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \frac{\partial\rho(\zeta)}{\partial\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{(\Phi(\zeta,z) - \rho(\zeta))(|\zeta - z|^2 + \rho(\zeta)\rho(z))}\right) \wedge \phi(\zeta) \wedge \omega(\zeta).$$

We abuse notation slightly and let $H(\zeta, z)$ be the integral kernel of H. Direct calculation shows that we can decompose

$$\begin{aligned} |H(\zeta,z)| &\leq \left|\bar{\partial}_\zeta\left(\frac{\frac{\partial\rho(\zeta)}{\partial\zeta_1}(\bar{\zeta}_2 - \bar{z}_2) - \frac{\partial\rho(\zeta)}{\partial\zeta_2}(\bar{\zeta}_1 - \bar{z}_1)}{(\Phi(\zeta,z) - \rho(\zeta))(|\zeta - z|^2 + \rho(\zeta)\rho(z))}\right)\right| \\ &\lesssim \frac{1}{|\Phi(\zeta,z) - \rho(\zeta)|^2(|\zeta - z|^2 + \rho(\zeta)\rho(z))^{1/2}} + \frac{1}{|\Phi(\zeta,z) - \rho(\zeta))(|\zeta - z|^2 + \rho(\zeta)\rho(z))}. \end{aligned} \tag{4.4.2}$$

Since $(\rho(\zeta) - \rho(z))^2 \lesssim |\zeta - z|^2$, this implies $\rho(\zeta)^2 \lesssim |\zeta - z|^2 + \rho(\zeta)\rho(z)$, hence $|\zeta - z| + |\rho(\zeta)| \lesssim (|\zeta - z|^2 + \rho(\zeta)\rho(z))^{1/2}$. Thus

$$|\Phi(\zeta,z) - \rho(\zeta)| \lesssim |\zeta - z| + |\rho(\zeta)| \lesssim (|\zeta - z|^2 + \rho(\zeta)\rho(z))^{1/2}. \tag{4.4.3}$$

Combining (4.4.2) and (4.4.3), we obtain

$$\begin{aligned} |H(\zeta,z)| &\lesssim \frac{1}{|\Phi(\zeta,z) - \rho(\zeta)|^2(|\zeta - z|^2 + \rho(\zeta)\rho(z))^{1/2}} \\ &\leq \frac{1}{|\Phi(\zeta,z) - \rho(\zeta)|^2|\zeta - z|} \\ &\leq \frac{1}{(|\operatorname{Re}\Phi(\zeta,z) - \rho(\zeta)|^2 + |\operatorname{Im}\Phi(\zeta,z)|^2)|\zeta_1 - z_1|}. \end{aligned} \tag{4.4.4}$$

We will show that

$$\iint_{(\zeta,z)\in\Omega\times\Omega} H(\zeta,z)\phi(\zeta)dV(\zeta,z) \lesssim \|\phi\|_{L^1(\Omega)} < \infty. \tag{4.4.5}$$

Then, apply the Fubini's Theorem to obtain

$$\begin{aligned}\|H\phi\|_{L^1(\Omega)} &= \int_{z\in\Omega} H\phi(z)dV(z)\\ &= \int_{z\in\Omega}\int_{\zeta\in\Omega} H(\zeta,z)\phi(\zeta)dV(\zeta)dV(z)\\ &= \iint_{(\zeta,z)\in\Omega\times\Omega} H(\zeta,z)\phi(\zeta)dV(\zeta,z) \lesssim \|\phi\|_{L^1(\Omega)}.\end{aligned} \tag{4.4.6}$$

In order to prove (5.4.8), we remark that it is enough to assume that $z,\zeta\in\Omega\cap B(0,\delta)=\{\rho(z)=P(z_1)+r(z)<0\}$ because if $\zeta_o, z\in\bar\Omega\setminus B(0,\delta/2)$, then the estimates following classically using the strict convexity of r. If one of $\{z,\zeta\}$ is in $B(0,\delta/2)$ and other is an element of $B(0,\delta)^c$, then the integrand of H is bounded and bounded away from 0, and the estimate is trivial. We will investigate the complex and real ellipsoid cases separately to show

$$\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} H(\zeta,z)\phi(\zeta)dV(\zeta,z) \lesssim \|\phi\|_{L^1(\Omega)}. \tag{4.4.7}$$

4.4.2 Complex Ellipsoid Case

In this subsection, Ω is defined by (4.1.4). Since the argument of F is $|\zeta_1|^2$, the chain rule shows that $\frac{\partial}{\partial\zeta_1}F(|\zeta_1|^2)=\bar\zeta_1F'(|\zeta_1|^2)$. Similarly to Khanh [Kha13, (4.1)], Lemma 4.4.2 shows that

$$\begin{aligned}&\operatorname{Re}\big\{\Phi(\zeta,z)\big\}-\rho(\zeta)\geq -\rho(z)+F(|z_1|^2)-F(|\zeta_1|^2)-2F'(|\zeta_1|^2)\operatorname{Re}\big\{\bar\zeta_1(z_1-\zeta_1)\big\}\\ &=-\rho(z)+F'(|\zeta_1|^2)|z_1-\zeta_1|^2+\Big(F(|z_1|^2)-F(|\zeta_1|^2)-F'(|\zeta_1|^2)\big(|z_1|^2-|\zeta_1|^2\big)\Big).\end{aligned} \tag{4.4.8}$$

Now, we consider two case of $F'(0)$. If $F'(0)\neq 0$, then there is a suitable $\delta>0$ such that $F'(|\zeta_1|^2)>0$ for any $|\zeta_1|<\delta$. Hence,

$$\operatorname{Re}\big\{\Phi(\zeta,z)\big\}-\rho(\zeta)\gtrsim -\rho(z)+|z_1-\zeta_1|^2,$$

and

$$|H(\zeta,z)|\leq \frac{1}{(|\rho(z)|^2+|\operatorname{Im}\Phi(\zeta,z)|^2+|\zeta_1-z_1|^4)|\zeta_1-z_1|}.$$

Our problem goes back to the case of strongly pseudoconvex domain, and the result is trivial. So we only assume that $F'(0)=0$. We will have

Lemma 4.4.3. *Let F be defined in Section 1 with the extra assumption $F'(0)=0$. Then*

$$|H(\zeta,z)|\lesssim\begin{cases}\dfrac{1}{(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(|z_1-\zeta_1|^2))|z_1-\zeta_1|} & \textit{if}\quad |\zeta_1|\geq|z_1-\zeta_1|,\\[2ex] \dfrac{1}{(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(\frac12|z_1|^2))|z_1|} & \textit{if}\quad |\zeta_1|\leq|z_1-\zeta_1|,\end{cases} \tag{4.4.9}$$

Proof. Using Lemma 4.2.1 for the expression in the parenthesis of in the second line of (4.4.8), we obtain

$$\operatorname{Re}\big\{\Phi(\zeta,z)\big\}-\rho(\zeta)\geq-\rho(z)+\begin{cases}F'(|\zeta_1|^2)|z_1-\zeta_1|^2 & \text{for all}\quad 0<|z_1|,|\zeta_1|<\delta,\\ F(|z_1|^2-|\zeta_1|^2) & \text{only for}\quad |\zeta_1|\leq|z_1|\leq\delta.\end{cases}\tag{4.4.10}$$

We compare $|\zeta_1|$ and $|z_1-\zeta_1|^2$.

Case 1: $|\zeta_1|\geq|z_1-\zeta_1|$. Using the first case (4.4.10) with the fact that F' is increasing and $F'(t)t\geq F(t)$ (since $\dfrac{F(t)}{t}$ is increasing), we obtain

$$\operatorname{Re}\big\{\Phi(\zeta,z)\big\}-\rho(\zeta)\geq-\rho(z)+F(|z_1-\zeta_1|^2).$$

The first line of (4.4.9) follows by above inequality and (4.4.4).

Case 2: $|\zeta_1|\leq|z_1-\zeta_1|$. In this case, we need to compare $|\zeta_1|$ and $\frac{1}{\sqrt{2}}|z_1|$. If $|\zeta_1|\geq\frac{1}{\sqrt{2}}|z_1|$, then same argument above in Case 1 provides that

$$\operatorname{Re}\big\{\Phi(\zeta,z)\big\}-\rho(\zeta)\geq-\rho(z)+F(\frac{1}{2}|z_1|^2).$$

Hence, we get the second line of (4.4.9). Otherwise, if $|\zeta_1|\leq\frac{1}{\sqrt{2}}|z_1|$, this implies $|z_1|\geq|\zeta_1|$ and $|z_1-\zeta_1|\geq(1-\frac{1}{\sqrt{2}})|z_1|$. So we can use the second case of (4.4.10) and obtain

$$\operatorname{Re}\big\{\Phi(\zeta,z)\big\}-\rho(\zeta)\geq-\rho(z)+F(|z_1|^2-|\zeta_1|^2)\geq-\rho(z)+F(\frac{1}{2}|z_1|^2),$$

and

$$(|\operatorname{Re}\Phi(\zeta,z)-\rho(\zeta)|^2+|\operatorname{Im}\Phi(\zeta,z)|^2)|\zeta_1-z_1|\gtrsim(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(\frac{1}{2}|z_1|^2))|z_1|.$$

This completes the proof. ◻

Proof of the Theorem 4.4.1.i. Using Lemma 4.4.3, we have

$$\begin{aligned}&\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2}H(\zeta,z)\phi(\zeta)dV(\zeta,z)\\ =&\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2\text{ and }|\zeta_1|\geq|z_1-\zeta_1|}\cdots+\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2\text{ and }|\zeta_1|\leq|z_1-\zeta_1|}\cdots\\ \lesssim&(I)+(II)\end{aligned}\tag{4.4.11}$$

where

$$\begin{aligned}(I)&:=\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2}\frac{|\phi(\zeta)|dV(\zeta,z)}{(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(|z_1-\zeta_1|^2))|z_1-\zeta_1|};\\ (II)&:=\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2}\frac{|\phi(\zeta)|dV(\zeta,z)}{(|\rho(z)+i\operatorname{Im}\Phi(\zeta,z)|^2+F^2(\frac{1}{2}|z_1|^2))|z_1|}.\end{aligned}\tag{4.4.12}$$

For the integral (I), we make the change variables $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, z_1 - \zeta_1, \rho(z) + i \operatorname{Im} \Phi(\zeta, z))$. Direct calculus the Jacobian of this transformation is the matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho(z)}{\partial(\operatorname{Re} z_1)} & \frac{\rho(z)}{\partial(\operatorname{Im} z_1)} & \frac{\rho(z)}{\partial(\operatorname{Re} z_2)} & \frac{\rho(z)}{\partial(\operatorname{Im} z_2)} \\ \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Re} \zeta_1)} & \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Im} \zeta_1)} & \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Re} \zeta_2)} & \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Im} \zeta_2)} & \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Re} z_1)} & \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Im} z_1)} & \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Re} z_2)} & \frac{\partial \operatorname{Im} \Phi(\zeta,z)}{\partial(\operatorname{Im} z_2)} \end{pmatrix}$$

To check that this coordinate change is legitimate, we compute

$$\det(J) = -\frac{\partial \operatorname{Im}(\Phi(\zeta, z))}{\partial \xi_2} \frac{\partial \rho(\zeta)}{\partial \eta_2} + \frac{\partial \operatorname{Im}(\Phi(\zeta, z))}{\partial \eta_2} \frac{\partial \rho(\zeta)}{\partial \xi_2}.$$

By a possible rotation of Ω, we can assume that $\nabla \rho(0) = (0, 0, 0, -1)$. Direct calculation then establishes that if δ is chosen sufficiently small (so that $\frac{\partial \rho(\zeta)}{\partial \xi_2}$ dominates the other parts of ρ and $|\zeta - z|$ are small), then $\det(J) \neq 0$. Since Φ is smooth, we can assume that there exists $\delta' > 0$ that depends on Ω and ρ so that

$$\begin{aligned}(I) &\lesssim \iint_{(\psi,w)\in(\Omega\cap B(0,\delta))\times B(0,\delta')} \frac{|\phi(\psi)|}{(|w_2|^2 + F^2(|w_1|^2)|w_1|} dV(\psi, w) \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \int_0^{\delta'} \frac{r_1 r_2 dr_2 dr_1}{(r_2^2 + F^2(r_1^2))r_1} \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \ln F(r_1^2) dr_1 < \infty.\end{aligned} \tag{4.4.13}$$

Here, the last inequality follows by the hypothesis of ϕ and F.

Repeating this argument with the change coordinates $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, \frac{1}{\sqrt{2}} z_1, \rho(z) + i \operatorname{Im} \Phi(\zeta, z))$ for the integral (II), we obtain the same conclusion. Therefore, this completes the estimate in complex case.

4.4.3 Real Ellipsoid Case

In this subsection, one will consider the case Ω is defined by (4.1.5). The argument from (4.4.8) now yields

$$\operatorname{Re}\{\Phi(\zeta, z)\} - \rho(\zeta) \geq -\rho(z) + F'(\xi_1^2)(x_1 - \xi_1)^2 + \left(F(x_1^2) - F(\xi_1^2) - F'(\xi_1^2)(x_1^2 - \xi_1^2)\right),$$

where $z_1 = x_1 + iy_1$, $\zeta_1 = \xi_1 + i\eta_1$. Following the setup in the complex case, with the same proof, one also have

Lemma 4.4.4. *Let F be defined in Section 1 with the extra assumption $F'(0) = 0$. Then*

$$|H(\zeta, z)| \lesssim \begin{cases} \dfrac{1}{(|\rho(z) + i\operatorname{Im}\Phi(\zeta, z)|^2 + F^2((x_1 - \xi_1)^2))(|x_1 - \xi_1| + |y_1 - \eta_1|)} & \text{if} \quad |\xi_1| \geq |x_1 - \xi_1|, \\ \dfrac{1}{(|\rho(z) + i\operatorname{Im}\Phi(\zeta, z)|^2 + F^2(\frac{1}{2}x_1^2))(\frac{1}{\sqrt{2}}|x_1| + |y_1 - \eta_1|)} & \text{if} \quad |\xi_1| \leq |x_1 - \xi_1|, \end{cases} \tag{4.4.14}$$

Proof of Theorem 4.4.1.ii. Using Lemma 4.4.4, we have

$$\iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} H(\zeta, z)\phi(\zeta)dV(\zeta, z) \lesssim (I) + (II), \tag{4.4.15}$$

where

$$\begin{aligned} (I) &:= \iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} \frac{|\phi(\zeta)|dV(\zeta, z)}{(|\rho(z) + i\operatorname{Im}\Phi(\zeta, z)|^2 + F^2((x_1 - \xi_1)^2))(|x_1 - \xi_1| + |y_1 - \eta_1|)}; \\ (II) &:= \iint_{(\zeta,z)\in(\Omega\cap B(0,\delta))^2} \frac{|\phi(\zeta)|dV(\zeta, z)}{(|\rho(z) + i\operatorname{Im}\Phi(\zeta, z)|^2 + F^2(\frac{1}{2}x_1^2))(\frac{1}{\sqrt{2}}|x_1| + |y_1 - \eta_1|)}. \end{aligned} \tag{4.4.16}$$

We make the change variables $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, z_1 - \xi_1, \rho(z) + i\operatorname{Im}\Phi(\zeta, z))$ for (I) and $(\psi, w) = (\psi_1, \psi_2, w_1, w_2) = (\zeta_1, \zeta_2, \frac{1}{\sqrt{2}}x_1 - i(y_1 - \eta_1), \rho(z) + i\operatorname{Im}\Phi(\zeta, z))$ for (II). Similar the Subsection 3.1, we can check that $\det(J) \neq 0$ for both integrals. Therefore

$$\begin{aligned} (I) + (II) &\lesssim \iint_{(\psi,w)\in(\Omega\cap B(0,\delta))\times B(0,\delta')} \frac{|\phi(\psi)|}{(|w_2|^2 + F^2((\operatorname{Re} w_1)^2)(|\operatorname{Re} w_1| + |\operatorname{Im} w_1|)} dV(\psi, w) \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'}\int_0^{\delta'}\int_0^{\delta'} \frac{r_2 dr_2 d(\operatorname{Re} w_1) d(\operatorname{Im} w_1)}{(r_2^2 + F^2((\operatorname{Re} w_1)^2))(|\operatorname{Re} w_1| + |\operatorname{Im} w_1|)} \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'}\int_0^{\delta'} \frac{\ln(F((\operatorname{Re} w_1)^2) d(\operatorname{Re} w_1) d(\operatorname{Im} w_1)}{|\operatorname{Re} w_1| + |\operatorname{Im} w_1|} \\ &\lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \ln(|\operatorname{Re} w_1|) \ln(F((\operatorname{Re} w_1)^2) d(\operatorname{Re} w_1) < \infty. \end{aligned} \tag{4.4.17}$$

Here, the last inequality follows by the hypothesis of ϕ and F. This completes our proof of Theorem 4.4.1.

4.4.4 Examples

In this section, we present an example to show that our estimates are optimal in the sense that the inequality $\|u\|_{L^q(\Omega)} \lesssim \|\phi\|_{L^p(\Omega)}$ cannot hold if $1 \leq p < q \leq \infty$. Specifically, let $0 < \alpha < 1$, $1 \leq p < q \leq \infty$, and

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : e^{1 - \frac{1}{|z_1|^\alpha}} + |z_2|^2 < 1\}. \tag{4.4.18}$$

We will show that there is a $\bar{\partial}$-closed $(0,1)$-form $\phi \in L^p_{0,1}(\Omega)$ for which there does not exist a function $u \in L^q(\Omega)$ so that $\bar{\partial} u = \phi$ in Ω. Indeed, let

$$\phi(z) = \frac{(1-\ln(1-z_2))^k}{(1-z_2)^{2/q}} d\bar{z}_1 \quad \text{and} \quad v(z) = \frac{(1-\ln(1-z_2))^k}{(1-z_2)^{2/q}} \tag{4.4.19}$$

where $k := [\frac{q+2}{q\alpha}] + 1 \in \mathbb{N}$. The function $\dfrac{(1-\ln(1-z_2))^k}{(1-z_2)^{1/q}}$ is holomorphic on Ω with the principle branch of the logarithm $0 < \arg(1-z_2) < 2\pi$. The form ϕ is a $\bar{\partial}$-closed $(0,1)$-form on Ω and function v is a solution of the equation $\bar{\partial} u = \phi$. Moreover, we observe that v is L^2-orthogonal to all holomorphic functions on Ω (by Mean Value Theorem). By directly calculating (see Lemma 4.4.5 in below), we obtain $\phi \in L^p_{1,0}(\Omega)$, $v \in L^p(\Omega)$ and $v \notin L^q(\Omega)$. Let P be the Bergman projection on Ω, i.e., the L^2-orthogonal projection onto all holomorphic functions on Ω. In recently, Khanh and Thu [KT13] have proven that P of domain of domain defined in (4.4.18) is a bounded operator form $L^q(\Omega)$ to $L^q(\Omega)$ for any $q > 1$ (see Appendix). Therefore if $u \in L^q(\Omega)$ is a solution to $\bar{\partial} u = \phi$, then $v = u - P(u)$ is in $L^q(\Omega)$. This is impossible. Therefore, there is no solution $u \in L^q(\Omega)$ with $q \geq p$.

Lemma 4.4.5. *Let ϕ and v be defined in* (4.4.19)*. Then, $\phi \in L^p_{1,0}(\Omega)$, $v \in L^p(\Omega)$ and $v \in L^q(\Omega)$.*

Proof. We now show that $\phi \in L^p_{0,1}(\Omega)$. We have

$$\begin{aligned}
\int_\Omega |\phi(z)|^p dV(v) &= \int_\Omega \frac{|1-\ln|1-z_2| + i\arg(1-z_2)|^{kp}}{|1-z_2|^{2p/q}} dV(z) \\
&\leq \int_{|z_2|<1} \frac{\left((1-\ln|1-z_2|)^2 + 4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}} \int_{|z_1|<(1-\ln(1-|z_2|^2))^{-1/\alpha}} 1\, dV(z_1)\, dV(z_2) \\
&\lesssim \int_{|z_2|<1} \frac{\left((1-\ln|1-z_2|)^2 + 4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}((1-\ln(1-|z_2|^2))^{2/\alpha}} dV(z_2) \\
&\lesssim \int_{|z_2|<1} \frac{\left((1-\ln|1-z_2|)^2 + 4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}} dV(z_2) \\
&\lesssim \int_{|z_2|<1, |z_2-1|\geq 1} \cdots + \int_{|z_2|<1, |z_2-1|<1} \cdots .
\end{aligned} \tag{4.4.20}$$

Since the function $\frac{\left((1-\ln|1-z_2|)^2+4\pi^2\right)^{kp/2}}{|1-z_2|^{2p/q}}$ is bounded on $\{|z_2| < 1, |z_2 - 1|\}$, so the first integral $\int_{|z_2|<1,|z_2-1|\geq 1} \cdots$ is also bounded. For the second integral, we have

$$\begin{aligned}
\int_{|z_2|<1, |z_2-1|<1} \cdots &\leq \int_{|z_2-1|<1} \cdots \\
&= \int_0^1 \frac{((1-\ln t)^2 + 4\pi^2)^{kp/2}}{t^{2p/q-1}} dt < \infty,
\end{aligned} \tag{4.4.21}$$

since $2p/q-1<1$ The proof of $v \in L^p(\Omega)$ is automatically follows by our computation that $\phi \in L^p_{0,1}(\Omega)$ since $|z_1|$ is bounded. Now, we prove that $v \notin L^q(\Omega)$. We have

$$\begin{aligned}\int_\Omega |v(z)|^q dV(z) &= \int_\Omega \frac{|1-\ln(1-z_2)|^{kq}\,|z_1|^q}{|1-z_2|^2}\,dV(z)\\ &= \int_{|z_2|<1} \frac{|1-\ln|1-z_2|+i\arg(1-z_2)|^{kq}}{|1-z_2|^2} \int_{|z_1|<(1-\ln(1-|z_2|^2))^{-1/\alpha}} |z_1|^q\,dV(z_1)\,dV(z_2)\\ &\geq \frac{2\pi\alpha}{q+2}\int_{|z_2|<1} \frac{|1-\ln|1-z_2||^{kq}}{|1-z_2|^2(1-\ln(1-|z_2|^2))^{\frac{q+2}{\alpha}}}\,dV(z_2)\\ &\gtrsim \int_{z_2\in D} \frac{|1-\ln|1-z_2||^{kq}}{|1-z_2|^2(1-\ln(1-|z_2|^2))^{\frac{q+2}{\alpha}}}\,dV(z_2),\end{aligned} \tag{4.4.22}$$

where

$$D=\{z_2=1+re^{i\theta}\in\mathcal{C}: 0<r<\frac{1}{3}, \frac{3\pi}{4}<\theta<\frac{5\pi}{4}\}\subset\{|z_2|<1, |z_2-1|<\frac{1}{3}\}\subset\{|z_2|<1\}.$$

The domain of the integral forces $1-|z_2|^2=-2r\cos\theta-r^2\geq\sqrt{2}r-r^2\geq r(\sqrt{2}-r)\geq r$ (since $\sqrt{2}=r\geq\sqrt{2}-\frac{1}{3}\geq 1$). So we obtain

$$\int_\Omega |v(z)|^q dV(z) \gtrsim \int_0^{\frac{1}{3}} \frac{(1-\ln r)^{kq-\frac{q+2}{\alpha}}}{r}dr \geq \int_0^{\frac{1}{3}} \frac{dr}{r} \quad \text{(divergence)}. \tag{4.4.23}$$

Here, the last inequality follows by $kq-\frac{q+2}{\alpha}>0$ by the choice of k. $\square$

Chapter 5

Complex Monge-Ampère Equations in $\mathbb{C}^n$

In this chapter, we will discuss one of most important non-linear partial differential equations in several complex variables, Complex Monge-Ampère Equations.
During the last four decades, complex Monge-Ampère equations have been the subject of intensive studies. These equations have a simplest form which is a fully non-linear equation of elliptic type

$$\mathrm{CMA}(u) := \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) = h,$$

where the solution u should be a (continuous) plurisubharmonic functions in $\mathbb{C}^n$. The real Monge-Ampère equations have been studied deeply in relation to many problems in Riemann Geometry, and also in the modern applications of partial differential equations field. However, it seems to be difficult to solve them in a completely acceptable way. The notation of convex surfaces was applied by A.D.Alexandrov to provide the existence and uniqueness of solutions in a general sense to certain real Monge-Ampère equations. Nevertheless, there is a lacking for a suitable geometric interpretation of the complex Monge-Ampère equations. And the techniques used for real Monge-Ampère equations are not enough to consider the complex one.
In 1976, Bedford and Taylor [BT76] applied methods in pluripotential theory to construct plurisubharmonic solutions of the Dirichlet problem for complex Monge-Ampère equation with continuous data in a strictly pseudoconvex domain. In this well-known fundamental paper, there are two considerations. The right hand side is understood as a non-negative Borel measure when u is plurisubharmonic, and not necessarily C^2. The left hand side is the positive bidegree current, which is an essential ingredient was introduced by LeLong. In the book by S. Kolodziej [Kol05], the reader will find a detailed exposition of the complex Monge-Ampère equations and Pluripotential theory.
Now, we will make a short history for complex Monge-Ampère equations with regularities on pseudoconvex domains.
Let Ω be a bounded, weakly pseudoconvex domain of $\mathbb{C}^n$ with C^2-smooth boundary $b\Omega$. For

given functions $h \geq 0$ defined in Ω and ϕ defined on $b\Omega$, the Dirichlet problem of the complex Monge-Ampère consists in finding a continuous, plurisubharmonic function u on Ω such that

$$\begin{cases} \det(u_{ij}) = h & \text{in } \Omega \\ u = \phi & \text{on } b\Omega, \end{cases} \tag{5.0.1}$$

where $u_i = \dfrac{\partial u}{\partial z_i}$, $u_{ij} = \dfrac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ and (u_{ij}) is a $n \times n$-matrix with its $(i,j)^{th}$-entry is u_{ij}.

5.1 The operator $(dd^c)^n$

We will recall the notation of $(dd^c)^n$ and give some its properties. First of all, $d = \partial + \bar{\partial}$, and $d^c = i(\partial - \bar{\partial})$. For Ω an open set in $\mathbb{C}^n$, let $\mathcal{P}(\Omega)$ denote the space of all plurisubharmonic functions on Ω, and $C(\Omega)$, $C^k(\Omega)$ the usual spaces of continuous functions, kth order continuously differentiable functions. For $u \in C^2(\Omega)$, $\mathcal{H}(u) = \left[\dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right]$ denote the complex Hessian of u. Then

$$4^n n! \det(\mathcal{H}(u))\beta_n = dd^c u \wedge ... \wedge dd^c u = (dd^c u)^n,$$

where the volume form $\beta_n = \left(\dfrac{i}{2}\right)^n \prod_{j=1}^n dz_j \wedge d\overline{z}_j$. Next, we will define the left hand side via theory of currents introduced by Bedford and Taylor, as well as via a general measure theoretic construction given by Goffman and Serrin.
Let $\mathcal{D}_{(p,q)}(\Omega)$ denote the space of test forms in Ω of bidegree (p,q) equipped with Schwartz's topology.

Definition 5.1.1. Any continuous linear functional on the space $\mathcal{D}_{(p,q)}(\Omega)$ is called a *current* of bi-degree $(n-p, n-q)$ (equivalent: of bi-dimension (p,q)) in Ω.
We equip the space of currents of bi-degree $(n-p, n-q)$ with a weak-topology by follows: a sequence T_j of currents of bi-degree $(n-p, n-q)$ converges to T if and only if $\lim_{j\to\infty} T_j(\phi) = T(\phi)$ for any $\phi \in \mathcal{D}_{(p,q)}(\Omega)$.
Let T be a current of bi-dimension (p,p) in Ω, if we have

$$(T, \omega) \geq 0,$$

for any simple positive test form $\omega = i^p \omega_1 \wedge \overline{\omega}_1 \wedge ... \wedge \omega_p \wedge \overline{\omega}_p$, with ω_k's $\in C^\infty_{(1,0)}$, the T is called a positive current.

For two (p,p)-currents S, T, the inequality

$$S \leq T$$

means that $T - S$ is a positive current.

For an increasing ordered multi-index J, we denote by J' the unique increasing multi-index such that $J \cup J' = \{1, 2, ..., n\}$ such that $|J| + |J'| = n$. Let us denote by α_{JK} the form complementary to $dz_J \wedge d\bar{z}_K$, that is

$$\alpha_{JK} = \lambda dz_{J'} \wedge d\bar{z}_{K'},$$

where λ is chosen so that $dz_J \wedge d\bar{z}_K \wedge \alpha_{JK} = \beta_n$.
Let us observe that one can identify a current $T \in \mathcal{D}'_{(p,q)}(\Omega)$ with a differential form which has distribution coefficients

$$T = \sum\nolimits'_{|J|=n-p,|K|=n-q} T_{JK} dz_J \wedge d\bar{z}_K.$$

The coefficients T_{JK} are defined by

$$(T_{JK}, \phi) = (T, \phi\alpha_{JK}).$$

Moreover, all T_{JK} are non-negative Radon measures if T is positive. For a current T with measure coefficients, one can define a norm

$$||T||_E = \sum\nolimits'_{|J|=n-p,|K|=n-q} |T_{JK}|_E,$$

where $|T_{JK}|_E$ is the total variation of T_{JK} on a compact set E.
One may also define a wedge product of a current and a smooth form ω setting

$$(T \wedge \omega, \phi) := (T, \omega \wedge \phi)$$

for any test form ϕ. If T is positive and ω is a positive $(1,1)$-form, then $T \wedge \omega$ is again positive. In particular, for a positive (p,p)-current T, and a $(n-p, n-p)$ simple form, the current $T \wedge \omega$ is a non-negative Borel measure.
We differentiate currents according to the formula

$$(DT, \phi) = -(T, D\phi),$$

for a first order differential operator D.
Now, let $u \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$, then $dd^c u$ is a bounded, positive of bidimension $(1,1)$ current, and $u.dd^c u$ is a well-defined current, so is

$$dd^c u \wedge dd^c u := dd^c(u.dd^c u),$$

in the sense that

$$\int \phi.dd^c u \wedge dd^c u = \int u.dd^c \phi \wedge dd^c u.$$

The latter current is also closed and positive. By this way, we may defined closed positive currents

$$dd^c u \wedge dd^c u \wedge ... \wedge dd^c u,$$

for $u \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$. This definition is well-defined since the following theorem by Chern-Levine-Nirenberg

Theorem 5.1.2. *(Chern-Levine-Nirenberg Inequality).*
If $K \subset\subset U \subset\subset \Omega$, then for a constant $C = C(K, U, \Omega)$, the following holds

$$||dd^c u_0 \wedge dd^c u_1 \wedge ... \wedge dd^c u_k \wedge T||_K \leq C.||u_0||_U ||u_1||_U ... ||u_k||_U ||T||_U,$$

for any closed positive T and any set of $u_j \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$. Moreover,

$$||dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k||_K \leq C(K, \Omega)||u_1||_\Omega ||u_2||_\Omega ... ||u_k||_\Omega,$$

and

$$||u_0 \wedge dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k||_K \leq C(K, \Omega)||u_0||_{L^1(\Omega)} ||u_1||_\Omega ||u_2||_\Omega ... ||u_k||_\Omega.$$

This gives us a definition of $(dd^c u)^n$ for $u \in \mathcal{P}(\Omega) \cap L^\infty_{loc}(\Omega)$ as a positive Borel measure.

Let $\mathcal{C}$ denote the cone of $n \times n$ non-negative Hermitian matrixes and define on $\mathcal{C}$ a homogeneous super-additive functional

$$\mathcal{F}(A) = \det(A)^{1/n}, \quad A \in \mathcal{C}.$$

Now, let $\mu = (\mu_{jk})$ be a matrix of Borel measures on Ω such that for any Borel set $E \subset \Omega$, $(\mu_{jk}(E)) \in \mathcal{C}$. Choose a nonnegative Borel measure λ on Ω such that μ is absolutely continuous with respect λ, i.e, $d\mu = Ad\lambda$, where A is a Borel measurable function on Ω with values in the cone $\mathcal{C}$. Then, we define

$$\mathcal{F}(\mu) = \mathcal{F}(A)\lambda.$$

If $u \in \mathcal{P}(\Omega)$, the matrix of Borel measures $\left(\dfrac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right)$ takes values in the cone $\mathcal{C}$ since $dd^c u$ is a positive $(1,1)$ current. Then, we can define

$$\Phi(u) = 4(n!)^{1/n}\mathcal{F}\left(\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right]\right).$$

Note that for smooth u, we have

$$(dd^c u)^n = \Phi^n(u)dV,$$

that means $\Phi(u)$ is essentially $[(dd^c u)^n]^{1/n}$.

Proposition 5.1.3.
1. *$\Phi(tu) = t\Phi(u)$, for $t \geq 0$, and $\Phi(u + v) \geq \Phi(u) + \Phi(v)$.*
2. *If α is a test function, then $\Phi(u * \alpha) \geq \Phi(u) * \alpha$.*
3. *If a sequence of plurisubharmonic functions u_j tends ewakly to u, and $\Phi(u_j)$ is weakly convergent, then $\Phi(u) \geq \lim_{j\to\infty} \Phi(u_j)$.*
4. *For the standard regularization $\lim_{\epsilon\to 0} \Phi(u_\epsilon) = \Phi(u)$.*
5. *$\Phi(\max(u, v)) \geq \min(\Phi(u), \Phi(v))$.*

Therefore, for $h \geq 0$, $h \in C(\Omega)$, the complex Monge-Ampère equation (5.0.1) is understood by the following ways

$$(dd^c u)^n = hdV$$

in the sense of positive bidegree currents, or

$$\Phi(u) = h^{1/n} dV$$

in the Borel measure sense.
The fact that the unique solutions of (5.0.1) defined via these setups are coincide. In this chapter, we will concentrate solutions of (5.0.1) defined in the sense of currents.

5.2 Some well-known facts and the main result

When Ω is a smooth, bounded strongly pseudoconvex domain in $\mathcal{C}^n$, a great deal of work has been done about the existence, uniqueness and regularity of this problem (cf. []). The following we briefly review some significant, classical results.

1. The classical solvability of the Dirichlet problem in [BT76] was established by Bedford and Taylor. They proved that if Ω is a strongly pseudoconvex, bounded domain in $\mathbb{C}^n$ with C^2 boundary, and if $\phi \in Lip^{\alpha}(b\Omega)$, $0 \leq h^{\frac{1}{n}} \in Lip^{\frac{\Omega}{2}}(\overline{\Omega})$, where $0 < \alpha \leq 1$, then there is an unique solution of (5.0.1) $u \in Lip^{\frac{\alpha}{2}}(\overline{\Omega})$. The pluripotential theory allows the authors to study weak solutions for the right hand side being just a nonnegative Borel measure. This result is sharp.

2. In [CKNS85], the smoothness of solution of (5.0.1) was also established. In particular, on a bounded strongly pseudoconvex domain with smooth boundary, and $\phi \in C^{\infty}(b\Omega)$, then there exixsts uniquely solution $u \in C^{\infty}(\overline{\Omega})$ and also $u \in C^{1,1}(\overline{\Omega})$ when h satisfies some critical conditions. Their approach followed the one taken for the real Monge-Ampère equations.

3. More generally, Blocki also considered the Dirichlet problem (5.0.1) on hyperconvex domains in [Blo96]. In the paper, when data $\phi \in C(b\Omega)$ can be continuously extended to a plurisubharmonic function on Ω and the right hand is nonnegative, contiunous, then the plurisubharmonic solution exists uniquely and continuously. The author also gave an example to show this existence on some hyperconvex domain but not B-regular which was first considered by Bedford and Taylor in [BT76]. However, the Hölder continuity for the solution on these domains was not verified.

4. In [Co97], Coman showed how to connect some geometrical conditions on domains in $\mathbb{C}^2$ to the existence of plurisubharmonic upper envelope in Hölder spaces. In particular, the weakly pseudoconvexity of finite type m and the fact that the Perron-Bremermann function belongs to $Lip^{\frac{\alpha}{m}}$ with corresponding data in Lip^{α} are equivalent. Again, this means the condition of finite type plays the critical role in Hölder regularity for complex Monge-Ampère equations.

5. Recently, in [Li04], these results have been extended in the case that Ω is a weakly pseudoconvex, bounded domain in $\mathbb{C}^n$ of plurisubharmonic type m. In particular, when $0 < \alpha \leq \frac{2}{m}$, and $\phi \in Lip^{m\alpha}(b\Omega)$ and if $h^{\frac{1}{n}} \in Lip^{\alpha}$, then the unque existence of the solution for (5.0.1) $u \in Lip^{\alpha}(\overline{\Omega})$ holds. The author also gives the example on complex ellipsoid to show that this result is optimal. The critical point in the proof is based on the observation by Catlin and main results of Fornaess and Sibony in [FS89] about the existence of a family of weighted functions on a such domain.

The main purpose in this chapter is to generalize above results to pseudoconvex domains, not necessarily of finite type, but admitting an f-Property. The f-Property introduced in the followings is generalized the Catlin's family of weights which is sufficient for an f-estimate for the $\bar{\partial}$-Neumann problem [Cat87, KZ10]. This is new point in the theory of complex Monge-Ampère equations, the techniques follow from solving to Cauchy-Riemann equations applied to seek complex Monge-Ampère equations.

Definition 5.2.1. For a smooth, monotonic, increasing function $f : [1 + \infty) \to [1, +\infty)$ with $\dfrac{f(t)}{t^{1/2}}$ decreasing, we say that Ω has an f-Property if there exist a neighborhood U of $b\Omega$ and a family of functions $\{\phi_\delta\}$ such that

(i) the functions ϕ_δ are plurisubharmonic, $-1 \leq \phi_\delta \leq 0$ and C^2 on U;

(ii) $i\partial\bar{\partial}\phi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\phi_\delta| \lesssim \delta^{-1}$ for any $z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\}$, where r is a defining function of Ω.

Remark 5.2.2.

1. The f-Property is a consequence of the geometric finite type of pseudoconvex domains. In [Cat83, Cat87], Catlin proved that every smooth, pseudoconvex domain Ω of finite type m in $\mathcal{C}^n$ is of the f-Property with $f(t) = t^{\epsilon}$ with $\epsilon = m^{-n^2 m^{n^2}}$. Specially, if Ω is strongly pseudoconvex, or else it is pseudoconvex of finite type in $\mathcal{C}^2$, or else decoupled or convex in $\mathcal{C}^n$ then $\epsilon = \dfrac{1}{m}$ where m is the finite type (cf. [Cat89, Kha10, McN91a, McN92a]).

2. The relation of the general type (both finite and infinite type) and the f-Property has been studied by Khanh and Zampieri [Kha10, KZ12]. Moreover, they proved if $P_1, ..., P_n : \mathcal{C} \to \mathbb{R}^+$ are functions such that $\Delta P_j(z_j) \geq \dfrac{F(|x_j|)}{x_j^2}$ or $\dfrac{F(|y_j|)}{y_j^2}$ for any $j = 1, ..., n$, then the pseudoconvex ellipsoid

$$C = \{(z_1, \ldots, z_n) \in \mathcal{C}^n : \sum_{j=1}^{n} P_j(z_j) \leq 1\}$$

has f-Property with $f(t) = (F^*(t^{-1}))^{-1}$. Here we denote F^* is the inverse function of F.

In this paper, using the f-Property we prove a "weak" Hölder regularity for the solution of the Dirichlet problem of complex Monge-Ampère equation. For convenience, we recall a suitable definition of the Hölder continuous spaces in the last chapter, but here we define function be continuous up to the boundary $b\Omega$.

Definition 5.2.3. Let f be an increasing function such that $\lim_{t\to+\infty} f(t) = +\infty$. For $\Omega \subset \mathcal{C}^n$, define the f-Hölder space on $\bar{\Omega}$ by

$$\Lambda^f(\bar{\Omega}) = \{u : \|u\|_\infty + \sup_{z,w\in\overline{\Omega}} f(|z-w|^{-1}) \cdot |u(z) - u(w)| < \infty\},$$

and set

$$\|u\|_f = \|u\|_\infty + \sup_{z,w\in\overline{\Omega}} f(|z-w|^{-1}) \cdot |u(z) - u(w)|.$$

Note that the f-Hölder space includes the standard Hölder space $\Lambda_\alpha(\bar{\Omega})$ by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha < 1$. Our main result is

Theorem 5.2.4. *(joint work with Khanh, T. V. [HK13]) Let f be in Definition 5.2.1 such that $g(t)^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty$. Assume that Ω is a bounded, pseudoconvex domain admitting the f-Property. Then, for any $0 < \alpha \le 1$, if $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and $h \ge 0$ on Ω with $h^{\frac{1}{n}} \in \Lambda^{g^\alpha}(\overline{\Omega})$, then the following Dirichlet problem of complex Monge-Ampère equation*

$$\begin{cases} \det(u_{ij}) = h & in \quad \Omega, \\ u = \phi & on \quad \partial\Omega, \end{cases} \tag{5.2.1}$$

has an unique plurisubharmonic solution $u \in \Lambda^{g^\alpha}(\overline{\Omega})$.

We organize the paper as follows. In Section 2, we will establish the sufficiently fine defining functions, as a consequence of f-Property, to construct the unique solution of (5.0.1), this solution should be contiunous and plurisubharmonic. We will also see that this condition is a generalization of the one named by plurisubharmonic of finite type m in [Li04]. Then, we will prove the main theorem 5.2.4 in the Section 3. Finally, an example in the Section 4 will show that our result can not be improved.

5.3 The f-Property

In this section, under the f-Property assumption we construct a strictly plurisubharmonic defining function with g^2-Hölder continuous, where g defined in the following theorem.

Theorem 5.3.1. *Let f be in Definition 5.2.1 such that $g(t)^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty$. Assume that Ω is a bounded, pseudoconvex domain admitting the f-Property. Then there exists a strictly plurisubharmonic defining function of Ω which belongs to g^2-Hölder space of $\bar{\Omega}$, that means, there is a plurisubharmonic function ρ such that*

1. $z \in \Omega$ *if and only if* $\rho(z) < 0$, $b\Omega = \{z \in \mathcal{C}^n : \rho(z) = 0\}$.

2. $\rho \in C^2(\Omega)$ *and* $i\partial\bar{\partial}\rho(L, \bar{L}) \geq |L|^2$.

3. ρ *is in* g^2*-Hölder space of* $\bar{\Omega}$*, that is,* $|\rho(z) - \rho(z')| \leq g(|z - z'|^{-1})^{-2}$ *for any* $z, z' \in \bar{\Omega}$.

The proof of Theorem 5.3.1 is based on the result about the existence of a family of plurisubhamonic peak functions, Theorem 5.3.2, which is proven by Khanh [Kha13]. Moreover, a remark that the f-Property with the requirement $\int_t^\infty \frac{da}{af(a)} < \infty$ is stronger than the sup-log condition, i.e , $\lim_{a\to\infty} \frac{f(a)}{\ln a} = \infty$, which implies that the Bergman metric has a lower bound with the rate $g(t) = \frac{f}{\ln}(t^{1-\eta})$, for some $\eta > 0$ [KZ12]. The following theorem is a generalization of the well-known result by Fornaess and Sibony in the case of finite type.

Theorem 5.3.2. *Under the assumptions of Theorem 5.3.1, for any* $w \in b\Omega$ *there is a* C^2*-plurisuhharmonic function* ψ_w *on* Ω *and peaking at* w*, that means,* $\psi_w(z) < 0$ *for all* $z \in \bar{\Omega} \setminus \{w\}$ *and* $\psi_w(w) = 0$*. Moreover, for any constant* $0 < \eta < 1$*, there are a positive constants* c *such that the followings hold*

1. $|\psi_w(z) - \psi_w(z')| \leq c|z - z'|^\eta$ *for any* $z, z' \in \bar{\Omega}$*,*

2. $\psi_w(z) \leq -G^\eta(|z - w|)$ *for any* $z \in \bar{\Omega}$*,*

where $G(\delta) := \big(g^*(\gamma\delta^{-1})\big)^{-1}$*. Here, the superscript* * *denotes the inverse function and* $\gamma > 0$ *sufficiently small.*

Remark 5.3.3. We also note that if Ω is strongly pseudoconvex, we can choose $\eta = 1$.

Before giving the proof of Theorem 5.3.1, we need the following technique lemma

Lemma 5.3.4. *Let* $\omega(t) := g(t^{-\frac{1}{\eta}})^{-2}$*. Then we have*

1. ω *is increasing function on* $(0, 1)$ *and* $h(0) = 0$*.*

2. *For a suitable choice of* $\eta > 0$*, there is* $\delta > 0$ *such that* ω *is concave upward on* $(0, \delta)$*.*

3. *The inequality*
$$|\omega(t) - \omega(s)| \leq \omega(|t - s|)$$
holds for any $t, s \in [0, \delta)$*.*

4. *For a constant* $c > 0$*, there is* $c' > 0$ *such that* $\omega(ct) \leq c'\omega(t)$*.*

Proof. Before going to proof the lemma, we give some calculations on function g. By the definition of g, i.e., $\frac{1}{g(t)} := \int_t^\infty \frac{da}{af(a)} < \infty$, we have

$$\frac{\dot{g}(t)}{g(t)} = \frac{g(t)}{tf(t)}. \tag{5.3.1}$$

and

$$\frac{\ddot{g}(t)}{\dot{g}(t)} = \frac{2\dot{g}(t)}{g(t)} - \frac{1}{t} - \frac{\dot{f}(t)}{f(t)}. \tag{5.3.2}$$

Since $\dfrac{f(t)}{t^{1/2}}$ is decreasing on $(1, +\infty)$, we obtain

$$\frac{f(t)}{g(t)} = f(t)\int_t^\infty \frac{da}{af(a)} = f(t)\int_t^\infty \frac{a^{1/2}}{f(a)}\cdot\frac{da}{a^{3/2}} \geq f(t)\frac{t^\epsilon}{f(t)}\int_t^\infty \frac{da}{a^{3/2}} = 2.$$

i.e., $\dfrac{g(t)}{f(t)} \leq \frac{1}{2}$, and also $\dfrac{t\dot{f}(t)}{f(t)} \leq \frac{1}{2}$. We notice that the equalities hold if and only if $f(t) = c\sqrt{t}$ for constant $c > 0$ (this holds if and only if Ω is strongly pseudoconvex), that means

$$\frac{g(t)}{f(t)} + \frac{t\dot{f}(t)}{f(t)} = 1.$$

Otherwise, if Ω is not strongly pseudoconvex, then

$$\frac{g(t)}{f(t)} + \frac{t\dot{f}(t)}{f(t)} < 1.$$

Now, we are ready for proving Lemma 5.3.4. We have $\dot{\omega}(t) = \frac{2}{\eta}t^{-\frac{1}{\eta}-1}\dot{g}(t^{-\frac{1}{\eta}})g^{-3}(t^{-\frac{1}{\eta}}) \geq 0$ and

$$\begin{aligned}\ddot{\omega}(t) &= \left(\frac{2}{\eta^2}t^{-\frac{1}{\eta}-2}\dot{g}(t^{-\frac{1}{\eta}})g^{-3}(t^{-\frac{1}{\eta}})\right)\left[-\left(\eta + 1 + \frac{t^{-\frac{1}{\eta}}\ddot{g}(t^{-\frac{1}{\eta}})}{\dot{g}(t^{-\frac{1}{\eta}})} - 3\frac{t^{-\frac{1}{\eta}}\dot{g}(t^{-\frac{1}{\eta}})}{g(t^{-\frac{1}{\eta}})}\right)\right] \\ &= \left(\frac{2}{\eta^2}t^{-\frac{1}{\eta}-2}\dot{g}(t^{-\frac{1}{\eta}})g^{-3}(t^{-\frac{1}{\eta}})\right)\left[-\left(\eta - \frac{t^{-\frac{1}{\eta}}g(t^{-\frac{1}{\eta}})}{f(t^{-\frac{1}{\eta}})} - \frac{t^{-\frac{1}{\eta}}\dot{f}(t^{-\frac{1}{\eta}})}{f(t^{-\frac{1}{\eta}})}\right)\right].\end{aligned} \tag{5.3.3}$$

If Ω is strongly pseudoconvex, we can choose $\eta = 1$ such that the estimates in Theorem 5.3.2 hold. Otherwise there is a constant $\eta < 1$ such that the bracket term $[\dots]$ in the last line of (5.3.3) is non-negative. Therefore, ω is concave upward.

Now we prove that $|\omega(t) - \omega(s)| \geq \omega(|t-s|)$ for any $t, s \in (0, \delta)$. Assume $t \geq s$, for a fixed $s \in [0, \delta)$ we set $k(t) := \omega(t) - \omega(s) - \omega(t-s)$. Since ω is concave upward, $\dot{k}(t) = \dot{\omega}(t) - \dot{\omega}(t-s) \geq 0$. That is k is increasing, so we obtain $k(t) \geq k(s) = 0$. That is the proof of inequality.

For the inequality (4), we notice that if $c \leq 1$ then $\omega(ct) \leq \omega(t)$ since ω is increasing. Otherwise, if $c > 1$ we use the fact that $\dfrac{g(t)}{t^{1/2}}$ is decreasing (this is obtained from $\dfrac{t\dot{g}(t)}{g(t)} = \dfrac{g(t)}{f(t)} \leq \dfrac{1}{2}$), we have

$$\omega(ct) = (ct)^{1/2}\frac{\omega(ct)}{(ct)^{1/2}} \leq (ct)^{1/2}\frac{\omega(t)}{(t)^{1/2}} = \frac{1}{\sqrt{c}}\omega(t).$$

This completes the proof of Lemma 5.3.4. $\square$

Now, we will prove the aim of this section .

Proof of Theorem 2.1. Fix $w \in b\Omega$, we define

$$\rho_w(z) = -\frac{2}{\gamma^2}\omega\left(-\psi_w(z)\right) + |z-w|^2. \tag{5.3.4}$$

We will show that the function $\rho_w(z)$ satisfies the following properties

1. $\rho_w(z) < 0$, for $z \in \Omega$, $\rho_w(w) = 0$,
2. $\rho_w \in C^2(\Omega)$ and $\partial\bar{\partial}\rho_w(L, \bar{L}) \geq |L|^2$ in distribution sense for $z \in \Omega$, and $L \in T^{1,0}_z\mathcal{C}^n$.
3. ρ_w is in g^2-Hölder space in $\bar{\Omega}$.

Proof of (1). From (1) in Theorem 5.3.2 and ω increasing, we have

$$\begin{aligned}\omega(-\psi_w(z)) &\geq \omega(G^\eta(|z-w|))\\ &= \left(g\left(G(|z-w|)^{-1}\right)\right)^2\\ &= \left(g(g^*((\gamma|z-w|)^{-1}))\right)^{-2}\\ &= \gamma^2|z-w|^2.\end{aligned} \tag{5.3.5}$$

Hence, we have

$$\rho_w(z) = -\frac{2}{\gamma^2}g\bigg((-\psi_w)^{\frac{-1}{\eta}}(z)\bigg)^{-2} + |z-w|^2 \leq -|z-w|^2 < 0,$$

where $w \in b\Omega$, and $z \in \Omega$. Moreover, since $\psi_w(w) = 0$ and $\omega(0) = 0$, that implies $\rho_w(w) = 0$ for any $w \in b\Omega$.

Proof of the assertation (2). Fix $w \in b\Omega$, the Levi form of $\omega(-\psi_w)$ on Ω is following

$$i\partial\bar{\partial}\omega(-\psi_w)(X, \bar{X}) = \dot{\omega} i\partial\bar{\partial}\psi_w(X, \bar{X}) - \ddot{\omega}|X\psi_w|^2 \geq 0 \tag{5.3.6}$$

Proof of (3). Lemma 5.3.4, we have

$$\begin{aligned}\big|\omega(-\psi_w(z)) - \omega(-\psi_w(z'))\big| &\leq \omega\left(|\psi_w(z) - \psi_w(z')|\right)\\ &\leq \omega(c|z-z'|^\eta)\\ &\leq c'\omega(|z-z'|^\eta) = g(|z-z'|^{-1})^{-2}.\end{aligned} \tag{5.3.7}$$

Finally, since Ω is bounded and $g(t) \lesssim t^{\frac{1}{2}}$, we can show that

$$\left||z-w|^2 - |z'-w|^2\right| \lesssim |z-z'| \lesssim g\big(|z-z'|^{-1}\big)^{-2}. \tag{5.3.8}$$

The last two inequalities verify that $\rho_w(z) \in \Lambda^{g^2}(\overline{\Omega})$ for any $w \in b\Omega$.

Now, we define

$$\rho(z) = \sup_{w \in b\Omega} \rho_w(z).$$

The above properties of ρ_w imply that the function ρ is plurisubharmonic in Ω since the well-known result by LeLong, and since $g(0) = 0$, and $g : [0, \infty] \to [0, \infty]$, then ρ is also g^2-Hölder continuous in $\bar{\Omega}$ due to the fact that, from theory of Modulus of continuity, the superior envelope of these such functions belongs to the same space. Moreover in the distribution sense we have

$$\partial\bar{\partial}\rho(L, \bar{L}) \geq |L|^2.$$

□

5.4 Proof of the main theorem 5.2.4

Let Ω be a bounded open set in $\mathbb{C}^n$, and $\mathcal{P}(\Omega)$ denote the space of plurisubharmonic functions on Ω. The following proof of Theorem 5.2.4 is adapted from the argument given by Bedford and Taylor [BT76, Theorem 6.2] for weakly pseudoconvex domains. Based on the approach in [BT76], we need the following proposition.

Proposition 5.4.1. *Let Ω be bounded, pseudoconvex domain. Assume that there is a strictly plurisubharmonic defining function ρ of Ω such that $\rho \in \Lambda^{g^2}(\bar{\Omega})$. Let $0 < \alpha \leq 1$, and $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and let $h \geq 0$ with $h^{1/n} \in \Lambda^{g^\alpha}(\overline{\Omega})$. Then, for all $\zeta \in b\Omega$, there exists $v_\zeta \in \Lambda^{g^\alpha}(\Omega) \cap \mathcal{P}(\Omega)$ such that*

(i) $v_\zeta(z) \leq \phi(z)$ for all $z \in b\Omega$, and $v_\zeta(\zeta) = \phi(\zeta)$;

(ii) $\|v_\zeta\|_{\Lambda^{g^\alpha}(\overline{\Omega})} \leq C_0$;

(iii) det $(H(v_\zeta)(z)) \geq h(z)$.

where C_0 is a positive constant depending only on Ω and $||\phi||_{\Lambda^{t^\alpha}(b\Omega)}$.

Proof. For each $\zeta \in b\Omega$, we may choose the family $\{v_\zeta\}$ by

$$v_\zeta(z) = \phi(\zeta) - K[-2\rho(z) + |z - \zeta|^2]^{\frac{\alpha}{2}}, \quad z \in \overline{\Omega}$$

where ρ is defined by Theorem 5.3.1, and K will be chosen step by step later.

It is easy to see that $v_\zeta(\zeta) = \phi(\zeta)$. Moreover, with the choice K such that $K \geq c_\phi$ where $c_\phi = \sup\limits_{z \neq \zeta \in b\Omega} \dfrac{|\phi(z) - \phi(\zeta)|}{|z - \zeta|^\alpha}$, we have

$$v_\zeta(z) \leq -K|z - \zeta|^\alpha + \phi(\zeta) \leq \phi(z), \quad \text{for all} \quad z \in b\Omega. \tag{5.4.1}$$

This proves (i).

For the proof of (ii), we have the following estimates

$$
\begin{aligned}
|v_\zeta(z) - v_\zeta(z')| &= \left| [-2\rho(z) + |z-\zeta|^2]^{\frac{\alpha}{2}} - [-2\rho(z') + |z'-\zeta|^2]^{\frac{\alpha}{2}} \right| \\
&\le \left| -2\rho(z) + |z-\zeta|^2 + 2\rho(z') - |z'-\zeta|^2 \right|^{\frac{\alpha}{2}} \\
&\le \left[2|\rho(z) - \rho(z')| + ||z-\zeta|^2 - |z'-\zeta|^2| \right]^{\frac{\alpha}{2}}. \\
&\lesssim g^{-\alpha}(|z-z'|^{-1}).
\end{aligned}
\tag{5.4.2}
$$

Here, the first inequality follows by the fact that $|t^{\frac{\alpha}{2}} - s^{\frac{\alpha}{2}}| \le |t-s|^{\frac{\alpha}{2}}$ for all t, s small and $\alpha \le 1$; the last inequality follows by Theorem 5.3.1 and (5.3.8). This implies $v_\zeta \in \Lambda^{g^\alpha}(\overline{\Omega})$ for all $\zeta \in b\Omega$. Moreover $\|v_\zeta\|_{\Lambda^{g^\alpha}(\overline{\Omega})}$ is independent on ζ.

To prove (iii), we compute $(v_\zeta)_{ij}$ on Ω

$$
\begin{aligned}
(v_\zeta(z))_{ij} =& K\frac{\alpha}{2}(-2\rho(z) + |z-\zeta|^2)^{\frac{\alpha}{2}-2}\Big[(-2\rho(z) + |z-\zeta|^2)(2\rho(z)_{ij} - \delta_{ij}) \\
&+ \big(1 - \frac{\alpha}{2}\big)(-2\rho_i + \bar{z}_i - \bar{\zeta}_i)\overline{(-2\rho_j + \bar{z}_j - \bar{\zeta}_j)}\Big].
\end{aligned}
\tag{5.4.3}
$$

Hence

$$
i\partial\bar\partial v_\zeta(X,X) \ge K\frac{\alpha}{2}(-2\rho(z)+|z-\zeta|^2)^{\frac{\alpha}{2}-1}(2i\partial\bar\partial\rho(X,X) - |X|^2) \ge K\frac{\alpha}{2}(-2\rho(z)+|z-\zeta|^2)^{\frac{\alpha}{2}-1}|X|^2.
$$

for any $X \in T^{1,0}\mathcal{C}^n$. Here the last inequality follows from Theorem 5.3.1(2).Thus v_ζ is plurisubharmonic and furthermore we obtain

$$
\det[(v_\zeta)_{ij}](z) \ge \left[K\frac{\alpha}{2}(-2\rho(z) + |z-\zeta|^2)^{\left(\frac{\alpha}{2}-1\right)} \right]^n. \tag{5.4.4}
$$

Now, let's choose

$$
K \ge \max\left\{ \frac{2}{\alpha} \max_{z\in\overline{\Omega}, \zeta\in b\Omega} (-2\rho(z) + |z-\zeta|^2)^{1-\frac{\alpha}{2}} \|h^{1/n}\|_{L^\infty(\Omega)}, c_\phi \right\}.
$$

Then

$$
\det[(v_\zeta)_{ij}](z) \ge \|h^{1/n}\|^n_{L^\infty(\Omega)} \ge (h^{1/n}(z))^n = h(z), \tag{5.4.5}
$$

for all $z \in \Omega$, and $\zeta \in b\Omega$. This completes the proof of Propoition 5.4.1.

$\square$

Before to give a proof of Theorem 5.2.4, we re-call the existence theorem for the problem (5.0.1) by Bedford and Taylor [BT76, Theorem 8.3, page 42].

Theorem 5.4.2 (Bedford-Taylor [BT76])**.** *Let Ω be a bounded open set in $\mathcal{C}^n$. Let $\phi \in C(\partial\Omega)$ and $0 \leq h \in C(\Omega)$. If the Perron Bremermman family*

$$\mathcal{B}(\phi, h) := \left\{ v \in \mathcal{P}(\Omega) \cap C(\Omega) : det[(v)_{ij}] \geq h \text{ and } \lim \sup_{z \to z_0} v(z) \leq \phi(z_0), \text{ for all } z_0 \in b\Omega \right\}.$$

is empty, and its upper envelope

$$u = \sup\{v : v \in \mathcal{B}(\phi, h)\} \tag{5.4.6}$$

is continuous on $\bar{\Omega}$ with $u = \phi$ on $\partial\Omega$, then u is a solution to the Dirichlet problem (5.0.1)*.*

Proof of Theorem 5.2.4. First, we see that the set $\mathcal{B}(\phi, h)$ is non-empty, in particular, it contains the family of $\{v_\zeta\}_{\zeta \in b\Omega}$ in Proposition 5.4.1. The proof of this theorem will be completed if the upper envelope defined in (5.4.6) has the properties

1. $u(\zeta) = \phi(\zeta)$ for all $\zeta \in \partial\Omega$;
2. $u \in \Lambda^{g^\alpha}(\bar{\Omega})$.

We note that the uniqueness of solution follows from the Minimum Principle.

Now, we define another upper envelope, for each $z \in \overline{\Omega}$,

$$v(z) := \sup_{\zeta \in b\Omega} \{v_\zeta(z)\}.$$

Since the first property of $\{v_\zeta\}$ in Proposition 5.4.1, we have

$$\begin{aligned} v(\zeta) \geq v_\zeta(\zeta) = \phi(\zeta) \quad &\text{for all } \zeta \in b\Omega, \\ v(z) \leq \phi(z) \quad &\text{for all } z \in b\Omega. \end{aligned} \tag{5.4.7}$$

and so $v = \phi$ on $b\Omega$.
Then, from the second property in Proposition 5.4.1

$$|v_\zeta(z) - v_\zeta(z')| \leq C_0(g^\alpha(|z - z'|^{-1}))^{-1} \quad \text{for all } z, z' \in \bar{\Omega}.$$

notice that C_0 is independent on ζ so taking the supremum in ζ, theory Modulus of continuity again implies that

$$|v(z) - v(z')| \leq C_0(g^\alpha(|z - z'|^{-1}))^{-1} \quad \text{for all } z, z' \in \bar{\Omega}.$$

By Proposition 2.8 in [BT76], the following inequality holds

$$\det[(v)_{ij}](z) \geq \inf_{\zeta \in b\Omega} \{\det[(v_\zeta)_{ij}](z)\} \geq h(z), \quad \text{for all } z \in \Omega.$$

Thus, we conclude that $v \in \mathcal{B}(\phi, h) \cap \Lambda^{g^\alpha}(\overline{\Omega})$ and $v(\zeta) = \phi(\zeta)$ for any $\zeta \in \partial\Omega$.

By a similar construction there exists a plurisuperharmonic function $w \in \Lambda^{g^\alpha}(\bar{\Omega})$ such that $w(\zeta) = \phi(\zeta)$ for any $\zeta \in \partial\Omega$. Thus, $v(z) \leq u(z) \leq w(z)$ for any $z \in \bar{\Omega}$, and hence $u(\zeta) = \phi(\zeta)$ for any $\zeta \in \partial\Omega$. We also obtain

$$|u(z) - u(\zeta)| \leq \max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}, \|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\}(g^\alpha(|z-\zeta|^{-1})^{-1} \quad \text{for any} \quad z \in \bar{\Omega}, \zeta \in \partial\Omega. \tag{5.4.8}$$

Here, the inequality follows by $w, v \in \Lambda^{g^\alpha}(\bar{\Omega})$ and $v(\zeta) = u(\zeta) = w(\zeta) = \phi(\zeta)$ for any $\zeta \in \partial\Omega$.

Finally, we want to show that (5.4.8) also holds for $\zeta \in \Omega$. For any small vector $\tau \in \mathbb{C}^n$, we define

$$V(z,\tau) = \begin{cases} u(z) & \text{if } z+\tau \notin \Omega, z \in \overline{\Omega}, \\ \max\{u(z), V_\tau(z)\}, & \text{if } z, z+\tau \in \Omega \end{cases}$$

where

$$V_\tau(z) = u(z+\tau) + \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1})$$

and here

$$K_1 \geq \max_{k\in\{1,\dots,n\}} \binom{n}{k}^{1/k} \|h^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}(\bar{\Omega})}, \quad K_2 \geq K_1|z|^2, \quad \text{and} \quad K_3 \geq \max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}, \|w\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\}.$$

We will show that $V(z,\tau) \in \mathcal{B}(\phi, h)$. Observe that $V(z,\tau) \in \mathcal{P}(\Omega)$ for all z, τ. Moreover, for $z \in \partial\Omega$, $z+\tau \in \Omega$, we have

$$\begin{aligned} V_\tau(z) - u(z) =& u(z+\tau) - u(z) + \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1}) \\ \leq& \max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}, \|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\} g^{-\alpha}(|\tau|^{-1}) + \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1}) \\ \leq& 0. \end{aligned} \tag{5.4.9}$$

Here the first inequality follows by (5.4.8) and the second follows by the choices of K_2 and K_3. This implies that $\limsup_{z\to\zeta} V(z,\tau) \leq \phi(\zeta)$ for all $\zeta \in \partial\Omega$. For the proof of $\det[V(z,\tau)_{ij}] \geq h(z)$, we need the following lemma.

Lemma 5.4.3. *Let* $(\alpha_{ij}) \geq 0$ *and* $\beta \in (0, +\infty)$. *Then*

$$det[\alpha_{ij} + \beta I] \geq \sum_{k=0}^{n} \beta^k \det(\alpha_{ij})^{(n-k)/n}$$

Proof of Lemma 5.4.3. Let $0 \le \lambda_1 \le \cdots, \lambda_n$ be the eigenvalue of (α_{ij}). We have

$$\begin{aligned}\det[\alpha_{ij}+\beta] &= \prod_{j=1}^{n}(\lambda_j+\beta)\\ &\ge \sum_{k=0}^{n}\left(\beta^k \prod_{j=k+1}^{n}\lambda_j\right)\\ &\ge \sum_{k=0}^{n}\left(\beta^k \det[\alpha_{ij}]^{(n-k)/n}\right).\end{aligned} \tag{5.4.10}$$

Here the last inequality follows by

$$\det[\alpha_{ij}] = \prod_{j=1}^{n}\lambda_j \le \left(\prod_{j=k+1}^{n}\lambda_j\right)^{n/(n-k)}.$$

□

Continuing the proof of Theorem 5.2.4, for any $z, z+\tau \in \Omega$ we have

$$\begin{aligned}\det[(V_\tau(z))_{ij}] &= \det[u_{ij}(z+\tau) + K_1 g^{-\alpha}(|\tau|^{-1})I]\\ &\ge \det[u_{ij}(z+\tau)] + \sum_{k=1}^{n} K_1^k [g^\alpha(|\tau|^{-1})]^{-k}.\det[u_{ij}(z+\tau)]^{\frac{n-k}{n}}\\ &\ge h(z+\tau) + \sum_{k=1}^{n} K_1^k [g^\alpha(|\tau|^{-1})]^{-k}.(h(z+\tau))^{\frac{n-k}{n}}.\end{aligned} \tag{5.4.11}$$

where the first inequality follows by Lemma 5.4.3. Since $h^{\frac{1}{n}} \in \Lambda^{g^\alpha}(\Omega)$, we obtain

$$h^{\frac{1}{n}}(z) - h^{\frac{1}{n}}(z+\tau) \le g^{-\alpha}(|\tau|^{-1})\|h^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}}, \quad \text{for any} \quad z, z+\tau \in \Omega,$$

and hence

$$h(z) \le h(z+\tau) + \sum_{k=1}^{n}\binom{n}{k} h(z+\tau)^{(n-k)/n}\left(g^{-\alpha}(|\tau|^{-1})\|h^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}}\right)^k \tag{5.4.12}$$

Combining (5.4.11), (5.4.12) with the choice of K_1, we get

$$\det[(V_\tau)_{ij}](z) \ge h(z), \quad \text{for any} \quad z, z+\tau \in \Omega.$$

We conclude that $V(z,\tau) \in \mathcal{B}(\phi, h)$. It follows that for all $z \in \Omega$, $V(z,\tau) \le u(z)$. If $z+\tau \in \Omega$, this yields

$$\begin{aligned}u(z+\tau) - u(z) \le& V(\tau, z) - \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1}) - u(z)\\ \le& \left(-K_1|z|^2 + K_2 + K_3\right) g^{-\alpha}(|\tau|^{-1})\\ \le& \left(K_2 + K_3\right) g^{-\alpha}(|\tau|^{-1})\end{aligned} \tag{5.4.13}$$

By reversing the role of z and $z+\tau$, we assert that $u \in \Lambda^{g^\alpha}(\overline{\Omega})$. This completes the proof. □

5.5 A special domain

In this section, we will give an example to show why the above estimates cannot be improved. For domains satisfied $t^{\frac{1}{m}}$-Property, the exmaple in [Li04] supports our claim. As a next step, we consider the problem on the following complex ellipsoid of exponential type in $\mathbb{C}^2$

$$E = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = \exp\left(1 - \frac{1}{|z_1|^\alpha}\right) + |z_2|^2 < 1 \right\}, \tag{5.5.1}$$

with $0 < \alpha \leq 1/2$.
In [Li04], the author also considered the complex Monge-Ampère equation on this domain. The unique solution stills be continuous, but it can not belong to any classical Hölder space. Here, we will give a new view point for this example, in particular, the solution should be in some weak Hölder spaces. The exponeniall complex ellipsoid E satisfies f-Property, with $f(\delta) := (1+\log\delta)^{\frac{1}{\alpha}}$ ([Kha10, KZ12]).

We consider the following complex Monge-Ampère equation on E:

$$\begin{cases} (dd^c u)^2 = 0 & \mathcal{T}on \quad E \\ u = \phi & \mathcal{T}on \quad bE, \end{cases} \tag{5.5.2}$$

where

$$\phi(z) = |z_1|^{2\alpha} \in \Lambda^{t^\alpha}(bE).$$

We can easily check that u is the unique solution of (5.5.2) on E.

$$u(z) = (F^*(1 - |z_2|^2))^\alpha, \quad z \in E,$$

where $F^*(\delta) = (1 - \log\delta)^{-\frac{1}{\alpha}}$. We want to prove that $u(z) \in \Lambda^{g^\alpha}(\overline{E})$, where g defined in the main theorem, and

$$g(\delta) = \left(\frac{1}{\alpha} - 1\right)(1 + \log\delta)^{\frac{1}{\alpha}-1}.$$

Now, let $\widetilde{F}(z) = (F^*(1 - |z|^2))^\alpha$, then since F^* is increasing and also concave

$$\begin{aligned} |\widetilde{F}(z) - \widetilde{F}(z')| &\leq \left(F^*\left(||z|^2 - |z'|^2|\right)\right)^\alpha \\ &\leq \left(F^*\left(|z - z'||z + z'|\right)\right)^\alpha \\ &\lesssim f(|z - z'|^{-1})^{-\alpha} \\ &\lesssim g(|z - z'|^{-1})^{-\alpha} \end{aligned} \tag{5.5.3}$$

where the third inequality follows from the fact that $F^*\left(\frac{\delta}{t}\right) \leq \frac{F^*(\delta)}{t}$ again. Therefore, this implies the first assertation, $u(z) \in \Lambda^{g^\alpha}(\overline{E})$.
□

Remark 5.5.1. This example again says that no matter how smooth is the boundary data ϕ, even $\phi \in C^\infty(\Omega)$, and that no matter how smooth is $h^{1/n}$ on $\overline{\Omega}$, the unique plurisubharmonic solution u may not belong to $\Lambda^g(\Omega)$, for any such g and f as the above example. However, as the mention in the introduction, in [CKNS85], the authors provided the hypoelliptcity for complex Monge-Ampère operator, although this case is actually of F-type, with $F(t) = t^2$. The factor is that the non-vanishing right hand side and the constant boundary data play a critical role in their consideration. Hence, elliptic regularity theory was applied successfully. In our case, the singularity is vanishing of the right hand side at some points. Naturally, the question is that if we could improve the result in [CKNS85] when the right hand side has only one zeros and this zeros is of finite order. For instance, we consider

$$\mathrm{CMA}(u) := \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) = h,$$

where $h \approx |z|^{2m}$, with $m > 1$. Clearly, 0 is the unique singularity of finite order $2m$ in the sense [CKNS85]. Let h is belongs to Sobolev space $H^s(\Omega)$, the open question is that if the solution $u \in H^r$, for some $r = r(s, m)$.

Appendix A

Functional Analysis

A.1 Spectral Theorem (continuous functional calculus)

Theorem A.1.1. *Let $\mathcal{A}$ be a bounded, self-adjoint operator on a Hilbert space H. Let $\sigma(\mathcal{A})$ denote the spectrum of $\mathcal{A}$. Then, there exists a unique map $\Phi : \mathcal{B}(\sigma(\mathcal{A})) \to \mathcal{L}(H)$ such that*

1. *Φ is an algebra homomorphism, that is $\Phi(fg) = \Phi(f)\Phi(g)$, and*

$$\Phi(1) = I; \Phi(\lambda f + g) = \lambda\Phi(f) + \Phi(g); \Phi(\bar{f}) = \Phi(f)^*.$$

2. *Φ is continuous.*

3. *If $f(x) = x$, then $\Phi(f) = \mathcal{A}$.*

Since $(\Box_b\phi, \phi)_{L^2_{0,q}(M)} \geq 0$, for any $\phi \in L^2_{0,q}(M)$, and whenever the *a priori* estimate 1.4.3 holds, $\sigma(\Box_b) \subset [0, \infty)$. Therefore, there exists a unique algebra homomorphism $\Phi : \mathcal{B}([0,\infty)) \to \mathcal{L}(L^2_{0,q}(M))$ is continuous. The corresponding bounded,linear operator for $e_s := e^{-sx}$ is denoted by $e^{-s\Box_b}$. Notice that $\frac{d}{ds}\big(- e^{-sx}\big) = xe^{-sx}$, so

$$\frac{d}{ds}\big(- e^{-s\Box_b}\big) = \Box_b\big[e^{-s\Box_b}\big].$$

A.2 Distributions in Product spaces

Definition A.2.1. Let Ω_1, Ω_2 contained in M. and let $u_1 \in C(\Omega_1), u_2 \in C(\Omega_2)$, then the function $u_1 \otimes u_2$ in $\Omega_1 \times \Omega_2 \subset M \times M$ is defined by

$$(u_1 \otimes u_2)(x_1, x_2) = u_1(x_1)u_2(x_2),$$

for $x_1 \in \Omega_1, x_2 \in \Omega_2$, is called the tensor product of u_1 and u_2

Theorem A.2.2. *If $u_j \in \mathcal{D}'(\Omega_j), j = 1, 2,$ then there is a unique distribution $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ such tha*

$$u(\phi_1 \otimes \phi_2) = u_1(\phi_1)u_2(\phi_2), \;\; \phi_j \in C_0^\infty(\Omega_j).$$

Also, we have

$$u(\phi) = u_1[u_2(x_1, x_2))] = u_2[u_1(x_1, x_2))], \;\; \phi \in C_0^\infty(\Omega_1 \times \Omega_2),$$

where u_j acts on the following function as a function of x_j only.

Theorem A.2.3. *(The Schwartz Kernel Theorem) Every $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ defines a linear map $\mathfrak{K}$ from $C_0^\infty(\Omega_2)$ to $\mathcal{D}'(\Omega_1)$ by*

$$< \mathfrak{K}\phi, \psi >= K(\phi \otimes \psi), \;\; \phi \in C_0^\infty(\Omega_2), \psi \in C_0^\infty(\Omega_1),$$

which is continous in the sense that $\phi_{\mathrm{j}} \to 0$ in $\mathcal{D}'(\Omega_1)$ if $\phi_j \to 0$ in $C_0^\infty(\Omega_2)$.
Conversely, to every such linear map $\mathfrak{K}$, there is a unique distrbution K such that

$$< \mathfrak{K}\phi, \psi >= K(\phi \otimes \psi), \;\; \phi \in C_0^\infty(\Omega_2), \psi \in C_0^\infty(\Omega_1).$$

A.3 Interpolation Theorem

Definition A.3.1. Let (X, μ) and (Y, ν) be two measure spaces and let T be a linear operator from a linear subspace of measurable functions on (X, μ) into measurable functions defined on (Y, ν). T is called an operator of type (p, q) if there exists a constant $M > 0$ such that

$$||Tf||_{L^q(Y)} \leq M.||f||_{L^p(X)} \tag{A.3.1}$$

for all $f \in L^p(X)$. The least M is called the (p, q)-norm of T.

Theorem A.3.2. *(Marcinkiewicz) Let (X, μ) and (Y, ν) be two measure spaces and let T be a linear operator from a linear subspace of measurable functions on (X, μ) into measurable functions defined on (Y, ν). Let p_0, p_1, q_0, q_1 be numbers such that $1 \leq p_i, q_i \leq \infty$ for $i = 0, 1$. Then, if T is of type (p_i, q_i) with (p_i, q_i)-norm M_i, $i = 0, 1$, then T is of type (p, q) and*

$$||Tf||_{L^{q_t}(Y)} \leq M_0^{1-t}.M_1^t.||f||_{L^{p_t}(X)}. \tag{A.3.2}$$

provided

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad and \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1},$$

with $0 < t < 1$.

Appendix B

On the globally analytic hypoellipticity for $\Box_b$-operator on compact CR manifolds

In this appendix, we will discuss the global analytic regularity of the $\Box_b$ operator on a general CR manifold of real $(2n-1)$ dimension, with $n \geq 3$. In particular, if M is a CR-manifold satisfying the conditions D_q^ϵ and $(CR - P_q)$, we consider the following equation

$$\Box_b v = f.$$

If f is globally analytic, we conclude that v is globally analytic as well. The methods applied in this paper are inspired from [Ta76], [Ta81].

B.0.1 Analytic Class and Some Geometrical Conditions

First, we recall the standard definition of analytic class.

Definition B.0.3. A smooth function f belongs to the analytic class $\mathbb{A}(M)$ at x_0 provided there exists a neighborhood $U \subset M$ of x_0 and constant C_f such that for all multi-indices $\alpha = (\alpha_1, ..., \alpha_{2n-1})$

$$|D_x^\alpha f(x)| \leq C_f^{|\alpha|+1}(|\alpha|)!,$$

for any $x \in U$, where the derivative symbol $D_x^\alpha = \big(X_1\big)_x^{\alpha_1} ... \big(X_{2n-2}\big)_x^{\alpha_{2n-2}} T_x^{\alpha_{2n-1}}$.
The following proposition is necessary

Proposition B.0.4. *The space $\mathbb{A}(M)$ is a vector space and a ring, with respect to the arithmetic product of functions. Moreover it is also closed under differentiation.*

The natural definition of analytic class of $(0,q)$-forms is: a $(0,q)$-forms $f = \sum_{|I|=q} f_J \overline{\omega}_I$ belongs to the analytic class $\mathbb{A}_q(M)$ if its each coefficient is in $\mathbb{A}(M)$.

In order to state our results, we need some critical geometrical conditions on M. The first one is the $D^\epsilon(q)$ condition defined in Chapter 1. Note that the $Y(q)$ condition, i.e., $0 < \sigma_q < \tau$, for all possible σ_q, implies the $D^\epsilon(q)$ condition.

Definition B.0.5. Seting correspondingly $0 \leq s^+(x), s^-(x), s^0(x) \leq n-1$ the number of positive, negative and zero eigenvalues of the Levi form at the point $x \in M$, we assume that $s^0(x) = s^0$ be a constant, uniformly in x. That means the Levi form has constant rank $s^+(x) + s^-(x) = n - s^0 - 1$, for all $x \in M$.

Proposition B.0.6. *If the $D^\epsilon(q)$ condition holds and the Levi matrix (c_{ij}) has constant rank $n-q$, the $Y(q)$ condition holds as well.*

Proof. It is easily to derive that $\tau \geq 0$. Now, if $\tau > 0$, we are done. Otherwise, we can see that $\sigma_q = 0$ for all possible one, since $D^\epsilon(q)$ holds. Now, the constant rank $n-q$ means that the Levi matrix has $(q-1)$ zero-eigenvalues, all the rest are non-zeros. This implies that there is some σ_q different from zero, and we have a contradiction. Therefore, the $Y(q)$ condition holds. □

Next, we will introduce a "good" vector field T' playing a critical role in our computing. on some open subset of M such that the Lie brackets of T' with $L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}$ are independent on T.

"Good"-(T) Condition. Suppose that T is real analytic, nowhere zero. M is called to satisfy the *good* $-T$ condition if there exist a finite sequence of $a_1, ..., a_{n-1}, b_1, ..., b_{n-1}$ such that the following vector field

$$T' = T + \sum_{j=1}^{n-1} a_j L_j + \sum_{j=1}^{n-1} b_j \overline{L}_j \tag{B.0.1}$$

has the same properties as T and

$$[T', \mathcal{L}] = 0 \quad \mod (L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}),$$

for all $\mathcal{L} \in span\{L_1, ..., L_{n-1}\}$.
For example, when the Levi matrix is invertible, the sequence of $a_1, ..., a_{n-1}, b_1, ..., b_{n-1}$ is determined from the coefficients of T in the commutator $[T, L_k]$ and $[T, \overline{L}_k]$, for $k = 1, ..., n-1$.
In the local frame, we can identify the operator $\Box_b$ with

$$\Box_b = \sum_{j,k=1}^{n-1} a_{jk} L_j \overline{L}_k + \sum_{j,k=1}^{n-1} a'_{j,k} \overline{L}_j L_k + \sum_{j=1}^{n-1} b_j L_j + \sum_{j=1}^{n-1} b'_j \overline{L}_j + a \tag{B.0.2}$$

The next condition was introduced in [KPZ12]. Let ϕ be a smooth function, denote by (ϕ_{ij}) the matrix of the Levi form $\mathcal{L}_\phi = \frac{1}{2}\big(\bar\partial \bar\partial_b - \bar\partial_b \bar\partial\big)(\phi)$, and by $\lambda_1^\phi \leq ... \leq \lambda_{n-1}^\phi$ the ordered eigenvalues of $\mathcal{L}_\phi$.

Definition B.0.7. Let x_0 be a point of M and q an index in the range $1 \leq q \leq n-1$. We say that M satisfies property $(CR-P_q)$ at x_0 if there exists a family of smooth weights $\{\phi^\epsilon\}$ in a neighborhood U of x_0 such that

$$\begin{cases} |\phi^\epsilon(x)| \leq 1, & x \in U, \\ \sum_{k=1}^q \lambda_{j_k}^{\phi^\epsilon}(x) \geq \epsilon^{-1}, & z \in U \text{ and } \mathrm{Ker}(\text{Levi form at } x) \neq 0.\ j_k \in \{1, ..., n-1\}. \end{cases}$$

It is obvious that $(CR-P_q)$ implies $(CR-P_k)$ for any $k \geq q$.

Lemma B.0.8. *Let M be a compact CR manifold of dimension $2n-1$, with $n \geq 3$. Assume that $(CR-P_q)$ and $D^\epsilon(q)$ hold for a fixed q with $1 \leq q \leq \frac{n-1}{2}$ over a covering $\{U\}$ of M. Then we have the following full compactness estimates: given $K > 0$, there is C, C_K such that*

$$\sum_{|J|=q}{}' \sum_{j=1}^{n-1} \big(||L_j u_J||^2_{L^2(M)} + ||\overline{L}_j u_J||^2_{L^2(M)}\big) + K.||u||^2_{L^2} \leq C.Q(u,u) + C_K||u||^2_{H^{-1}}, \tag{B.0.3}$$

for any $(0,k)$-form $u \in Dom(\Box_b)$ and $q \leq k \leq n-1-q$.

Proof. Let $x_0 \in M$, for a suitable neighborhood U of x_0, using the modified Kohn-Morrey-Hörmander Estimate in [KPZ12] combined with Kohn's microlocalization as above, we have

$$\begin{aligned} ||\bar\partial_b u||^2_{L^2(M)} + ||\bar\partial_b^* u||^2_{L^2(M)} \gtrsim ||\overline{L}u||^2_{L^2(M)} + \sum_{|K|=q-1} \sum_{i,j=1}^{n-1} \big(c_{ij} u_{iK}, u_{jK}\big) \\ + O\big(||\overline{L}u||.||u|| + ||u||^2\big), \end{aligned} \tag{B.0.4}$$

for any $u \in \Lambda^{0,q}(C_0^\infty(M))$. Now, as a consequence, with the sufficiently small $\epsilon > 0$. $D^\epsilon(q)$ implies that any sum of distinct q values among $\lambda_1, ..., \lambda_{n-1}$ is non-negative, i.e, $\sum_{k=1}^q \lambda_{j_k} \geq 0$, for $\lambda_{j_k} \in \{\lambda_1, ..., \lambda_{n-1}\}$. This property is called *weak q-convexity* (see [Ho91]). By a unitary change of coordinates, we can assume that the Levi matrix $(c_{ij})_{i,j=1}^{n-1}$ is diagonal. Hence, from these observations, the following inequality follows:

$$\sum_{|K|=q-1} \sum_{i,j=1}^{n-1} \big(c_{ij} u_{iK}, u_{jK}\big) \geq 0, \tag{B.0.5}$$

for any $(0,q)$-form u. This means

$$||\bar\partial_b u||^2_{L^2(M)} + ||\bar\partial_b^* u||^2_{L^2(M)} \gtrsim ||\overline{L}u||^2_{L^2(M)} + O\big(||\overline{L}u||.||u|| + ||u||^2\big), \tag{B.0.6}$$

for any $u = \sum_{|I|=q} u_I \overline{\omega}_I$. Furthermore, following the setup by Ho [Ho91], the estimate (B.0.6) is also true for any $u = \sum_{|I|=k} u_I \overline{\omega}_I$, with $n-1 \geq k \geq q$.

Next, arguments in [Appendix, [Koe02]] (also hold for the considering T' instead of T) apply to show that

$$||\bar{\partial}_b u||^2_{L^2(M)} + ||\bar{\partial}^*_b u||^2_{L^2(M)} \gtrsim ||Lu||^2_{L^2(M)} + ||\overline{L}u||^2_{L^2(M)} + O\big(||\overline{L}u||.||u|| + ||u||^2\big), \qquad \text{(B.0.7)}$$

and we know that if $D^\epsilon(q)$ condition holds, then $D^\epsilon(k)$ does also, with the range $q \leq k \leq n-1-q$. Therefore, again, the estimate (B.0.7) is true for any $(0,k)$-forms, with $q \leq k \leq n-1-q$. Finally, applying the above inequality (B.0.5) to the proof of the main result in [KPZ12], the same compactness estimate holds, and then we have the full estimate (B.0.3) as desired. □

We also have the closed range property when M satisfies the conditions of $(CR-P_q)$ and $D^\epsilon(q)$.

Corollary B.0.9. *Let M be a compact CR manifold of dimension $2n-1$, with $n \geq 3$. Assume that $(CR-P_q)$ and $D^\epsilon(q)$ hold for a fixed q with $1 \leq q \leq \frac{n-1}{2}$ over a covering $\{U_j\}$ of M. Then the hypothesis of closed-range for the operator $\bar{\partial}_b$ holds in the spaces $L^2_{0,k}(M)$, where $q \leq k \leq n-1-q$.*

Proof. We want to show that

$$||u||_{L^2_{0,k}(M)} \leq C||\bar{\partial}_b u||_{L^2_{0,k}(M)}, \qquad \text{(B.0.8)}$$

for all $(0,k)$-form $u \in Dom(\bar{\partial}_b) \cap Ker(\bar{\partial}_b)^\perp$, where $q \leq k \leq n-1-q$.
If (B.0.8) does not hold, we can find a sequence of $(0,k)$-forms $\{u_j\} \subset Dom(\bar{\partial}_b) \cap Ker(\bar{\partial}_b)^\perp$, $||u_j||_{L^2} = 1$ such that

$$||u_j||_{L^2_{0,k}(M)} > j||\bar{\partial}_b u_j||_{L^2_{0,k}(M)}, \qquad \text{(B.0.9)}$$

this implies

$$\lim_{j\to\infty} ||\bar{\partial}_b u_j||_{L^2_{0,k}(M)} = 0.$$

Now, take a subsequential L^2-weak limit u_0 of $\{u_{j_l}\} \subset \{u_j\}$, this limit belongs to $Ker(\bar{\partial}_b) \cap Ker(\bar{\partial}_b)^\perp$, so $u_0 = 0$ and $||u_0||^2_{H^{-1}} = 0$. Finally, to get contradiction, take the limit in the fully compactness estimate with u_{j_l}, and notice that $\lim_{j\to\infty} ||u_j||_{L^2} = 1$ as it's definition. □

In fact that, the assertion of closed range for $\bar{\partial}_b$ operator can be proved if we assume only $(CR-P_q)$ and weak q-convexity (see in [Ho91]). But, here we need more than closed range property.

Now, we can state the main theorem.

Theorem B.0.10. *Let M be a compact CR-manifold of dimension $2n-1$ $(n \geq 3)$ satisfying $(CR-P_q)$ and $D^\epsilon(q)$ condition, and that is also analytic. Assume that the good $-(T)$ condition holds and the vector fields $\{L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}, T\}$ also belong to the analytic class. Then if u is in $\Lambda^{0,q'}(C^\infty_0 M))$, $q \leq q' \leq n-1-q$, and*

$$\Box_b u = f,$$

with $f \in \mathbb{A}_{q'}(M)$, then u belongs to the analytic class $\mathbb{A}_{q'}(M)$.

Before proving the theorem, we will introduce a "suitable" partition of unity globally defined on M. This family exists by using localizing functions due to Ehrenpreis (1960).

Definition B.0.11. A family of functions $\{\psi_j\}_{j=1,2...,N}$ in $C_0^\infty(M)$ such that $\sum_{j=1}\psi_j = 1$ on M is said to be an analytically localizing family provided there exists a constant C independent of j such that

$$|\psi_j^{(k)}(x)| \leq C^{k+1}j^k,$$

for $k \leq 3j$.

B.1 Proof of the Theorem B.0.10

Proof. We follow the approach by Tartakoff [Ta76]..
We have to show that, for any multi-indexes $\alpha = (\alpha_1, ..., \alpha_{2n-1})$ and $k \geq 0$, there is a constant C_u depending on u such that

$$|D_x^\alpha u(x)| \leq C_u^{|\alpha|+1}|\alpha|!. \tag{B.1.1}$$

Equivalently, we must prove that there exists an constant C_u (of course, it is different from the previous one), such that for all $a = |\alpha|$, (see Proposition 1.4.2, [Ro93])

$$|\mathrm{Op}(a)u(x)| \leq C_u^{a+1}a!, \tag{B.1.2}$$

where $\mathrm{Op}(a)$ is any a-th order differential operator formed by a successive applications of the L_j's, $\overline{L}_j$'s, and T acting in x-variables.
The key point in our proof is that for every $K > 0$, there is a constant $C_K > 0$ such that

$$\begin{aligned}\sum_{j=1}^{2n-2} ||\big(X_j\big)_x u||^2_{L^2_{0,q}(M)} + K.||u||^2 \\ \leq C_0\big|\big\langle \Box_b u, u\big\rangle_{L^2_{0,q}(M)}\big| + C_K.||u||^2_{H^{-1}_{0,q}(M)}.\end{aligned} \tag{B.1.3}$$

By Sobolev's Lemma, it is sufficient to provide that for given $\omega_1 \subset\subset \omega_2 \subset\subset (0,\infty)\times M$, and $\psi \in C_0^\infty(\omega_2)$, $\psi = 1$ on ω_1, the following holds

$$||\psi\mathrm{Op}(a)u||_{L^2} \leq C_u^{a+1}(a!). \tag{B.1.4}$$

We also denote by $\mathrm{Op}(l,a)$ any Op(a) with exactly l terms of L_j or $\overline{L}_j$ (without T-direction). We have

$$\mathrm{Op}(l,a) = W\mathrm{Op}(k-1,a-1) \text{ modulo operators of the form } \mathrm{Op}(l,a-j),\ j \geq 1,$$

where $W \in \{L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}\}$. Since T is the weakest of the tangential vector fields in the estimate, we will begin with terms of pure powers of T, i.e, $\psi T^p u$. From the *good* $-$ (T)

condition, it is suffices to proceed on the T'-direction.
Now, from (B.1.3), we obtain that

$$\begin{aligned} I^2_{p,\psi} := \sum_{j=1}^{n-1} ||W_j\psi T'^p u||^2_{L^2_{0,q}(M)} + K.||\psi T'^p u||^2 \\ &\leq C_0\big|\big\langle \Box_b \psi T'^p u, \psi T'^p u\big\rangle_{L^2_{0,q}(M)}\big| + C_K.||\psi T'^p u||^2_{H^{-1}_{0,q}(M)} \\ &\leq C_0\big|\big\langle \psi T'^p \Box_b u, \psi T'^p u\big\rangle_{L^2_{0,q}(M)}\big| + C_K.||\psi T'^p u||^2_{H^{-1}_{0,q}(M)} \\ &+ C_0\big|\big\langle \psi\big[\Box_b,\psi\big]T'^p u, T'^p u\big\rangle_{L^2_{0,q}(M)}\big| + C_0\big|\big\langle \big[\Box_b, T'^p\big]u, \psi^2 T'^p u\big\rangle_{L^2_{0,q}(M)}\big|. \end{aligned} \tag{B.1.5}$$

The first term in the right-hand side of (B.1.5) is bounded from above by $C_\epsilon||\psi T'^p f||^2_{L^2_{0,q}(M)} + \epsilon||\psi T'^p u||^2_{L^2_{0,q}(M)}$, for sufficiently small $\epsilon > 0$. The term $\epsilon||\psi T'^p u||^2_{L^2_{0,q}(M)}$ can be absorbed in the left-hand side of B.1.5.
Now, the third term in the right-hand side of (B.1.5) is following

$$\begin{aligned} \big|Re\big\langle \psi\big[\Box_b,\psi\big]T'^p u, T'^p u\big\rangle_{L^2_{0,q}(M)}\big| &\leq \big|\big\langle [\psi,\big[\Box_b,\psi\big]]T'^p u, T'^p u\big\rangle_{L^2_{0,q}(M)}\big| \\ &\leq C.||\psi' T'^p u||^2_{L^2_{0,q}(M)}. \end{aligned} \tag{B.1.6}$$

The appearance of ψ' is the reason that our problem must be global, not local. So, we have to assume that T, and then T'^p, is globally defined.
Now, the last term in the right-hand side of (B.1.5) is more complicated. We will need an expression for the more complicated bracket

$$\big[\Box_b, T'^p\big] = \sum_{p'=0}^{p-1} T'^{p'}\big[\Box_b, T'\big]T'^{p-p'-1}.$$

By Property (B.0.1) and (B.0.2), we have

$$\big[\Box_b, T'\big] = \sum_{j,k}((T)(\mathcal{A}-\text{coef.})_{j,k})W_jW_k + \sum_j(\mathcal{A}-\text{coef.})_j W_j + (\mathcal{A}-\text{coef.}),$$

where the term $(T')(\mathcal{A}-\text{coef.})_{j,k})$ denotes (at most) of first derivative in x-variables of $(\mathcal{A}-\text{coef.})_{j,k})$.
Hence,

$$\big[\Box_b, T'^p\big] = \sum_{p'=1}^{p} \underline{\binom{p'}{p}} \sum_{j,k} T'^{p'}(\mathcal{A}-\text{coef.})_{j,k}))W_jW_kT'^{p-p'} + ...$$

where "..." are the similar terms with lower order of W and no more T', and the underlining of $\binom{p-1}{p'}$ means for each p', there are possibly $\binom{p}{p'}$ terms of the indicated form.

Now, for example, we will concentrate the terms of highest power of T', i.e , $p.aW_1W_2T'^{p-1}$, applying a weighted Schwartz inequality to these terms, we obtain

$$\left|\left\langle p.aW_1W_2T'^{p-1}, \psi^2T'^pu\right\rangle\right| \leq p^2C_\epsilon||W\psi T'^{p-1}u||^2_{L^2_{0,q}(M)} + \epsilon||W\psi T'^pu||^2_{L^2_{0,q}(M)}, \tag{B.1.7}$$

Again, the term $\epsilon||W\psi T'^pu||^2_{L^2_{0,q}(M)}$ is absorbed in the left-hand side of (B.1.3). More generally, we have

$$\begin{aligned}\left|\left\langle [\Box_b, T'^p]u, \psi^2T'^pu\right\rangle_{L^2_{0,q}(M)}\right| \leq \widetilde{C}_\epsilon \sum_{p'=1}^{p}\left(\underline{\binom{p}{p'}}R_{coef}R^{p'}_{coef}(p'!)||W\psi T'^{p-p'}u||_{L^2_{0,q}(M)}\right)^2 \\ + \epsilon\text{-terms of } \underbrace{||W\psi T'^pu||^2_{L^2_{0,q}(M)}}_{\text{absorbed again}}.\end{aligned} \tag{B.1.8}$$

Iterating the principle terms, we get lower order terms

$$\begin{aligned}\left|\left\langle [\Box_b, T'^p]u, \psi^2T'^pu\right\rangle^{\frac{1}{2}}_{L^2_{0,q}(M)}\right| \leq \widetilde{C}\sum_{p'=1}^{p}\underline{\binom{p}{p'}}R_{coef}R^{p'}_{coef}(p'!)\Big\{\left\langle \psi T'^{p-p'}f, \psi T'^{p-p'}u\right\rangle^{\frac{1}{2}}_{L^2} + ||\psi'T'^{p-p'}u||_{L^2} \\ + C'_K||\psi T'^{p-p'}u||_{H^{-1}} + \sum_{k=1}^{p-p'}\underline{\binom{p-p'}{k}}R_{coef}R^k_{coef}(k!)||W\psi T'^{p-p'-k}u||_{L^2}\Big\}.\end{aligned} \tag{B.1.9}$$

Now, we need to estimate the terms making the global regularity, $||\psi'T'^{p-p'}u||_{L^2}$, for $p' = 1, ..., p$. To do this, by assuming T' globally defined, we recall the family of standard partition of unity $\{\psi_j\} \in C_0^\infty(M)$, $\sum_{j=1}^N \psi_j = 1$ on M. We can assume that $0 \leq \psi_j \leq 1$, $j = 1, ..., N$, then

$$||\psi'T'^{p-p'}u||_{L^2} \leq \sum_{j=1}^N ||\psi'\psi_jT'^{p-p'}u||_{L^2} \leq \underbrace{\sup_{j=1,...,N}|\psi'_j|}_{C_\psi}\sum_{j=1}^N||\psi_jT'^{p-p'}u||_{L^2}. \tag{B.1.10}$$

Hence, the compactness estimate for $T'^p u$ can be rewritten as follows

$$
\begin{aligned}
I_{p,\psi}^2 &= \sum_{i=1}^{N}\Big\{\sum_{j=1}^{n-1} ||W_j\psi_i T'^p u||_{L^2_{0,q}(M)} + K.||\psi_i T'^p u||_{L^2}\Big\} \\
&\leq C.\sum_{i=1}^{N} ||\psi_i T'^p f||_{L^2} + C_K \sum_{i=1}^{N} ||\psi_i T'^p u||_{H^{-1}} \\
&+ \widetilde{C'}_\psi \sum_{i=1}^{N}\Big\{\sum_{p'=1}^{p}\binom{p}{p'} R_{coef}R_{coef}^{p'}(p'!)\Big\{\langle \psi_i T'^{p-p'} f, \psi_i T'^{p-p'} u\rangle_{L^2}^{\frac{1}{2}} + ||\psi_i T'^{p-p'} u||_{L^2} \\
&+ \sum_{k=1}^{p-p'}\binom{p-p'}{k} R_{coef}R_{coef}^{k}(k!)||W\psi_i T'^{p-p'-k} u||_{L^2}\Big\}\Big\} + C'_K \sum_{i=1}^{N}||\psi_i T'^{p-p'} u||_{H^{-1}}. \qquad \text{(B.1.11)} \\
&\leq C.\sum_{i=1}^{N} ||\psi_i T'^p f||_{L^2} + C_K \sum_{i=1}^{N} ||\psi_i T'^p u||_{H^{-1}} \\
&+ \widetilde{C'}_\psi \sum_{i=1}^{N}\Big\{\sum_{p'=1}^{p}\binom{p}{p'} R_{coef}R_{coef}^{p'}(p'!)\Big\{||\psi_i T'^{p-p'} f||_{L^2} + ||\psi_i T'^{p-p'} u||_{L^2} \\
&+ \sum_{k=1}^{p-p'}\binom{p-p'}{k} R_{coef}R_{coef}^{k}(k!)||W\psi_i T'^{p-p'-k} u||_{L^2}\Big\}\Big\} \\
&+ C'_K \sum_{i=1}^{N}\sum_{p'=1}^{p}\binom{p}{p'} R_{coef}R_{coef}^{p'}(p'!)||\psi_i T'^{p-p'} u||_{H^{-1}}.
\end{aligned}
$$

where the new constants in the above estimates are independent of u and p, and K is sufficiently large.
On the other hand,

$$
\begin{aligned}
C_K.||\psi_j T'^p u||^2_{H^{-1}_{0,q}(M)} &\leq C_K.||T'\psi_j T'^{p-1} u||^2_{H^{-1}_{0,q}(M)} + C_K.||[T',\psi_j]T'^{p-1}u||^2_{H^{-1}_{0,q}(M)} \\
&\leq C_{K,\psi}.\sum_{i=1}^{N}||\psi_i T'^{p-1}u||^2_{L^2_{0,q}(M)} \leq C_{K,\psi} I^2_{p-1,\psi}.
\end{aligned}
$$

Now, we can see that, the order of direction T in the right hand side is less than p, only the data term of f is not. Therefore, in all, from (B.1.5), with new constant independent of u and p

$$
I^2_{p,\psi} \leq C.R_f R_f^p(p!) + C_\psi \sum_{p'=1}^{p}\sum_{k=0}^{p-p'} R^2_{coef}R_{coef}^{p'+k}\binom{p}{p'}\binom{p-p'}{k}(p'!)(k!)I_{p-p'-k,\psi}. \qquad \text{(B.1.12)}
$$

Now, when $p=0$, we can see that $I_0 \leq R_u$, so by induction in $0 \leq q < p$,

$$
I_q \leq C_u C_u^q(q!),
$$

we have

$$I_{p,\psi}^2 \leq C.R_f R_f^p(p!) + C_\psi \sum_{p'=1}^{p}\sum_{k=0}^{p-p'} R_{coef}^2 R_{coef}^{p'+k}\binom{p}{p'}\binom{p-p'}{k}(p'!)(k!)C_u C_u^{p-p'-k}((p-p'-k)!). \tag{B.1.13}$$

Choosing C_u is large enough, we obtain

$$||\psi T'^p u|| \leq C_u C_u^p(p!). \tag{B.1.14}$$

And hence, the main estimate (B.1.14) in our approach is true for T-derivative.
By this estimate, we will analyze the term of full-order $\mathrm{Op}(l,p)$ with $l \geq 1$. We know that

$$\psi \mathrm{Op}(l,p) = W\psi \mathrm{Op}(l-1,p-1)$$

modulo operators of the forms $\psi\mathrm{Op}(l,p-j)u$, $j = 1,2,...$, which have the lower order of T' direction. Hence, principally, we must consider the term $W\psi\mathrm{Op}(l-1,p-1)$. This is dominated by

$$\begin{aligned} &||\psi\mathrm{Op}(l-1,p-1)f||_{L^2}^2 + ||\psi'\mathrm{Op}(l-1,p-1)u||_{L^2}^2 \\ &\qquad + |\big\langle[\Box_b,\mathrm{Op}(l-1,p-1)]u, \psi^2\mathrm{Op}(l-1,p-1)u\big\rangle_{L^2}|. \end{aligned} \tag{B.1.15}$$

The most important term is the last one, and as before, we obtain

$$\begin{aligned} |\big\langle[\Box_b,&\mathrm{Op}(l-1,p-1)]u, \psi^2\mathrm{Op}(l-1,p-1)u|_{L^2} \\ &\leq C_\epsilon \sum_{p'=1}^{p}\left(\underline{\binom{p}{p'}}||\psi\mathrm{Op}(l,p-p')u||_{L^2}\right)^2 \\ &+ \epsilon \text{ -terms of the form } ||\psi\mathrm{Op}(l,p)u||_{L^2}^2, \end{aligned} \tag{B.1.16}$$

again, the terms with ϵ is absorbed in the left, and the terms with large constant are bounded by induction hypothesis.
Up to all, modulo with the term of f and less harmful terms (the terms with lower power can be estimated by induction), we have

$$\begin{aligned} \sum_{l=1}^{p}||\psi\,\mathrm{Op}(l,p)u||_{L^2} \leq \underline{p}.C.\sum_{l=1}^{p}&\underbrace{||\psi\,\mathrm{Op}(l,p-1)u||_{L^2}}_{\text{inductive hypothesis}} \\ &+ C.\sum_{l=1}^{p}||\psi'\,\mathrm{Op}(l,p-1)u||_{L^2} + C.\underbrace{||\psi T^p u||_{L^2}}_{\leq R_u R_u^p(p!)}. \end{aligned} \tag{B.1.17}$$

Notice that we can not choose some ψ in the partition of unity as before since there is some W in the construction $\psi'\,\mathrm{Op}(l,p-1)u$ are possible not global, so we will iterate the estimate above for this term with ψ chosen among partition of unity by Tartakoff in (B.0.11). In particular, leading us to estimate the following terms

$$I_{p-p',\psi^{(p')}} = ||\psi^{(p')}\mathrm{Op}(l, p-p')u||, \;\; p' = 1, ..., p. \tag{B.1.18}$$

By the construction of the partition of unity, when $p' = p$, there is $\psi = \psi_p$

$$I_0(\psi^{(p)}) = ||\psi^{(p)}u|| \leq C_u.C_u^p(p!).$$

Now, let $p_0 \geq 0$, we will estimate $||\psi^{(p_0)}\mathrm{Op}(l, p-p_0)u||$ by the inductive hypothesis at the time $p \geq p' > p_o \geq 0$

$$I_{p-p',\psi^{(p')}} = ||\psi^{(p')}\mathrm{Op}(l, p-p')u|| \leq C_u C_u^{p'}(p'!).\widetilde{C}_u^{p-p'}((p-p')!). \tag{B.1.19}$$

Again, modulo some less harmful terms, as before

$$\begin{aligned}\sum_{l=1}^{p} ||\psi^{(p_0)}\,\mathrm{Op}(l, p-p_0)||_{L^2} &\leq \underline{(p-p_0)}.C.\sum_{l=1}^{p} \underbrace{||\psi^{(p_0)}\,\mathrm{Op}(l, p-p_0-1)u||_{L^2}}_{\leq C' C'^{P_0}(p_0!)^{\delta} C"^{p-p_0-1}((p-p_0-1)!)} \\ &+ C.\sum_{l=1}^{p} u||\psi^{p_0+1}\,\mathrm{Op}(l, p-p_0-1)u||_{L^2} + C.\underbrace{||\psi T^{p-p_0}u||_{L^2}}_{\leq R_u R_u^{p-p_0}((p-p_0)!)}.\end{aligned} \tag{B.1.20}$$

Therefore, up to all, with the new constant (sufficiently large) depending of u,

$$\sum_{l=1}^{p} ||\psi\,\mathrm{Op}(l, p)u||_{L^2} \leq C_u C_u^p(p!), \;\; \forall p.. \tag{B.1.21}$$

This completes the proof. $\square$

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