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**NONLINEAR PDEs IN ERGODIC CONTROL,
MEAN FIELD GAMES AND PRESCRIBED
CURVATURE PROBLEMS**

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Abstract

This thesis is concerned with nonlinear elliptic PDEs and system of PDEs arising in various problems of stochastic control, differential games, specifically Mean Field Games, and differential geometry. It is divided in three parts.

The first part is focused on stochastic ergodic control problems where both the state and the control space is \mathbb{R}^d . The interest is in giving conditions on the fixed drift, the cost function and the Lagrangian function that are sufficient for synthesizing an optimal control of feedback type. In order to obtain such conditions, an approach that combines the Lyapunov method and the approximation of the problem on bounded sets with reflection of the diffusions at the boundary is proposed. A general framework is developed first, and then particular cases are considered, which show how Lyapunov functions can be constructed from the solutions of the approximating problems.

The second part is devoted to the study of Mean Field Games, a recent theory which aims at modeling and analyzing complex decision processes involving a very large number of indistinguishable rational agents. The attention is given to existence results for the multi-population MFG system of PDEs with homogeneous Neumann boundary conditions, that are obtained combining elliptic a-priori estimates and fixed point arguments. A model of segregation between human populations, inspired by ideas of T. Schelling is then proposed. The model, that fits into the theoretical framework developed in the thesis, is analyzed from the qualitative point of view using numerical finite-difference techniques. The phenomenon of segregation between the population densities arises in the numerical experiments on the particular mean field game model, assuming mild ethnocentric attitude of people as in the original model of Schelling.

In the last part of the thesis some results on existence and uniqueness of solutions for the prescribed k -th principal curvature equation are presented. The Dirichlet problem for such a family of degenerate elliptic fully nonlinear partial differential equations is solved using the theory of Viscosity solutions, by implementing a version of the Perron method which involves semiconvex subsolutions; the restriction to this class of functions is sufficient for proving a Lipschitz estimate on the elliptic operator with respect to the gradient entry which is also required for obtaining the comparison principle. Existence and uniqueness are stated under general assumptions, and examples of data which satisfy the general hypotheses are provided.

Questa tesi ha come oggetto di studio EDP ellittiche nonlineari e sistemi di EDP che si presentano in problemi di controllo stocastico, giochi differenziali, in particolare Mean Field Games e geometria differenziale. I risultati contenuti si possono suddividere in tre parti.

Nella prima parte si pone l'attenzione su problemi di controllo ergodico stocastico dove lo spazio degli stati e dei controlli coincide con l'intero \mathbb{R}^d . L'interesse è posto sul formulare condizioni sul drift, il funzionale di costo e la Lagrangiana sufficienti a sintetizzare un controllo ottimo di tipo feedback. Al fine di ottenere tali condizioni, viene proposto un approccio che

combina il metodo delle funzioni di Lyapunov e l'approssimazione del problema su domini limitati con riflessione delle traiettorie al bordo. Le tecniche vengono formulate in termini generali e successivamente sono presi in considerazione esempi specifici, che mostrano come opportune funzioni di Lyapunov possono essere costruite a partire dalle soluzioni dei problemi approssimanti.

La seconda parte è incentrata sullo studio di Mean Fielda Games, una recente teoria che mira a elaborare modelli per analizzare processi di decisione in cui è coinvolto un grande numero di agenti indistinguibili. Sono ottenuti nella tesi alcuni risultati di esistenza di soluzioni per sistemi MFG a più popolazioni con condizioni al bordo omogenee di tipo Neumann, attraverso stime a-priori ellittiche e argomenti di punto fisso. Viene in seguito proposto un modello di segregazione tra popolazioni umane che prende ispirazione da alcune idee di T. Schelling. Tale modello si inserisce nel contesto teorico sviluppato nella tesi, e viene analizzato dal punto di vista qualitativo tramite tecniche numeriche alle differenze finite. Il fenomeno di segregazione tra popolazioni si riscontra negli esperimenti numerici svolti sul particolare modello mean field, assumendo l'ipotesi di moderata mentalità etnocentrica delle persone, similmente all'originale modello di Schelling.

L'ultima parte della tesi riguarda alcuni risultati di esistenza e unicità di soluzioni per l'equazione di k -esima curvatura principale prescritta. Il problema di Dirichlet per tale famiglia di equazioni ellittiche degeneri nonlineari è risolto implementando la teoria delle soluzioni di Viscosità, applicando in particolare una versione del metodo di Perron basata su soluzioni semiconvesse; la restrizione a tale classe di funzioni risulta sufficiente per dimostrare una stima di tipo Lipschitz sull'operatore ellittico, essenziale per ottenere un principio di confronto. Esistenza e unicità di soluzioni sono formulate in termini generali; vengono forniti infine esempi in cui condizioni particolari sui dati soddisfano tali ipotesi.

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Notations

\mathbb{R}^d	The d -dimensional Euclidean space, $d \geq 1$.
$B_r(x), B_r$	The open ball of radius r centered at x . If x is omitted, we assume that $x = 0$.
Ω	will always be a bounded domain of \mathbb{R}^d .
$\text{USC}(\overline{\Omega})$	The space of upper semicontinuous functions on $\overline{\Omega}$.
$\text{LSC}(\overline{\Omega})$	The space of lower semicontinuous functions on $\overline{\Omega}$.
$C(\Omega)$	The space of continuous functions on Ω .
$C_{\text{av}}(\Omega)$	The space of functions $u \in C(\Omega)$ such that $\int_{\Omega} u = 0$.
$C^k(\Omega)$	The space of continuous functions on Ω with continuous derivatives of order j , $j = 1, \dots, k$.
$\mathcal{P}(\Omega)$	The space of functions $m \in L^1(\Omega)$ such that $\int_{\Omega} m = 1$ (probability densities).
$\partial_{x_i} f, Df$	Partial derivatives with respect to the i -th variable and gradient vector of f .
$D^2 f$	Hessian matrix of f .
$n = n(x)$	The outer normal vector at a point $x \in \partial\Omega$.
$\partial_n f$	Normal derivative $n \cdot Df$.
\mathcal{S}	The space of square $d \times d$ symmetric matrices with real entries.
$\text{diag}[a_1, \dots, a_d]$	Diagonal matrix A with entries $A_{ii} = a_i$ for all $i = 1, \dots, d$.
$\lambda_i(A)$	The i -th eigenvalue of $A \in \mathcal{S}$, $i = 1, \dots, d$, with ordering $\lambda_1(A) \leq \dots \leq \lambda_d(A)$.
$\kappa_i[u](x)$	The i -th principal curvature of the graph of a function u at a point $(x, u(x))$, $i = 1, \dots, d$. Curvatures will be ordered: $\kappa_1[u](x) \leq \dots \leq \kappa_{d-1}[u]$
$\kappa_{\Omega, i}(x)$	If $\Omega \in C^2$, the i -th principal curvature of $\partial\Omega$ at $x \in \partial\Omega$, $i = 1, \dots, d-1$, ordered such that $\kappa_{\Omega, 1}(x) \leq \dots \leq \kappa_{\Omega, d-1}$.
(H)	See page 6.
(H0)	See page 13.
(H1)	See page 14.
(M)	See page 16.
(V _{NL})	See page 41.
(h0)	See page 86.
(h1), (h2)	See page 90.

Introduction

This thesis deals with various problems arising in stochastic control, differential games, specifically Mean Field Games, and differential geometry, and leading to nonlinear elliptic PDEs or system of PDEs.

It is divided in three parts. In the first part we study some ergodic control problems where the state space is the whole \mathbb{R}^d . In the second part we present some existence results for multi-population ergodic mean field game systems with Neumann boundary conditions; we also carry out some numerical experiments on a mean field game derived from a population model inspired by the work of T. Schelling. In the third part we investigate existence and uniqueness for the prescribed k -th principal curvature equation. The first two parts are related, being contributions to stochastic control theory and differential games theory, while the last part is more on its own and employs the theory of viscosity solutions for fully nonlinear degenerate elliptic equations.

Part I - Ergodic Control in \mathbb{R}^d .

The first problem we focus on is a stochastic control problem where we aim at minimizing a long-time average cost, namely

$$\begin{aligned} & \text{Minimize } \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^x \int_0^T [L(\alpha_t) + f(X_t)] dt & (1) \\ & \text{Subject to } X_t = x + \int_0^t [b(X_s) - \alpha_s] ds + \sqrt{2\nu} B_t, \quad t \geq 0, \end{aligned}$$

where the minimization is among a set of admissible control processes α_t which take their values in \mathbb{R}^d (unbounded controls). Here B_t is a Brownian motion on some probability space, $L : \mathbb{R}^d \rightarrow \mathbb{R}$ is the Lagrangian function (the cost paid by using the control α_t), $f : \mathbb{R}^d \rightarrow [0, \infty)$ the cost paid by the agent being at position X_t and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ a fixed drift which, together with α_t and the Brownian motion, influences the dynamics of X_t .

The goal is to synthesize a so-called optimal Markov policy, a control process which is defined as a function of the trajectory itself. Precisely, we seek for a measurable (and possibly regular) $\alpha^* : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that the control process

$$\alpha_t^* := \alpha^*(X_t^*),$$

where X_t^* is the solution of the stochastic differential equation

$$X_t = x + \int_0^t [b(X_s) - \alpha^*(X_s)] ds + \sqrt{2\nu} B_t,$$

is a minimizer of (1). This problem has been studied in [4, 18] with control processes taking values in compact domains and in [10] in the deterministic case. The situation where controls might not be bounded has been treated recently and a deep studied has been carried out in

a series of papers by Kaise, Sheu and Ichihara [46, 47, 48, 53]. In this thesis the main focus is to understand which are possible conditions on the data b, f, L sufficient for the solvability of (1) through a synthesis of a Markovian control; moreover, we study the long-time properties (ergodicity) of the optimal trajectory X_t^* . With respect to the works just mentioned, which concentrate on other phenomena arising in such problems (convergence of finite-horizon problems as the final time goes to infinity, criticality, ...), we derive some very precise conditions on the data, using a different approximation technique.

A natural way to approach the minimization problem is to solve the Hamilton-Jacobi-Bellman equation

$$-\nu\Delta u(x) - b(x) \cdot Du(x) + H(Du(x)) + \lambda = f(x) \quad \text{in } \mathbb{R}^d, \quad (2)$$

where the Hamiltonian H is the Legendre transform of L . A solution u to (2) provides indeed a candidate for the optimal Markov policy, namely $\alpha^*(x) = -D_p H(Du(x))$ and the corresponding λ , which is unknown a priori, is the optimal value of (1). Equation 2 is considered by Bensoussan and Frehse [17] and Kaise, Sheu and Ichihara, who prove that usually there exists a continuum of λ such that (2) has a solution; however, only one value of λ allows us to define an optimal control α^* for which X_t^* has good long-time behavior, that is precisely what we are looking for.

We overcome the difficulty of not knowing this particular λ and the lack of compactness of the domain \mathbb{R}^d with the following method: we define a suitable family of bounded domains \mathcal{O}_R for every $R > 0$, then solve the approximating equations

$$\begin{cases} -\nu\Delta u_R - b(x) \cdot Du_R + H(Du_R) + \lambda_R = f(x) & \text{in } \mathcal{O}_R, \\ \partial_n u_R = 0 & \text{on } \partial\mathcal{O}_R. \end{cases} \quad (3)$$

and pass to the limit as $R \rightarrow \infty$. We expect, up to some subsequence, that $\lambda_R \rightarrow \lambda$, $u_R \rightarrow u$ and (u, λ) solves (2).

This kind of approximation is not standard (it has been only mentioned, as far as we know, in [19] to recover some already known result and in [63] to study uncontrolled diffusions on \mathbb{R}^d) but it is effective and works under mild assumptions on H, f (see Theorem 2.6). It is also useful for proving the ergodicity of the optimal process X_t^* . In general, we recall that a process X_t is ergodic if it has a unique invariant measure π :

$$\int_{\mathbb{R}^d} f(x) d\pi(x) = \int_{\mathbb{R}^d} \mathbb{E}^x f(X_t) d\pi(x), \quad (4)$$

for all $t \geq 0$ and bounded continuous functions f (see, for example, [54]). The stationary distribution is found by solving the Kolmogorov equation

$$\nu\Delta m + \operatorname{div}[m(-b + D_p H(Du))] = 0 \quad \text{in } \mathbb{R}^d, \quad \int_{\mathbb{R}^d} m = 1, \quad (5)$$

and m is the density of π ; it is also possible to prove that the law of X_t converges to m as t goes to infinity. Our approximation argument has a counterpart for (5), because we are able to find a solution m by taking the limit of solutions m_R of (5) in the bounded domain \mathcal{O}_R with suitable Neumann conditions at the boundary (Theorem 2.8). In order for this convergence to take place we require that some property on the family m_R holds:

$$\sup_R \int_{\{s \leq |x| \leq R\}} m_R \rightarrow 0 \quad \text{as } s \rightarrow +\infty, \quad (6)$$

which basically assures that the masses m_R do not “disperse” but remain concentrated in a bounded domain as $R \rightarrow \infty$.

If (6) holds and the assumptions on the data for the existence of u are satisfied, we are able, through the existence of m , to solve the minimization problem (1) with an optimal Markovian control using a verification argument. Our problem is then reduced to find some sufficient condition for (6). We recall that the property of ergodicity for a stochastic process is strictly linked to the existence of a Lyapunov function, i.e. a $W \in C^2(\mathbb{R}^d)$ such that $W \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and for all x

$$-\nu\Delta W(x) - (b(x) - D_p H(Du(x))) \cdot DW(x) \geq 1.$$

The difficulty of constructing a Lyapunov function for the optimal process X_t^* is that information on Du (boundedness/growth rate) is needed and usual a-priori estimates provided by Bernstein methods might not be sufficient. We are able to prove the existence of such W , thus to verify (6) and solve the ergodic control problem, using convenient functions of u_R , obtaining fine properties via maximum principle and comparison, and passing to the limit as $R \rightarrow \infty$. If $H(p)$ is $C|p|^m$, $C > 0$, $m > 1$, or possibly is a perturbation of it, we treat in a unified way the following cases:

- The drift b is inward pointing and coercive, Theorem 2.19 (covering the Ornstein-Uhlenbeck case treated in [36] and [35]),
- The drift b is inward pointing and bounded and the cost f is also bounded, Theorem 2.22 (new),
- The trajectory with null control $\alpha_t = 0$ is ergodic and the cost is bounded and increasing as $x \rightarrow \infty$, Theorem 2.23 (new),
- The cost f is coercive and $m \geq 2$, Theorem 2.24 (results that partially overlap [46, 48]).

This program is carried out in Chapter 2. We point out that the drift b and the cost f determine the qualitative behavior of the optimal process driven by $b - D_p H(Du)$, and we think that the interest of these results lies in the fact that we are able to construct an optimal ergodic process starting just from explicit conditions on b , f , rather than assuming directly ergodicity of X_t^* or the existence of a Lyapunov function.

We mention that our approach is quite natural and has a precise meaning from the point of view of stochastic control: equation (3) is associated to a minimization problem of type (1) where X_t is subject to the stochastic differential equation with *reflection* at the boundary $\partial\mathcal{O}_R$

$$\begin{aligned} X_t &= x + \int_0^t b(X_s, \alpha_s) ds + \sqrt{2\nu} B_t - \int_0^t n(X_s) dl_s \quad X_t \in \overline{\mathcal{O}_R} \\ l_t &= \int_0^t \chi_{\partial\mathcal{O}_R}(X_s) dl_s \\ l(0) &= 0 \quad \text{and } l \text{ (the local time) is nondecreasing,} \end{aligned} \tag{7}$$

rather than the corresponding stochastic equation on the whole space \mathbb{R}^d . Such a problem has been considered, for example, in [19, 23] and we give a brief overview of that in Chapter 1. It becomes clear that if the optimal trajectory for the reflected process has an invariant measure concentrated in a bounded set (that is heuristically required by condition (6)), it approximates reliably the optimal trajectory for the problem on \mathbb{R}^d if \mathcal{O}_R is big enough.

We finally point out that one of our motivation for considering ergodic problems in \mathbb{R}^d was to prove some existence result for the Mean Field Game system (9) in \mathbb{R}^d . Our approximation scheme can be adapted to treat this problem if existence for (9) is proven in bounded domains of type \mathcal{O}_R with Neumann boundary conditions, that is one of the topic we deal with in the second part of the thesis.

Part II - Multi-population MFG.

In the second part of the thesis we concentrate on Mean Field Games, a recent area of research introduced by Lasry and Lions [56, 57, 58, 65] and independently by Huang, Caines and Malhamé [44, 45]. Mean field games (briefly MFG) theory is devoted to the study of games with a very large number of “identical” players, each one having a very little influence on the overall system. Consider, specifically, an N -persons game where the i -th player controls his own state

$$X_t^i = x^i + \int_0^t b^i(X_s^i, \alpha_s^i) ds + \sqrt{2\nu} B_t^i, \quad (8)$$

and aims at minimizing the long-time average cost

$$J^i(x^i, \alpha_t^1, \dots, \alpha_t^N) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^x \int_0^T [L^i(X_t^i, \alpha_t^i) + F^{i,N}(X_t^1, \dots, X_t^N)] dt.$$

Here the noises B^i of different players are independent. Suppose that the cost paid for being at position x is a function of the current position and the *empirical density of the other players* only, namely

$$F^{i,N}(x^1, \dots, x^N) = V \left[x^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j} \right],$$

where $V : \mathbb{R}^d \times \mathcal{P} \rightarrow \mathbb{R}$ is a sufficiently smooth operator, so every player is indistinguishable and $b^i = b, L^i = L$ for all i . In this setting Lasry and Lions proved that a Nash equilibrium for the N -persons game, which is given in feedback form by solving a $2N$ system of elliptic partial differential equations, converges as $N \rightarrow \infty$ to a strategy which is the same for every player and can be synthesized by solving a system of *two* equations only:

$$\begin{cases} -\nu \Delta u + H(x, Du) + \lambda = V[x, m] \\ -\nu \Delta m - \operatorname{div}(D_p H(x, Du)m) = 0, \quad \int m = 1 \end{cases} \quad (9)$$

The result is very deep, as it states the possibility to capture the behavior of the system just by studying the interaction between a single player and the overall distribution, instead of considering interactions between every player. Moreover, we are able to investigate how the individual rational criterion V reflects into macroscopic phenomena.

Lasry and Lions consider the stationary system (9) in the periodic setting, in the sense that H, V are periodic in the space variable and (8) is set on the torus. This to avoid some technical difficulties and focus on the main features of the problem. However, periodic conditions might not be natural for some models: we would like to consider some game where the dynamics of an *average* player (players are identical) is given by a controlled stochastic differential equation on a bounded domain Ω with *reflection* at the boundary (see (7)). Moreover, we are interested in M -population games ($M > 1$), where all the players belonging to the k -th population are indistinguishable for all $k = 1, \dots, M$, and the cost they aim at minimizing depends on k , i.e.

$$V = V^k [x_k, m_1, \dots, m_M].$$

Here V^k is paid by the average player of the k -th population and might depend on the distribution of all the populations $m = (m_1, \dots, m_M)$. In this setting, we are then lead to consider the system of $2M$ equations

$$\begin{cases} -\nu \Delta u_k(x) + H^k(x, Du_k(x)) + \lambda_k = V^k[x, m], & \forall x \in \Omega, \\ -\nu \Delta m_k(x) - \operatorname{div}(D_p H^k(x, Du_k(x))m_k(x)) = 0, \\ \partial_n u_k(x) = 0, & \forall x \in \partial\Omega, \\ \nu \partial_n m_k(x) + m_k D_p H^k(x, Du_k(x)) \cdot n = 0. \end{cases} \quad (10)$$

We derive heuristically (10) in Chapter 3 and briefly present some result of [34] where the derivation from Nash equilibria of MN -player games is carried out rigorously in the periodic setting.

In Chapter 4 we prove some existence and uniqueness results for (10). If V^k are non-local (smoothing) operator, we carry out the procedure suggested in [58], which exploits the fixed point structure to obtain a classical solution; in our situation we employ standard elliptic estimates, with some care for the presence of Neumann boundary conditions (Theorem 4.2). Similar results are obtained if V^k are local continuous functions of $x, m_1(x), \dots, m_M(x)$ and are *uniformly bounded*, Theorem 4.5. In the local unbounded case we show how to prove some a-priori estimates using the full structure of (10) (it has an adjoint structure, Evans and Gomes [31]) combined with the integral Bernstein method of Lions [60] and some estimates on the Kolmogorov equation. Such a-priori bounds, stated in Proposition 4.9, can be used to prove existence for the MFG system and hold if the costs V^k satisfy precise growth assumptions with respect to m_1, \dots, m_M . As an example, in the one-population case we solve (10) with $V(m) = am^\beta$, $a > 0$ and

$$\beta < 1/(d-3) \quad \text{if } d \geq 4$$

and $\beta < \infty$ otherwise. We mention that the existence problem in one-population, stationary and periodic setting is studied under general assumptions in [38], submitted while this thesis was being written. It is also considered by [29], with quadratic Hamiltonian H .

As for uniqueness for (10), it is not expected in general. If we consider, as an example, two populations competing, i.e. aiming at avoiding individuals of the other population, we may think that any configuration where the two distributions have disjoint support (or are simply concentrated in different parts of the domain) should provide a “solution” of (10). We adapt the uniqueness argument of [58] for single-population systems to formulate a sufficient condition, which is satisfied in the specific situations where basically every V^k is strongly increasing with respect to m_k (Theorem 4.14). We finally show a simple example where non-uniqueness of solutions occurs.

As we already mentioned, existence results for (10) can be used to prove existence of solutions for the ergodic MFG system in the whole space \mathbb{R}^d , implementing the methods for ergodic control described in Part I. Results for this problem are natural byproducts of this thesis and are currently under investigation, but they are not included here.

Another motivation for considering multi-population mean field games in bounded domains is investigating some population models coming from social sciences. In particular, we focus our attention on a model of segregation that the Nobel laureate in economics T. C. Schelling proposed in [71]. In this simple model, individuals of two populations interact moving on a chessboard; the rule is that a player stays in his place if the number of individuals of his own kind in a neighborhood is above some threshold, otherwise he moves to a free house nearby. Simple simulations show that any random initial state evolves into a configuration where clusters of individuals of the same kind are evident, even if the “happiness” threshold is fairly small. Schelling concludes that segregation between races might spontaneously occur in urban agglomerates even if the degree of racism is relatively low.

In Chapter 5, we adapt the ideas of this model to a two-population (ergodic) mean field game. Precisely, we suppose that the cost paid by an average player of the k -th population ($k = 1, 2$) is

$$V^k[x, m_1, m_2] = \left(\frac{m_k(x)}{m_1(x) + m_2(x) + \eta_k} - a_k \right)^-, \quad (11)$$

where x is his position, m the distribution, a_k the happiness threshold and η_k a very small constant. Heuristically, the cost function V^k is zero if the ratio $m_k(x)/(m_1(x) + m_2(x))$ is

above a_k and it is positive otherwise. It can be derived naturally from a $(N + N)$ -persons game where the i -th player of the first population pays

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \left(\frac{\#\{x_j \in \mathcal{U}(x_i) : j \neq i\}}{\#\{x_j \in \mathcal{U}(x_i) : j \neq i\} + \#\{y_j \in \mathcal{U}(x_i)\} + \eta_1(N-1)} - a_1 \right)^-,$$

where x_i, y_i are the players of the first and second population, respectively, and $\mathcal{U}(z)$ is a neighborhood of z . In Chapter 5 we derive other cost functionals adapted to the ideas of Schelling; we focus mostly on the local version (11), for which we consider the mean field game system

$$\begin{cases} -\nu\Delta u_k + |Du_k|^2/2 + \lambda_k = V^k(x, m_1(x), m_2(x)) & \Omega \\ -\nu\Delta m_k - \operatorname{div}(Du_k m_k) = 0, & \\ \partial_n u_k = 0, \quad \partial_n m_k = 0, & \partial\Omega \end{cases} \quad (12)$$

System (12) fits into the theory developed in Chapter 4, so existence of solutions is guaranteed, but uniqueness should not be expected to hold. Actually, since V^k has no explicit dependence on x , constant m and u provide a solution; if some “nice” potential function is added, say the right hand side of the HJB equations is substituted by $V^k(x, m_1(x), m_2(x)) + f(x)$, existence holds as well. In Section 5.3 we carry out some numerical experiments looking for non-constant solutions that show some kind of segregation phenomena. Such tests have been made in collaboration with Yves Achdou.

We mention that a non-stationary mean field game multi-population model of congestion is also investigated numerically in [55].

Numerical analysis for stationary one-population MFG is developed in [3], where the authors suggest to study the long-time behavior of solutions of the non-stationary *forward-forward* system

$$\begin{cases} \partial_t u_k - \nu\Delta u_k + |Du_k|^2/2 = V^k[m_1, m_2](x) & (0, T) \times \Omega \\ \partial_t m_k - \nu\Delta m_k - \operatorname{div}(Du_k m_k) = 0, & \\ \partial_n u_k = 0, \quad \partial_n m_k = 0, & (0, T) \times \partial\Omega \\ u_k(t=0) = u_{k,0}, \quad m_k(t=0) = m_{k,0}(x), & \end{cases} \quad (13)$$

indicating that (at least in the one-population case), any initial $m_{k,0}$ should evolve into some stable distribution \bar{m}_k as $T \rightarrow \infty$, $u_k(x, T) - \lambda_k T \rightarrow \bar{u}_k$ for some λ_k , and $\bar{u}_k, \bar{m}_k, \lambda_k$ solving (12); this overcomes the issue of not knowing the coefficients λ_k a-priori. We implement a finite difference method for approximating (13), for which we obtain the numerical counterpart

$$\begin{cases} \frac{U_{i,j}^{k,n+1} - U_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h U^{k,n+1})_{i,j} + g([D_h U^{k,n}]_{i,j}) \\ \quad + Dg([D_h U^{k,n}]_{i,j}) \cdot ([D_h U^{k,n+1}]_{i,j} - [D_h U^{k,n}]_{i,j}) \\ \quad = V^k(M_{i,j}^{1,n}, M_{i,j}^{2,n}), & (14) \\ \frac{M_{i,j}^{k,n+1} - M_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h M^{k,n+1})_{i,j} - \mathcal{B}(U^{k,n+1}, M^{k,n+1}) = 0, \end{cases}$$

Here $U_{i,j}^{k,n}, M_{i,j}^{k,n}$ represent the approximations of u_k, m_k respectively at a point $x_{i,j}$ of a uniform mesh of $[0, 1]^2$ with step h (space dimension is $d = 2$) at discrete time $t_n = \Delta n$; Δ_h and D_h are finite difference versions of the Laplacian and the gradient vector, g the numerical hamiltonian and \mathcal{B} comes from the $\operatorname{div}()$ term of Kolmogorov equations. Once a $M_{i,j}^{k,0}$ with total mass $h^2 \sum_{i,j} M_{i,j}^{k,0} = 1$ is fixed (we take $U_{i,j}^{k,0} = 0$) a linear system of equations is solved for each time step until t_n reaches a prescribed final time T .

We point out that the numerical computations we made are just experiments, since we do not have any rigorous proof of convergence of $m_k(\cdot, T) \rightarrow \bar{m}_k(\cdot)$ as $T \rightarrow \infty$ nor convergence of the scheme $M^k \rightarrow m_k$ as $h, \Delta \rightarrow 0$. We observe that, starting from $M^{1,0}$ and $M^{2,0}$ with disjoint support (we want to avoid constant solutions), after some time the two distributions stabilize to some configuration which is “segregated” in the sense that M^1 and M^2 are concentrated in separate parts of the domain $[0, 1]^2$. In our tests the happiness threshold a is set to 0.4 for both populations, so our mean field model shows similar segregation phenomena arising from mild ethnocentric attitude as the original discrete and deterministic model by Schelling. An evidence of long-time convergence is given by the fact that numerically $\partial_t m_k$ vanishes as T increases; in some cases the distributions pass through apparently stable configurations that quickly turn into other regimes.

Part III - Prescribed Curvature Equations.

The third part of this thesis is concerned with the Dirichlet problem for the k -th prescribed principal curvature equation

$$\kappa_k[u](x) = f(x) \quad \forall x \in \Omega, \quad (15)$$

where $\kappa_1[u](x) \leq \dots \leq \kappa_d[u](x)$ denote the principal curvatures of the graph of u at point $(x, u(x))$. Equation (15) can be rewritten explicitly as

$$F^k(Du(x), D^2u(x)) = f(x) \quad \forall x \in \Omega, \quad (16)$$

where

$$F^k(Du, D^2u) := \lambda_k \left(\frac{1}{\sqrt{1 + |Du|^2}} \left(I - \frac{Du \otimes Du}{1 + |Du|^2} \right) D^2u \right),$$

$\lambda_k(A)$ being the k -th eigenvalue of the matrix A , which is a fully non-linear (degenerate) elliptic equation.

The standard theory of prescribed curvature equations mainly focuses on the case

$$\sigma_K(\kappa_1[u], \dots, \kappa_d[u]) = f, \quad (17)$$

where σ_K is the K -th elementary symmetric function, and was developed to embrace intermediate situations between the mean curvature ($K = 1$) and the Gaussian curvature ($K = d$); see [25, 51, 66, 73, 74]. Similarly, equations of type $\sigma_K(\lambda_1(D^2u), \dots, \lambda_d(D^2u)) = f$ have been considered (we refer to the recent survey [52] and references therein).

Our equation (15), which as far as we know has never been treated in the literature, appears to be more “degenerate” than (17), as fewer information on u are prescribed by f . Indeed, it is not even non-totally degenerate (we refer to [13] for such kind of equations) in the sense that $-F^k(p, M + rI) \leq -F^k(p, M) - \eta r$ for all $M \in \mathcal{S}$ and for some $\eta = \eta(p) > 0$, which vanishes as $p \rightarrow \infty$. Moreover the operator F^k is not everywhere differentiable as it is the k -th eigenvalue of a matrix. We implement the theory of *viscosity solutions*, which is the natural framework for this kind of problem due to the degeneracy of the equation and the expected non-smoothness of solutions.

Even if F^k is elliptic in the whole set of $C^2(\Omega)$ functions we look for C -semiconvex solutions, in order to exploit boundedness of the gradient on compact subsets of Ω (and so recover the non-total degeneracy of the operator) and the bound from below of the hessian matrix; we recall that u is C -semiconvex if $u + C/2|x|^2$ is convex. Moreover we study the Dirichlet problem in the *generalized* sense, where the solution at the boundary has to coincide with the datum $g \in C(\partial\Omega)$ or to satisfy the equation (16). To summarize, for us a solution

u of the Dirichlet problem, briefly a C -solution, is a C -semiconvex subsolution, in the sense that if $u - \phi$ has a maximum point at $x_0 \in \bar{\Omega}$ for some $\phi \in C^2(\bar{\Omega})$, then

$$\begin{aligned} -F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0) &\leq 0 \quad \text{if } x \in \Omega \\ \min\{-F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0), u(x_0) - g(x_0)\} &\leq 0 \quad \text{if } x \in \partial\Omega \end{aligned}$$

and it is a C -supersolution, namely, if $u - \phi$ has a minimum point at $x_0 \in \bar{\Omega}$ for some $\phi \in C^2(\bar{\Omega})$ such that

$$-C < \lambda_1(D^2\phi(x_0))$$

then

$$\begin{aligned} -F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0) &\geq 0 \quad \text{if } x \in \Omega \\ \max\{-F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0), u(x_0) - g(x_0)\} &\geq 0 \quad \text{if } x \in \partial\Omega. \end{aligned}$$

Our modified definition of viscosity supersolution is reminiscent of [14, 50] and is motivated by the restriction we adopt to C -semiconvex subsolutions.

We introduce the problem and our notion of solution in Chapter 6. In Chapter 7 we prove a comparison principle and an existence result for the Dirichlet problem under abstract assumptions, stating in Section 7.3 some sufficient conditions for those assumptions to hold.

For the comparison principle (Theorem 7.2) we mainly require continuity of the data. We avoid the doubling of variables using an argument of [15] and a *local* perturbation technique by which a strict subsolution is produced starting from a subsolution. This is necessary due to the lack of strict monotonicity of the operator with respect to the u variable. The method is indeed localized around a maximum point of the difference between a subsolution and a supersolution (it cannot be carried out on the whole domain as in [13]), and exploits the Lipschitzianity of $F^k(p, M)$ with respect to the gradient entry p uniform in M in suitable subset of \mathcal{S} . This non-trivial fact on Lipschitz regularity of F^k is formulated in Proposition 7.1.

Concerning the existence for the Dirichlet problem, stated by Theorem 7.4, it is proved under the assumption of existence of a C -semiconvex subsolution and a C -supersolution, that makes the usual Perron family well-defined, and an implicit condition on the geometry of $\partial\Omega$ and f , that assures that there is no loss of boundary data (so that the Dirichlet problem is solved in standard sense). We follow the lines of the existence proof by Da Lio [32], that we can adapt to our case with nonstandard sub. and supersolutions.

We are able to construct explicit sub. and supersolutions if Ω is contained in a set of type

$$\Gamma_{\tilde{k}}^R := \left\{ x \in \mathbb{R}^d : \sum_{i=1}^{\tilde{k}} x_i^2 < R^2 \right\}, \quad (18)$$

$\tilde{k} = \max\{k, d - k + 1\}$, where R is inversely proportional to the maximum of $|f|$ on $\bar{\Omega}$. Moreover, it is possible to check that the abstract assumptions on the geometry of Ω are satisfied if, for example,

$$-\kappa_{\Omega, d-k}(x) < f(x) < \kappa_{\Omega, k-1}(x) \quad \forall x \in \partial\Omega, \quad (19)$$

denoting by $\kappa_{\Omega, j}(x)$ the j -th curvature of $\partial\Omega$ at a point x . We point out that our condition for existence involve f and the shape of Ω , whereas g is an arbitrary continuous function on the boundary and we do not exclude that for particular g ($g = 0$ for example) it is possible to relax the hypotheses. Furthermore, we can exhibit some situations where (18) and (19)

are not far from being optimal, employing some well-known results and counterexamples on the Gaussian curvature equation to which $\kappa_1[u] = f$ is naturally linked.

Finally, by an example we underline that our notion of solution does not coincide with the standard definition of viscosity solution and it is possible to have existence of both kind of solutions with the same data Ω, f, g . The interesting feature of C -solutions is that they are unique.

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Part I

Ergodic Control

Chapter 1

Ergodic Control for Reflected Processes

This chapter is an introduction to the ergodic control problem for reflected processes, the starting point of this dissertation. In particular, we will see how the control problem is linked to an Hamilton-Jacobi-Bellman (HJB, in short) equation with Neumann boundary conditions through a verification theorem; then, we will recall the property of ergodicity of a reflected process, and the Kolmogorov equation for the invariant measure. All the results presented here will be used extensively in the sequel.

1.1 Derivation of the HJB Equation

Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain of class C^2 , $d \geq 1$. Denote with $n(x) \in \mathbb{R}^d$ the outward normal to the boundary $\partial\Omega$ at $x \in \partial\Omega$ and let

$$b \in C^1(\bar{\Omega} \times \mathbb{R}^d, \mathbb{R}^d)$$

be a given controlled drift term.

We fix a *stochastic system*: a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a d -dimensional Brownian motion B_t adapted to the filtration \mathcal{F}_t , and consider a progressively measurable process α_t that takes values in \mathbb{R}^d .

For any initial condition $x \in \Omega$ the state of the system, that we will denote by $X_t = X_t^{\alpha, x}$, is given by a solution, for all $t > 0$, of

$$\begin{aligned} X_t &= x + \int_0^t b(X_s, \alpha_s) ds + \sqrt{2\nu} B_t - \int_0^t n(X_s) dl_s \quad X_t \in \bar{\Omega} \\ l_t &= \int_0^t \chi_{\partial\Omega}(X_s) dl_s \\ l(0) &= 0 \quad l \text{ is nondecreasing,} \\ X_t, l_t &\text{ are continuous } \mathcal{F}_t\text{-adapted processes.} \end{aligned} \tag{1.1}$$

In the literature X_t is a (controlled) reflected diffusion process, l the local time; solving the Skorokhod problem means to find a solution of (1.1). The process X_t is driven by the drift term $b(X_t, \alpha_t)$, the Brownian motion B_t , and is reflected with velocity $-n(X_t)$ as it hits the boundary $\partial\Omega$; consequently, $X_t \in \bar{\Omega}$ at every time t , in other words it never leaves the closure of the domain. It is known that this problem has a unique solution, see for example [64], [49], [61].

We now define, for a control α_t , the so-called long-time average cost functional as

$$J(x, \alpha_t) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^x \int_0^T [L(\alpha_t, X_t) + f(X_t)] dt, \tag{1.2}$$

where X_t is subject to (1.1) and

$$f \in C^1(\bar{\Omega}), L \in C^1(\mathbb{R}^d \times \mathbb{R}^d).$$

f is related to the cost paid by the controller for being at position X_t , while L takes into account the “effort” of using the control α_t .

If for all $T > 0$

$$\mathbb{E}^x \int_0^T [L(\alpha_t, X_t) + f(X_t)] dt < \infty,$$

we will say that α_t is an *admissible controll process*, and write

$$\alpha_t \in \mathcal{A}.$$

An important sub-class of controls is the one of *feedback* or *Markovian* controls: given a Borel measurable functions $v : \bar{\Omega} \rightarrow \mathbb{R}^d$ set

$$v_t = v(X_t^{v_t}).$$

By this notation, we mean that (1.1) is solved with drift $b(\cdot, v(\cdot))$ and the control v_t is defined as $v_t = v(X_t)$. We will often identify the control v_t with the function v .

Once x is fixed, our aim is to minimize the functional J within the set of admissible controls \mathcal{A} , namely to find $\alpha_t^* \in \mathcal{A}$ such that

$$J(x, \alpha_t^*) = \inf_{\mathcal{A}} J(x, \alpha_t).$$

We now define the Hamiltonian $H : \bar{\Omega} \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$H(x, p) = \sup_{\alpha \in \mathbb{R}^d} \mathcal{H}(x, p, \alpha), \tag{1.3}$$

where

$$\mathcal{H}(x, p, \alpha) = -L(\alpha, x) - b(x, \alpha) \cdot p.$$

We suppose that the supremum is finite for all x and p ; this is true under mild assumptions on L and b , which are usually satisfied in interesting situations (for example when b is a linear function of α and L is strictly convex).

The optimization problem is then associated in a natural way by a *verification theorem* to the *Hamilton-Jacobi-Bellman* partial differential equation

$$-\nu \Delta u(x) + H(x, Du(x)) + \lambda = f(x) \quad \forall x \in \Omega, \tag{1.4}$$

with homogeneous Neumann boundary conditions

$$\partial_n u(x) = 0 \quad \forall x \in \partial\Omega. \tag{1.5}$$

In particular, it is possible to prove the

Theorem 1.1. *If there exist $u \in C^2(\bar{\Omega})$ and $\lambda \in \mathbb{R}$ that solve (1.4) and (1.5), then*

- i) $\lambda \leq J(x, \alpha_t)$ for all $\alpha_t \in \mathcal{A}$ and $x \in \Omega$.*
- ii) if v_t^* is an admissible feedback control such that*

$$v^*(x) \in \operatorname{argmax}_{\alpha} \mathcal{H}^i(x, Du(x), \alpha),$$

for all $x \in \Omega$, then $\lambda = J(x, v_t^)$. Thus v_t^* is optimal.*

Proof. By the definition of H as a supremum, for all $\alpha \in \mathbb{R}^d$ and $x \in \Omega$ we have

$$-\nu\Delta u(x) - L(\alpha, x) - b(x, \alpha) \cdot Du(x) + \lambda \leq f(x), \quad (1.6)$$

so, setting $\alpha = \alpha_t$ and $x = X_t$ we obtain

$$-\nu\Delta u(X_t) - b(X_t, \alpha_t) \cdot Du(X_t) + \lambda \leq f(X_t) + L(\alpha_t, X_t). \quad (1.7)$$

It is proven in [72] that for all $\phi \in C^2(\bar{\Omega})$ such that $\partial_n \phi = 0$ on $\partial\Omega$,

$$M_t := \phi(X_t) - \int_0^t [\nu\Delta\phi(X_t) + b(X_t, \alpha_t) \cdot D\phi(X_t)]dt$$

is a martingale¹, hence, setting $u = \phi$ and taking expectation yields for all $x, T > 0$

$$\mathbb{E}^x u(X_T) - \mathbb{E}^x \int_0^T [\nu\Delta u(X_t) + b(X_t, \alpha_t) \cdot Du(X_t)]dt = M_0 = u(x),$$

and by plugging (1.6) and rearranging terms

$$\lambda T \leq \mathbb{E}^x \int_0^T [f(X_t) + L(\alpha_t, X_t)]dt + \mathbb{E}^x u(X_T) - u(x).$$

The term $\mathbb{E}^x u(X_T) - u(x)$ is bounded for all x, T so if we divide by T and take the $\liminf_{T \rightarrow \infty}$ we obtain $\lambda \leq J(x, \alpha_t)$.

In order to prove *ii*), we note that (1.6) becomes an equality if $\alpha \in \operatorname{argmax}\mathcal{H}(x, Du(x), \alpha)$, so if we use v_t^* defined above instead of a generic admissible control α_t , the equality $\lambda = J(x, v_t^*)$ follows. \square

The verification theorem states the possibility of synthesizing an optimal control of Markovian type by solving a quasilinear elliptic equation. We mention that the problem is studied also (with probabilistic methods) in polyhedral domains in [23].

1.2 An Existence Result for the HJB equation

The problem of finding a solution to (1.4) has been considered by many authors, and the literature on quasilinear uniformly elliptic equation is in general very wide. The Neumann problem (1.4)-(1.5) is studied in [19, 60, 69], in [21] in polyhedral domains and in [62] in the first order case ($\nu = 0$).

For this particular Hamilton-Jacobi-Bellman partial differential equation, the unknown consists in a pair (u, λ) . The value of λ is usually unique, in the sense that there exists a precise value of λ such that (1.4) has a solution u ; this value might be difficult to be determined a-priori. It has been observed (see [5]) that an efficient strategy to prove existence is to pass to the limit on approximating problems where there is only one unknown. In particular, for all $\alpha > 0$ we may consider solutions u_α of

$$-\nu\Delta u_\alpha(x) + H(x, Du_\alpha(x)) + \alpha u_\alpha = f(x) \quad \forall x \in \Omega. \quad (1.8)$$

This family of HJB equations is also associated to a control problem, where the cost functional is a long-time evaluation of $L + f$ against a discount factor $e^{-\alpha t}$. Under some assumptions,

¹ $\mathbb{E}^x[M_t|\mathcal{F}_s] = M_s$ for all $0 \leq s < t$.

which basically guarantee uniform bounds on Du_α and the existence of converging subsequences αu_α , it is possible to prove that

$$\alpha u_\alpha \rightarrow \lambda, \quad u_\alpha - \frac{1}{|\Omega|} \int_\Omega u_\alpha \rightarrow u$$

as $\alpha \rightarrow 0$ uniformly in x , and u, λ solve (1.4).

This method is successfully implemented in [60]. The author proves some estimates on Du_α using the method of Bernstein, which consists in taking derivatives of the original equation and using maximum principles on the resulting elliptic equation for Du_α . The main result, which will be used in the sequel, requires the following set of assumptions:

(H) 1. There exist $B \in W^{1,\infty}(\Omega, \mathbb{R}^d), C > 0$, such that

$$H(x, p) \geq B(x) \cdot p - C \quad \forall x \in \Omega, p \in \mathbb{R}^d,$$

2. For all $x \in \Omega, p \in \mathbb{R}^d$

$$\partial_p H(x, p) \cdot p - H(x, p) \geq -C$$

for some $C > 0$,

3. There exist $\mu > 0, \theta \in (0, 1), R_0 > 0$ such that

$$\partial_x H \cdot p + \frac{\theta}{d} H^2 + \mu |p|^2 + \mu [\partial_p H \cdot p - H] > 0$$

for all $|p| \geq R_0, x \in \Omega$, and

$$\partial_x H \cdot p + \frac{\theta}{d} H^2 + \mu |p|^2 + \mu [\partial_p H \cdot p - H] + t |p|^2 [\partial_p H \cdot n(x)] > 0$$

for all $|p| \geq R_0, x$ in a neighborhood of $\partial\Omega$ and $t \in [0, C_0]$, where C_0 is the maximum among the negative part of the principal curvatures of $\partial\Omega$.

If (H) holds the approximating problems (1.8) with Neumann boundary condition (1.5) have solutions u_α , and it is possible to pass to the limit.

Theorem 1.2. *Suppose that Ω is a bounded C^2 domain, $f \in W^{1,\infty}(\Omega), \|f\|_{W^{1,\infty}(\Omega)} \leq \alpha$ for some $\alpha > 0$ and that (H) holds. Then, there exist $u \in C^2(\overline{\Omega})$ and $\lambda \in \mathbb{R}$ that solve (1.4) and (1.5). Moreover,*

$$|\lambda| \leq \|f\|_{L^\infty(\Omega)} \tag{1.9}$$

and

$$\|u\|_{W^{1,\infty}(\Omega)} \leq C \tag{1.10}$$

where C is a positive constant that depends on Ω, H and α .

Proof. See [60], Theorem II.1. □

Remark 1.3. Given any solution (u, λ) of (1.4), (1.5), then $(u + c, \lambda)$ is still a solution for all $c \in \mathbb{R}$. The verification theorem produces a (candidate) feedback for every solution u to the HJB equation, so there is apparently a continuum of optimal feedback strategies. This does not happen because the optimal v^* is constructed starting from Du . It is then common to choose c so that $\int_\Omega u = 0$ or $u(0) = 0$.

1.3 Ergodic property of Reflected Processes

Ergodic theory studies the long-time behavior of processes. In some cases, dynamical systems evolve into stable configurations; in such situations it is interesting to characterize the *invariant measure* of the process, a measure that does not change under the flow of the process and is a limit of different starting configurations. In the deterministic case the limit at infinity is taken pointwise, but in the stochastic case we have to consider convergence in law.

Our aim is to prove in this section that reflected diffusions (1.1) have a “nice” long-time behavior, meaning that the law of the process converges to a unique invariant measure. This is well known and has been studied by many authors, see for example [76] and [6] for some results on non-smooth (polyhedral) domains. We will follow here the presentation of [18], Chapter II.4. Consider first the Cauchy problem

$$\begin{cases} \partial_t z - \nu \Delta z - b \cdot Dz = 0 & \text{in } \Omega \\ \partial_n z|_{\partial\Omega} = 0 \\ z(x, 0) = f(x), \end{cases} \quad (1.11)$$

where f is a Borel measurable bounded function. Here we take $b(\cdot) = b(\cdot, v(\cdot))$, assuming also that the feedback strategy v that we fix is bounded. We remark that we are interested in the *optimal* process, that is associated to a feedback v^* which can be constructed starting from a solution of a HJB equation (see Theorem 1.1) and has usually a bounded gradient.

Since the Cauchy problem is well-posed, we can define a linear operator P , acting on the space of bounded Borel functions on Ω in the following way:

$$Pf(x) := z(x, T),$$

where $T > 0$ is fixed. Then, it is possible to prove that (see [18], section II.4.2) there exists a unique probability measure π on Ω such that

$$\left| P^n f(x) - \int_{\Omega} f(x) d\pi(x) \right| \rightarrow 0 \quad (1.12)$$

as $n \rightarrow \infty$. Moreover π is the invariant probability measure for the process X_t , in the sense that

Definition 1.4. *A probability measure π on Ω is said to be invariant if*

$$\int_{\Omega} f(x) d\pi(x) = \int_{\Omega} Pf(x) d\pi(x). \quad (1.13)$$

We note that (1.13) can be restated as

$$\int_{\Omega} f(x) d\pi(x) = \int_{\Omega} \mathbb{E}^x f(X_T) d\pi(x), \quad (1.14)$$

being the equality $Pf(x) = \mathbb{E}^x f(X_T)$ easily verified using Ito formula.

We observe that the invariant measure is linked in a natural way to the solution of the *Kolmogorov* partial differential equation

$$\begin{cases} \nu \Delta m - \operatorname{div}(bm) = 0 & \text{in } \Omega \\ \nu \partial_n m = m b \cdot n & \text{on } \partial\Omega, \quad \int_{\Omega} m = 1. \end{cases} \quad (1.15)$$

Indeed, suppose that a weak solution m to (1.15) is given; multiplying the equation in (1.11) by m , integrating by parts in space and integrating in time from 0 to t leads to the equality

$$\int_{\Omega} z(x, t)m(x)dx = \int_{\Omega} f(x)m(x)dx \quad \forall t.$$

Since $P^n f(x) = z(x, nT)$, setting $t = nT$ and passing to the limit as $n \rightarrow \infty$ in (1.12) we prove that

$$\int_{\Omega} f(x)d\pi(x) \int_{\Omega} m(x)dx = \int_{\Omega} f(x)m(x)dx.$$

The function f can be chosen arbitrarily and m has unit mass in Ω , so we obtain that m is the *density* of the invariant measure π :

$$\pi(dx) = m(x)dx.$$

A way to characterize the invariant measure of the reflected process driven by b is then to consider solutions of (1.15). In particular,

Theorem 1.5. *Suppose that Ω is a C^2 bounded domain and $b \in L^\infty(\Omega, \mathbb{R}^d)$. Then, (1.15) has a unique weak solution $m \in W^{1,p}(\Omega)$ for all $p \geq 1$, in the sense that*

$$\nu \int_{\Omega} Dm \cdot D\phi = \int_{\Omega} mb \cdot D\phi \quad \forall \phi \in W^{1,2}(\Omega). \quad (1.16)$$

Moreover, $m \in C(\bar{\Omega})$ and satisfies

$$\delta^{-1} \leq m(x) \leq \delta \quad \forall x \in \bar{\Omega}$$

for some $\delta > 0$ depending only on $\|b\|_{L^\infty(\Omega)}$.

Proof. See [18], Theorems II.4.4, II.4.5, II.4.7. \square

We conclude by proving a bound on $\|m\|_{L^\infty(\Omega)}$ that depends just on the L^p norms of m and b ; it will be useful in the sequel.

Proposition 1.6. *Let $r > d$, $q > \frac{r}{r-1}$ and suppose that $b \in C(\bar{\Omega})$ satisfies $b \cdot n = 0$ on $\partial\Omega$. Moreover, $\|m\|_{L^q(\Omega)} \leq K$, $\|b\|_{L^r(\Omega)} \leq K$ for some $K > 0$. If m is a solution of (1.16), then*

$$\|m\|_{L^\infty(\Omega)} \leq C \quad (1.17)$$

for some $C = C(K, \nu, d, \Omega)$.

Proof. From [70], Theorem 3.1, we know that the following a-priori estimate on m holds:

$$\|m\|_{W^{1,p}(\Omega)} \leq C(\|\nu\Delta m\|_{W^{-1,p}(\Omega)} + \|m\|_{W^{-1,p}(\Omega)}),$$

for all $p > 1$ and a constant C that depends on p, ν, d, Ω . Using equation (1.16) and Holder inequality, for all test functions $\phi \in C^\infty(\bar{\Omega})$,

$$\left| \int_{\Omega} \nu\Delta m \phi \right| \leq \int_{\Omega} |mb \cdot D\phi| \leq \|m\|_{L^q(\Omega)} \|b\|_{L^r(\Omega)} \|D\phi\|_{L^{p'}(\Omega)}$$

and similarly

$$\left| \int_{\Omega} m \phi \right| \leq \int_{\Omega} |m\phi| \leq \|m\|_{L^q(\Omega)} \left(\int_{\Omega} dx \right)^{1/r} \|\phi\|_{L^{p'}(\Omega)},$$

setting p, p' such that

$$\frac{1}{r} + \frac{1}{q} + \frac{1}{p'} = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

That leads to

$$\|m\|_{W^{1,p}(\Omega)} \leq C,$$

as $p > 1$ by the choice of r, q . Plus, C depends only on K and fixed data of the problem. If $p > d$ we are done by using Sobolev embeddings (by which m is continuous and bounded on the whole domain). Else,

$$\|m\|_{L^{p^*}(\Omega)} \leq C,$$

with $p^* \geq q + \epsilon$ for some $\epsilon > 0$ that does not depend on q by the hypothesis $r > d$. Iterating the last two estimates and setting $q = p^*$ at each time, a bootstrap argument let us conclude in a finite number of steps that

$$\|m\|_{L^\infty(\Omega)} \leq C.$$

□

Chapter 2

Ergodic Control in the Whole Space

In this chapter we will prove some results on the existence of an optimal feedback control for ergodic problems where the state space is the whole \mathbb{R}^d . After a brief introduction, we will show a general method for solving this problem, based on approximation on bounded domains (Section 2.1); results of the previous chapter will be used. Then, we will present some particular cases where the abstract assumptions of the first part are satisfied and thus the results apply (Section 2.2).

In Chapter 1 we focused on controlled diffusions on bounded domains with reflection at the boundary. We are now interested in studying the controlled process¹

$$dX_t = [b(X_t) - \alpha_t]dt + \sqrt{2\nu}dB_t, \quad X_0 = x \in \mathbb{R}^d. \quad (2.1)$$

Our goal is to minimize the long-time average cost

$$J(x, \alpha_t) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^x \int_0^T [L(\alpha_t) + f(X_t)] dt,$$

within a set of admissible controls, where X_t is subject to (2.1).

Such kind of minimization problem on the whole space has been considered, for example, in [4, 17, 22]. Partial differential equation techniques or probabilistic methods can be implemented, and the main difficulty is the lack of compactness of the state space. Recently, this topic has been reconsidered by other authors [47, 48, 53], who realized that interesting phenomena arise if some parameter is put in front of the cost functional. Here we will focus on the sufficient conditions on the data b, f that should be satisfied in order for the ergodic problem to have a solution.

Once a stochastic system is fixed, i.e. $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a d -dimensional Brownian motion B_t , we will need some minimal regularity assumptions on b :

$$b \in C^1(\mathbb{R}^d, \mathbb{R}^d) \quad (2.2)$$

for some $C > 0$. Moreover, we postpone the precise definition of admissible control, but already mention that they should satisfy

1. α_t is \mathcal{F}_t -progressively measurable.
2. The SDE (2.1) has a unique strong solution which is defined for all $T > 0$ a.s., namely $X_t = X_t^{\alpha_t}$ does not explode in finite time.

¹With respect to Chapter 1, the dependence of drift b in the stochastic differential equation with respect to X_t and α_t is split. This simplifies computations, but the methods described in this chapter can be adapted to more general drifts b .

As in the case of controlled reflected diffusions, this optimization problem is naturally linked to the Hamilton-Jacobi-Bellman elliptic partial differential equation

$$-\nu\Delta u - b(x) \cdot Du + H(Du) + \lambda = f(x) \quad \text{in } \mathbb{R}^d, \quad (2.3)$$

where the hamiltonian H is given by the Legendre transform

$$H(p) = \sup_{\alpha \in \mathbb{R}^d} [\alpha \cdot p - L(\alpha)].$$

Indeed, a verification argument (see Theorem 1.1 for bounded domain case) states that a regular solution \bar{u} of (2.3) produces a candidate optimal control of feedback type, once it is checked that it is also admissible; in particular, if X_t is the solution of

$$dX_t = [b(X_t) - D_p H(D\bar{u}(X_t))]dt + \sqrt{2\nu}dB_t, \quad (2.4)$$

then the minimum of $J(x, \alpha_t)$ should be attained by $\alpha_t^* := -D_p H(D\bar{u}(X_t))$ and be equal to the constant λ .

Note that for diffusions in \mathbb{R}^d we are not requiring any boundary condition for solutions of (2.3), even though we will study their behavior at infinity in order to build an admissible optimal Markov process. Another difference with respect to the bounded case is that the process (2.4) might not be ergodic, in particular the optimal trajectory does not necessarily have an invariant measure. Being the state space not compact, the dynamics could drift away from the origin, so the study of its ergodicity properties is of primary importance. We address in particular to situations where the optimal process has an invariant measure, i.e. there exists a positive solution of the Kolmogorov equation (see the discussion in Section 1.3 for the link between the equation and the invariant measure)

$$\nu\Delta m + \text{div}[m(-b(x) + D_p H(Du))] = 0 \quad \text{in } \mathbb{R}^d. \quad (2.5)$$

2.1 A general recipe for solving the ergodic control problem on \mathbb{R}^d .

Our aim is to build a general setup for

- Proving the existence of a (classical) solution of (2.3),
- Solving the Kolmogorov equation (2.5),
- Synthesizing a feedback control α_t^* and verifying the equality $\lambda = J(x, \alpha_t^*) = \inf_{\alpha} J(x, \alpha_t)$.

We will state this precisely in Definition 2.13. To do so we use the machinery of control on bounded sets developed in Chapter 1. Precisely, we solve the control problem on bounded domains \mathcal{O}_R , truncating the trajectory defined by (2.1) by reflection on $\partial\mathcal{O}_R$, then we let $R \rightarrow \infty$ and \mathcal{O}_R invade the whole space \mathbb{R}^d . In order to preserve the ergodicity property of the approximating problems and guarantee the existence of an invariant measure for the limit process, optimal for the original control problem, we implement the method of Lyapunov functions.

Such an approximation process overcomes the problem that there are two unknowns in (2.3) and λ cannot be determined a priori. Here λ will be the limit of the λ_R coming from optimal reflected processes. This technique has been used in the uncontrolled case by [63] to study ergodicity of diffusions on the whole space. It has been mentioned also in [19], where

some results for ergodic control in \mathbb{R}^d obtained previously in [17] with a different kind of approximation are recovered. In [17] existence for (2.3) is obtained by passing to the limit in control problems on the whole space with positive discount factor (where the only unknown u is present).

It has been shown in [53] that the value λ for which equations of type (2.3) have a solution u is not unique (in contrast to the bounded domain case); also in [47] this kind of phenomenon is examined, and it is pointed out that there usually exists a unique value of λ for which the diffusion defined by the corresponding u is ergodic. We mention that a different way of approximation is used (Dirichlet problems on compact domains are solved).

2.1.1 The approximating Neumann problems.

We describe now the approximating problems on compact domains. First we gather some assumptions on L, b, f .

(H0) *i)* $L \in C^2(\mathbb{R}^d \setminus \{0\})$ is strictly convex, $f \in C^1(\mathbb{R}^d)$, $f \geq 0$, $|Df|$ is bounded on \mathbb{R}^d and b satisfies (2.2).

ii) There exist some $L_0 > 0$ and $\gamma^* > 1$ such that

$$L_0|q|^{\gamma^*} \leq L(q) \leq L_0^{-1}|q|^{\gamma^*} \quad \forall x \in \mathbb{R}^d.$$

The Lagrangian that we have in mind is $L(q) = L_0|q|^{\gamma^*}$, $L_0 > 0, \gamma^* > 1$. From (H0) some properties on the hamiltonian H follow directly:

Proposition 2.1. *If (H0) holds, then there exist $C_0, \tilde{C}_0 > 0$ such that for all $p \in \mathbb{R}^d$,*

i) $H \in C^2(\mathbb{R}^d \setminus \{0\})$ and it is strictly convex,

ii) $C_0^{-1}|p|^\gamma - C_0 \leq H(p)$,

iii) $|D_p H(p)| \leq C_0(1 + |p|^{\gamma-1})$,

iv) $D_p H(p) \cdot p - H(p) \geq \tilde{C}_0|p|^\gamma$,

where $\gamma = \gamma^*/(\gamma^* - 1)$.

Proof. See, for example, [46], Theorem 3.4 for *i)*, *ii)*, *iii)*. For *iv)*, we note that $\gamma^*(\gamma - 1) = \gamma$ and $H(p) + L(q) = p \cdot q$ if $q = D_p H(p)$, so

$$D_p H(p) \cdot p - H(p) = L(D_p H(p)) \geq L_0|D_p H(p)|^{\gamma^*} \geq \tilde{C}_0|p|^{\gamma^*(\gamma-1)}$$

for some positive constant \tilde{C}_0 . □

Suppose that a function W is given (it will be constructed later); we consider the family of Neumann problems

$$\begin{cases} -\nu \Delta u_R - b(x) \cdot Du_R + H(Du_R) + \lambda_R = f(x) & \text{in } \mathcal{O}_R, \\ \partial_n u_R = 0 & \text{on } \partial \mathcal{O}_R. \end{cases} \quad (2.6)$$

where the family of domains \mathcal{O}_R is defined in the following

Definition 2.2. *Let $W \in C^2(\mathbb{R}^d)$ and $R = R_j \nearrow +\infty$. We define $\mathcal{O}_R = \mathcal{O}_{W,R}$ as*

$$\mathcal{O}_R := \{x \in \mathbb{R}^d : W(x) \leq R\}$$

As stated in Theorem 1.1, (2.6) is related to a control problem for a reflected stochastic differential equation (1.1) with cost (1.2) to be minimized. For this HJB equation we guarantee existence on every domain \mathcal{O}_R .

Theorem 2.3. *Suppose that (H0) holds and let \mathcal{O}_R be the family of domains of Definition 2.2. Then, for all R there exist (unique) $u_R \in C^2(\overline{\mathcal{O}_R})$, $\lambda_R \in \mathbb{R}^d$ satisfying (2.6) and $u(0) = 0$.*

Proof. See Theorem 1.2: let $\tilde{H}(x, p) = -b(x) \cdot p + H(p) - f(x)$, then the hypotheses (H) hold. \square

As for the Kolmogorov equation on \mathcal{O}_R , we look for a solution of the Neumann problem in weak sense:

$$\nu \int_{\mathcal{O}_R} Dm_R \cdot D\phi + \int_{\mathcal{O}_R} m_R(-b(x) + D_p H(Du_R)) \cdot D\phi = 0, \quad \forall \phi \in W^{1,2}(\mathcal{O}_R) \quad (2.7)$$

We recall that m_R is the invariant measure of the optimal reflected process driven by drift $b - D_p H(Du_R)$.

Theorem 2.4. *Suppose that (H0) holds and let \mathcal{O}_R be the family of domains of Definition 2.2. Then, for all R there exists a unique positive $m_R \in W^{1,2}(\mathcal{O}_R)$ solution of (2.7) such that $\int_{\mathcal{O}_R} m_R = 1$.*

Proof. This is Theorem 1.5, since by Theorem 2.7 the quantity $D_p H(Du_R)$ exists, is continuous and is bounded on $\overline{\mathcal{O}_R}$. \square

We note that $-b(x) + D_p H(Du_R) \in C^1(\mathcal{O}_R)$, then by standard elliptic regularity theory (see [37] for example) $m \in C^{2,\alpha}(\mathcal{O}_R)$, and the Kolmogorov equation is solved in classical sense.

2.1.2 Convergence of the problems as $R \rightarrow \infty$.

Convergence for the HJB equation as $R \rightarrow \infty$ takes place under mild hypotheses. A fundamental requirement is that for some control, the corresponding process has the ergodic property; this enables us the possibility to control uniformly the ergodic constants λ_R .

(H1) There exist $k : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable, $\alpha \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ and $C_1, C'_1 > 0$ such that

$$f(x) \leq C_1(C'_1 + k(x)), \quad L(\alpha(x)) \leq C_1(C'_1 + k(x)),$$

and there exists a $W \in C^2(\mathbb{R}^d)$ such that $W \rightarrow +\infty$ as $|x| \rightarrow +\infty$,

$$-\nu \Delta W - (b(x) - \alpha(x)) \cdot DW \geq k(x) \text{ in } \mathbb{R}^d. \quad (2.8)$$

Condition (2.8) is a well known sufficient requirement for ergodicity of the process driven by drift $b - \alpha$, see [54], [11], [63]; it is also ‘‘almost’’ necessary.

We now estimate λ_R and $|Du_R|$ independently on R .

Proposition 2.5. *Under the assumption (H1),*

$$|\lambda_R| \leq C$$

for all R .

Proof. Let $G(x) := -(b(x) - \alpha(x))$. Let $\mu \in W^{1,2}(\Omega)$ be the (weak) solution of

$$\nu \int_{\mathcal{O}_R} D\mu \cdot D\psi + \int_{\mathcal{O}_R} \mu G \cdot D\psi = 0, \quad \forall \psi \in W^{1,2}(\mathcal{O}_R), \quad \int_{\mathcal{O}_R} \mu = 1, \quad (2.9)$$

namely μ is the invariant measure of the process driven by the drift $b(x) - \alpha(x)$. We know also that μ is positive and continuous on the whole \mathcal{O}_R (Theorem 1.5). We choose $\psi = W$ as a test function, and integrate by parts the first integral to get

$$-\nu \int_{\mathcal{O}_R} \mu \Delta W + \int_{\mathcal{O}_R} \mu G \cdot DW + \int_{\partial \mathcal{O}_R} \mu D_n W = 0.$$

Since $D_\nu W = |DW|^2 \geq 0$ on $\partial \mathcal{O}_R$,

$$\int_{\mathcal{O}_R} \mu (-\nu \Delta W + G \cdot DW) \leq 0,$$

hence

$$\int_{\mathcal{O}_R} \mu k(x) \leq 0.$$

Let now $u = u_R$ be the solution of (2.6) on \mathcal{O}_R subject to Neumann boundary conditions. u satisfies $\forall x \in \mathcal{O}_R$

$$-\nu \Delta u(x) - b(x) \cdot Du(x) + \sup_{a \in \mathbb{R}^d} \{Du(x) \cdot a - L(a)\} + \lambda = f(x),$$

so

$$-\nu \Delta u(x) + Du(x) \cdot G(x) + \lambda \leq f(x) + L(\alpha(x)).$$

We multiply the inequality by μ and integrate over \mathcal{O}_R to get

$$-\nu \int_{\mathcal{O}_R} \Delta u \mu + \int_{\mathcal{O}_R} \mu G \cdot Du + \lambda \leq \int_{\mathcal{O}_R} (f(x) + L(\alpha(x))) \mu.$$

Integrating by parts and using u as a test function in (2.9) yields

$$\lambda \leq \int_{\mathcal{O}_R} (f(x) + L(\alpha(x))) \mu \leq \int_{\mathcal{O}_R} 2C_1(C'_1 + k(x)) \mu \leq 2C_1 C'_1,$$

that is the upper bound for λ . By the maximum principle we also have $\lambda \geq \min_{\mathcal{O}_R} f \geq 0$. \square

We now are ready to state the theorem on the convergence of u_R to a solution of (2.3), guaranteed by the estimate of Proposition 2.5 and interior Bernstein estimates on Du_R stated in Theorem A.2.

Theorem 2.6. *Suppose that (H0), (H1) hold. Then, there exists a subsequence $R_j \rightarrow +\infty$ such that $u_{R_j} \rightarrow \bar{u} \in \mathcal{C}^{2,\beta}(\mathbb{R}^d)$ locally in $\mathcal{C}^{2,\beta}$, $\lambda_{R_j} \rightarrow \bar{\lambda} \in [0, +\infty)$ for some $0 < \beta < 1$. Moreover, $(\bar{u}, \bar{\lambda})$ satisfy (2.3).*

Proof. By Proposition 2.5, λ_R is bounded, so we can extract a converging subsequence (that we will still denote by R) $\lambda_R \rightarrow \bar{\lambda}$. Let B_ρ be a ball of \mathbb{R}^d , then $B_{2\rho} \subset \mathcal{O}_R$ if $R \gg 0$ ($W \rightarrow \infty$, so $\bigcup_R \mathcal{O}_R = \mathbb{R}^d$). By virtue of Theorem A.2, $|Du_R|$ is bounded on B_ρ by a constant that does not depend upon R . Moreover,

$$\sup_{B_\rho} |u_R(x)| \leq C_\rho,$$

C_ρ will denote through the proof a constant that depends on ρ but not on R . Let now $\tau \in C^2(B_\rho)$, $\tau = 1$ on $B_{\rho/2}$ and identically zero on ∂B_ρ be a localization function. Then

$$\begin{aligned} \nu|\Delta(\tau u_R)| &= |\nu(\Delta\tau)u_R + 2\nu D\tau \cdot Du_R + (\nu\Delta u_R\tau)| \\ &\leq \nu|(\Delta\tau)u_R| + 2\nu|D\tau \cdot Du_R| + \tau|b \cdot Du_R| + H(Du_R) + \lambda - f| \leq C_\rho \end{aligned} \quad (2.10)$$

for all $x \in B_\rho$; $\tau u_R = 0$ on ∂B_ρ , so by standard elliptic estimates (see Theorem 8.33 [37])

$$\|u_R\|_{C^{1,\beta'}(B_{\rho/2})} \leq \|\tau u_R\|_{C^{1,\beta'}(B_\rho)} \leq C_\rho$$

for some β' . Therefore $\|Du_R\|_{C^{0,\beta'}(B_{\rho/2})}$ is bounded uniformly, and then, again by elliptic Schauder estimates (corollary 6.3 [37]), $\|u_R\|_{C^{2,\beta'}(B_{\rho/2})} \leq C_\rho$. since u_R, Du_R, D^2u_R are equibounded and equicontinuous on every $B_{\rho/2}$ with constants depending only on ρ , by Ascoli-Arzelà theorem (and a diagonalization process) it is possible to extract a subsequence $u_R \rightarrow \bar{u}$, which converges locally in $C^{2,\beta}$ on \mathbb{R}^d for some $\beta < \beta'$.

Moreover, $f - \lambda_R \rightarrow f - \bar{\lambda}$ uniformly on \mathbb{R}^d , so \bar{u} satisfies the equation (2.3) on the whole space. □

For the convergence of the invariant measures m_R of the approximating reflected processes to an invariant measure on the whole space, we first need some local estimates.

Proposition 2.7. *Let $\rho > 0$ such that $B_{8\rho} \subset \mathcal{O}_R$. Let m_R be the solution of (2.7) such that $\int_{\mathcal{O}_R} m_R = 1$. Then there exists $C > 0$ depending on $N, \rho, \sup_{B_{8\rho}} |b|, \sup_{B_{8\rho}} |H_p(Du_R)|, p > 1$, such that*

$$\|m_R\|_{W^{2,p}(B_\rho)} \leq K.$$

Proof. First we note that $\inf_{B_{2\rho}} m_R \leq |B_{2\rho}|^{-1}$; indeed, if not

$$1 < \int_{B_{2\rho}} \inf_{B_{2\rho}} m_R \leq \int_{B_{2\rho}} m_R \leq \int_{\mathcal{O}_R} m_R = 1,$$

which is a contradiction. Then, by Harnack inequality ([37], Theorem 8.20)

$$\sup_{B_{2\rho}} m_R \leq C \inf_{B_{2\rho}} m_R \leq C|B_{2\rho}|^{-1}, \quad (2.11)$$

where C is a positive constant that does not depend on R (and may possibly increase during the proof). Since m_R is a solution of the elliptic equation (2.7), by standard L^p estimates ([37], Theorem 9.11)

$$\|m_R\|_{W^{2,p}(B_\rho)} \leq C\|m_R\|_{L^p(B_{2\rho})} \leq C'$$

for any $p > 1$, by (2.11). □

We also need m_R not to “disperse too much” as the domain gets bigger. Precisely, we suppose that

(M)

$$\sup_R \int_{\{s \leq |x| \leq R\}} m_R \rightarrow 0 \quad \text{as } s \rightarrow +\infty.$$

This enable us to prove the following result on the convergence of m_R to a solution of the Kolmogorov equation on the whole space.

Theorem 2.8. *Let $q > 1$, R_j be the sequence in Theorem 2.6 and suppose that (H0), (H1), (M) are satisfied. Then, up to a subsequence, $m_{R_j} \rightarrow \bar{m}$ in $W_{\text{loc}}^{1,q}(\mathbb{R}^d)$ for some $\bar{m} \in W_{\text{loc}}^{1,q}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \bar{m} = 1$. Moreover \bar{m} solves*

$$\int_{\mathbb{R}^d} D\bar{m} \cdot D\phi + \int_{\mathbb{R}^d} \bar{m}(-b + H_p(D\bar{u})) \cdot D\phi = 0 \quad \forall \phi \in C_0^2(\mathbb{R}^d), \quad (2.12)$$

where \bar{u} is the solution of (2.3).

Proof. We may suppose that $m_R = m_{R_j}$ is defined on the whole space \mathbb{R}^d , setting $m_R = 0$ on $\mathbb{R}^d \setminus B_{R_j}$. For every fixed ball B_ρ , m_R is bounded in $W^{2,p}(B_\rho)$ by Proposition 2.7, since $|Du_R|$ is uniformly bounded on B_ρ and H_p is continuous. Hence, the Rellich-Kondrachov theorem assures that there exists a subsequence that converges in $W^{1,q}(B_\rho)$ (if p is chosen large enough) to some \bar{m} ; through a diagonalization process it is possible to extend the convergence locally in $W^{1,q}$ on the whole space \mathbb{R}^d .

We now have to verify that \bar{m} is a probability measure. Indeed, let $\epsilon > 0$ and, thanks to (M), s sufficiently large such that

$$\sup_R \int_{B_R \cap \{|x| \geq s\}} m_R \leq \epsilon.$$

Notice also that for all R

$$\int_{\{|x| \leq s\}} m_R + \int_{B_R \cap \{|x| \geq s\}} m_R = 1,$$

hence

$$\begin{aligned} \int_{\{|x| \leq s\}} \bar{m} &= \int_{\{|x| \leq s\}} (\bar{m} - m_R) + \int_{\{|x| \leq s\}} m_R \\ &= 1 + \int_{\{|x| \leq s\}} (\bar{m} - m_R) - \int_{B_R \cap \{|x| \geq s\}} m_R = 1 + e, \end{aligned}$$

where $|e| \leq 2\epsilon$ if we choose $R \gg 0$. Letting $s \rightarrow +\infty$, since ϵ is arbitrarily close to zero

$$\int_{\mathbb{R}^d} \bar{m} = 1.$$

We eventually show that \bar{m} solves the Kolmogorov equation. m_R is a solution of (2.7), so if $\phi \in C_0^2(\mathbb{R}^d)$

$$\nu \int_{\mathcal{O}_R} Dm_R \cdot D\phi + \int_{\mathcal{O}_R} m_R(-b(x) + D_p H(Du_R)) \cdot D\phi = 0.$$

Since $Du_R \rightarrow D\bar{u}$ locally uniformly, we can pass to the limit as $R \rightarrow +\infty$ to obtain

$$\nu \int_{\mathcal{O}_R} D\bar{m} \cdot D\phi + \int_{\mathcal{O}_R} \bar{m}(-b(x) + D_p H(D\bar{u})) \cdot D\phi = 0.$$

□

Remark 2.9. We note that the condition

$$\exists s_0, C_M > 0 \text{ such that } \int_{\{|x| \leq s_0\}} m_R \geq C_M \quad \forall R \geq s_0,$$

which is weaker than (M), is sufficient for carrying out the proof of the existence of an invariant *probability* measure. Indeed, from the local convergence of m_R it follows that $m_R \rightarrow \bar{m}$ in $L^1(B_{s_0})$, so $\int_{\{|x| \leq s_0\}} \bar{m} \geq C_M$; moreover $\int_{\mathbb{R}^d} \bar{m} \leq 1$, so if we set $\hat{m} = \bar{m} / \int_{\mathbb{R}^d} \bar{m}$, \hat{m} is a probability measure and solves the Kolmogorov equation with drift $b - D_p H(D\bar{u})$.

Since we are adjusting the limit of m_R by a constant that is possibly different from one, the invariant measure of the optimal trajectory on \mathbb{R}^d might not be precisely the limit of the invariant measures of the approximating problems. Contrarily, this is true if (M) holds.

We state now a general criterion for solving the ergodic control problem and producing the synthesis of an optimal feedback control. We suppose that (H0), (H1), (M) hold, so there exist a solution \bar{u} of the HJB equation (2.3) on \mathbb{R}^d and a solution \bar{m} of the Kolmogorov equation (2.12). We underline that, at this stage, \bar{u} provides just a candidate for an optimal control in feedback form, since we have to check that it belongs to a reasonable class of admissible controls (see the following Definition 2.10). Moreover, \bar{m} is only a candidate for the invariant measure of the optimal process, as uniqueness for (2.12) is not guaranteed without further assumptions².

First, some control on the growth at infinity of u is needed. Suppose in particular that for some $C > 0$

$$|\bar{u}(x)| \leq C(1 + |x|) \quad \forall x \in \mathbb{R}^d. \quad (2.13)$$

Then, under this assumption a natural choice for the class of admissible controls is:

Definition 2.10. *We say that a \mathcal{F}_t -progressively measurable control α_t is admissible if the corresponding process $X_t = X_t^{\alpha_t}$ does not explode in finite time and*

$$\liminf_{t \rightarrow \infty} \frac{\mathbb{E}^x |X_t|}{t} = 0 \quad (2.14)$$

holds.

Any solution \bar{u} enables the synthesis of a feedback control. A control of feedback type is given by a locally Lipschitz function $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^d$; the solution $X_t = X_t^\alpha$ of

$$dX_t = [b(X_t) - \alpha(X_t)]dt + \sqrt{2\nu}dB_t, \quad X_0 = x$$

let us define the control

$$\alpha_t = \alpha(X_t).$$

We usually denote a control α_t given in feedback form by the function α itself.

We look for an optimal process which is ergodic, namely having an invariant measure:

Definition 2.11. *A stochastic process is ergodic if there exists a probability measure π such that*

$$\int_{\mathbb{R}^d} f(x) d\pi(x) = \int_{\mathbb{R}^d} \mathbb{E}^x(f(X_t)) d\pi(x).$$

for all $t > 0$ and for all bounded $f \in C(\mathbb{R}^d)$.

It is known that a sufficient condition for a diffusion X_t solving

$$dX_t = b(X_t)dt + \sqrt{2\nu}dB_t, \quad X_0 = x \in \mathbb{R}^d.$$

²See [20] for a recent survey on the issue of uniqueness for the Kolmogorov equation with unbounded drift and [67] for details on the link between solutions of the Kolmogorov equation and invariant measures of diffusion processes on the whole space.

to be ergodic is the existence of a *Lyapunov function*, i.e. a function $W \in C^2(\mathbb{R}^d \setminus B_R)$ such that $W \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and

$$-\nu\Delta W(x) - b(x) \cdot DW(x) \geq 1 \text{ for all } x \in \mathbb{R}^d \setminus B_R$$

for some $R \geq 0$. For this fact see [68, 11, 63].

We now show that

Theorem 2.12. *Suppose that (H0), (H1), (M), (2.13) hold and α_t is admissible. Then*

$$\bar{\lambda} \leq J(x, \alpha_t) \quad \forall x \in \mathbb{R}^d.$$

Suppose also that the process X_t^ corresponding to the control*

$$\alpha_t^* := D_p H(D\bar{u}(X_t^*))$$

given in feedback form is ergodic, and

$$\int_{\mathbb{R}^d} |x| \bar{m}(x) dx < \infty, \quad (2.15)$$

where \bar{m} is the invariant distribution of X_t^ , then α_t^* is admissible and*

$$\bar{\lambda} = J(x, \alpha_t^*) \quad \forall x \in \mathbb{R}^d. \quad (2.16)$$

Proof. We apply Ito's rule and use the dynamic programming equation to obtain

$$\bar{u}(x) + \bar{\lambda}T \leq \mathbb{E}^x \bar{u}(X_T) + \mathbb{E}^x \int_0^T [L(\alpha_t) + f(X_t)] dt. \quad (2.17)$$

Since $|\mathbb{E}^x \bar{u}(X_T)| \leq C\mathbb{E}^x |X_T| + C$, dividing by T and taking the \liminf as $T \rightarrow \infty$ proves the first assertion.

For the second part, we first note that X_t^* does not blow up in finite time because it is ergodic. Moreover, the solution \bar{m} of the Kolmogorov equation is its (unique) invariant measure (see, for example, [68], Theorem 4.8.4). Since $\int_{\mathbb{R}^d} |x| \bar{m} < \infty$, we can apply the Ergodic Theorem ([48] Theorem 2.6) to obtain

$$\mathbb{E}^x |X_T^*| \rightarrow \int_{\mathbb{R}^d} |x| \bar{m}(x) dx < \infty \quad \text{as } T \rightarrow \infty, \quad (2.18)$$

that proves that α_t^* is admissible. Moreover, we can conclude using (2.18) that $|\bar{u}(x)|/T + |\mathbb{E}^x \bar{u}(X_T^*)/T| \leq |\bar{u}(x)|/T + C(\mathbb{E}^x |X_T^*| + 1)/T \rightarrow 0$, and since (2.17) becomes an equality choosing α_t^* as the control, (2.16) follows. \square

Solving a control problem means for us finding solutions to a HJB equation and a Kolmogorov equation, so that it is possible to construct an optimal feedback control that makes the optimal process ergodic. We collect these goals in the following

Definition 2.13. *The ergodic control problem (for (2.1)-(1.2)) has a solution if*

- (E) *i) There exists a solution $(\bar{u}, \bar{\lambda}) \in C^2(\mathbb{R}^d) \times \mathbb{R}$ of the Hamilton-Jacobi-Bellman equation (2.3),*
- ii) There exists a positive solution $\bar{m} \in W_{\text{loc}}^{1,q}(\mathbb{R}^d)$ of the Kolmogorov equation (2.12) such that $\int_{\mathbb{R}^d} \bar{m} = 1$,*
- iii) $\bar{\lambda} \leq J(x, \alpha_t)$ for all x and admissible control α_t and $\bar{\lambda} = J(x, \alpha_t^*)$ for the control of feedback type $\alpha_t^* = D_p H(D\bar{u}(X_t^*))$.*

2.1.3 Some additional results.

We will state now some further estimates for the approximating problems, that can be derived naturally by exploiting the *adjoint* structure of equations (2.6) and (2.7).

Proposition 2.14. *Suppose that (H0) and (H1) hold. Then*

$$\int_{\mathcal{O}_R} f m_R + \tilde{C}_0 \int_{\mathcal{O}_R} |Du_R|^\gamma m_R \leq C \quad (2.19)$$

for some $C > 0$ that does not depend on R .

Proof. Set $u = u_R$. We first multiply (1.4) by $m = m_R$ and integrate over \mathcal{O}_R , to get

$$-\nu \int_{\mathcal{O}_R} \Delta u m - \int_{\mathcal{O}_R} b \cdot Du m + \int_{\mathcal{O}_R} H(Du) m + \lambda = \int_{\mathcal{O}_R} f m.$$

Then, we use u as a test function in (2.7) and integrate by parts

$$\nu \int_{\mathcal{O}_R} \Delta u m = \int_{\mathcal{O}_R} m D_p H(Du) \cdot Du - \int_{\mathcal{O}_R} mb \cdot Du.$$

Substituting,

$$\int_{\mathcal{O}_R} f m + \tilde{C}_0 \int_{\mathcal{O}_R} |Du|^\gamma m \leq \int_{\mathcal{O}_R} f m + \int_{\mathcal{O}_R} m [D_p H(Du) \cdot Du - H(Du)] = \lambda.$$

and (2.14) follows. \square

Under some additional conditions it is also possible to bound the $W^{1,2}(\mathbb{R}^d)$ norm of \sqrt{m} .
Let

(H2) *i)* if $x \in \partial\mathcal{O}_R$ for some R , then $b(x) \cdot n_{\partial\mathcal{O}_R}(x) \leq 0$.

ii) $\operatorname{div} b(x) \geq -C_3$ for all $x \in \mathbb{R}^d$.

Proposition 2.15. *Suppose that (H2) holds and*

$$\|Du_R\| \leq K \quad (2.20)$$

or

$$[D_p H(p)]^2 \leq K(1 + |p|^\gamma) \quad (2.21)$$

for some $K > 0$. Then,

$$\|\sqrt{m_R}\|_{W^{1,2}(\mathcal{O}_R)} \leq C$$

for some $C > 0$ that does not depend on R .

Proof. We use $\ln m = \ln m_R$ as a test function in (2.7) (we recall that $m > 0$ on \mathcal{O}_R)

$$\nu \int_{\mathcal{O}_R} \frac{|Dm|^2}{m} - \int_{\mathcal{O}_R} b \cdot Dm + \int_{\mathcal{O}_R} D_p H(Du) \cdot Dm = 0.$$

By the assumptions on b ,

$$-\int_{\mathcal{O}_R} b \cdot Dm = -\int_{\partial\mathcal{O}_R} mb \cdot n_{\partial\mathcal{O}_R} + \int_{\mathcal{O}_R} m \operatorname{div} b \geq -C$$

Hence,

$$\nu \int_{\mathcal{O}_R} \frac{|Dm|^2}{m} \leq \frac{\nu}{2} \int_{\mathcal{O}_R} \frac{|Dm|^2}{m} + \frac{1}{2\nu} \int_{\mathcal{O}_R} [D_p H(p)]^2 m + C \leq \frac{\nu}{2} \int_{\mathcal{O}_R} \frac{|Dm|^2}{m} + C,$$

using Young's inequality and bounding $\frac{1}{2\nu} \int_{\mathcal{O}_R} [D_p H(p)]^2 m$ with the aim of (2.20) or (2.21)+(2.14). We conclude by noticing that

$$2 \int_{\mathcal{O}_R} |D\sqrt{m}|^2 = \frac{1}{2} \int_{\mathcal{O}_R} \frac{|Dm|^2}{m} \leq C$$

and, since $\int_{\mathcal{O}_R} |\sqrt{m}|^2 = 1$, the estimate on $\|\sqrt{m}\|_{W^{1,2}(\mathcal{O}_R)}$ follows.

Notice that (2.21) holds, under our assumptions (H0) on L , if $1 < \gamma \leq 2$. □

2.2 The ergodic problem in some special cases.

In this section we give several explicit conditions on the data b, f such that the abstract assumptions (H1) and (M) hold true. In such particular situations the general results of the previous section apply and provide the solution of the ergodic control problem in the sense of definition 2.13.

First, we require f, b to have *polynomial* growth, i.e. they satisfy throughout the section

$$f(x), |b(x)| \leq D(1 + |x|^a) \tag{2.22}$$

for some $D, a > 0$.

Lemma 2.16. *If (H0) and (2.22) are true then (H1) holds.*

Proof. Let

$$W(x) = |x|^{2\beta}, \quad \alpha(x) = b(x) + \chi(x),$$

with $\beta > 0$ that will be chosen large enough, χ a smooth vector field on \mathbb{R}^d such that $\chi(0) = 0$, $|\chi| \leq 1$ and $\chi(x) = \frac{x}{|x|}$ on $\mathbb{R}^d \setminus B_1(0)$. By simple computations, for $j = 1, \dots, n$

$$\begin{aligned} D_{x_j} W &= 2\beta x_j |x|^{2(\beta-1)} \\ D_{x_j x_j} W &= 2\beta |x|^{2(\beta-1)} + 4\beta(\beta-1)x_j^2 |x|^{2(\beta-2)}, \end{aligned}$$

so

$$\begin{aligned} -\nu \Delta W - (b(x) - \alpha(x)) \cdot DW &= -\nu \Delta W + \chi \cdot DW \\ &= -2\beta |x|^{2(\beta-1)} - 4\beta(\beta-1) |x|^{2(\beta-1)} + 2\beta \chi \cdot x |x|^{2(\beta-1)} \\ &\geq -\delta + 2\beta |x|^{2\beta-1} =: k(x) \end{aligned}$$

for all $x \in \mathbb{R}^d$ and for some positive constant δ . Now,

$$f(x) \leq D(1 + |x|^a) \leq C_1(C'_1 - \delta + 2\beta |x|^{2\beta-1})$$

and, by (H0)

$$L(\alpha(x)) \leq D'(1 + |x|^{a\gamma^*}) \leq C_1(C'_1 - \delta + 2\beta |x|^{2\beta-1})$$

if $C_1 = \max\{D, D'\}$ and $C'_1 \gg 0, 2\beta - 1 > a$. □

Since the Lyapunov function W constructed in the previous lemma is *radial*, from now on we assume

$$\mathcal{O}_R = B_R.$$

In order to derive existence and some integrability properties of the invariant measure \bar{m} , we will use the method of Lyapunov functions. We describe this tool in the

Proposition 2.17. *Let $\tilde{R} > 0$. Suppose that (H0) holds and for all $R > \tilde{R}$ there exists $W_R \in C^2(\overline{B_R})$ such that*

$$-\nu\Delta W_R + (-b(x) + D_p H(Du_R)) \cdot DW_R \geq g(x) \quad \forall x \in B_R \setminus B_{\tilde{R}},$$

for some measurable $g : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} g(x) \rightarrow +\infty & \text{as } |x| \rightarrow +\infty, \\ |g(x)| \leq C & \forall x \in B_{\tilde{R}} \end{cases}$$

and

$$\partial_n W_R(x) \geq 0 \quad \forall x \in \partial B_R.$$

Then, (M) holds and

$$\int_{\mathbb{R}^d} g\bar{m} \, dx < \infty.$$

Proof. We plug W_R as a test function in (2.7) and integrate by parts to get

$$\begin{aligned} 0 &= \nu \int_{B_R} Dm_R \cdot DW_R + \int_{B_R} m_R (-b(x) + D_p H(Du_R)) \cdot DW_R \\ &= \nu \int_{\partial B_R} m_R \partial_n W_R + \int_{B_R} m_R [-\nu\Delta W_R + (-b(x) + D_p H(Du_R)) \cdot DW_R] \\ &\geq \int_{B_R} gm_R \geq \int_{B_R \setminus B_{\tilde{R}}} gm_R - C \int_{B_{\tilde{R}}} m_R, \end{aligned}$$

so

$$\int_{B_R \setminus B_{\tilde{R}}} gm_R \leq C \int_{B_{\tilde{R}}} m_R, \quad (2.23)$$

From (2.23) we deduce that $\int_{B_R \setminus B_{\tilde{R}}} gm_R \leq C$ for all R , hence if $s > \tilde{R}$

$$\left(\inf_{|x| \geq s} g \right) \int_{B_R \cap \{|x| \geq s\}} m_R \leq \int_{B_R \cap \{|x| \geq s\}} gm_R \leq \int_{B_R \setminus B_{\tilde{R}}} gm_R \leq C,$$

so

$$\sup_R \int_{B_R \cap \{|x| \geq s\}} m_R \leq \frac{C}{\inf_{|x| \geq s} g} \rightarrow 0 \quad \text{as } s \rightarrow \infty$$

and (M) holds. To conclude, fix any ball $B \subset \mathbb{R}^d$, then $B \subset B_R$ if R is sufficiently large. Since $\int_{B \setminus B_{\tilde{R}}} gm_R \leq C$ for every R (large), using Fatou's lemma and pointwise convergence of m_R to \bar{m} as $R \rightarrow \infty$ we get

$$\int_{B \setminus B_{\tilde{R}}} g\bar{m} \leq C.$$

The ball B was chosen arbitrarily, hence $\int_{\mathbb{R}^d \setminus B_{\tilde{R}}} g\bar{m} \leq C$ too; to conclude the proof, we note that

$$\int_{\mathbb{R}^d} g\bar{m} = \int_{\mathbb{R}^d \setminus B_{\tilde{R}}} g\bar{m} + \int_{B_{\tilde{R}}} g\bar{m} \leq 2C.$$

□

In order to prove bounds on u (typically (2.13) is needed in our case) we apply the standard comparison principle. We stress that the homogeneous Neumann conditions at the boundary for u play a key role when comparing u with other sub(super)solutions of the approximating problem (2.6).

Proposition 2.18. *Suppose that $v \in C^2(\mathbb{R}^d)$ satisfies $v(0) = 0$,*

$$-\nu\Delta v - b(x) \cdot Dv + H(Dv) + \lambda_R - f(x) \leq (\geq) 0 \quad \text{on } \mathbb{R}^d \setminus B_{\tilde{R}} \text{ for all } R > \tilde{R}$$

for some fixed $\tilde{R} > 0$ and

$$\partial_n v \leq (\geq) 0 \quad \text{on } \partial B_R \text{ for all } R > \tilde{R}.$$

Then there exists some positive C that does not depend on R such that

$$u_R \geq v - C (\leq v + C) \quad \text{on } B_R \text{ for all } R > \tilde{R}.$$

Proof. We prove the assertion with the \geq sign, the argument is similar if the signs are reversed. Observe first that $|u_R(x)|$ is bounded in $B_{\tilde{R}}$, independently upon R , by Bernstein estimates (Proposition A.2); since $u_R(0) = 0$ we may choose C large enough so that

$$u_R \geq v - C \quad \text{on } \overline{B_{\tilde{R}}}. \quad (2.24)$$

Since $-\nu\Delta u_R - b(x) \cdot Du_R + H(Du_R) + \lambda_R - f(x) = 0$ on B_R , the maximum of $v - u_R$ on $\overline{B_R} \setminus \overline{B_{\tilde{R}}}$ is attained on $\partial B_R \cup \partial B_{\tilde{R}}$. By the Hopf lemma a maximum point cannot be on ∂B_R , and $v - u_R \leq C$ on $\partial B_{\tilde{R}}$ by (2.24), so $v - u_R \leq C$ on $\overline{B_R} \setminus \overline{B_{\tilde{R}}}$ and therefore $v - u_R \leq C$ on $\overline{B_R}$. \square

2.2.1 Bounded cost. Case 1: b is inward pointing and coercive.

We now suppose that f is bounded, i.e. there exists $M_f > 0$ such that

$$0 \leq f(x) \leq M_f \quad \forall x \in \mathbb{R}^d. \quad (2.25)$$

We also require that, at least outside a compact set, the drift is inward pointing, and its effect grows as $|x| \rightarrow \infty$.

$$\limsup_{|x| \rightarrow \infty} b(x) \cdot \frac{x}{|x|} = -\infty. \quad (2.26)$$

A similar case has been studied in [36], where f need not to be bounded but b has the special form $b(x) = -\beta x$, $\beta > 0$ (Ornstein-Uhlenbeck diffusion). We will prove in the subsection the following

Theorem 2.19. *Suppose that (H0) is true and f, b satisfy (2.22), (2.25), (2.26). Then the ergodic control problem has a solution in the sense of Definition 2.13.*

Proof. We have to verify (E). As (H1) is true because of (2.22), we first have to check (M) to prove the existence of \bar{u} and \bar{m} using Theorems 2.6 and 2.8.

First we verify that u_R as at most linear growth, i.e. there exists $C > 0$ such that

$$-|x| - C \leq u_R(x) \leq |x| + C \quad \forall x \in B_R \quad (2.27)$$

for all R . Indeed, let $v \in C^2(\mathbb{R}^d)$ be such that $v(0) = 0$ and $v(x) = |x|$ for all $x \in \mathbb{R}^d \setminus B_1$. Since $Dv = |x|$, $\Delta|x| = (n-1)/|x|$ outside B_1 by a straightforward computation we obtain

$$\begin{aligned} & -\nu\Delta v - b \cdot Dv + H(Dv) + \lambda_R - f \\ & = -\nu(n-1)/|x| - b \cdot x/|x| + H(x/|x|) + \lambda_R - f \rightarrow +\infty, \end{aligned}$$

as $|x| \rightarrow \infty$, because all the terms appearing on the right hand side are bounded (uniformly w.r.t. R) on \mathbb{R}^d except $b \cdot x/|x| \rightarrow +\infty$. In the same way, if $v = -|x|$ outside B_1 ,

$$-\nu \Delta v - b \cdot Dv + H(Dv) + \lambda_R - f \rightarrow -\infty,$$

so Proposition 2.18 gives the desired estimate on u_R .

We now show that (M) holds and

$$\int_{\mathbb{R}^d} |x| \bar{m} dx < \infty,$$

using the family of Lyapunov functions

$$W_R = (u_R + |x|)|x|.$$

A computation yields

$$\begin{aligned} & -\nu \Delta W_R + (-b + D_p H(Du_R)) \cdot DW_R = \\ & |x| \left[-H(Du_R) + f - \lambda_R - \frac{2x}{|x|^2} \cdot Du_R - \frac{2n}{|x|} - \frac{u_R}{|x|} \Delta |x| - b \cdot \frac{x}{|x|} \frac{u_R + 2|x|}{|x|} \right. \\ & \quad \left. + D_p H(Du_R) \cdot Du_R + D_p H(Du_R) \cdot \frac{x}{|x|} \frac{u_R + 2|x|}{|x|} \right] =: |x| h(x). \end{aligned}$$

Now, $\frac{1}{2} \leq \frac{u_R + 2|x|}{|x|} \leq 4$ for $|x| \geq \tilde{R}$ independent on R , so

$$-b \cdot \frac{x}{|x|} \frac{u_R + 2|x|}{|x|} \rightarrow +\infty$$

as $|x| \rightarrow \infty$. Moreover

$$\begin{aligned} & D_p H(Du_R) \cdot Du_R - H(Du_R) + D_p H(Du_R) \cdot \frac{x}{|x|} \frac{u_R + 2|x|}{|x|} - \frac{2x}{|x|^2} \cdot Du_R \\ & \geq \tilde{C}_0 |Du_R|^\gamma - |D_p H(Du_R)| \frac{u_R + 2|x|}{|x|} - \frac{2}{|x|} |Du_R| \\ & \geq \tilde{C}_0 |Du_R|^\gamma - 4C_0(1 + |Du_R|^{\gamma-1}) - \frac{2}{|x|} |Du_R| \geq -K_0 \end{aligned}$$

for some $K_0 > 0$ and for all R , $|x| \geq \tilde{R}$. Eventually $f, \lambda_R, u_R/|x|$ are bounded on \mathbb{R}^d , again independently on R , so

$$f - \lambda_R - \frac{2n}{|x|} - \frac{u_R}{|x|} \Delta |x| \geq -K_1,$$

$K_1 > 0$. Then, $h(x) \geq 1$ if $|x| \geq \tilde{R}$. Notice also that $\partial_{x/|x|} W_R \geq 0$ for all x large, so $\int_{\mathbb{R}^d} |x| \bar{m} dx < \infty$ and (M) follows from Proposition 2.17.

Due to (2.27) and the integrability property of \bar{m} also (E) iii) holds by virtue of Theorem 2.12. \square

2.2.2 Bounded cost. Case 2: b is inward pointing and bounded

We now study the case of f bounded, so (2.25) is true, but drop (2.26), and suppose that b is bounded:

$$|b(x)| \leq M_b \quad \forall x \in \mathbb{R}^d. \quad (2.28)$$

As far as we know, the ergodic problem in this case has never been treated in the literature.

Example 2.20. To better understand the problem, we first study the situation where f, b are *radial* functions:

$$f = f(r), \quad b = b(r)\mathbf{e}_r \quad \forall r \in (0, +\infty)$$

and write the HJB equation in polar coordinates, in the special case of quadratic hamiltonian $H(p) = |p|^2/2$ and with $\nu = 1$.

$$-u''(r) - \frac{n-1}{r}u'(r) - b(r)u'(r) + \frac{1}{2}(u'(r))^2 + \lambda = f(r), \quad \text{in } (0, R).$$

If we set $v = u' \in C^1((0, R))$ and recall that the HJ equation has homogeneous boundary conditions, v solves

$$-v'(r) - \frac{n-1}{r}v(r) - b(r)v(r) + \frac{1}{2}v^2(r) + \lambda = f(r), \quad \text{in } (0, R), \text{ and } v(R) = 0. \quad (2.29)$$

We omit to underline the dependance of v, λ upon R . We want to obtain some fine estimates on v with respect to the magnitude of b (as $R \rightarrow +\infty$). Let $c(r) = c_R(r) = \lambda_R - f(r)$; by the maximum principle, if $0 \leq f(r) \leq M$ for all r , then

$$|c(r)| \leq M, \quad \forall r, R > 0.$$

Suppose now that for some $\bar{r} > 0$

$$b(r) < -\sqrt{2M} \quad \forall r > \bar{r},$$

then also

$$\tilde{b}(r) := b(r) + \frac{n-1}{r} < -\sqrt{2M} \quad \forall r > \bar{r},$$

if \bar{r} is large enough. Let $w(t) = -v(-t)$, (2.29) becomes

$$w'(t) = \frac{1}{2}w^2(t) + \tilde{b}(-t)w(t) + c(-t), \quad \text{in } (-R, 0), \text{ and } w(R) = 0.$$

Then, if we set

$$w_{1,2}(t) = -\tilde{b}(-t) \pm \sqrt{\tilde{b}^2(-t) - 2c(-t)}, \quad t < -\bar{r}$$

we notice that

$$w_1(t) < \sqrt{2M}, \quad w_2(t) > \sqrt{2M}, \quad \forall t < -\bar{r},$$

hence

$$\frac{1}{2}w^2(t) + \tilde{b}(-t)w(t) + c(-t) \leq 0, \quad \forall (t, w) \in (-\infty, \bar{r}) \times (\sqrt{2M} - \epsilon, \sqrt{2M} + \epsilon)$$

for some $\epsilon > 0$. We may now exploit this general lemma:

Lemma 2.21. Let $F \in C^1(\mathbb{R} \times \mathbb{R})$ and $v \in C^1((-R, -\bar{r}))$ be a solution of

$$w'(x) = F(x, w(x)), \quad \text{in } (-R, \bar{r}), \text{ and } w(-R) = 0.$$

If $F(x, v) \leq 0$ for all $(t, w) \in (-R, \bar{r}) \times (\bar{M} - \epsilon, \bar{M} + \epsilon)$, then

$$w(x) \leq \bar{M} - \epsilon \quad \forall t \in (-R, -\bar{r}).$$

Proof. Left to the reader as an elementary calculus exercise. \square

It follows that

$$Du(x) \cdot \frac{x}{|x|} = u'(r) = -w(-r) \geq -\sqrt{2M} + \epsilon \quad \forall |x| \in (\bar{r}, R).$$

The (candidate) optimal process is driven by the drift $b(x) - Du(x)$, and by the last estimate $(b(x) - Du(x)) \cdot \frac{x}{|x|} \leq -\epsilon$; it's then easy to see via Lyapunov functions that the process is ergodic. This simple analysis suggests that, in order for X_t^* to be ergodic, b should be sufficiently strong to overcome the effect of $D_p H(Du)$.

We treat the general (non-radial) case by constructing suitable Lyapunov functions. We will fix the hamiltonian to

$$H(p) = \frac{1}{\gamma} |p|^\gamma, \quad (2.30)$$

because we want to compute sharp constants. The computations that follow can be easily generalized anyway to Hamiltonians that come from Lagrangians satisfying merely (H0), for which the properties listed in Proposition 2.1 hold.

Our condition on b will be that for some (small) $\delta > 0$

$$b(x) \cdot \frac{x}{|x|} \leq -\Lambda^{\gamma-1} - \delta, \quad \Lambda = \left(\frac{\gamma}{\gamma-1} M \right)^{\frac{1}{\gamma}} \quad (2.31)$$

outside a ball $B_{\tilde{R}}$, where $M = \max_{\mathbb{R}^d} f$.

Theorem 2.22. *Suppose that H has the form (2.30) and $f, b \in C^1(\mathbb{R}^d)$ satisfy (2.25), (2.28) and (2.31). Then the ergodic problem has a solution in the sense of Definition 2.13.*

Proof. (H0) is true due to the special form of H and f, b are bounded, so they have polynomial growth and (H1) holds. We have to prove that (M) is true in order to exploit Theorems 2.6 and 2.8, and to do so we construct a suitable family of Lyapunov functions. We first perturb u_R with a convenient $\psi \in C^2(\mathbb{R}^d)$: let $R > 0$ and

$$w_R = u_R + \psi, \quad \psi(x) = \Lambda|x|, \quad x \in B_R \setminus B_1.$$

We recall that $D\psi = \Lambda x/|x|$, $\Delta\psi = \Lambda(n-1)/|x|$, and $(1 - 1/\gamma)y^\gamma - \Lambda y^{\gamma-1} \geq -(1/\gamma)\Lambda^\gamma$ for all $\gamma > 1$ and $y \geq 0$, hence, using the definition of Λ ,

$$\begin{aligned} & -\Delta w_R - b \cdot Dw_R + |Du_R|^{\gamma-2} Du_R \cdot DW w_R = \\ & -\Delta u_R - b \cdot Du_R + |Du_R|^\gamma - \Delta\psi - b \cdot D\psi + |Du_R|^{\gamma-2} Du_R \cdot D\psi = \\ & \left(1 - \frac{1}{\gamma}\right) |Du_R|^\gamma + f - \lambda - \frac{\Lambda(n-1)}{|x|} - \Lambda b \cdot \frac{x}{|x|} + \Lambda |Du_R|^{\gamma-2} Du_R \cdot \frac{x}{|x|} \geq \\ & f - \lambda - \frac{1}{\gamma} \Lambda^\gamma - \frac{\Lambda(n-1)}{|x|} - \Lambda b \cdot \frac{x}{|x|} \geq \delta \Lambda + M - \lambda - \frac{\Lambda(n-1)}{|x|} \geq -\frac{\Lambda(n-1)}{|x|} + \delta \Lambda, \end{aligned}$$

Therefore,

$$-\nu \Delta w_R - b \cdot Dw_R + |Du_R|^{\gamma-2} Du_R \cdot Dw_R \geq \delta \Lambda / 2 \quad \text{if } |x| > \tilde{R},$$

\tilde{R} not depending on R . Now, let

$$W_R = e^{kw_R},$$

with k small enough to guarantee $k|Dw_R|^2 \leq \delta\Lambda/4$ on \mathbb{R}^d ; this can be done because of the bound on $|Du_R|$ on \mathbb{R}^d given by Bernstein estimates. Hence,

$$\begin{aligned} & -\nu\Delta W_R + (-b + |Du_R|^{\gamma-2}Du_R) \cdot DW_R \\ & = kW_R[-\nu\Delta w_R + (-b + |Du_R|^{\gamma-2}Du_R) \cdot Dw_R - k|Dw_R|^2] \geq \frac{k\delta\Lambda}{4}W_R \end{aligned}$$

for all $|x| > \tilde{R}$.

We now prove a bound from below of W_R . Let $v = (-\Lambda + \eta)|x|$, η to be determined later; a computation yields

$$\begin{aligned} & -\nu\Delta v - b \cdot Dv + \frac{1}{\gamma}|Dv|^\gamma + \lambda_R - f(x) = \\ & \quad \frac{\nu(-\Lambda + \eta)(n-1)}{|x|} + (\Lambda - \eta)b \cdot \frac{x}{|x|} + \frac{1}{\gamma}|\Lambda - \eta|^\gamma + \lambda_R - f \leq \\ & \quad \frac{\nu(-\Lambda + \eta)(n-1)}{|x|} - \delta\Lambda + C\eta + o(\eta) + \lambda_R - M \leq 0 \end{aligned}$$

if η is small (it can be chosen independently upon R), again exploiting the definition of Λ in (2.31). Moreover $\partial_n v \leq 0$ on every ∂B_R , so by Proposition 2.18

$$u_R \geq v - C \quad \text{on } B_R \text{ for all } R > \tilde{R}$$

for some constant $C > 0$. This implies that $w_R = u_R + \Lambda|x| \geq \eta|x| - C$ outside $B_{\tilde{R}}$, and therefore

$$W_R \geq e^{\epsilon|x|} \quad \forall |x| \geq \tilde{R}$$

for some positive ϵ . Notice also that $\partial_n W_R \geq 0$ on ∂B_R , so by virtue of Proposition 2.17

$$\int_{\mathbb{R}^d} e^{\epsilon|x|} \bar{m} < \infty.$$

Moreover $D\bar{u}$ is uniformly bounded on \mathbb{R}^d , so $|\bar{u}| \leq C|x|$ for some positive constant C . Since $\int_{\mathbb{R}^d} |x| \bar{m} < \infty$, the ergodic control problem has a solution because of the assertions of Theorem 2.12. \square

2.2.3 Bounded cost. Case 3: $dX_t = b(X_t) + \sqrt{2\nu}dB_t$ is ergodic.

If an uncontrolled process driven by a drift b is ergodic, one may ask if the corresponding control problem with drift $b - \alpha_t$ has an ergodic optimal trajectory. This should depend on the shape of the cost, in particular if the controller is lead or not to move away from the origin to minimize the cost paid.

We suppose that the data are bounded, and the make mild assumption that $L(0) = 0, L \geq 0$ ($\Rightarrow H(0) = 0$); moreover, we ask that if the control $\alpha \equiv 0$, then the trajectory has an invariant measure; in terms of Lyapunov functions, a sufficient condition can be stated as

$$\begin{aligned} & \exists W \in C^2(\mathbb{R}^d) \text{ s.t. } W \rightarrow +\infty \text{ as } |x| \rightarrow +\infty, \\ & \quad -\nu\Delta W - b(x) \cdot DW \geq 1 \text{ for } |x| \text{ large.} \end{aligned} \quad (2.32)$$

In [63] some weak sufficient conditions for which (2.32) holds are stated; b might also vanish at infinity, so extra assumptions on the cost are needed, for example that

$$0 \leq f(x) < M_f \quad \forall x \in \mathbb{R}^d, \quad f(x) \rightarrow M_f \quad \text{as } |x| \rightarrow \infty. \quad (2.33)$$

Theorem 2.23. *Suppose that (H0) holds, $L(0) = 0, L \geq 0$, f satisfies (2.33), b satisfies (2.28) and (2.32). Then the ergodic problem has a solution in the sense of Definition 2.13.*

Proof. As in the previous cases, we have to prove (M) to show existence of \bar{u}, \bar{m} . We first want to check that $f - \lambda_R$ is positive outside some ball, so u_R itself can be used to construct Lyapunov functions.

Let μ_R be the solution of the Kolmogorov equation

$$\nu \int_{B_R} D\mu_R D\phi - \int_{B_R} \mu_R b \cdot D\phi = 0,$$

such that $\int_{B_R} \mu_R = 1$, if we use $\phi = W$ as a test function, integrating by parts we obtain

$$\begin{aligned} 0 \geq \int_{B_R} \mu_R (-\Delta W - b(x) \cdot DW) &\geq \int_{B_R \cap \{|x| \geq s_0\}} \mu_R - D_0 \int_{B_R \cap \{|x| \leq s_0\}} \mu_R \\ &= 1 - (D_0 + 1) \int_{B_R \cap \{|x| \leq s_0\}} \mu_R \end{aligned}$$

for some $s_0, D_0 \leq 0$, so

$$\int_{\{|x| \leq s_0\}} \mu_R \geq \frac{1}{1 + D_0}, \quad \forall R \geq s_0.$$

Since f satisfies (2.33), we have

$$\begin{aligned} \max_{|x| \leq s_0} f &\leq M - \delta, \\ \min_{|x| > s_1} f &\geq M - \frac{\delta}{2(1 + D_0)}, \end{aligned} \tag{2.34}$$

for some $\delta, s_1 > 0$.

Since u_R solves the Hamilton-Jacobi-Bellman equation,

$$0 \geq -\nu \Delta u_R - b \cdot Du_R + \lambda_R - f(x),$$

hence

$$0 \geq \nu \int_{B_R} D\mu_R Du_R - \int_{B_R} \mu_R b \cdot Du_R - \int_{B_R} \mu_R f + \lambda_R.$$

Then,

$$\begin{aligned} \lambda &\leq \int_{B_R} f \mu_R = \int_{|x| \leq s_0} f \mu_R + \int_{|x| > s_0} f \mu_R \\ &\leq (\max_{|x| \leq s_0} f) \int_{|x| \leq s_0} \mu_R + M \int_{|x| > s_0} \mu_R \\ &\leq (M - \delta) \int_{|x| \leq s_0} \mu_R + M \left(1 - \int_{|x| \leq s_0} \mu_R \right) = \\ &M - \delta \int_{|x| \leq s_0} \mu_R \leq M - \frac{\delta}{1 + D_0}, \end{aligned} \tag{2.35}$$

exploiting the first line of (2.34). Notice now that

$$\begin{aligned} -\nu \Delta u_R + (-b + D_p H(Du_R)) \cdot Du_R &\geq \\ -\nu \Delta u_R - b \cdot Du_R + H(Du_R) &= f(x) - \lambda_R \geq \frac{\delta}{2(1 + D_0)} \end{aligned}$$

if $|x| > s_1$.

Our family of Lyapunov functions for this problem will be $W_R = e^{ku_R}$, and we choose k sufficiently small to have $k|Du_R|^2 \leq \frac{\delta}{4(1+D_0)}$ on \mathbb{R}^d ; this can be done because $|Du_R|$ is bounded on B_R by a constant that does not depend on R by virtue of Theorem A.2 (f, b are bounded). By computation

$$\begin{aligned} -\nu\Delta W_R + (-b + D_p H(Du_R)) \cdot DW_R = \\ W_R[-\nu\Delta u_R + (-b + D_p H(Du_R)) \cdot Du_R - k|Du_R|^2] \geq W_R \frac{\delta}{4(1+D_0)} \end{aligned} \quad (2.36)$$

for all $|x| > s_1$, and $\partial_n W_R = kW_R \partial_n u_R = 0$ on ∂B_R for all $R(> s_1)$. To conclude, we prove that W_R grows as $|x| \rightarrow \infty$; let $\psi_R \in C^2([0, R])$ such that $\psi(r) = r$ for all $r \in [0, R-1]$, $\psi'(R) = 0$, $|\psi'|, |\psi''| \leq 1$ on $[0, R]$. We have that, setting $v_R = \eta\psi_R$, $\eta > 0$,

$$\begin{aligned} -\nu\Delta v_R - b \cdot Dv_R + H(Dv_R) + \lambda_R - f = \\ -\nu\eta \left[\psi''(|x|) + \psi'(|x|) \frac{n-1}{|x|} \right] - \eta b \cdot \frac{x}{|x|} \psi'(|x|) + H \left(\eta \frac{x}{|x|} \psi'(|x|) \right) + \lambda_R - f(x) \leq \\ -\frac{\delta}{2(1+D_0)} + o(\eta) \leq 0 \end{aligned}$$

if η is small enough and $|x| > s_1$. $\partial_n v_R = 0$ on ∂B_R , so by comparison principle, as in Proposition 2.18, we obtain

$$u_R(x) \geq -C + \eta\psi_R(|x|) \geq \frac{\eta}{2}|x| \quad (2.37)$$

if $|x| > s_1$ (possibly increasing s_1).

Combining (2.36) and (2.37), by virtue of Proposition 2.17 we obtain that (M) holds and

$$\int_{\mathbb{R}^d} e^{k\eta|x|/2} \bar{m} dx < \infty.$$

Notice finally that $D\bar{u}$ is uniformly bounded on \mathbb{R}^d by Bernstein estimates (see Theorem A.2), so $|\bar{u}| \leq C|x|$ for some positive constant C . Since $\int_{\mathbb{R}^d} |x|\bar{m} < \infty$, the ergodic control problem has a solution by virtue of Theorem 2.12. \square

2.2.4 Coercive cost.

The Hamilton-Jacobi-Bellman equation (2.3) in the case of

$$\inf_{|x| \geq s} f(x) \rightarrow +\infty \quad \text{as } s \rightarrow +\infty \quad (2.38)$$

has been discussed in literature, for example, in [17]. In our framework, under this condition it is very easy to prove that (M) holds, and so that the HJB and Kolmogorov equations have solutions. If the Hamiltonian is also superquadratic and

$$b \cdot \frac{x}{|x|} \leq C_1 \quad (2.39)$$

for some $C_1 > 0$ it is possible to prove (E) iii), i.e., the solution of the partial differential equation provides an optimal feedback, therefore completing the results of [17]. In [46] the ergodic control problem is studied for general superlinear Hamiltonians but in the particular case of $f(x) = x^\beta$.

Theorem 2.24. *Suppose that (H0) is true, the constant γ^* in (H0) satisfies $1 < \gamma^* \leq 2$ and f, b satisfy (2.22), (2.28), (2.38) and (2.39). Then the ergodic problem has a solution in the sense of Definition 2.13 within the set of admissible controls α_t such that*

$$\liminf_{T \rightarrow \infty} \frac{\mathbb{E}^x |X_T|^\beta}{T} = 0,$$

where $\beta > 1 + a/\gamma$ and a is the constant in (2.22).

Proof. (H1) holds by the assumption of polynomial growth of data f, b . Moreover, by Proposition 2.14 one has

$$\left(\inf_{\{|x| \geq s\}} f \right) \int_{B_R \cap \{|x| \geq s\}} m \leq \int_{B_R \cap \{|x| \geq s\}} f m \leq \int_{B_R} f m \leq \lambda_R \leq C,$$

with C not depending upon R . Hence

$$\sup_R \int_{B_R \cap \{|x| \geq s\}} m \leq C \left(\inf_{\{|x| \geq s\}} f \right)^{-1}$$

and (M) is proved using (2.38) since $(\inf_{\{|x| \geq s\}} f) \rightarrow +\infty$ as $s \rightarrow +\infty$, so (E) i), ii) hold.

In order to prove (E) iii) we first show some bounds on u_R . First note that the conjugate γ of γ^* satisfies $\gamma \geq 2$; set $v = |x|^\beta$, $\beta > 1 + a/\gamma$. By computation

$$\begin{aligned} & -\nu \Delta v - b \cdot Dv + H(Dv) + \lambda_R - f \\ & \geq -C_1 |x|^{\beta-2} - \beta b \cdot \frac{x}{|x|} |x|^{\beta-1} + C_0^{-1} \beta^\gamma |x|^{\gamma(\beta-1)} + \lambda_R - D(1 + |x|^a) \\ & \geq -C |x|^{\beta-2} - \beta C_1 \delta |x|^{\beta-1} + C_0^{-1} \beta^\gamma |x|^{\gamma(\beta-1)} + \lambda_R - D(1 + |x|^a) \geq 0 \end{aligned}$$

for all $|x| \geq s$ for some $s, C_2 > 0$, using (2.39). By Proposition 2.18 we can conclude that for some $C > 0$ and for all R large

$$u_R \leq |x|^\beta + C, \tag{2.40}$$

also valid for \bar{u} . Arguing in the same way as the proof of Theorem 2.23, using Proposition 2.18 and $v_R = \eta \psi_R$ with $\eta > 0$ as a subsolution, we also have the bound from below

$$\bar{u}(x) \geq \frac{\eta}{2} |x|.$$

Indeed,

$$\begin{aligned} & -\nu \Delta v_R - b \cdot Dv_R + H(Dv_R) + \lambda_R - f = \\ & -\nu \eta \left[\psi''(|x|) + \psi'(|x|) \frac{n-1}{|x|} \right] - \eta b \cdot \frac{x}{|x|} \psi'(|x|) + H \left(\eta \frac{x}{|x|} \psi'(|x|) \right) + \lambda_R - f(x) \leq 0 \end{aligned}$$

if x is large, because every term that appears in the last sum is bounded except $f \rightarrow +\infty$.

Let now $W_R = e^{k u_R}$, $0 < k\nu \leq \tilde{C}_0$, where \tilde{C}_0 is the constant that appears in Proposition 2.1. By computation

$$\begin{aligned} & -\nu \Delta W_R + (-b + D_p H(Du_R)) \cdot DW_R = \\ & k W_R [-\nu \Delta u_R + (-b + D_p H(Du_R)) \cdot Du_R - k\nu |Du_R|^2] \geq \\ & k W_R [f - \lambda_R + \tilde{C}_0 |Du_R|^\gamma - k\nu |Du_R|^2] \geq \\ & k W_R [f - \lambda_R - 1] \geq k W_R = k e^{\frac{k\eta}{2} |x|} \end{aligned} \tag{2.41}$$

for all $|x| > s_1$, and $\partial_n W_R = kW_R \partial_n u_R = 0$ on ∂B_R for all $R(> s_1)$. Then

$$\int_{\mathbb{R}^d} e^{k\eta|x|/2} \bar{m} dx < \infty$$

by Proposition 2.17. This shows in particular that $|x|^\beta \bar{m}$ is integrable. We cannot use Theorem 2.12 to derive directly (E) iii) because the sublinearity of \bar{u} (2.13) is replaced by (2.40), but all the assertions stated follow by replacing $|x|$ with $|x|^\beta$ in the proof. \square

Part II

Multi-population Mean Field Games

Chapter 3

Brief introduction to Multi-population MFG

Mean Field Games theory is a branch of Dynamic Games and it has been developed to model and analyze complex decision processes involving a large number of indistinguishable rational agents. As the number of individuals grows, it is unlikely for a single player to collect detailed information about all the other players, and the influence of every individual is very little on the system at macroscopic level; therefore, optimal strategies are based upon the overall distribution of the other players, that is codified in the theory as an external mean field. In Mean Field Games (briefly MFG) it is shown that if every player is identical, in the sense that they are indistinguishable and interchangeable, it is sufficient to focus on the dynamics of a single *average* player. An interesting consequence of this approach is that one can observe the large scale effects on the entire population caused by the common behavior of the “small” individuals.

From a practical point of view, in standard differential games theory, a game with a huge number of players is associated to a system of coupled partial differential equations with a huge number of equations and unknowns. In mean field games theory one just needs to consider a system of two equations, which completely characterizes the dynamics of the average player.

The idea of considering equilibria in a continuum of players was first documented by the seminal works of Aumann [7]. However, the modern version of the theory in the differential games setting has been implemented only very recently: the starting point is a series of papers by Lasry and Lions [56, 57, 58], who introduced the terminology of mean field games, and by Huang, Caines and Malhamé [45], who used the words “Nash certainty equivalence principle”. Then, the literature on this topic has grown very fast, following mainly two different approaches: the first one, most focused on PDE techniques, is inspired by Lasry and Lions, while the second one, based on the work of Huang, Caines and Malhamé is more on the pure control-theoretic side and investigates many applications to engineering, economy and social science.

We will present the methods introduced by Lasry and Lions, as the common tool of this dissertation are partial differential equations. In this context, the community has been mainly concentrated on the study of single population systems, where every individual is identical, i.e. everyone belongs to the same population. A natural generalization of this framework is to consider systems where more than one population is present, and every small player acts on the base of the population he belongs to. This setting is investigated by Bardi and Feleqi [8, 34] and by Achdou, who carried out some numerical experiments. We will examine in depth this topic, showing some new existence and uniqueness results and implementing the

theory on a segregation model (Chapter 5).

The criterion of the agents will be to minimize a long-time average cost, the *ergodic* case in the control theory jargon, that leads to stationary problems and stationary equations. We will not deal with the evolutionary case, which is treated for example in [28, 29, 40, 45, 57, 58].

Moreover, the simplest state space has a periodic structure. Periodicity allows to concentrate on the main features of the problem and avoid some technical difficulties. However, it can be restrictive for many models; we will consider the case where dynamics are subject to *reflection* at the boundary, that has never been studied in the literature, at least to our knowledge.

We will show in this chapter how to derive the multipopulation MFG system, the starting point of this part of the dissertation.

3.1 A Heuristic Interpretation of the ergodic MFG system

We give the interpretation of the MFG system following the presentation of Cardaliaguet [28], done in the one population case. The main point is guessing the equations that Nash equilibria of differential games with infinitely many players should satisfy and then show how the resulting solutions of these equations allow to solve differential games with finitely many players.

Another approach is to look at the limit of Nash equilibria in differential games with a large number of players and try to pass to the limit as this number tends to infinity; this approach is harder, but can be carried out in the case of ergodic mean field games, as we will see in Section 3.2.

Let M be a natural number which indicates the number of the populations and Ω a bounded domain of \mathbb{R}^d the common state space¹. Suppose that the average player of the i -th population ($i = 1, \dots, M$), i.e. the player representing every individual of its own population, controls the *reflected* stochastic differential equation (see also (1.1))

$$\begin{aligned} X_t^i &= x^i + \int_0^t b^i(X_s^i, \alpha_s^i) ds + \sqrt{2\nu_i} B_t^i - \int_0^t n(X_s) dl_s \\ l_t^i &= \int_0^t \chi_{\partial\Omega}(X_s^i) dl_s^i \\ l^i(0) &= 0 \quad l \text{ is nondecreasing,} \end{aligned} \tag{3.1}$$

where the drift b^i may depend on the population, α_t^i is the control and x_i is the initial condition; B^i , $i = 1, \dots, M$, are independent d -dimensional Brownian motions and $\nu_i > 0$ is the parameter associated to the effect of the noise. The setting is the same as Section 1.1, but we are in the presence of M agents, everyone controlling his own state through a stochastic differential equation.

We now denote with $m \in \mathcal{P}^M$

$$m = (m_1, \dots, m_M)$$

the vector of the distributions of all the average players, that, by now, is supposed to be fixed. We shall consider m_i as *stationary* distributions, as we will see.

Every average agent aims at minimizing a long time average cost

$$J^i(x^i, \alpha_t^i, m) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^x \int_0^T [L^i(X_t^i, \alpha_t^i) + V^i[m](X_t^i)] dt. \tag{3.2}$$

We stress that in this control problem, the costs $V^i : \mathcal{P}^M \times \bar{\Omega} \rightarrow \mathbb{R}$ depend on the distribution vector m in addition to the position X_t^i .

¹We could consider different state spaces Ω^i , but for simplicity we choose $\Omega^i = \Omega$ for all i

Suppose now that u^i, λ^i solve

$$-\nu^i \Delta u^i(x) + H^i(x, Du^i(x)) + \lambda^i = V^i[m](x) \quad \forall x \in \Omega, \quad (3.3)$$

with homogeneous Neumann boundary conditions, then Theorem 1.1 tells us that the optimal control for the i -th average player is given in feedback form by

$$\alpha^{i,*}(x) \in \operatorname{argmax}_{\alpha} \mathcal{H}^i(x, Du^i(x), \alpha),$$

where $\mathcal{H}^i(x, p, \alpha) = -L^i(x, \alpha) - b^i(x, \alpha) \cdot p$. If we suppose that $H(x, \cdot)$ is strictly convex for all x , then $b^i(x, \alpha^{i,*}(x)) = -D_p H^i(x, Du^i(x))$, so the drift term of the stochastic differential equation is completely determined by the gradient Du^i .

The MFG main assumption is that all the players of a population argue in the same way, in particular they all move with optimal velocity $b^i(x, \alpha^{i,*}(x))$. In Section 1.3 we have shown that the laws m_t^i of the optimal processes $X_t^{i,*}$ converge weakly to stationary distributions $m^{i,*}$, which solve the Kolmogorov equations

$$\begin{cases} \nu \Delta m^{i,*} + \operatorname{div}(D_p H^i(x, Du^i(x)) m^{i,*}) = 0 & \text{in } \Omega \\ \nu \partial_n m^{i,*} + m^{i,*} D_p H^i(x, Du^i(x)) \cdot n = 0 & \text{on } \partial\Omega, \quad \int_{\Omega} m^{i,*} = 1. \end{cases} \quad (3.4)$$

In other words, the long-time law of every agent of the i -th population is $m^{i,*}$, independently on his initial distribution.

From a vector of stationary m we are then led to a vector of optimal distributions m^* , that is the optimal allocation of players with respect to the given configuration. In an optimal regime it should be that

$$m = m^*.$$

If we put everything together, we deduce that in a mean field game situation, where every player belonging to the same population is identical and behaves optimally with respect to the distribution of the other players, the vector of distributions m should satisfy,

$$\begin{cases} -\nu_i \Delta u_i + H^i(x, Du_i) + \lambda_i = V^i[m](x) \\ -\nu_i \Delta m_i - \operatorname{div}(D_p H^i(x, Du_i) m_i) = 0, \end{cases} \quad (3.5)$$

where u determines the optimal feedback strategies. Since the boundary is reflecting, we have also to take into consideration the boundary conditions

$$D_n u_i = 0, \quad \nu_i D_n m_i + m_i D_p H^i(x, Du_i) \cdot n = 0 \quad \text{on } \partial\Omega.$$

3.1.1 The Special Case of Quadratic Hamiltonian

If the hamiltonians are of the special quadratic form

$$H^i(x, p) = R_i |p|^2 - H_0^i(x), \quad (3.6)$$

with $R_i > 0$ and $H_0^i \in C^1(\Omega)$, by the change of variables $v_i = e^{-\frac{R_i u_i}{\nu_i}}$ the original system becomes

$$-\nu_i \Delta v_i + (V^i[v_1^2, \dots, v_M^2](x) + H_0^i(x) - \lambda_i) v_i = 0,$$

subject to boundary conditions $D_n v_i = 0$ and $\int_{\Omega} v_i^2 = 1$, as $m_i = v_i^2$.

Hamiltonians of type (3.6) come from Lagrangians that have separate dependency on α and x , quadratic in α ; $H_0^i(x)$ is associated to the cost paid for being at position x . This simple situation is very natural in many models, where the effort for moving at speed $|\alpha|$ is quadratic, and there is some physical potential acting on the state space.

The corresponding system of $2M$ equations is reduced to a system of M *semilinear* equations. In this framework we shall cite [9, 29, 40].

3.2 Nash equilibria of games with N players as $N \rightarrow \infty$

In the previous section we derived heuristically the system (3.5) of equations that should capture, in some sense, the behavior of a huge number of small identical and rational individuals. We show now that this derivation can be made rigorous, considering Nash equilibria of N -persons differential game and letting $N \rightarrow \infty$. This program has been presented in the seminal paper by Lasry and Lions [58] in one-population games and with detailed proofs by Feleqi [34] for multi-population games, in a periodic setting. We believe that in domains with reflection, where the associated system of equations has Neumann conditions instead of periodic conditions at the boundary, similar arguments should produce the same results.

To motivate more rigorously system (3.5), we briefly present the results of Feleqi [34] with two populations, so $i = 1, 2$. Each population consists in N players, that control their state by the stochastic differential equation on the torus $\mathbb{T} = [0, 1]^d$

$$X_t^{k,i} = x^{k,i} + \int_0^t b^i(X_s^{k,i}, \alpha_s^{k,i}) ds + \sqrt{2\nu_i} B_t^{k,i}, \quad (3.7)$$

where $k = 1, \dots, N$ denotes the k -th player, all the Brownian motions $B^{k,i}$ are independent and every control $\alpha^{k,i}$ is adapted to $B^{k,i}$. Every player aims at minimizing a long-time average cost

$$J^{k,i}(x^{k,i}, \alpha^{1,1}, \dots, \alpha^{N,2}) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}^x \int_0^T \left[L^i(X_t^{k,i}, \alpha_t^i) + f^{k,i}(X_t^{1,1}, \dots, X_t^{N,1}, X_t^{1,2}, \dots, X_t^{N,2}) \right] dt.$$

We suppose that players are *identical*, in the sense that for all $k = 1, \dots, N$

$$f^{k,1}(x^{1,1}, \dots, x^{N,1}, x^{1,2}, \dots, x^{N,2}) = V^1 \left(\frac{1}{N-1} \sum_{j \neq k} \delta_{x^{j,1}}, \frac{1}{N} \sum \delta_{x^{j,2}} \right) (x^{k,1})$$

$$f^{k,2}(x^{1,1}, \dots, x^{N,1}, x^{1,2}, \dots, x^{N,2}) = V^2 \left(\frac{1}{N} \sum \delta_{x^{j,1}}, \frac{1}{N-1} \sum_{j \neq k} \delta_{x^{j,2}} \right) (x^{k,2}),$$

where $V^i : \mathcal{P} \times \mathcal{P} \rightarrow W^{1,\infty}(\mathbb{T})$ are continuous and bounded operators. Note that the costs are symmetric and depend *only* on the distribution of the other players.

First, it is possible to prove the existence of feedback Nash equilibria (Theorem 4, [34])

Theorem 3.1. *There exist $v^{i,k} \in C^2(\mathbb{T})$ and $\alpha^{\bar{k},i}$ such that*

$$\alpha^{\bar{k},i}(x) = \alpha^{\bar{k},i}(Dv^{i,k}(x))$$

$k = 1, \dots, N$ and $i = 1, 2$ is a Nash equilibria (in feedback form). Moreover there exist $m^{i,k} \in W^{1,p}(\mathbb{T})$ for all $p \geq 1$ that are invariant measures respectively of the process (3.7).

The theorem is proven by solving a suitable system of N Hamilton-Jacobi-Bellman and N Kolmogorov equations.

It is shown by some a-priori estimates that the families $v^{i,k}$ and $m^{i,k}$ are compact, and it is possible to extract some limits as the number of players grows. In particular,

Theorem 3.2. *As $N \rightarrow \infty$*

$$\sup_{1 \leq k, h \leq N} (\|v^{1,k} - v^{1,h}\|_{C^2(\mathbb{T})} + \|v^{2,k} - v^{2,h}\|_{C^2(\mathbb{T})}) \rightarrow 0,$$

$$\sup_{1 \leq k, h \leq N} (\|m^{1,k} - m^{1,h}\|_{W^{1,p}(\mathbb{T})} + \|m^{2,k} - m^{2,h}\|_{W^{1,p}(\mathbb{T})}) \rightarrow 0$$

Moreover, up to subsequences, $v^{i,k} \rightarrow v^i$ in $C^2(\mathbb{T})$ and $m^{i,k} \rightarrow m^i$, for some v^i, m^i , $i = 1, 2$. The limits (v^i, m^i) solve the mean field game system (3.5).

This result proves that (3.5) is the right system to study when considering an “infinite” number of players, since solutions are limits of feedback strategies of N -persons games as $N \rightarrow \infty$.

As mentioned at the beginning of this chapter, one may wonder if it is also possible to prove a “converse”: given a solution (u^i, m^i) how can we define a strategy for a game with finite number of (identical) players? The answer is usually that the feedback produced by (3.5) gives an ϵ -Nash equilibrium for the game with N players. More precisely, for any $\epsilon > 0$ fixed, if every player uses that strategy, the corresponding dynamics turns out to be ϵ -optimal if $N \geq N_0 = N_0(\epsilon)$. This is made rigorous, for example, in [45] and [28], where non-stationary problems are considered: in this case, this is the best result achieved, since passing directly to the limit in Nash strategies as we presented in this section for stationary problems is still an open issue.

Chapter 4

Existence and Uniqueness for the MFG system

In this chapter we will prove some existence and uniqueness results for the multi-population mean field games system that we derived in the previous chapter. We will study in particular the system

$$\begin{cases} -\nu_i \Delta u_i(x) + H^i(x, Du_i(x)) + \lambda_i = V^i[m](x), & \forall x \in \Omega & i), \\ -\nu_i \Delta m_i(x) - \operatorname{div}(D_p H^i(x, Du_i(x)) m_i(x)) = 0 & \forall x \in \Omega & ii), \\ D_n u_i(x) = 0, & \forall x \in \partial\Omega & iii), \\ \nu_i D_n m_i(x) + m_i D_p H^i(x, Du_i(x)) \cdot n = 0 & \forall x \in \partial\Omega & iv). \end{cases} \quad (\text{MFG})$$

We will deal initially with the case of regularizing costs V^i , then with the more delicate situation where V^i are local functions of m . The assumptions on the hamiltonians will be sufficient to guarantee existence for the Hamilton-Jacobi-Bellman equations in (MFG).

Existence for such a system has been stated with periodic boundary conditions in [56, 58], without detailed proofs in a single-population setting and in [34] with full details and more general assumptions (multi-population case). We mention also [38], submitted while this dissertation was being written.

Solutions u_i of (MFG), $i)$ are defined up to an additive constant; moreover, we require the m_i to be probability measures. We summarize our minimal requirements for u_i, λ_i, m_i in the following

Definition 4.1. *A solution of (MFG) will be a $3M$ -uple (u, λ, m) such that*

$$u_i \in C_{\text{av}}(\Omega), \quad \lambda_i \in \mathbb{R}, \quad m_i \in \mathcal{P} \cap W^{1,2}(\Omega) \quad \forall i = 1, \dots, M,$$

u, λ solve $i), iii)$ in the viscosity sense and m solves $ii), iv)$ in the weak (distributional) sense.

Depending on the regularity assumptions on the data H^i, V^i , solutions might be more regular.

4.1 Existence: Non-local Costs

Suppose that the costs V^i satisfy the following hypothesis

- (V_{NL}) 1. $\|V^i[m]\|_{W^{1,\infty}(\Omega)} \leq \alpha$ for all $m \in [W^{1,p}(\Omega)]^M \cap \mathcal{P}^M$ and for some $\alpha > 0, p > d$.
 2. $m^{(n)} \rightarrow m$ uniformly on $\bar{\Omega} \Rightarrow V^i[m^{(n)}] \rightarrow V^i[m]$ uniformly on $\bar{\Omega}$.

The main result of this section is the following existence statement:

Theorem 4.2. *If (V_{NL}) and (H) hold (see Section 1.2, page 6) for every H^i , then there exists a solution (u, λ, m) of (MFG). Moreover $u_i \in C^2(\bar{\Omega})$ and $m_i \in W^{1,p}(\Omega)$ for all $p \geq 1$.*

The proof exploits the fixed point structure of the MFG system. Once m is fixed, it is possible to solve $i)$, $iii)$ and find u ; Du then enters into the Kolmogorov equations $ii)$ and solutions μ can be found. By suitable a-priori estimates the Schauder fixed point theorem assures the existence of a fixed point of the map that associates m to μ : such a fixed point provides a solution of (MFG).

Proof. Let $m_i \in W_{1,p}(\Omega) \cap \mathcal{P}$, $i = 1, \dots, M$ be fixed, p the constant that appears in (V_{NL}) ; we set $F_i(x) = V^i[m_1, \dots, m_M](x)$, and by Proposition 1.2 there exist solutions $(v_i, \lambda_i) \in C^2(\bar{\Omega}) \times \mathbb{R}$, of

$$-\nu_i \Delta v_i + H^i(x, Dv_i) + \lambda_i = F_i(x), \quad (4.1)$$

respectively, together with the estimates $\|v_i\|_{W^{1,\infty}(\Omega)} \leq C_i$, where the constants do not depend on m . If we let $g(x) = D_p H^i(x, Dv_i)$, Proposition 1.5 guarantees existence (and uniqueness) of solutions $\mu_i \in W^{1,p}(\Omega) \cap \mathcal{P}$ of

$$-\nu_i \Delta \mu_i - \operatorname{div}(D_p H^i(x, Dv_i) \mu_i) = 0$$

satisfying Neumann boundary conditions, with estimates $\|\mu_i\|_{W^{1,p}(\Omega)} \leq \hat{C}_i$. Let now

$$K_i = \mathcal{P} \cap \{m \in W^{1,p}(\Omega) : \|m\|_{W^{1,p}(\Omega)} \leq \hat{C}_i\},$$

one has $\mu_i \in K_i$ independently upon the m chosen, as V^i is bounded by hypothesis. It is consequently well defined the map

$$\Gamma : (m_1, \dots, m_M) \in \mathcal{K} := K_1 \times \dots \times K_M \mapsto (v_1, \dots, v_M) \mapsto (\mu_1, \dots, \mu_M) \in \mathcal{K}. \quad (4.2)$$

Being every K_i compactly imbedded in $C(\bar{\Omega})$ ([37], Theorem 7.26), by showing that Γ is continuous (with respect to $C(\bar{\Omega}) \times \dots \times C(\bar{\Omega})$ topology), it is possible to apply the Schauder theorem ([37], Theorem 11.1) in order to obtain a fixed point. A fixed point (m_1, \dots, m_M) will be a solution of (MFG), together with (u_1, \dots, u_2) obtained by solving the equations for u_i .

So, let $\{m^{(n)}\}$ be a sequence in \mathcal{K} converging uniformly to some $m \in \mathcal{K}$; we first want to show that $v^{(n)} \rightarrow v$. Each $v_i^{(n)}$ solve

$$-\nu_i \Delta v_i^{(n)} + H^i(x, Dv_i^{(n)}) + \lambda_i^{(n)} = V^i[m^{(n)}](x) \quad \text{on } \Omega, \quad (4.3)$$

while each v_i is a solution of

$$-\nu_i \Delta v_i + H^i(x, Dv_i) + \lambda_i = V^i[m](x) \quad \text{on } \Omega; \quad (4.4)$$

we also know, thanks to (1.9) and (1.10), that the constants λ_i and $\lambda_i^{(n)}$ are bounded in absolute value by α , and $\|v_i^{(n)}\|_{W^{1,\infty}(\Omega)}, \|v_i\|_{W^{1,\infty}(\Omega)} \leq C_i$. We now consider any uniformly convergent subsequence $(v_1^{(n)}, v_2^{(n)}) \rightarrow (\bar{v}_1, \bar{v}_2)$ (by Ascoli-Arzelà there exists at least one). We begin by proving that $\lambda_i^{(n)} \rightarrow \lambda_i$ (reasoning as in [5]): we consider (with $i = 1$ fixed) some further converging subsequence $\lambda_1^{(n)} \rightarrow \bar{\lambda}_1$ and suppose by contradiction that $\bar{\lambda}_1 \neq \lambda_1$. Since by hypothesis $V^i[m_1^{(n)}, m_2^{(n)}] \rightarrow V^i[m_1, m_2]$ uniformly, we deduce that \bar{v}_1 is a solution in the viscosity sense of the limit equation

$$-\nu_1 \Delta \bar{v}_1 + H^1(x, D\bar{v}_1) + \bar{\lambda}_1 = V^1[m_1, m_2](x) \quad \text{on } \Omega,$$

with Neumann boundary conditions satisfied in generalized sense. Without loss of generality $\bar{\lambda}_1 > \lambda_1$ and $\bar{v}_1(y) > v_1(y)$ at some $y \in \Omega$, possibly adding a positive constant to \bar{v}_1 ; hence, there exists $\delta > 0$ such that

$$\begin{aligned} -\nu_1 \Delta \bar{v}_1 + H^1(x, D\bar{v}_1) - V^1 + \delta \bar{v}_1 &= \delta \bar{v}_1 - \bar{\lambda}_1 \\ &\leq \delta v_1 - \lambda_1 = -\nu_1 \Delta v_1 + H^1(x, Dv_1) - V^1 + \delta v_1. \end{aligned}$$

By comparison principle ([30]) it follows that $\bar{v}_1 \leq v_1$ in Ω , that is a contradiction. So, $\bar{\lambda}_1 = \lambda_1$ and $\lambda_1^{(n)} \rightarrow \lambda_1$.

We now show that $\bar{v}_1 = v_1$. By subtracting (4.4) to (4.3), we obtain

$$\begin{aligned} V^1[m_1^{(n)}, m_2^{(n)}](x) - V^1[m_1, m_2](x) - (\lambda_1^{(n)} - \lambda_1) \\ &= -\nu_1 \Delta(v_1^{(n)} - v_1) + H^1(x, Dv_1^{(n)}) - H^1(x, Dv_1) \\ &\geq -\nu_1 \Delta(v_1^{(n)} - v_1) + \frac{\partial H^1}{\partial p}(x, \xi)(Dv_1^{(n)} - Dv_1) \geq \\ &\quad -\nu_1 \Delta(v_1^{(n)} - v_1) - \left| \frac{\partial H^1}{\partial p}(x, \xi) \right| |Dv_1^{(n)} - Dv_1| \geq \\ &\quad \quad \quad -\nu_1 \Delta(v_1^{(n)} - v_1) - C|Dv_1^{(n)} - Dv_1|. \end{aligned}$$

where $\xi = \xi(x) \in [Dv_1^{(n)}(x), Dv_1(x)]$ and $C = \sup_{x \in \Omega, |\xi| \leq C_1} \left| \frac{\partial H^1}{\partial p}(x, \xi) \right|$. Set $w_1^{(n)} = v_1^{(n)} - v_1$, taking the limit as $n \rightarrow +\infty$, $w_1^{(n)} \rightarrow w_1 = \bar{v}_1 - v_1$ uniformly,

$$\begin{cases} -\nu_1 \Delta w_1 - C|Dw_1| \leq 0 & \text{su } \Omega \\ \frac{\partial w_1^{(n)}}{\partial n} = 0 & \text{su } \partial\Omega, \end{cases} \quad (4.5)$$

again with Neumann boundary conditions that have to be intended in generalized sense. We want w_1 to be everywhere constant, and we suppose that w_1 reaches its maximum (that we may assume to be positive, by eventually adding a positive constant to w_1) on $\bar{\Omega}$ at a point *inside* the domain: in this case, the strong maximum principle in [12] implies that w_1 is constant. Furthermore, if that maximum was reached at some $x \in \partial\Omega$ we would have a contradiction (as in the proof of Theorem 2.1 in [16]); indeed, letting $M = u(x)$, we would have $u(y) < M$ for every $y \in \Omega$. We know that there exist $r > 0$ and a smooth function ϕ such that ¹

$$\begin{cases} -\nu_1 \Delta \phi - C|D\phi| > 0 & \text{su } B_r(x) \\ \frac{\partial \phi}{\partial n} > 0 & \text{su } \partial\Omega \cap B_r(x), \end{cases} \quad (4.6)$$

where $\phi(x) = 0$ and $\phi(y) > 0$ for all $y \in B_r(x) \cap \bar{\Omega} \setminus \{x\}$. x would be a local maximum point of $w_1 - \phi$, that is impossible by (4.6) and the definition of viscosity subsolution, so $\bar{v}_1 - v_1$ is constant on Ω and $\bar{v}_1, v_1 \in C_{\text{av}}(\Omega)$, hence $\bar{v}_1 = v_1$; by the same argument $\bar{v}_i = v_i$ for $i = 2, \dots, M$.

Since the limit \bar{v} is unique, we deduce that the entire sequence $(v_1^{(n)}, v_2^{(n)})$ converges to v . Let now $\mu_i^{(n)} \in W^{1,\infty}(\Omega) \cap \mathcal{P}$ be solutions of

$$-\nu_i \Delta \mu_i^{(n)} - \text{div}(D_p H^i(x, Dv_i) \mu_i^{(n)}) = 0.$$

¹Take, for example, $\phi = e^{-\rho s^2} - e^{-\rho|x-x_0|^2}$, with $\rho > 0$ large enough and $B_s(x_0)$ the external sphere $\bar{\Omega}$ at x .

We prove (to obtain the continuity of Γ) that $\mu_i^{(n)} \rightarrow \mu_i$ uniformly, where $\mu = \Gamma(m)$. Notice that

$$\nu_i \Delta v_i^{(n)} = G_i(x) := H^i(x, Dv_i^{(n)}) + \lambda_i^{(n)} - V^i[m_1^{(n)}, m_2^{(n)}](x),$$

and the estimate $\|G_i\|_{L^\infty(\Omega)} \leq \hat{C}$ holds for some $\hat{C} > 0$ independent on n , hence, by the $C^{1,\alpha}$ interior elliptic estimates (for example [37], theorem 8.32) one has $\|v_i^{(n)}\|_{C^{1,\alpha}(\Omega')} < \infty$ on every $\Omega' \subset\subset \Omega$. Fix $i = 1$, then $Dv_1^{(n)} \rightarrow Dv_1$ uniformly on compacts in Ω and that easily implies that every converging subsequence $\mu_1^{(n)} \rightarrow \bar{\mu}_1$ is a (weak) solution of

$$-\nu_1 \Delta \mu - \operatorname{div}(D_p H^1(x, Dv_1) \mu) = 0,$$

that has μ_1 as a unique solution in \mathcal{P} . Similarly, $\mu_i^{(n)} \rightarrow \mu_i$ for every $i = 1, \dots, M$. \square

Example 4.3. An easy example of costs satisfying (V_{NL}) is given by

$$V^i[m](x) := W^i(m_1 \star \varphi(x), \dots, m_M \star \varphi(x)) \quad \forall x \in \bar{\Omega},$$

where $W^i \in C^1(\mathbb{R}^M)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a regularizing kernel. We recall that

$$m_i \star \varphi(x) = \int_{\Omega} \varphi(x-y) m_i(y) dy \quad \forall x \in \bar{\Omega},$$

so

$$|m_i \star \varphi(x)| \leq (\sup_{\mathbb{R}^d} \varphi) \|m_i\|_{L^1(\Omega)} = (\sup_{\mathbb{R}^d} \varphi) \quad \forall x \in \Omega, m \in \mathcal{P}.$$

Moreover $D(m_i \star \varphi) = m_i \star (D\varphi)$, hence

$$|D(m_i \star \varphi)(x)| \leq (\sup_{\mathbb{R}^d} D\varphi) \|m_i\|_{L^1(\Omega)} = (\sup_{\mathbb{R}^d} \varphi) \quad \forall x \in \Omega, m \in \mathcal{P}.$$

The two inequalities show that (V_{NL}) , 1. holds.

Given $m^{(n)} \rightarrow m$ uniformly on $\bar{\Omega}$, we have that

$$\begin{aligned} |(m_i^{(n)} - m_i) \star \varphi(x)| &\leq \int_{\Omega} |\varphi(x-y)(m_i^{(n)}(y) - m_i(y))| dy \\ &\leq \|\varphi\|_{L^1(\mathbb{R}^d)} \sup_{\bar{\Omega}} |(m_i^{(n)} - m_i)(y)| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$ uniformly w.r.t $x \in \bar{\Omega}$, so also (V_{NL}) , 2. holds.

Example 4.4. Another family of admissible costs, similar to the preceding example, that will be useful in the next section is given by

$$V^i[m](x) := W^i(m_1, \dots, m_M) \star \varphi(x) \quad \forall x \in \bar{\Omega},$$

where $W^i \in C^0(\mathbb{R}^M)$ is bounded:

$$|W^i(y)| \leq L \quad \forall y \in \mathbb{R}^M, i = 1, \dots, M$$

for some $L > 0$, and $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a regularizing kernel. We have

$$|V^i[m](x)| \leq L \|\varphi\|_{L^1(\mathbb{R}^d)} \quad \forall x \in \bar{\Omega},$$

and

$$|DV^i[m](x)| = |W^i(m) \star D\varphi(x)| \leq L \|D\varphi\|_{L^1(\mathbb{R}^d)} \quad \forall x \in \bar{\Omega}$$

uniformly with respect to $m \in W^{1,p}(\Omega)$. Moreover, if $m^{(n)} \rightarrow m$ uniformly on $\bar{\Omega}$ also $W(m^{(n)}) \rightarrow W(m)$ uniformly on $\bar{\Omega}$; in particular $V^i[m^{(n)}] \rightarrow V^i[m]$ uniformly, hence (V_{NL}) holds.

4.2 Existence: Local Costs

Non-local costs V^i are quite natural from the game point of view: in this case, the cost paid by the average player by being at some point x of the domain depends on the distribution of the average players in a *neighborhood* of x . In other words, he chooses his strategy by looking around, and has some feeling of what is happening nearby.

One could also imagine some situation where the player has information about the others that are very close to himself (a very crowded place, ...) and the cost that he pays should depend *only* on his position x ; from a mathematical point of view, the functionals $V^i[m](x)$ become local functions, i.e. $V^i[m](x) = V^i(m(x))$. In this case one loses the smoothing effect of V^i and uniform regularity of $V^i(m(x))$ with respect to m is not guaranteed anymore; the fixed point scheme we implemented for existence in the last section might fail if we are not able to produce suitable a-priori estimates for solutions. Moreover, we expect solutions to have less regularity; for this reason, we require u_i and m_i to solve the system (at least) in the weak sense stated in Definition 4.1.

Through all this section, the hamiltonians H^i will have the particular form

$$H^i(x, p) = R_i |p|^\gamma - H_0^i(x), \quad R_i > 0, \gamma > 1, \quad (4.7)$$

with $H_0^i \in C^2(\overline{\Omega})$. This enable us to derive some L^p estimates on u_i through the method of Bernstein (see Appendix A); although more general hamiltonians could be considered (say, $H^i(x, p) = R_i(x)|p|^\gamma$ for example), we keep this form for simplicity.

The first result on existence is proved under the assumption of bounded costs; this assures an uniform L^∞ boundedness of $V^i[m]$ with respect to m .

Theorem 4.5. *Let Ω be a C^2 convex domain. If H has the form (4.7) with $1 < \gamma \leq 2$ and $V^i[m] = W^i(m_1, \dots, m_M)$, such that $W^i \in C^0(\mathbb{R}^M)$ and*

$$|W^i(y)| \leq L \quad \forall y \in \mathbb{R}^M, i = 1, \dots, M$$

for some $L > 0$, then there exists a solution (u, λ, m) of (MFG). Moreover $u_i \in C^{1,\delta}(\overline{\Omega})$ and $m_i \in W^{1,p}(\Omega)$ for some $0 < \delta < 1$ and for all $p \geq 1$.

Existence is carried out by taking the limit of approximating problems: let

$$V_\epsilon^i[m](x) := W^i(m_1, \dots, m_M) \star \varphi_\epsilon(x) \quad \forall x \in \overline{\Omega},$$

as in example 4.4, where $\varphi_\epsilon(x) = \epsilon^{-d} \varphi(x/\epsilon)$ and φ is a non-negative smooth kernel with compact support and $\|\varphi\|_{L^1(\mathbb{R}^d)} = 1$. Then, since H satisfies (H) and (V_{NL}) holds with costs $V_\epsilon^i[m]$, by Theorem 4.2 there exist $(u_\epsilon, \lambda_\epsilon, m_\epsilon) \in [C^2(\overline{\Omega})]^M \times \mathbb{R}^M \times [W^{1,p}(\Omega)]^M$ solution of

$$\begin{cases} -\nu_i \Delta(u_\epsilon)_i(x) + H^i(x, D(u_\epsilon)_i(x)) + (\lambda_\epsilon)_i = V_\epsilon^i[m_\epsilon](x), & \forall x \in \Omega & i), \\ -\nu_i \Delta(m_\epsilon)_i(x) - \operatorname{div}(D_p H^i(x, D(u_\epsilon)_i(x))(m_\epsilon)_i(x)) = 0 & \forall x \in \Omega & ii), \\ D_n(u_\epsilon)_i(x) = 0, & \forall x \in \partial\Omega & iii), \\ D_n(m_\epsilon)_i(x) = 0 & \forall x \in \partial\Omega & iv). \end{cases} \quad (4.8)$$

for all $\epsilon > 0$. We then let $\epsilon \rightarrow 0$.

Proof. Let $(u_\epsilon)_i \in C^2(\overline{\Omega})$ be a solution of (4.8) *i), iii)*, then by Theorem A.3,

$$\|D(u_\epsilon)_i\|_{L^r} \leq C$$

for some $r > d$ and $C > 0$ that does not depend on $\epsilon > 0$ (in particular it depends on L and $\max_{\Omega} H_0^i$). By Sobolev imbedding theorems

$$\|(u_{\epsilon})_i\|_{C^0(\bar{\Omega})} \leq C,$$

possibly increasing $C > 0$. Using this L^{∞} bound, by the estimate stated in Theorem A.1 it is possible to conclude that

$$\|(u_{\epsilon})_i\|_{C^{1,\delta'}(\bar{\Omega})} \leq C, \quad (4.9)$$

for some $0 < \delta' < 1$.

By (1.9) also $(\lambda_{\epsilon})_i$ is bounded independently on ϵ . Moreover, for any $p \geq d$ and $\delta < \delta'$ fixed,

$$\|(m_{\epsilon})_i\|_{W^{1,p}} \leq C,$$

so, up to subsequences,

$$\begin{aligned} (\lambda_{\epsilon})_i &\rightarrow \lambda_i \in \mathbb{R} \\ (u_{\epsilon})_i &\rightarrow u_i \in C^{1,\delta}(\bar{\Omega}) \\ (m_{\epsilon})_i &\rightarrow m_i \in C^0(\bar{\Omega}) \\ (m_{\epsilon})_i &\rightarrow m_i \in W^{1,p}(\Omega) \end{aligned}$$

as $\epsilon \rightarrow 0$ for all $i = 1, \dots, M$. By uniform convergence of $D(u_{\epsilon})_i$ and weak convergence of $(m_{\epsilon})_i$, passing to the limit in Kolmogorov equations let us conclude that u_i and m_i solve (MFG) *ii*), *iv*) in weak sense. In order to pass to the limit in the HJB equations, we need to prove that $V_{\epsilon}^i[m_{\epsilon}] \rightarrow V^i[m]$ locally uniformly. Indeed,

$$\begin{aligned} V_{\epsilon}^i[m_{\epsilon}] - V^i[m] &= W^i((m_{\epsilon})_1, \dots, (m_{\epsilon})_M) \star \varphi_{\epsilon} - W^i(m_1, \dots, m_M) \star \varphi \\ &= [W^i((m_{\epsilon})_1, \dots, (m_{\epsilon})_M) - W^i(m_1, \dots, m_M)] \star \varphi_{\epsilon} \\ &\quad + W^i(m_1, \dots, m_M) \star \varphi_{\epsilon} - W^i(m_1, \dots, m_M) \star \varphi, \end{aligned}$$

and both terms of the sum converge to zero locally uniformly. Hence (u_i, λ_i) solves (MFG) *i*), *iii*) in viscosity sense. □

4.2.1 Some Estimates for the Unbounded Costs Case.

When costs V^i are not bounded (from above) it is still possible to obtain a-priori estimates on solutions. If $\gamma = 2$, i.e. the Hamiltonian is quadratic, Cardaliaguet et al. [29] proved that the stationary system (MFG) in the one population case ($M = 1$) has a solution if V is a C^1 non-decreasing function of m , without any assumption on its growth rate. An essential tool used in their proofs is the Hopf-Cole change of variables, that reduces (MFG) to a semilinear equation; if the hamiltonian is not quadratic there is no similar change of variables that simplifies or reduces the non-linearity of the equations.

In the general situation where $\gamma \neq 2$, the combination of standard and new estimates leads to some useful a-priori bounds on solutions under some assumptions on the growth of V^i . As pointed out in Achdou, [2], Remark 7, (without proof) it is possible to obtain the following

Proposition 4.6. *Suppose that Ω is a convex C^2 bounded domain. Suppose that H has the form (4.7) and $V^i[m] = V^i(m)$ satisfy*

$$V^i(m_1, \dots, m_M) m_i \geq \delta |V^i(m_1, \dots, m_M)|^q - D \quad \forall m_1, \dots, m_M \geq 0 \quad (4.10)$$

for some $\delta, D > 0, q > d$. Then, if (u, λ, m) is a solution of (MFG),

$$\|Du_i\|_{L^r(\Omega)} \leq C, \quad \lambda_i \leq C$$

for some $C > 0$.

Proof. We integrate the equations for u_i , to get

$$R_i \int_{\Omega} |Du_i|^\gamma + \int_{\Omega} \lambda_i = \int_{\Omega} V^i(m) + \int_{\Omega} H_0^i(x),$$

so

$$\lambda_i \leq \frac{1}{|\Omega|} \int_{\Omega} V^i(m) + \frac{1}{|\Omega|} \int_{\Omega} H_0^i(x). \quad (4.11)$$

Then, we exploit the structure of mean field game systems by multiplying the equations for u_i by m_i and vice-versa, and integrate

$$\begin{cases} \int -\nu_i \Delta u_i m_i + R_i |Du_i|^\gamma m_i + \lambda_i = \int V^i(m) m_i + \int H_0^i m_i \\ \int -\nu_i \Delta m_i u_i - \operatorname{div}(D_p H^i(x, Du_i) m_i) u_i = 0. \end{cases}$$

Integrating by parts yields to

$$\int_{\Omega} V^i(m) m_i + \int_{\Omega} H_0^i(x) m_i \leq \lambda_i,$$

that, together with (4.11) implies

$$\int_{\Omega} V^i(m) m_i \leq \eta \int_{\Omega} V^i(m) + C$$

for some $\eta, C > 0$. By assumptions,

$$\delta \int_{\Omega} |V^i(m)|^q - D|\Omega| \leq \eta \int_{\Omega} V^i(m) + C \leq \eta |\Omega|^{1/q'} \left(\int_{\Omega} |V^i(m)|^q \right)^{1/q} + C,$$

so

$$\|V^i(m)\|_{L^q(\Omega)} \leq C$$

for some other $C > 0$. From integral Bernstein estimate (Theorem A.3) the bound on $\|Du_i\|_{L^r(\Omega)}$ follows, and using (4.11) we deduce an analogue bound for λ_i . \square

By this proposition we can infer the boundedness of $\|H^i(x, Du_i)\|_{L^q(\Omega)}$ and $\|V^i(m)\|_{L^q(\Omega)}$ for some $q > d$, so, using standard L^p elliptic estimates, also $\|u_i\|_{W^{2,q}} \leq C$. Sobolev imbedding theorems assure that $\|Du\|_{L^\infty(\Omega)}$ is controlled, hence also $\|m_i\|_{W^{1,p}}$ is for all $p \geq 1$, being m_i a solution of the Kolmogorov equation.

Example 4.7. In the single population framework, a typical model of unbounded cost is

$$V(m) = m^\beta, \quad \beta > 0.$$

In this case, (4.10) is satisfied if

$$\beta < \begin{cases} \infty & d = 1, \\ \frac{1}{d-1} & d \geq 2. \end{cases}$$

Indeed, (4.10) reads $m^{\beta+1} \geq m^{\beta q}$ with $\delta = 1, D = 0$ and if $\beta < \frac{1}{d-1}$ then $\frac{\beta+1}{\beta} > d$.

It has been indicated by some authors (for example [31]), that (MFG) has an adjoint structure, in the sense that the linearization of the right hand side of *i*, *iii*) is the linear adjoint of *ii*), *iv*). This suggests that some informations and apriori estimates can be extracted by simple integration and computations.

For simplicity, we will derive some result for the two-populations case (namely $M = 2$) with costs of a special form:

$$V^i(m_1, m_2) = W_{i1}(m_1) + W_{i2}(m_2), \quad i = 1, 2 \quad (4.12)$$

with entries $W_{ij} \in C^1((0, +\infty))$. Reasoning as in [31], Proposition 4, where the boundary conditions are periodic, we obtain the

Lemma 4.8. *Let Ω be convex and $(u, \lambda, m) \in [C^4(\Omega)]^2 \times \mathbb{R}^2 \times [C^2(\overline{\Omega})]^2$, be a solution of (MFG), with V^i of the form (4.12) and H^i of the form (4.7). Then,*

$$\int_{\Omega} [W'_{11}(m_1)|Dm_1|^2 + (W'_{12}(m_2) + W'_{21}(m_1))Dm_1 \cdot Dm_2 + W'_{22}(m_2)|Dm_2|^2] \leq C, \quad (4.13)$$

for some $C = C(\nu, H, V) > 0$.

Proof. We consider first the two equations for population $i = 1$, and apply Δ to the equation for u_1 :

$$\begin{aligned} -\nu \Delta \Delta u_1 + \text{tr}(D_p^2 H^1(x, Du_1)(D^2 u_1)^2) + D_p H^1(x, Du_1) \cdot D(\Delta u_1) \\ = \Delta_x H^1(x, Du_1) + \text{div}(W'_{11}(m_1)Dm_1 + W'_{12}(m_2)Dm_2) \end{aligned}$$

(no mixed derivatives of H appear by the particular choice of the hamiltonian (4.7)). Multiplying by m_1 , integrating by parts, and using the Kolmogorov equation for m_1 (with the boundary conditions) we obtain

$$\begin{aligned} \delta_1(Du_1) \int_{\Omega} |D^2 u_1|^2 m_1 + \int_{\Omega} (W'_{11}(m_1)|Dm_1|^2 + W'_{12}(m_2)Dm_2 \cdot Dm_1) \\ \leq \|\Delta_x H^1(x, Du_1)\|_{L^\infty(\Omega)} + \int_{\partial\Omega} m_1 D_n H^1(x, Du_1) \leq C, \end{aligned}$$

also exploiting the convexity of Ω , that implies $D_n |Du_1|^\gamma \leq 0$ ($\delta_1(Du_1) \geq 0$ is a constant associated to the convexity of H^1 with respect to the p variable at Du_1). Operating in the same way for the other population ($i = 2$),

$$\begin{aligned} \delta_2 \int_{\Omega} |D^2 u_2|^2 m_2 + \int_{\Omega} (W'_{21}(m_1)Dm_1 \cdot Dm_2 + W'_{22}(m_2)|Dm_2|^2) \\ \leq \|\Delta_x H^2(x, Du_2)\|_{L^\infty(\Omega)} + \int_{\partial\Omega} m_2 D_n H^2(x, Du_2) \leq C, \end{aligned}$$

that summed up to the preceding inequality gives (4.13). \square

We see that if W_{ij} are increasing, in particular their derivatives are strictly positive, (4.13) yields an uniform estimate on some Sobolev norm of m_i . This can be exploited to obtain the

Proposition 4.9. *Let Ω be convex. Suppose that $(u, \lambda, m) \in [C^4(\Omega)]^2 \times \mathbb{R}^2 \times [C^2(\overline{\Omega})]^2$ is a solution of (MFG), with H^i of the form (4.7) and V^i of the form (4.12) such that $\forall m_1, m_2 > 0$*

$$i) \quad \epsilon(m_1^\gamma |v|^2 + m_2^\gamma |w|^2) \leq W'_{11}(m_1)|v|^2 + (W'_{12}(m_2) + W'_{21}(m_1))v \cdot w + W'_{22}(m_2)|w|^2 \text{ for all } v, w \in \mathbb{R}^n,$$

$$ii) \quad -\overline{W} \leq W_{i1}(m_1) + W_{i2}(m_2) \leq C(1 + m_1^\eta + m_2^\eta), \quad i = 1, 2$$

for some $\gamma \geq 0$ and $\epsilon, \eta, \overline{W} > 0$ such that

$$\eta < \begin{cases} (\gamma + 2)/(d - 2) & \text{if } d \geq 3 \\ +\infty & \text{else} \end{cases}$$

Then,

$$\|Du_i\|_{L^r(\Omega)} \leq C, \quad \|m_i\|_{L^\infty(\Omega)} \leq C, \quad \lambda_i \leq C \quad (4.14)$$

Proof. We know that if (u_i, m_i) is a solution of the system, by hypothesis i) and Lemma 4.8

$$\epsilon \int_{\Omega} (m_1^\gamma |Dm_1|^2 + m_2^\gamma |Dm_2|^2) \leq C$$

for some $C > 0$; hence

$$\int_{\Omega} m_i^{\frac{d(\gamma+2)}{d-2}} \leq C, \quad i = 1, 2 \quad (4.15)$$

by Sobolev inequality if $d \geq 3$, otherwise $\|m_i\|_{L^p(\Omega)}$ is bounded for every $p \geq 1$.

Since $W_{ij} \geq -\overline{W}$, the ergodic constants λ_i are bounded from below by the same constant (by maximum principle). Moreover,

$$\int_{\Omega} (W_{i1}^q(m_1) + W_{i2}^q(m_2)) \leq C(1 + \int_{\Omega} (m_1^{\eta q} + m_2^{\eta q})) \leq C$$

for some $q > d$ (if $d \leq 2$ this is true just by requiring W_{ij} to have polynomial growth), so, thanks to estimate of Theorem A.3, it follows that

$$\|Du_1\|_{L^r(\Omega)} + \|Du_2\|_{L^r(\Omega)} \leq C$$

for every r fixed, and consequently

$$\|D_p H^1(x, Du_1(x))\|_{L^{\bar{r}}(\Omega)} + \|D_p H^2(x, Du_2(x))\|_{L^{\bar{r}}(\Omega)} \leq C \quad (4.16)$$

with \bar{r} as large as we need. By (4.15) and (4.16), we deduce using Proposition 1.6 that

$$\|m_1\|_{L^\infty(\Omega)} + \|m_2\|_{L^\infty(\Omega)} \leq C,$$

which also gives an uniform upper bound for the constants λ_i and for $W_{ij}(m_j)$. \square

Note that in both Propositions we do not assume that H is subquadratic as in Theorem 4.5.

Remark 4.10. If for some $M > 0$ conditions i), ii) of Proposition 4.9 hold for all $v, w \in \mathbb{R}^n$, $m_1, m_2 \geq M$ and

$$0 < W'_{11}(m_1)|v|^2 + (W'_{12}(m_2) + W'_{21}(m_1))v \cdot w + W'_{22}(m_2)|w|^2$$

for all $v, w \in \mathbb{R}^n$, $m_1, m_2 < M$ then the bound assertions of the Proposition still hold.

Remark 4.11. Going back to the one-population Example 4.7, we note that hypotheses of Proposition 4.9 are satisfied in a more general situation. Indeed, if $W_{12}, W_{21}, W_{22} = 0$ and

$$W_{11}(m_1) = m_1^\beta,$$

then $\gamma = \beta - 1$ and $\eta = \beta$, so if $n \geq 3$ it has to be $\beta < \frac{\beta+1}{d-2}$; so, the sufficient condition turns out to be

$$\beta < 1/(d-3) \quad \text{if } d \geq 4$$

and $\beta < \infty$ otherwise.

Example 4.12. If we now let

$$\begin{aligned} W_{11}(m_1) + W_{12}(m_2) &= am_1 + bm_2 \\ W_{21}(m_1) + W_{22}(m_2) &= cm_1 + am_2, \end{aligned} \quad (4.17)$$

with $a, b, c \in \mathbb{R}$ the mean field system can be interpreted as two populations interacting that behave in the same way with respect to themselves; the hypotheses of Proposition 4.9 are satisfied (at least in low dimension, when linear growth with respect to m_i is admissible) if

$$a > 0, \quad b, c \geq 0, \quad b + c < 2a.$$

Remark 4.13. At the present time a full proof of existence for the MFG system under the assumptions of Proposition 4.9 is not available. Even if the a-priori estimates hold, it is not clear how to set up approximating problems: mimicking the proof of Theorem 4.5 is not possible, as the estimates are not directly extendable to the system with regularized costs $V_\epsilon^i = V^i \star \varphi_\epsilon$. Moreover, such estimates rely on coerciveness of the Hamiltonians and of the costs V^i , avoiding the possibility of implementing a natural continuation method. We aim to move around this technical obstacle in the near future.

4.3 Uniqueness

In the one-population case, uniqueness of solutions for (MFG) is true when the hamiltonian is convex and the cost V^i is increasing with respect to the density m ; from the game point of view, players tend to *segregate* and to escape from regions where the density m is high. This has been proven by Lasry and Lions [58] using deeply the structure of the mean field system and an argument that is not standard in classical theory of elliptic systems. Their argument can be generalized naturally to multi-population systems:

Theorem 4.14. *Suppose that the following L^2 monotonicity condition on the costs V^i holds*

$$\int_{\Omega} \sum_{i=1}^M (V^i[m](x) - V^i[\bar{m}](x)) (m_i(x) - \bar{m}_i(x)) dx \geq 0 \quad \forall m, \bar{m} \quad (4.18)$$

and that the hamiltonians $H^i(x, \cdot)$ are strictly convex for every $x \in \Omega$. Then, uniqueness of (classical) solutions for (MFG) holds.

Proof. Let (u, λ, m) and $(\bar{u}, \bar{\lambda}, \bar{m})$ be two solutions of (MFG). We multiply *i*) by $(m_i - \bar{m}_i)$ and *ii*) by $(u_i - \bar{u}_i)$, subtract, integrate by parts, use the fact that $m_i, \bar{m}_i \in \mathcal{P}$ and sum for $i = 1, \dots, M$ to get

$$\begin{aligned} & - \sum_{i=1}^M \int_{\Omega} m_i [H^i(x, D\bar{u}_i) - H^i(x, Du_i) - D_p H^i(x, Du_i) \cdot (D\bar{u}_i - Du_i)] dx \\ & - \sum_{i=1}^M \int_{\Omega} \bar{m}_i [H^i(x, Du_i) - H^i(x, D\bar{u}_i) - D_p H^i(x, D\bar{u}_i) \cdot (Du_i - D\bar{u}_i)] dx \\ & = \int_{\Omega} \sum_{i=1}^M (V^i[m](x) - V^i[\bar{m}](x)) (m_i(x) - \bar{m}_i(x)) dx \end{aligned}$$

The left hand side of the equation is non-positive (H^i are convex with respect to p) and the right hand side is non-negative by assumption, so they both have to be zero. Moreover $m_i > 0$ on Ω and H^i are strictly convex, i.e.

$$H^i(x, p + q) - H^i(x, p) - D_p H^i(x, p) \cdot q = 0 \Rightarrow q = 0 \quad \forall x \in \Omega, p, q \in \mathbb{R}^d,$$

so $Du_i = D\bar{u}_i$ on Ω for all i ; hence u_i and \bar{u}_i differ by a constant, but they have to be equal because they both belong to $C_{\text{av}}(\Omega)$, and therefore also $\lambda_i = \bar{\lambda}_i$. Since uniqueness holds for Kolmogorov equations, we conclude that $m_i = \bar{m}_i$. □

For example 4.12 one has uniqueness of solutions if

$$a > 0, \quad |b + c| \leq 2a.$$

This hints to the fact that strict increase of every cost V_i with respect to m_i (i.e. every population tries to avoid high concentrations of members of the same kind) is not sufficient for uniqueness, unless it is (in some sense) the leading effect.

Condition 4.18 seems to be quite restrictive and not verified in many models; one expects then, in multi-population systems, to observe non-uniqueness of solutions. It is indeed possible to construct an example of system that has multiple solutions.

Example 4.15. Let us consider a system with costs as in (4.17), with $b = c$, and quadratic hamiltonians; from Section 3.1.1 we know that it reduces to

$$\begin{aligned} -\Delta v_1 + (av_1^2 + bv_2^2 - \lambda_1)v_1 &= 0 \\ -\Delta v_2 + (bv_1^2 + av_2^2 - \lambda_2)v_2 &= 0 \end{aligned}$$

by choosing $\nu_i = 1$, $H_i^0 = 0$. A particular solution is given by $(v_1, v_2) = (\varphi, \varphi)$, $\lambda_i = \lambda$, where φ, λ solve

$$-\Delta\varphi = \lambda\varphi - (a + b)\varphi^3 \quad \varphi > 0, \quad \int \varphi^2 = 1,$$

with $D_n\varphi = 0$ on the boundary. As it is pointed out in [58] (p. 11) the last equation has (at least) two solutions in dimension $d = 1$ if $-(a + b)$ is positive and large enough, that is true, for example, when $a > 0$ and $a + b \ll 0$.

Chapter 5

A Model of Segregation

In this chapter we will present a population model proposed by the Nobel Prize winner Thomas Schelling. We will derive a two-population mean field game from that model, considering one shot games and passing to the limit on the number of players. We will design then a finite difference method to approximate solutions of the corresponding ergodic mean field game and show that, to some extent, segregation between populations occurs as in the original discrete model of Schelling.

5.1 One-shot MFG Inspired by a model of Schelling

In the 1960s Schelling was studying segregated neighborhoods, as was easy in America, he noticed, “to find neighborhoods that were mostly or entirely black or white, and correspondingly difficult to find neighborhoods where neither race made up more than, say, three fourths of the total”¹.

He formulated a simple model that with relatively mild assumptions on each individual’s nearest neighbor preferences, shows up segregated configurations, i.e. big chunks of people of the same population, starting from any random distribution. The model, proposed in [71], has been studied a lot in the social science and economics community as it equates in some sense global aggregation (ethnocentric attitude of people) to segregation.

Schelling’s city is a chessboard where each individual is able to move. Every individual is surrounded by eight neighbors and decide wether to move or not depending on how many individuals of his own kind are around himself; through simulations, it is possible to observe that even if the least number of desired same individuals is low (say three), after a finite time the configuration becomes stable, and many areas of the same “color” are evident.

Here a big number of players belonging to two different populations behave with the same criterion, and this setting should be ideal for the design of a mean field game. In the sequel we try to translate the discrete model of Schelling in a game theory environment, starting with a so-called one shot game with $2N$ players. We see what happens as $N \rightarrow \infty$ and formulate a MFG that is the basis of some numerical simulations.

5.1.1 A game with $(N + N)$ players.

We consider a one-shot, two-populations game, where (x_1, \dots, x_N) are the players of the first population and (y_1, \dots, y_N) are the players of the second one. Every player x_i moves in a

¹From J. Rauch, “Seeing Around Corners”. The Atlantic magazine, 2002.
<http://www.theatlantic.com/issues/2002/04/rauch.htm>.

compact space $\bar{\Omega} \subset \mathbb{R}^d$ and pays a cost $F_i^{1,N}$; similarly, every player y_i moves in the same space and pays a cost that will be denoted by $F_i^{2,N}$.

In the model of Schelling every player has just two choices (to move or not to move), but in a game setting we have to design a *cost* paid by every player. The main hypothesis is that each one aims at being near some other players of his own population. In particular, he is completely happy if the ratio of the individuals of his own kind among the overall neighbors is above some threshold. Hence, the cost paid by each one may be written in the following way:

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \left(\frac{\#\{x_j \in \mathcal{U}(x_i) : j \neq i\}}{\#\{x_j \in \mathcal{U}(x_i) : j \neq i\} + \#\{y_j \in \mathcal{U}(x_i)\} + \eta_1(N-1)} - a_1 \right)^-,$$

where $\mathcal{U}(x)$ is some neighborhood of x (for example $B_r(x) \cap \bar{\Omega}$) and $a_1 \in [0, 1]$ the threshold. $\eta_1 > 0$ is a small constant that makes the cost continuous, but it also has a meaning in the game formulation: suppose that a player is surrounded just by individuals of his own kind, i.e. $\#\{y_j \in \mathcal{U}(x_i)\} = 0$; as long as

$$\#\{x_j \in \mathcal{U}(x_i) : j \neq i\} \geq \frac{a_1 \eta_1}{1 - a_1} (N - 1)$$

the cost paid by the player is zero, so he's completely happy. Instead, if the players nearby are too few (below a percentage of the total population that is proportional to η_1) he pays a cost that becomes maximum if no one is in his neighborhood. Therefore, it may be uncomfortable to stay in a desert area.

We rewrite

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = G(\#\{x_j \in \mathcal{U}(x_i) : j \neq i\}, \#\{y_j \in \mathcal{U}(x_i)\}; a_1, \eta_1(N-1)),$$

where $G : [0, +\infty) \times [0, +\infty) \times [0, 1] \times (0, 1) \rightarrow [0, +\infty)$

$$G(r, s; a, t) = \left(\frac{r}{r + s + t} - a \right)^-, \quad (5.1)$$

is a continuous function with respect to r, s , for a, t fixed. Similarly, for the players of the other population, we set

$$F_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) = G(\#\{y_j \in \mathcal{U}(y_i) : j \neq i\}, \#\{x_j \in \mathcal{U}(y_i)\}; a_2, \eta_2(N-1)),$$

where $a_2 \in [0, 1]$ still represents the ‘‘happiness’’ threshold (possibly $\neq a_1$).

We notice that every player of the same population behaves with the same criterion, as the costs can be rewritten as

$$F_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = V^{1,N} \left[\frac{1}{N-1} \sum_{i \neq j} \delta_{x_j}, \frac{1}{N} \sum \delta_{y_j} \right] (x_i), \quad (5.2)$$

setting

$$V^{1,N}[m_1, m_2](x_i) = G \left((N-1) \int_{\mathcal{U}(x_i)} m_1, N \int_{\mathcal{U}(x_i)} m_2; a_1, \eta_1(N-1) \right).$$

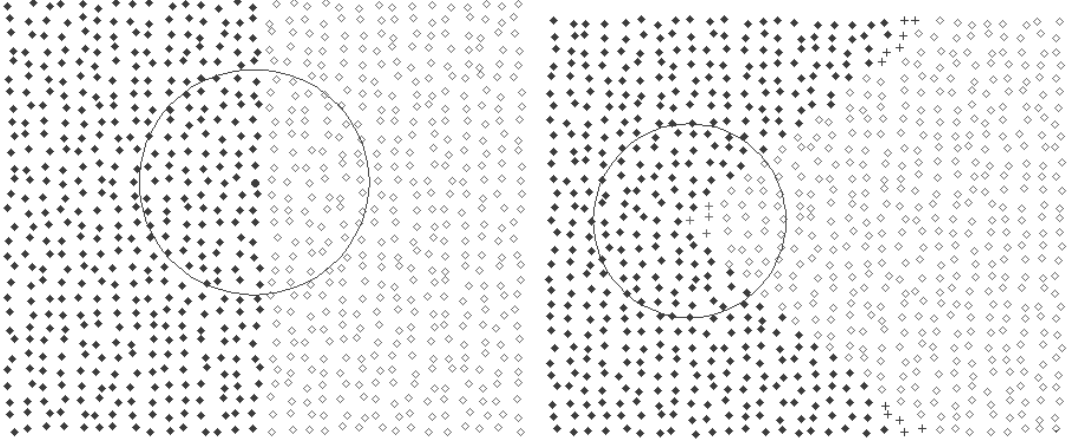


Figure 5.1: Two configurations for the one-shot game; filled and non-filled diamonds represent players respectively of the first and second population. Circles are boundaries of players' neighborhoods. Here $\alpha_1 = \alpha_2 = 0.45$. Left: A Nash equilibrium. Right: A configuration which is not a Nash equilibrium ('+' players pay a positive cost).

In the same way,

$$\begin{aligned}
 F_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) &= V^{2,N} \left[\frac{1}{N} \sum \delta_{x_j}, \frac{1}{N-1} \sum_{i \neq j} \delta_{y_j}, \right] (y_i) \\
 &= G \left((N-1) \int_{\mathcal{U}(y_i)} \frac{1}{N-1} \sum_{i \neq j} \delta_{y_j}, N \int_{\mathcal{U}(y_i)} \frac{1}{N} \sum \delta_{x_j}; a_2, \eta_2(N-1) \right).
 \end{aligned}$$

We say that $(\bar{x}_1, \dots, \bar{x}_N, \bar{y}_1, \dots, \bar{y}_N)$ is a *fully segregated configuration* if $\forall i = 1, \dots, N$,

$$\begin{aligned}
 \#\{\bar{x}_j \in \mathcal{U}(\bar{x}_i) : j \neq i\} &\geq \frac{a_1 \eta_1}{1 - a_1} (N-1) \\
 \#\{\bar{y}_j \in \mathcal{U}(\bar{x}_i)\} &= 0 \\
 \#\{\bar{y}_j \in \mathcal{U}(\bar{y}_i) : j \neq i\} &\geq \frac{a_2 \eta_2}{1 - a_2} (N-1) \\
 \#\{\bar{x}_j \in \mathcal{U}(\bar{y}_i)\} &= 0
 \end{aligned}$$

In this situation, every player has in his neighborhood a sufficient number of players of his own population and none of the other population, so the cost paid by everyone is identically zero: *every fully segregated configuration is a Nash equilibrium (in pure strategies)* for the $(N + N)$ -players game, i.e. $\forall i, \forall z, w \in \bar{\Omega}$

$$\begin{aligned}
 F_i^{1,N}(\bar{x}_1, \dots, \bar{x}_{i-1}, z, \bar{x}_{i+1}, \dots, \bar{x}_N, \bar{y}_1, \dots, \bar{y}_N) &\geq F_i^{1,N}(\bar{x}_1, \dots, \bar{x}_N, \bar{y}_1, \dots, \bar{y}_N) \\
 F_i^{2,N}(\bar{x}_1, \dots, \bar{x}_N, \bar{y}_1, \dots, \bar{y}_{i-1}, w, \bar{y}_{i+1}, \dots, \bar{y}_N) &\geq F_i^{2,N}(\bar{x}_1, \dots, \bar{x}_N, \bar{y}_1, \dots, \bar{y}_N)
 \end{aligned}$$

Fully segregated configurations are not the only possible Nash equilibria, since every player can tolerate some people of the other population nearby. On the other hand, even if the two populations are distributed (intuitively) in different areas of the domain, the layer that separates two groups of individuals cannot be of arbitrary shape (see Figure 5.1): this is a consequence of the ability of each player to take a look at the whole neighborhood.

5.1.2 Limits of Nash equilibria

Suppose now that $(\bar{x}_1, \dots, \bar{x}_N, \bar{y}_1, \dots, \bar{y}_N)$ is a Nash equilibrium for our one-shot game. We assume the existence of such an equilibrium in pure strategies, recalling that in general it might not exist. Setting

$$\bar{m}_1^N = \frac{1}{N} \sum_{j=1}^N \delta_{\bar{x}_j^N}, \quad \bar{m}_2^N = \frac{1}{N} \sum_{j=1}^N \delta_{\bar{y}_j^N},$$

we ask whether the limits of \bar{m}_k^N as $N \rightarrow +\infty$ have some properties and solve some minimization problem. First, we shall regularize the costs $V^{k,N}$, since they are not continuous operators on the set of probability measures; to do so, we notice that, for $m \in \mathcal{P}(\bar{\Omega})$

$$\int_{\mathcal{U}(x)} dm(y) = \int_{\bar{\Omega}} \chi_{\mathcal{U}(x)}(y) dm(y),$$

so we may replace every integral on \mathcal{U} by an integral with a smooth kernel K inside,

$$\begin{aligned} V^{1,N} [m_1, m_2] (x) \\ = G \left((N-1) \int_{\bar{\Omega}} K(x, y) dm_1(y), N \int_{\bar{\Omega}} K(x, y) dm_2(y); a_1, \eta_1(N-1) \right), \end{aligned}$$

where $K(x, y)$ is a regularized version of $\chi_{\mathcal{U}(x)}(y)$. The function $\bar{\Omega} \times \mathcal{P}(\bar{\Omega}) \ni (x, m) \mapsto \int_{\bar{\Omega}} K(x, y) dm(y)$ becomes then (uniformly) continuous, and so we have the same property for $V^{1,N}$. In an analogous way we set

$$\begin{aligned} V^{2,N} [m_1, m_2] (x) \\ = G \left((N-1) \int_{\bar{\Omega}} K(x, y) dm_2(y), N \int_{\bar{\Omega}} K(x, y) dm_2(y); a_2, \eta_2(N-1) \right). \end{aligned}$$

As in [28], Theorem 2.4, we obtain

Proposition 5.1. *Up to subsequences, the sequences of measures (\bar{m}_1^N) , (\bar{m}_2^N) converge respectively to $\bar{m}_1, \bar{m}_2 \in \mathcal{P}(\bar{\Omega})$ such that*

$$\int_{\bar{\Omega}} V^k[\bar{m}_1, \bar{m}_2](x) d\bar{m}_k(x) = \inf_{\mu \in \mathcal{P}(\bar{\Omega})} \int_{\bar{\Omega}} V^k[\bar{m}_1, \bar{m}_2](x) d\mu(x), \quad k = 1, 2 \quad (5.3)$$

where

$$V^k[\bar{m}_1, \bar{m}_2](x) = G \left(\int_{\bar{\Omega}} K(x, y) d\bar{m}_1(y), \int_{\bar{\Omega}} K(x, y) d\bar{m}_2(y); a_k, \eta_k \right). \quad (5.4)$$

Proof. By compactness, $\bar{m}_k^N \rightarrow \bar{m}_k$ (up to subsequences); we need to prove that \bar{m}_k satisfy (5.3). By definition of Nash equilibrium the measure $\delta_{\bar{x}_i^N}$ is a minimum of

$$\inf_{\mu \in \mathcal{P}(\bar{\Omega})} \int_{\bar{\Omega}} V^{1,N} \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}, \frac{1}{N} \sum_j \delta_{\bar{y}_j^N} \right] (x) d\mu(x),$$

and, as $G(\gamma r, \gamma s; a, t) = G(r, s; a, \gamma^{-1}t)$ for all $\gamma \neq 0$,

$$\begin{aligned} V^{1,N} \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{\bar{x}_j^N}, \frac{1}{N} \sum_j \delta_{\bar{y}_j^N} \right] (x) = \\ G \left(\int K(x, y) \frac{1}{N-1} \sum_{i \neq j} \delta_{x_j}, \int K(x, y) \frac{1}{N-1} \sum \delta_{y_j}; a_1, \eta_1 \right) \end{aligned}$$

Since

$$d\left(\frac{1}{N-1}\sum_{j \neq i} \delta_{x_j^N}, \bar{m}_1^N\right) \leq \frac{2}{N}, \quad d\left(\frac{1}{N-1}\sum_j \delta_{y_j^N}, \bar{m}_2^N\right) \leq \frac{1}{N},$$

for $\epsilon > 0$ fixed, by continuity of G

$$\int_{\bar{\Omega}} G\left(\int K(x, y) d\bar{m}_1^N(y), \int K(x, y) d\bar{m}_2^N(y); a_1, \eta_1\right) d\delta_{x_i^N}(x) \leq \inf_{\mu \in \mathcal{P}(\bar{\Omega})} \int_{\bar{\Omega}} G\left(\int K(x, y) d\bar{m}_1^N(y), \int K(x, y) d\bar{m}_2^N(y); a_1, \eta_1\right) d\mu(x) + \epsilon,$$

as soon as $N \gg 0$. Taking the sum for $i = 1, \dots, N$, dividing by N and passing to the limit, we obtain (5.3) with $k = 1$. The argument for $k = 2$ is similar. \square

As in [28], Remark 2.5, the mean field “system” (5.3) is equivalent to saying that the support of \bar{m}_k is contained in the set of minima of $V^k[\bar{m}_1, \bar{m}_2](x)$. This shows that there are plenty of equilibrium solutions and we do not expect in general the full convergence of sequences (\bar{m}_k^N) .

Moreover,

$$G(r, s; a, t) = \begin{cases} \frac{(a-1)r+as+at}{r+s+t} & \text{if } (a-1)r + as + at \geq 0 \\ 0 & \text{else,} \end{cases}$$

hence, if $(a-1)r + as + at \geq 0$

$$\partial_r G = \frac{-s-t}{(r+s+t)^2}, \quad \partial_s G = \frac{r}{(r+s+t)^2}$$

and so

$$-\frac{a^2(s+t)}{(r+s+t)^2} \leq \partial_r G < 0, \quad 0 \leq \partial_s G \leq \frac{a^2}{r}.$$

Given a point $x \in \Omega$ where the average cost for the first population V^1 is non-zero, it is decreasing with respect to $\int K \bar{m}_1$ and increasing w.r.t. $\int K \bar{m}_2$, so the average player would like to have more neighbors of his population and less of the other (as we expect). The same is for V^2 , it decreases as $\int K \bar{m}_2$ increases.

5.1.3 An improvement and a more sophisticated game.

In the discrete model of Schelling there is a structural impossibility of overcrowding: every player occupies a position in a chessboard, and every slot can host only one player. In our continuous model, the individuals do not suffer this constraint, and may even concentrate at a single point of the domain. In order to avoid this phenomenon, which is un-likely in real life situations, we shall introduce an “overcrowding” term in the costs $F_i^{k,N}$:

$$\begin{aligned} \hat{F}_i^{1,N}(x_1, \dots, y_N) &= F_i^{1,N} + C_1[(\#\{x_j \in \mathcal{U}(x_i)\} + \#\{y_j \in \mathcal{U}(x_i)\})/N - b_1]^+, \\ \hat{F}_i^{2,N}(x_1, \dots, y_N) &= F_i^{2,N} + C_2[(\#\{x_j \in \mathcal{U}(y_i)\} + \#\{y_j \in \mathcal{U}(y_i)\})/N - b_2]^+, \end{aligned}$$

for every $i = 1, \dots, N$, so every player starts paying a positive cost when the total number of players in his neighbor breaks the threshold $b_k N$, where $b_k \geq 0$ represents the maximum percentage with respect to the whole population that is tolerated. C_k are positive constants, possibly large: when the concentration of players is too high in some regions, the discomfort

should be attributed to overcrowding and not to an unsatisfactory ratio between the total number of individuals of the two populations (the $F_i^{k,N}$ term).

Passing to the limit as $N \rightarrow \infty$, we obtain the “mean field” costs

$$\begin{aligned}\hat{V}^1[m_1, m_2](x) &= V^1[m_1, m_2](x) + C_1 \left[\int_{\mathcal{U}(x)} (m_1 + m_2) - b_1 \right]^+, \\ \hat{V}^2[m_1, m_2](x) &= V^2[m_1, m_2](x) + C_2 \left[\int_{\mathcal{U}(x)} (m_1 + m_2) - b_2 \right]^+\end{aligned}$$

for the two populations.

When making a decision, an individual may be influenced also by the mood of other individuals of his own kind. In the model of human behavior that we have introduced, we may assume that every player takes into account the cost paid by players nearby, by adding each one to his own cost. In the framework of the $(N + N)$ -players game, we then modify the total cost associated to every player in the following way:

$$\begin{aligned}\bar{F}_i^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) &= \frac{1}{N} \sum_{l: x_l \in \mathcal{V}(x_i)} F_l^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) \\ \bar{F}_i^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N) &= \frac{1}{N} \sum_{l: y_l \in \mathcal{V}(y_i)} F_l^{2,N}(x_1, \dots, x_N, y_1, \dots, y_N).\end{aligned}$$

Here $\mathcal{V}(x)$ is a neighborhood of x in Ω . We may suppose that the opinion of other neighbors is “weighted” by a function that depends upon the distance from the individual:

$$\bar{F}_i^{k,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \frac{1}{N} \sum_{l=1}^N F_l^{k,N}(x_1, \dots, x_N, y_1, \dots, y_N) W(x_i, x_l), \quad (5.5)$$

$k = 1, 2$, where $W : \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}$ is continuous and $W(x_i, \cdot)$ should have support in $\mathcal{V}(x_i)$; this makes the model more realistic and adds regularity properties to the costs $\bar{F}_i^{k,N}$.

Hence, by (5.2)

$$\begin{aligned}\bar{F}_i^{k,N}(x_1, \dots, x_N, y_1, \dots, y_N) &= \frac{1}{N} \sum_{l=1}^N V^{k,N}(x_l) W(x_i, x_l) = \\ &= \frac{1}{N} \sum_{l=1}^N \left(\int_{\bar{\Omega}} W(x_i, \tau) V^{k,N}(\tau) \delta_{x_l}(d\tau) \right) = \int_{\bar{\Omega}} W(x_i, \tau) V^{k,N}(\tau) \left(\frac{1}{N} \sum_{l=1}^N \delta_{x_l} \right) (d\tau),\end{aligned}$$

recalling that $V^{k,N}(x)$ depend also upon the average densities

$$\frac{1}{N-1} \sum_{i \neq j} \delta_{x_j}, \frac{1}{N} \sum \delta_{y_j}, \dots$$

Then, we obtain an analog of proposition 5.1:

Proposition 5.2. *Let $\bar{m}_1^N = \frac{1}{N} \sum_{j=1}^N \delta_{\bar{x}_j^N}$, $\bar{m}_2^N = \frac{1}{N} \sum_{j=1}^N \delta_{\bar{y}_j^N}$, where $(\bar{x}_1, \dots, \bar{y}_N)$ is a Nash equilibrium for the one-shot game with costs (5.5). Then, up to subsequences (\bar{m}_1^N) , (\bar{m}_2^N) converge respectively to $\bar{m}_1, \bar{m}_2 \in \mathcal{P}(\bar{\Omega})$ such that*

$$\int_{\bar{\Omega}} \bar{V}^k[\bar{m}_1, \bar{m}_2](x) d\bar{m}_I(x) = \inf_{\mu \in \mathcal{P}(\bar{\Omega})} \int_{\bar{\Omega}} \bar{V}^k[\bar{m}_1, \bar{m}_2](x) d\mu(x), \quad k = 1, 2 \quad (5.6)$$

where

$$\bar{V}^k[\bar{m}_1, \bar{m}_2](x) = \int_{\bar{\Omega}} W(x, \tau) V^{k,N}[\bar{m}_1, \bar{m}_2](\tau) d\bar{m}_k(\tau) \quad (5.7)$$

Remark 5.3. As mentioned at the beginning of this section, Nash equilibria for the one-shot games might not exist. In order to overcome this issue, we should allow players to use *mixed strategies*, i.e., to minimize over elements of $\mathcal{P}(\bar{\Omega})$ instead of minimizing over elements of $\bar{\Omega}$. The Nash Theorem guarantees that Nash equilibria in mixed strategies always exist, and it is possible to prove a version of Propositions 5.1, 5.2 using such equilibria by following similar lines, as in [28], section 2.3.

5.1.4 The local limits.

Suppose now that the kernel K in (5.4) takes the form

$$K(x, y) = \varphi_\rho(x - y),$$

where $\varphi_\rho(x) = \rho^{-n} \varphi(x/\rho)$ and φ is a mollifier.

If $m_i \in L^1(\Omega)$, $\int K(\cdot, y) dm_k(y) \rightarrow m_k(\cdot)$ a.e., so

$$\begin{aligned} V^1[m_1, m_2](x) &\rightarrow G(m_1(x), m_2(x); a_1, \eta_1) \\ &= \left(\frac{m_1(x)}{m_1(x) + m_2(x) + \eta_1} - a_1 \right)^- =: V_\ell^1[m_1, m_2](x), \end{aligned}$$

and

$$\begin{aligned} V^2[m_1, m_2](x) &\rightarrow G(m_2(x), m_1(x); a_2, \eta_2) \\ &= \left(\frac{m_2(x)}{m_1(x) + m_2(x) + \eta_2} - a_2 \right)^- =: V_\ell^2[m_1, m_2](x), \end{aligned}$$

as $\rho \rightarrow 0$. We will consider in the sequel a stationary mean field game system with the costs above, which are local with respect to m_i and do not have a regularizing effect.

If the kernel W in (5.7) is also of the form

$$W(x, y) = \varphi_\rho(x - y),$$

we will take into account as well the local limit of the costs (of the second game we have introduced in the preceding paragraph) as $\rho \rightarrow 0$, that are

$$\bar{V}_\ell^k[m_1, m_2](x) = m_k(x) V_\ell^k[m_1, m_2](x) \quad \forall x \in \bar{\Omega}.$$

5.2 The Mean Field Game with ergodic costs.

In the last section, we derived the costs of a mean field game resembling the population model of Schelling by taking the limit of Nash equilibria of one-shot games with $2N$ -players. We shall now investigate the same mean field game in the a differential context, in particular considering the state of the average player of the k -th population governed by the controlled stochastic differential equation with reflection (see (1.1))

$$dX_t^k = \alpha_i dt + \sqrt{2\nu} dB_t^i - n(X_t^k) dl_t$$

aiming at minimizing the long-time-average cost

$$J^k(X_0^k, \alpha^1, \alpha^2, m_1, m_2) = \liminf_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E} \left[\int_0^T |\alpha_t^k|^2/2 + V^k[m_1, m_2](X_t^k) dt \right],$$

where m^k are the average distributions of the two populations. We are setting the mean field problem in the theoretic framework that we developed in Chapter 3. If we denote with m_k the distributions and with u_k the value functions of the average players, we find (see Section 3.1) that they solve the mean field game system

$$\begin{cases} -\nu \Delta u_k + |Du_k|^2/2 + \lambda_k = V^k[m_1, m_2](x) & \text{in } \Omega, k = 1, 2 \\ -\nu \Delta m_k - \operatorname{div}(Du_k m_k) = 0, \\ \partial_n u_k = 0, \quad \partial_n m_k = 0, & \text{on } \partial\Omega \end{cases} \quad (5.8)$$

for some $\lambda_k \in \mathbb{R}$, where the costs V^k might be replaced with \bar{V}^k or the local versions V_ℓ^1 and \bar{V}_ℓ^k defined above.

5.2.1 The deterministic case in space dimension one.

In the deterministic case and space dimension $d = 1$ it is possible to compute some explicit solution of (5.8). Suppose that the state space is a closed interval $\bar{\Omega} = [a, b] \subset \mathbb{R}$ and that there is no Brownian motion perturbing the dynamic of the average player ($\nu = 0$); (5.8) simplifies to

$$\begin{cases} \frac{(u'_k)^2}{2} + \lambda_k = V^k[m_1, m_2](x) \\ (u'_k m_k)' = 0, \end{cases} \quad (5.9)$$

with Neumann boundary conditions that should be intended in weak (viscosity) sense, as it is natural when taking the limit for $\nu \rightarrow 0$.

It is possible to construct explicit solutions for this system. For simplicity, we will consider the non-smoothened costs

$$\begin{aligned} V^1[m_1, m_2] &= G\left(\int_{\mathcal{U}(x)} m_1, \int_{\mathcal{U}(x)} m_2; a_1, \eta_1\right), \\ V^2[m_1, m_2] &= G\left(\int_{\mathcal{U}(x)} m_2, \int_{\mathcal{U}(x)} m_1; a_2, \eta_2\right), \end{aligned}$$

where G is defined in (5.1) as

$$G(r, s; a, t) = \left(\frac{r}{r+s+t} - a \right)^-.$$

Example 5.4.

$$m_k = \frac{1}{b-a}, \quad u_k = 0, \quad \lambda_k = V^k[m_1, m_2], \quad k = 1, 2$$

provides a solution: the two populations are distributed uniformly and the cost functions are everywhere zero if the two thresholds a_k are not large (say, below .5 if η is negligible).

Example 5.5. A family of “completely segregated” solutions may be written down explicitly. Suppose that $\mathcal{U}(x) = (x-r, x+r) \cap [a, b]$ and r is small, and let $a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b$ such that $x_{k+1} - x_k > r$ for $k = 0, \dots, 4$. Set

$$m_1(x) = \frac{1}{x_2 - x_1} \chi_{[x_1, x_2]}(x), \quad m_2(x) = \frac{1}{x_4 - x_3} \chi_{[x_3, x_4]}(x) \quad \forall x \in [a, b].$$

Then, $\int_{\mathcal{U}(x)} m_1$ and $\int_{\mathcal{U}(x)} m_2$ are continuous functions which have support in $(x_1 - r, x_2 + r)$ and $(x_3 - r, x_4 + r)$ respectively. $V^1[m_1, m_2](\cdot)$ is also continuous, and vanishes in $[x_1, x_2]$ (if $a_1 < 1$ and η_1 is small enough); indeed, $\int_{\mathcal{U}(x)} m_2 = 0$, so $\int_{\mathcal{U}(x)} m_1 / \int_{\mathcal{U}(x)} (m_1 + m_2) = 1$. The same is for V^2 , so we define

$$\lambda_k = 0, \quad u_k(x) = \int_a^x (2V^k[m_1, m_2](\sigma))^{1/2} d\sigma, \quad \forall x \in [a, b], k = 1, 2.$$

u_k simply verify the two HJB equations of (5.9). Moreover, they satisfy the Neumann boundary conditions $u'_k(a) = u'_k(b) = 0$ in the viscosity sense² (but not in classical sense, as $(u'_k)^2 = 2V^k \neq 0$ on the boundary of $[a, b]$); indeed, suppose that ϕ is a test function such that $u_1 - \phi$ has a local maximum at $x = b$. If we set $s = (2V^1[m_1, m_2](b))^{1/2}$ it follows that $\phi'(b) \leq s$. If $\phi'(b) \geq -s$ then $(\phi'(b))^2 \leq s^2$, so

$$\min\{(\phi'(b))^2 - 2V^1[m_1, m_2](b), \phi'(b)\} \leq 0.$$

Similarly, if $u_1 - \phi$ has a local minimum at $x = b$,

$$\max\{(\phi'(b))^2 - 2V^1[m_1, m_2](b), \phi'(b)\} \geq 0,$$

and in the same way it also holds that $u'_1(a) = u'_2(a) = u'_2(b) = 0$ in the viscosity sense.

It remains to be checked that m_k are (weak) solutions of the two Kolmogorov equations. To do so, we notice that m_1 is zero outside $[x_1, x_2]$; in $[x_1, x_2]$, however, $V^1[m_1, m_2](x) = 0$, hence $u'_1(x) = 0$. Similarly, $m_2(x)$ or $u'_2(x)$ vanishes, so $(u'_k m_k)' = 0$.

It is not difficult to show that solutions of other kinds exist, and the support of m_1 and m_2 may overlap.

5.3 Numerical Experiments

In this section we will present some numerical experiments for the system (5.8). General abstract results show that there exists at least a solution for this system, but we would like to see if some phenomenon of “segregation” occurs as it happens in the deterministic case that we discussed in the previous subsection. We observe that constant m and u produce a solution of (5.8), but we expect that it is possible to build other solutions, in particular with distributions m_1 and m_2 that are concentrated in different parts of the domain.

Finite difference methods have been implemented for mean field game systems in [3, 1]. In [2] some more results on the convergence of the methods are discussed and in [40] the special case of quadratic hamiltonian is considered. For the first order case we mention the work [26].

Usually, finite difference schemes are developed for the non-stationary version of the mean field system:

$$\begin{cases} \partial_t u_k - \nu \Delta u_k + |Du_k|^2/2 = V^k[m_1, m_2](x) & \text{in } (0, T) \times \Omega \\ -\partial_t m_k - \nu \Delta m_k - \operatorname{div}(Du_k m_k) = 0, & \text{in } (0, T) \times \Omega \\ \partial_n u_k = 0, \quad \partial_n m_k = 0, & \text{in } (0, T) \times \partial\Omega \end{cases} \quad (5.10)$$

and $u_k = u_k(t, x)$, $m_k = m_k(t, x)$ have to satisfy initial and terminal conditions

$$u_k(0, x) = u_{k,0}(x), \quad m_k(T, x) = m_{k,T}(x).$$

²A function $u \in C([a, b])$ satisfies the homogeneous Neumann boundary conditions in the viscosity sense in a if, for all test functions $\phi \in C^2$ such that $u - \phi$ has a local maximum at a , then $\min\{(\phi'(a))^2 - 2V^1[m_1, m_2](a), \phi'(a)\} \leq 0$, and for all $\phi \in C^2$ such that $u - \phi$ has a local minimum at a , then $\max\{(\phi'(a))^2 - 2V^1[m_1, m_2](a), \phi'(a)\} \geq 0$.

The main difficulty in solving numerically (5.10) is the mixed forward-backward in time structure, which is a particular feature of non-stationary MFG systems. Our aim is to construct some approximate solution of (5.8); here we do not have anymore the time variable, but the issue arising is that we have to find the correct values of λ_k for which there exists a corresponding vector of u, m that provide a solution of (3.5).

A strategy for finding a solution of (5.8) is suggested by [29]. They prove that in the single population case, setting

$$v^T(t, x) = u^T(tT, x), \quad \mu^T(t, x) = m^T(tT, x), \quad \forall x, t \in (0, 1)$$

where u^T, m^T is the solution of (5.10) on $[0, T]$, then v^T/T converges to $(1-t)\lambda$ while μ^T converges to m , where λ, m solve (5.8). One should be then able to approximate a solution of the stationary system by solving a non-stationary system with very large time horizon T . We mentioned before that (5.10) is computationally hard to treat; moreover long-time convergence is proved only in the quadratic hamiltonian case and with non-decreasing cost: in this situation uniqueness holds and it is deeply used in the arguments in [29]. It is not clear whether the methods presented by the authors might be adapted or not to (5.10), where uniqueness of solutions is not true in general.

Another strategy, suggested by [3], is to take the long time limit of the *forward-forward* mean field game system

$$\begin{cases} \partial_t u_k - \nu \Delta u_k + |Du_k|^2/2 = V^k[m_1, m_2](x) & (0, T) \times \Omega \\ \partial_t m_k - \nu \Delta m_k - \operatorname{div}(Du_k m_k) = 0, & \\ \partial_n u_k = 0, \quad \partial_n m_k = 0, & (0, T) \times \partial\Omega \\ u_k(t=0) = u_{k,0}, \quad m_k(t=0) = m_{k,0}(x). & \end{cases} \quad (5.11)$$

This method is reminiscent of long time approximations for the cell problem in homogenization: we expect that there exists some $\lambda_k \in \mathbb{R}$ such that $u_k^T(x, T) - \lambda_k T$ and $m_k^T(x, T)$ converge to some u_k, m_k solving (5.8). Although this is not proven rigorously in our setting, Gueant studies some single-population examples where the cost V has not the monotonicity property (so there is no uniqueness of solutions) and justifies the approach (see [41]), that is also successfully implemented in [3]. We will use this method that should be more efficient than the first one we presented, because the forward-forward system is computationally simpler to approximate than the backward-forward one. Still, what we are going to present are just numerical experiments, since no rigorous proof of any convergence is available at this stage.

We mention that it is possible to simplify (5.8), (5.10) and (5.11) through the Hopf-Cole change of variable, because the Hamiltonians are all quadratic (see Section 3.1.1). We prefer to study the original system, in order to develop a method which can be useful also for problems having more general hamiltonian functions.

5.3.1 The numerical scheme

We will develop the scheme for (5.11) in space dimension $d = 2$ as in [3]. Consider on $\Omega = [0, 1]^2$ a uniform grid with mesh step h , assuming that $1/h$ is an integer N_h , and denote with $x_{i,j}$ a generic point of the grid; let N_T be a positive integer and $\Delta t = T/N_T$, $t_n = n\Delta t$. The values of u_k and m_k at $x_{i,j}$, t_n will be approximated by $U_{i,j}^{k,n}$ and $M_{i,j}^{k,n}$ respectively, $k = 1, 2$, $i, j = 1, \dots, N_h$ and $n = 0, \dots, N_T$.

We introduce the finite difference operators

$$(D_1^+ U)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h}, \quad (D_2^+ U)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h}$$

and the numerical Hamiltonian $g : \mathbb{R}^4 \rightarrow \mathbb{R}$ defined by

$$g(q_1, q_2, q_3, q_4) = [(q_1^-)^2 + (q_3^-)^2 + (q_2^+)^2 + (q_4^+)^2]/2.$$

Denoting by

$$[D_h U]_{i,j} = ((D_1^+ U)_{i,j}, (D_1^+ U)_{i-1,j}, (D_2^+ U)_{i,j}, (D_2^+ U)_{i,j-1}),$$

the finite difference approximation of the Hamiltonian function $|Du|^2/2$ will be $g([D_h U]_{i,j})$.

We choose the classical five-points discrete version of the Laplacian

$$(\Delta_h U)_{i,j} = -\frac{1}{h^2}(4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1}).$$

In order to approximate the Kolmogorov equations in (5.11), we consider their weak formulation. Given any test function ϕ , the divergence term involved can be rewritten as

$$-\int_{\Omega} \operatorname{div}(m_k \partial_p H(Du_k)) \phi = \int_{\Omega} m \partial_p H(Du_k) \cdot D\phi,$$

which will be approximated by (boundary terms disappear by Neumann conditions)

$$h^2 \sum_{i,j} M_{i,j}^{k,n} Dg([D_h U^{k,n}]_{i,j}) \cdot [D_h \Phi]_{i,j},$$

where Φ is the finite difference version of ϕ . By introducing the compact notation

$$\mathcal{B}_{i,j}(U, M) = \frac{1}{h} \left(\begin{array}{l} M_{i,j} \partial_{q_1} g([D_h U^{k,n}]_{i,j}) - M_{i-1,j} \partial_{q_1} g([D_h U^{k,n}]_{i-1,j}) \\ + M_{i+1,j} \partial_{q_2} g([D_h U^{k,n}]_{i+1,j}) - M_{i,j} \partial_{q_2} g([D_h U^{k,n}]_{i,j}) \\ + M_{i,j} \partial_{q_3} g([D_h U^{k,n}]_{i,j}) - M_{i,j-1} \partial_{q_3} g([D_h U^{k,n}]_{i,j-1}) \\ + M_{i,j+1} \partial_{q_4} g([D_h U^{k,n}]_{i,j+1}) - M_{i,j} \partial_{q_4} g([D_h U^{k,n}]_{i,j}) \end{array} \right),$$

we can finally write the discrete version of (5.11)

$$\left\{ \begin{array}{l} \frac{U_{i,j}^{k,n+1} - U_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h U^{k,n+1})_{i,j} + g([D_h U^{k,n+1}]_{i,j}) = V^k(M_{i,j}^{1,n+1}, M_{i,j}^{2,n+1}), \\ \frac{M_{i,j}^{k,n+1} - M_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h M^{k,n+1})_{i,j} - \mathcal{B}(U^{k,n+1}, M^{k,n+1}) = 0. \end{array} \right. \quad (5.12)$$

The system above has to be satisfied for internal points, i.e. $2 \leq i, j \leq N_h - 1$. The finite difference version of the homogeneous Neumann boundary conditions is for all n, k

$$\begin{aligned} U_{1,j}^{k,n} &= U_{2,j}^{k,n}, & U_{N_h-1,j}^{k,n} &= U_{N_h,j}^{k,n} & \forall j &= 2, \dots, N_h - 1 \\ U_{i,1}^{k,n} &= U_{i,2}^{k,n}, & U_{i,N_h-1}^{k,n} &= U_{i,N_h}^{k,n} & \forall i &= 2, \dots, N_h - 1 \\ U_{1,1}^{k,n} &= U_{2,2}^{k,n}, & U_{N_h,1}^{k,n} &= U_{N_h-1,2}^{k,n}, \\ U_{1,N_h}^{k,n} &= U_{2,N_h-1}^{k,n}, & U_{N_h,N_h}^{k,n} &= U_{N_h-1,N_h-1}^{k,n}, \end{aligned}$$

and the same for $M^{k,n}$; we also require

$$M_{i,j}^{k,n} \geq 0$$

since M represents a mass distribution.

In [3] it is proven (see Theorem 5) that (5.12) has a solution when periodic boundary conditions are imposed. We expect that it is true also with Neumann conditions, since similar arguments can be implemented.

The scheme we presented is implicit, each time step consists in solving a coupled system of nonlinear equations for $U^{k,n+1}, M^{k,n+1}$ given $U^{k,n}, M^{k,n}$. This can be done for example by means of a Newton method, increasing possibly the time step when the asymptotic regime is reached. It has been indicated in [3], Remark 11, that to solve the system of nonlinear equations it is satisfactory to perform just one step of the Newton method, since they observed that in general one step reduces the residual substantially. Thus, we implemented an *approximate* version of (5.12), namely

$$\begin{cases} \frac{U_{i,j}^{k,n+1} - U_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h U^{k,n+1})_{i,j} + g([D_h U^{k,n}]_{i,j}) \\ \quad + Dg([D_h U^{k,n}]_{i,j}) \cdot ([D_h U^{k,n+1}]_{i,j} - [D_h U^{k,n}]_{i,j}) \\ \quad = V^k(M_{i,j}^{1,n}, M_{i,j}^{2,n}), \\ \frac{M_{i,j}^{k,n+1} - M_{i,j}^{k,n}}{\Delta t} - \nu(\Delta_h M^{k,n+1})_{i,j} - \mathcal{B}(U^{k,n+1}, M^{k,n+1}) = 0. \end{cases} \quad (5.13)$$

In this formulation, at each time step a coupled system of linear equations is solved. We set the initial data

$$U^{k,0} = 0, \quad M^{k,0} = M_0^k$$

and impose

$$h^2 \sum_{i,j} (M_0^k)_{i,j} = 1, \quad k = 1, 2.$$

We expect that, starting from different M_0^k , the distributions M^{k,N_T} evolve into some stationary configuration as T grows.

5.3.2 One-dimensional simulations.

We choose, for all the following experiments, the local version of the costs obtained in Section 5.1.4

$$V^k(m_1, m_2) = \left(\frac{m_k}{m_1 + m_2 + \eta_k} - a_k \right)^-, \quad (5.14)$$

with $\eta_1 = \eta_2 = 10^{-8}$ and thresholds $a_k = 0.4$.

We start by showing some numerical results in one-dimension, i.e. the initial data m_0^k are constant with respect to the y variable (the j entry in the approximated version M_0^k), to have some flavor of the phenomena occurring in a simplified and faster computational environment.

Test 1. We set $\nu = 0.05$, $N_h = 100$, $a_1 = a_2 = 0.4$, $dt = 0.02$ and the final time is $T = 10$. Starting from an initial configuration where the first and the second population are located on the left and right side respectively of the domain, the distributions evolve quickly into some stable regime, where they are much more “segregated” and overlap only in the center of the domain, Figure 5.2. The distribution of the other population M^2 is very close to zero where some meaningful mass M^1 is present. Also, it seems that U^k stabilize (Figure 5.3), in the sense that $U^{k,n} \approx W^k + \lambda_k t_n$, and W^k, λ_k are expected to be approximations of u_k, λ_k solving (5.8). In Figure 5.4 we tried to evidence this kind of convergence, plotting the sup norm of the approximation of $\frac{d}{dt} m^k(\cdot, t)$ as t goes to infinity: it is clear that it goes to zero in a very fast way. Moreover, $h \sum_i U_i^{k,n} / t_n$ approaches some value that should be the optimal value λ_k . Finally, although the scheme does not preserve the total mass $h \sum_i M_i^{1,n}$ at each time step, the mesh size is sufficiently big to guarantee that it is always close to one during the entire simulation.

Test 2. Here $\nu = 0.05$, $N_h = 100$, $a_1 = a_2 = 0.4$, $dt = 0.02$, final time is $T = 30$. Again the initial distributions M_0^k are located in different parts of the domain, M_0^1 is near

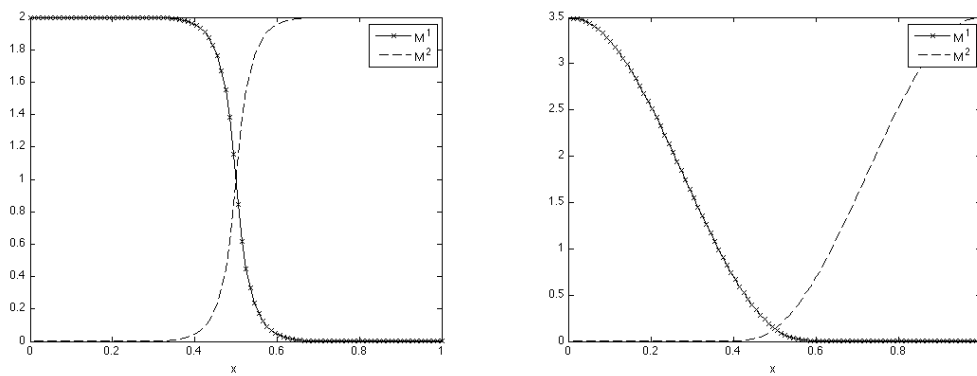


Figure 5.2: Test 1, M_0^k . Left: $t = 0$. Right: $t = 5$.

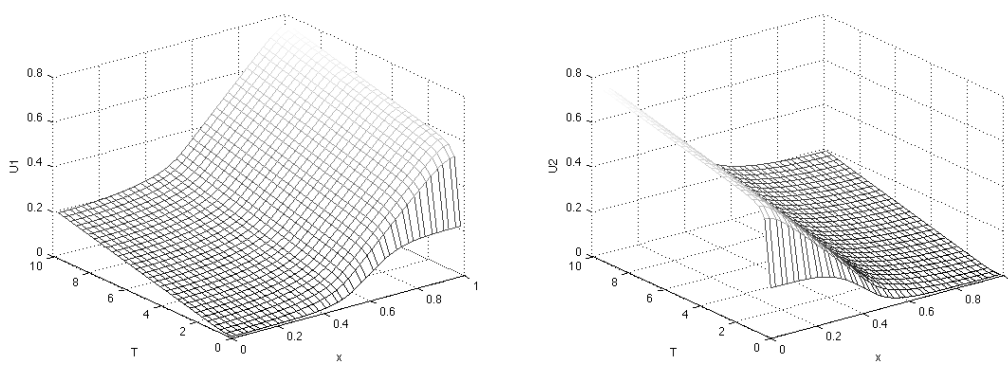


Figure 5.3: Test 1, Left: U^1 . Right: U^2 .

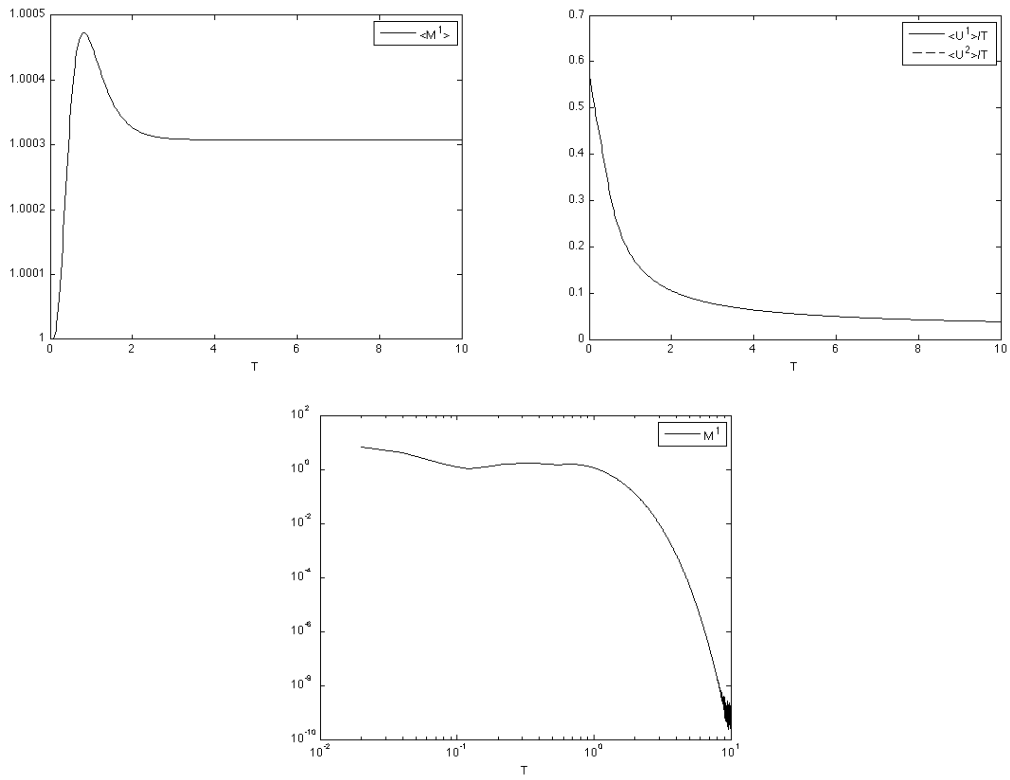


Figure 5.4: Test 1, Top Left: $h \sum_i M_i^{1,n}$. Top Right: $h \sum_i U_i^{1,n} / ndt$. Bottom $\max_i |M_i^{1,n} - M_i^{1,n-1}| / dt$

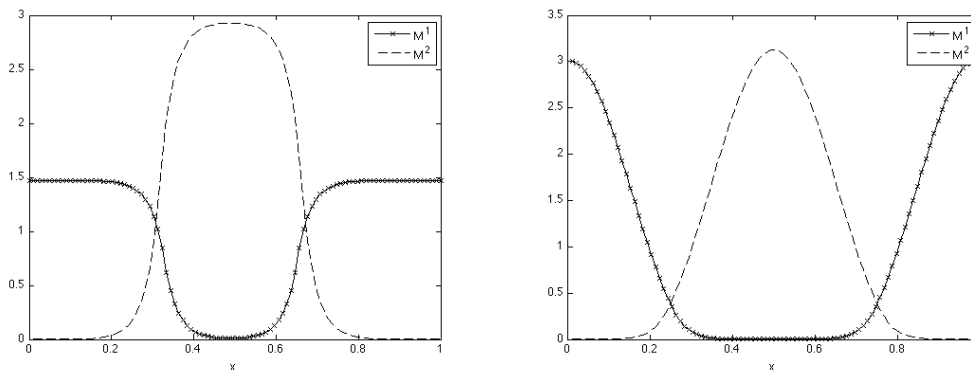


Figure 5.5: Test 2, M_0^k . Left: $t = 0$. Right: $t = 30$.

the boundary and M_0^1 around the center: we expect them to evolve into some segregated equilibrium which is not the constant one. As in Test 1 this happens (see Figure 5.5), in the sense that some stable regime is attained, where the two distributions avoid each other and overlap around two points of the domain (0.25 and 0.75). The difference with respect to Test 1 is that convergence appears to be slower (that is why we run the simulation up to time $T = 30$), but still initial masses are almost preserved, $\max_{[0,1]} |\frac{d}{dt} m^k(x,t)|$ vanishes and $h \sum_i U_i^{k,n} / t_n$ stabilizes.

Test 3. In this test we wanted to produce more complicated segregated solutions. Parameters are $\nu = 0.02$, $N_h = 200$, $a_1 = a_2 = 0.4$, $dt = 0.005$, final time is $T = 20$, initial distributions are already segregated but supports are more “disconnected” than Test 1,2. In Figure 5.7 we observe that they initially *seem* to stabilize to an equilibrium, but mass M^1 around the central peak slowly moves near the boundary; then, at $t_n \approx 8$, the way M^k are distributed drastically changes and evolves into the same regime as Test 2. In Figures 5.8 and 5.9 we observe this sudden transition on various experimental parameters.

5.3.3 Two-dimensional simulations.

We now set $\Omega = [0,1]^2$. We will see that, as in one-dimensional experiments, initial configurations M_0^k evolve into stable equilibria. Since in the two dimensional case computations are more time-consuming, we reduce the mesh size.

Test 4. Set $\nu = 0.01$, $N_h = 30$, $a_1 = a_2 = 0.4$, $dt = 0.01$ and the final time is $T = 15$. The usual strategy is to start from initial M_0^k that have disjoint support (in this test M_0^1 and M_0^2 are symmetric), Figure 5.10. We observe that at time $T = 15$ the two mass distributions are stable and concentrated at different corners of the domain; $h^2 \sum_i U^1 k, n_i / t_n$ and $h^2 \sum_i U^2 k, n_i / t_n$ stabilize around the same value (the problem is symmetric) and the total mass is close to one, although it deviates more than in one-dimensional tests, maybe because of the coarser mesh. Convergence to zero of the approximate time derivatives $|M^{k,n} - M^{k,n-1}| / dt$ is observable, even if the sup-norm starts oscillating at some time, small fluctuations arise around corners of $\partial\Omega$. See Figure 5.11.

Test 5. In the last test, $\nu = 0.01$, $N_h = 30$, $a_1 = a_2 = 0.4$, $dt = 0.01$ and the final time is $T = 110$. We have to choose a final time significantly bigger than the previous tests because, starting from M_0^k in Figure 5.12, the distributions take much longer to stabilize to an equilibrium. In Figure 5.12 we draw the final state, where the two masses are concentrated at opposite corners of Ω , however, they evolve through *apparent* stationary regimes that become suddenly unstable, even if they start segregated and with a good degree of symmetry. In

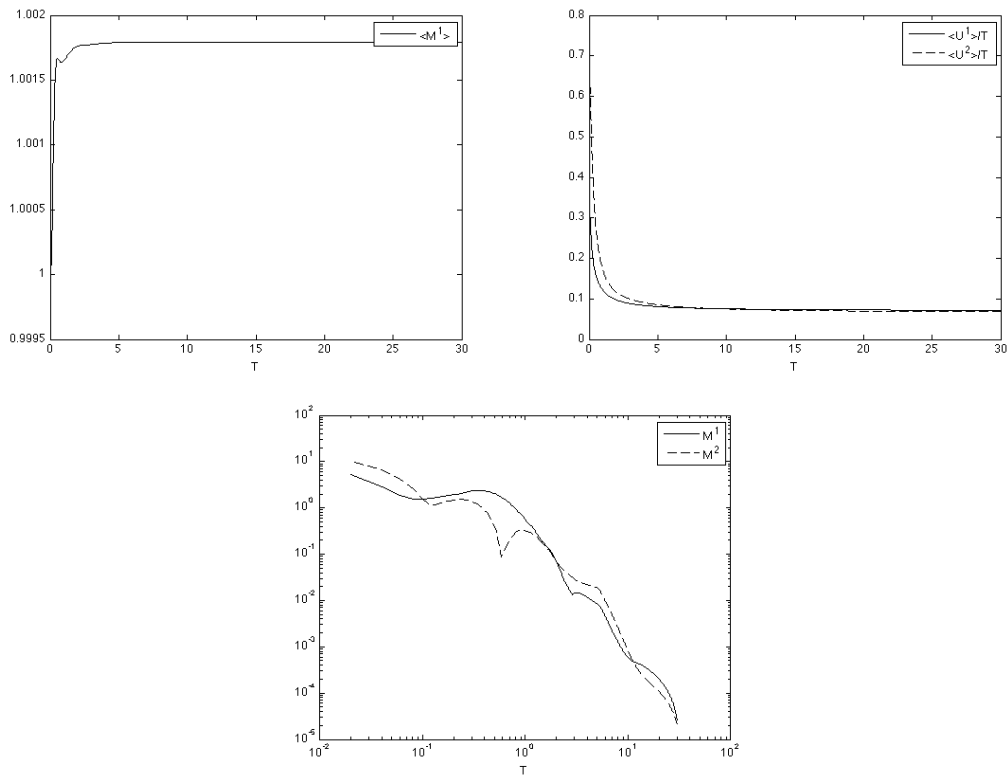


Figure 5.6: Test 2, Top Left: $h \sum_i M_i^{1,n}$. Top Right: $h \sum_i U_i^{1,n} / ndt$. Bottom: $\max_i |M_i^{1,n} - M_i^{1,n-1}| / dt$.

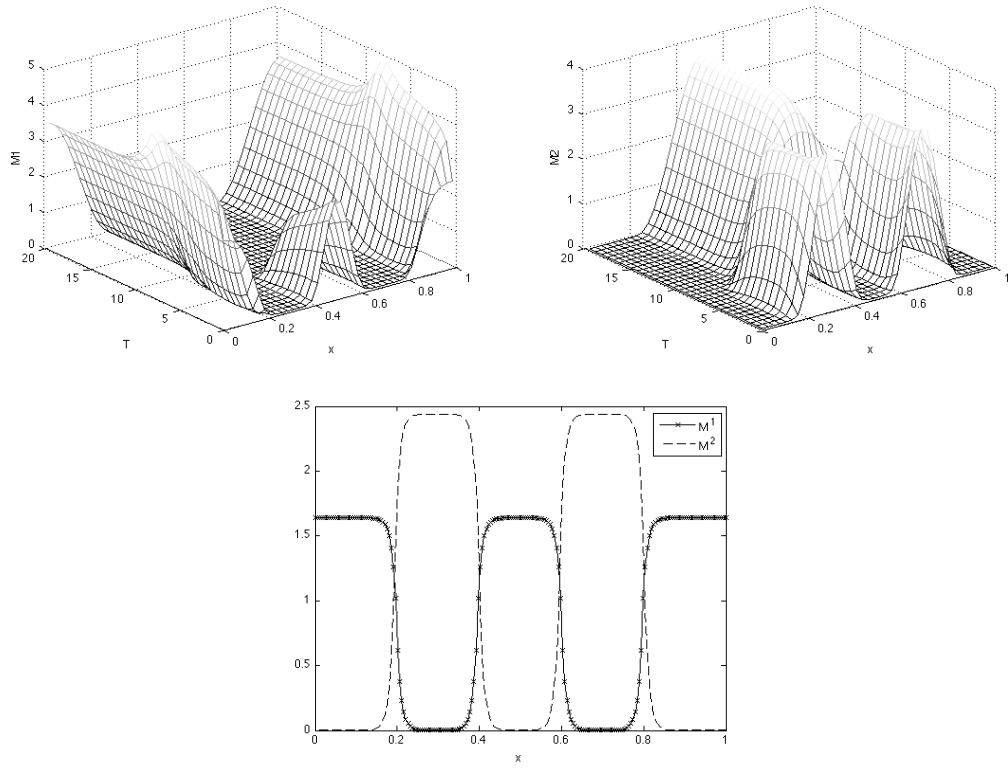


Figure 5.7: Test 3, Top Left, Right: M^1 , M^2 . Bottom, the initial distributions M_0^k

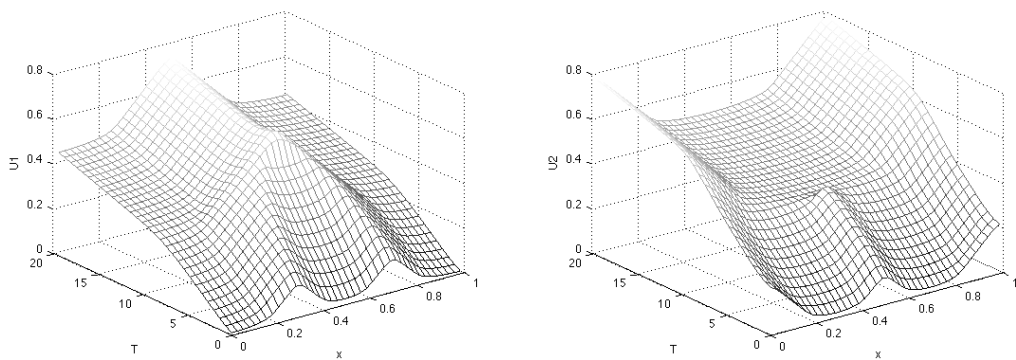


Figure 5.8: Test 3, Left: U^1 . Right: U^2 .

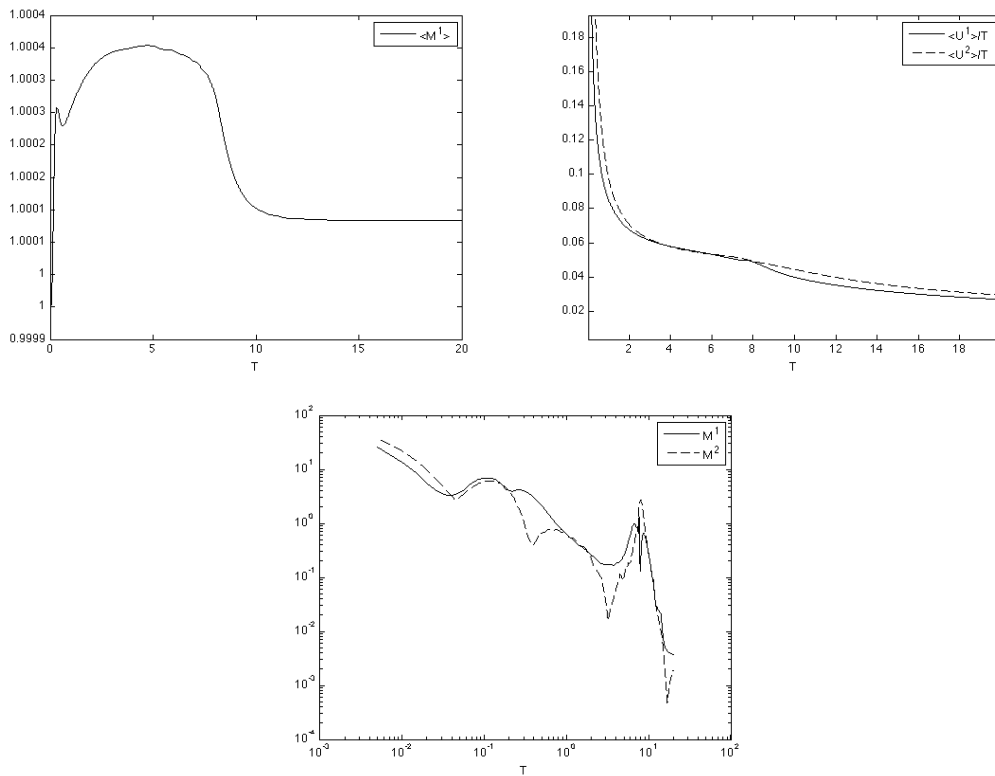


Figure 5.9: Test 3, Top Left: $h \sum_i M_i^{1,n}$. Top Right: $h \sum_i U_i^{1,n} / ndt$. Bottom $\max_i |M_i^{1,n} - M_i^{1,n-1}| / dt$

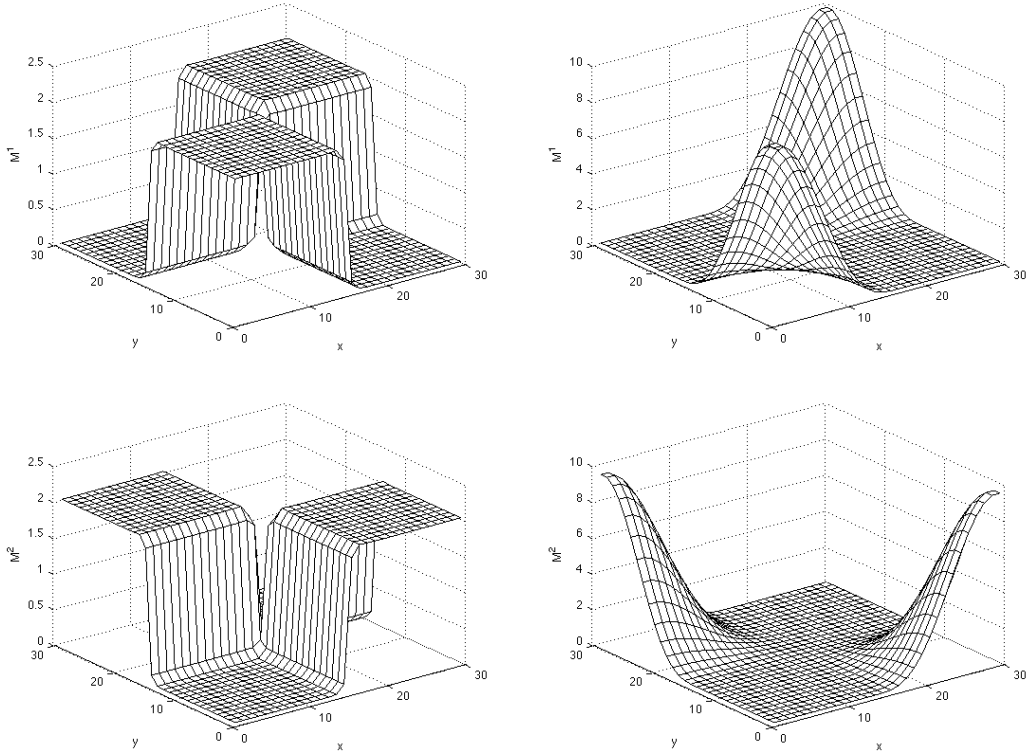


Figure 5.10: Test 4, Top Left, Right: M^1 at $t = 0, t = 15$. Bottom Left, Right: M^2 at $t = 0, t = 15$

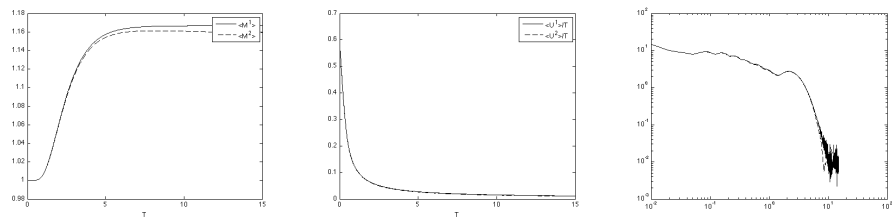


Figure 5.11: Test 4, Left: $h^2 \sum_i M_i^{k,n}$. Center: $h^2 \sum_i U_i^{k,n} / ndt$. Right $\max_i |M_i^{k,n} - M_i^{k,n-1}| / dt$

Figure 5.13 we are able to see quite well when the qualitative change of the shape of M^k happens: $|M^{k,n} - M^{k,n-1}|/dt$ becomes comparable to one (three peaks are evident), the total mass moves away from the equilibrium. Also $h^2 \sum_i U_i^{k,n}/t_n$ change its asymptotic, even though its decreasing. The fast transformation of behavior of solution might be caused by the non-smooth way costs are defined: the cost paid by a player is zero or positive if the density of the other population is below or above some threshold.

Conclusions. Our aim of the numerical tests was to produce some non-constant solution of (5.8) using a finite difference method to capture the long time behaviour (5.11). Moreover, we sought “segregated” configurations, i.e. M^1 and M^2 concentrated in different parts of the domain. In all tests, approximate solutions of (5.11) show convergence to some equilibrium after some time, and M^1 is always localized away from M^2 (and vice-versa). In the deterministic case ($\nu = 0$, Section 5.2.1) we showed that a rich family of segregated solutions can be constructed; we do not know if this is true also in the stochastic case we have studied, but we have not been able to capture numerically such hypothetical variety of solutions, the approximate method always stabilizes to some “low-potential” configuration. By that, we mean that the value $h^2 \sum_i U_i^{k,n}/t_n$, which should approximate the value function λ_k , decreases in time and tend to settle to a minimal value; we recall that λ_k is the cost paid by an average player of the k -th population which behaves optimally with respect to fixed distributions m_1, \dots, m_M . In general there might be different values of λ_k for which solutions u_k, m_k exist (and so multiple mean-field Nash equilibria), but the numerical solution seems to approach lower values of such λ_k and higher values appear more “unstable”. On that, we observe that stable solutions tend to minimize the surface where M^1 and M^2 overlap.

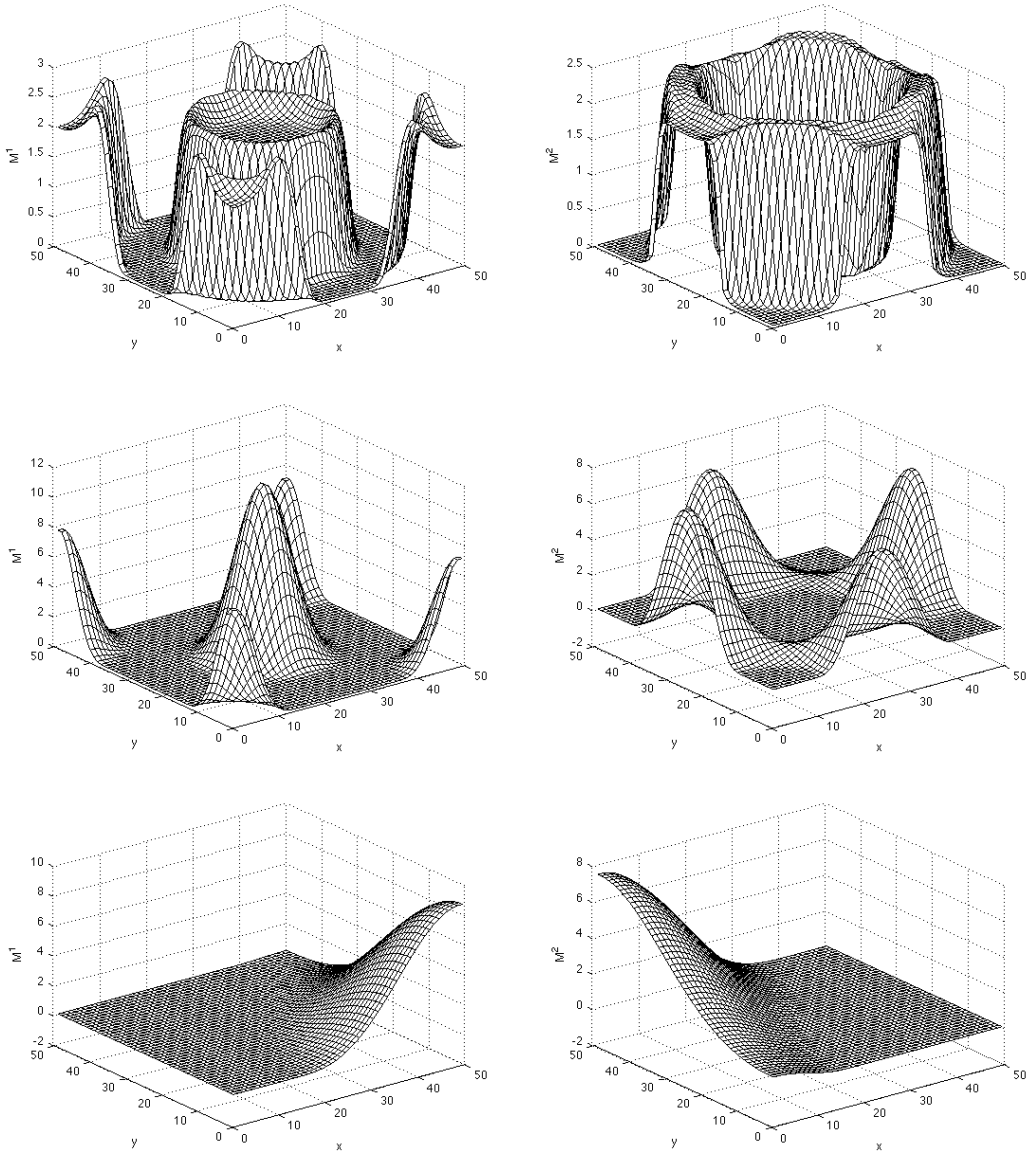


Figure 5.12: Test 5, Top, mid, bottom: M^k at $t = 0, t = 10, t = 110$. Left, right: M^1, M^2 .

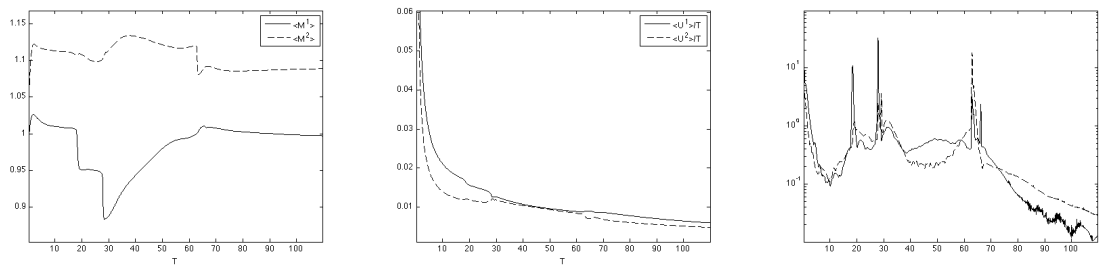


Figure 5.13: Test 5, Left: $h^2 \sum_i M_i^{k,n}$. Center: $h^2 \sum_i U_i^{k,n} / ndt$. Right $\max_i |M_i^{k,n} - M_i^{k,n-1}| / dt$

Part III

The k -th Prescribed Curvature Equation

Chapter 6

Introduction to Curvature Equations

The last part of this dissertation is concerned with the Dirichlet problem for degenerate elliptic fully non-linear equations of type

$$F(Du, D^2u) = f(x) \quad \text{in } \Omega \subset \mathbb{R}^d. \quad (6.1)$$

In particular, we are interested in studying *curvature equations*, that have the form

$$\mathcal{F}(\kappa_1[u], \dots, \kappa_d[u]) = f, \quad (6.2)$$

where $(\kappa_1[u], \dots, \kappa_d[u])$ denote the vector of ordered *principal curvatures* of the graph of the function u and $\mathcal{F} : \mathbb{R}^d \rightarrow \mathbb{R}$. As we will see later, every $\kappa_i[u]$ depends on Du and D^2u , thus (6.2) is really of type (6.1).

Although we are very far from having a general theory for equations of the form (6.2), particular cases of specific functions \mathcal{F} have been deeply studied and are now well understood. If $\mathcal{F}(\kappa_1, \dots, \kappa_d) = \sum_{i=1}^d \kappa_i$, $\mathcal{F}(\kappa_1, \dots, \kappa_d)$ becomes the *mean curvature* operator, for which the literature is huge (as an example we cite the classical book [37]). If $\mathcal{F}(\kappa_1, \dots, \kappa_d) = \prod_{i=1}^d \kappa_i$ we are facing the standard *Prescribed Gaussian curvature* equation, for which the Dirichlet problem is treated for example in [24, 37, 39, 75].

An interesting and more general situation is the case of symmetric functions of curvatures, i.e. $\mathcal{F}(\kappa_1, \dots, \kappa_d) = \sigma_K(\kappa_1, \dots, \kappa_d)$ (or perturbations of σ_K); In this framework we shall mention the works [25, 51, 52, 66, 73] and references therein. The features of these kind of equations are the high degree of nonlinearity with respect to the second order entry, their ellipticity only in some restricted class of unknowns u and the degeneracy, usually when the prescribing function f approaches zero.

Symmetry of σ_K is surely a key point which is exploited when dealing with these equations. For this reason, we will investigate some case without similar symmetry assumptions, that can be also interesting from a purely geometric point of view. In particular, we will focus on the *k-th prescribed principal curvature equation*, namely

$$\kappa_k[u] = f \quad (6.3)$$

for some $k = 1, \dots, d$ fixed. Our aim will be to solve the Dirichlet problem for this equation, possibly under suitable assumptions on the data, using the modern viscosity theory started by Crandall, Lions and Evans (for details we refer to the User's Guide to viscosity solutions [30]).

6.1 Definitions and Preliminaries

Let $A = A(p)$ be the matrix defined by

$$A(p) = \frac{1}{v(p)} \left(I - \frac{p \otimes p}{v^2(p)} \right),$$

$$v(p) = \sqrt{1 + |p|^2} \quad \forall p \in \mathbb{R}^d.$$

It is possible to define through the matrix A the *principal curvatures* of the graphic of a function u .

Definition 6.1. *Let Ω be a domain of \mathbb{R}^d and $u \in C^2(\Omega)$. The vector $\kappa[u](x)$ of principal curvatures of the graph of u at a point $x \in \Omega$ is the vector of (ordered) eigenvalues of the matrix*

$$A(Du(x))D^2u(x).$$

Note that $A(Du(x))D^2u(x)$ has indeed d real eigenvalues: $A(p)$ is positive definite and its square root is

$$P(p) = \frac{1}{v^{1/2}(p)} \left(I - \frac{p \otimes p}{v(p)(1 + v(p))} \right) \in \mathcal{S},$$

Through the definition of eigenvalue it is possible to check that $A(Du(x))D^2u(x)$ has the same eigenvalues of $P(Du(x))D^2u(x)P(Du(x))$, which is of course symmetric.

Principal curvatures can be defined in a more abstract way for differentiable manifolds through the so-called second fundamental form (see, for example, [33], p. 129).

We consider then the set of equations (6.3), that can be written in the form (6.1) as

$$F^k(Du(x), D^2u(x)) = f(x) \quad \text{in } \Omega, \tag{6.4}$$

where $F^k : \mathbb{R}^d \times \mathcal{S} \rightarrow \mathbb{R}$ is

$$F^k(p, M) = \lambda_k(A(p)M)$$

for every $k = 1, \dots, d$. Explicitly, (6.4) becomes

$$\lambda_k \left(\frac{1}{\sqrt{1 + |Du(x)|^2}} \left(I - \frac{Du(x) \otimes Du(x)}{1 + |Du(x)|^2} \right) D^2u(x) \right) = f(x)$$

We collect now some properties of $A(p)$ and the operators F^k .

Lemma 6.2. *For all $p \in \mathbb{R}^d$ we have $A(p) \in \mathcal{S}$ and the vector of eigenvalues of $A(p)$ is*

$$\lambda(A(p)) = \{v^{-3}(p), v^{-1}(p), \dots, v^{-1}(p)\}.$$

Moreover, let \mathcal{K} be a compact subset of \mathbb{R}^d . Then,

$$i) \quad (1/\alpha)I \leq A(p) \leq I \text{ for all } p \in \mathcal{K},$$

$$ii) \quad \|A(p) - A(r)\| \leq L|p - r| \text{ for all } p, r \in \mathcal{K},$$

for some $\alpha, L > 0$.

Proof. Let $p \in \mathbb{R}^d$, for some orthogonal matrix \mathcal{O} we have

$$p \otimes p = \mathcal{O}^{-1} \text{diag}[0, \dots, 0, |p|^2] \mathcal{O},$$

so $A(p)$ is equivalent to

$$\begin{aligned} \frac{1}{v(p)} \text{diag} \left[1, \dots, 1, 1 - \frac{|p|^2}{1 + |p|^2} \right] &= \frac{1}{v(p)} \text{diag} \left[1, \dots, 1, \frac{1}{1 + |p|^2} \right] \\ &= \text{diag} \left[\frac{1}{v(p)}, \dots, \frac{1}{v(p)}, \frac{1}{v^3(p)} \right]. \end{aligned} \quad (6.5)$$

Then *i)* and *ii)* follow from the representation (6.5). \square

Lemma 6.3. *The operator $-F^k$ defined in (6.4) is (degenerate) elliptic, i.e.*

$$-F^k(p, M + P) \leq -F^k(p, M)$$

for all $p \in \mathbb{R}^d$, $M \in \mathcal{S}$, $P \geq 0$.

Proof. We have, for all $p \in \mathbb{R}^d$, $M \in \mathcal{S}$, $P \geq 0$

$$\begin{aligned} -F^k(p, M + P) &= -\lambda_k(A(p)M + A(p)P) \leq \\ &= -\lambda_k(A(p)M) - \lambda_1(A(p)P) \leq -\lambda_k(A(p)M) - \frac{\lambda_1(P)}{v^3(p)}, \end{aligned} \quad (6.6)$$

using Weyl's inequality (Theorem B.2) for the first inequality and Ostrowski's Theorem (Theorem B.3) for the second one, recalling from Lemma 6.2 that $\lambda_1(A(p)) = v^{-3}(p)$. We then conclude that $-F^k(p, M + P) \leq -F^k(p, M)$. \square

Remark 6.4. We observe that, substituting $P = rI$ in (6.6),

$$-F^k(p, M + rI) \leq -F^k(p, M) - \frac{r}{v^3(p)}$$

for all $r \geq 0$. Inequalities in (6.6) cannot be improved in general, so $-F^k$ is far from being uniformly elliptic; since $v^3(p) \rightarrow +\infty$ as $|p| \rightarrow +\infty$ we may add that it is very degenerate. However, $v^{-3}(p)$ is bounded away from zero if $|p|$ is bounded, so in this case $-F^k$ becomes *non-totally degenerate elliptic*, see (7) in [13].

We define subsolutions of (6.4) in the standard viscosity sense:

Definition 6.5. $u \in \text{USC}(\overline{\Omega})$ is a (viscosity) subsolution of $-F^k(Du, D^2u) + f \leq 0$ in Ω if for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a maximum point at $x_0 \in \Omega$ we have $-F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0) \leq 0$.

In order to prove a comparison principle for equation (6.4), we will need the concept of C -semiconvexity; a semiconvex function is indeed locally Lipschitz (among other properties that will be used), and as we observed in Remark 6.4, F^k becomes (locally) non-totally degenerate in this class of subsolutions.

Definition 6.6. $u \in \text{USC}(\overline{\Omega})$ is C -semiconvex in Ω if it is a viscosity subsolution of

$$-\lambda_1(D^2u) - C \leq 0$$

in Ω .

Even if we will restrict ourselves to consider C -semiconvex subsolutions, which enjoy nice regularity properties, there might be in general viscosity subsolutions of (6.4) that are not semiconvex:

Example 6.7. Consider on $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ the function $u(x) = -|x_1|^{1/2}$. Let, for all $\epsilon > 0$, $u_\epsilon \in C^2(\Omega)$ depending only on x_1 be such that

$$u_\epsilon = u \text{ on } \Omega \setminus \{|x_1| \geq \epsilon\} \text{ and } u_\epsilon = u_\epsilon(x_1) \leq u \text{ on } \Omega.$$

We have that

$$F^2(Du_\epsilon, D^2u_\epsilon) = \max \left\{ \frac{(u_\epsilon)_{x_1 x_1}}{(1 + (u_\epsilon)_{x_1}^2)^{3/2}}, 0 \right\} \geq 0 \quad \text{on } \Omega,$$

so u_ϵ is a subsolution of $-F^2(Du, D^2u) \leq 0$. Note that

$$u(x) = \sup_{\epsilon > 0} u_\epsilon(x) \quad \forall x \in \Omega,$$

so by standard arguments u^* is a viscosity subsolution of $-F^2(Du, D^2u) \leq 0$, but u is continuous so $u = u^*$ and therefore u is a subsolution, which is not C -semiconvex for any $C \geq 0$ nor Lipschitz continuous.

Our subclass of subsolutions consists of functions that have locally bounded gradient and hessian controlled from below. These properties are required for proving a comparison principle and even if this class is rich, it might not be the optimal one.

As for supersolutions, we weaken the standard definition of viscosity supersolution, restricting the space of test functions.

Definition 6.8. A function $u \in \text{LSC}(\bar{\Omega})$ is a (viscosity) (C -)supersolution of $-F^k(Du, D^2u) + f \geq 0$ in Ω if for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a minimum point at $x_0 \in \Omega$ and

$$-C < \lambda_1(D^2\phi(x_0))$$

we have $-F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0) \geq 0$.

Definition 6.9. u is a (viscosity) (C -)solution of (6.4) in Ω if it is C -semiconvex, u is a subsolution of $-F^k + f \leq 0$ and u is a C -supersolution of $-F^k + f \geq 0$ in Ω .

Remark 6.10. Note that u is a C -solution if and only if it is both a sub. and supersolution in the standard viscosity sense of

$$\max\{-\lambda_1(D^2u) - C, -F^k(Du, D^2u) + f\} = 0. \quad (6.7)$$

This is reminiscent of the definition of sub. and supersolution for Monge-Ampere type equations in [50] and [14].

If u is a *strictly* C -semiconvex viscosity solution of (6.7), then it is twice differentiable almost everywhere due to Alexandrov Theorem and $-\lambda_1(D^2u) - C < 0$, thus $-F^k(Du, D^2u) + f = 0$ holds a.e. (everywhere if $u \in C^2(\Omega)$). However, a C -solution u is not in general a standard viscosity solution of (6.4), and vice-versa.

We will solve the Dirichlet problem for (6.4) first in viscosity sense (see Section 7, [30]): given

$$g : \partial\Omega \rightarrow \mathbb{R}$$

let us define $G^k : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S} \rightarrow \mathbb{R}$ as

$$G^k(x, r, p, X) = \begin{cases} -F^k(p, X) + f(x) & \text{if } x \in \Omega, \\ r - g(x) & \text{if } x \in \partial\Omega \end{cases} \quad (6.8)$$

Definition 6.11. $u \in \text{USC}(\overline{\Omega})$ is a (viscosity) subsolution of $(G^k)_* \leq 0$ in $\overline{\Omega}$ if it is a subsolution of $-F^k(Du, D^2u) + f \leq 0$ in Ω and for all $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a maximum point at $x_0 \in \partial\Omega$ we have

$$\min\{-F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0), u(x_0) - g(x_0)\} \leq 0 \quad (6.9)$$

Definition 6.12. $u \in \text{LSC}(\overline{\Omega})$ is a (viscosity) (C-)supersolution of $(G^k)^* \geq 0$ in $\overline{\Omega}$ if it is a supersolution of $-F^k(Du, D^2u) + f \geq 0$ in Ω and for all $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a minimum point at $x_0 \in \partial\Omega$ and

$$-C < \lambda_1(D^2\phi(x_0))$$

we have

$$\max\{-F^k(D\phi(x_0), D^2\phi(x_0)) + f(x_0), u(x_0) - g(x_0)\} \geq 0 \quad (6.10)$$

Definition 6.13. A function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is a discontinuous (viscosity) solution of the Dirichlet problem $G^k = 0$ in $\overline{\Omega}$ if for some $C \geq 0$ it is C -semiconvex in $\overline{\Omega}$, u^* is a subsolution of $(G^k)_* \leq 0$ and u_* is a C -supersolution of $(G^k)^* \geq 0$ in $\overline{\Omega}$.

A (viscosity) solution of $G^k = 0$ is a discontinuous solution of the Dirichlet problem $G^k = 0$ in $\overline{\Omega}$ such that $u \in C(\overline{\Omega})$ and

$$u = g \quad \text{on } \partial\Omega.$$

Chapter 7

Existence and Uniqueness

In this chapter we will prove a comparison result for the prescribed k -th principal curvature equation (6.4) and existence for the Dirichlet problem under some abstract assumptions. Then we will provide some sufficient conditions on the data for these assumption to hold, so that existence and uniqueness results apply.

7.1 The Comparison Principle.

The first step in studying a fully non-linear equation using viscosity tools is stating some comparison principle, which guarantees uniqueness and is useful for powering up the existence machinery. In order to apply standard results of [30], since (6.4) is not strictly monotone with respect to the u entry (the operator does not depend directly on u), a possibility is to produce a perturbation of subsolutions which are strict subsolutions, i.e. for any subsolution u given find a sequence u_ϵ of strict subsolutions that converges uniformly to u as ϵ goes to zero (see 5.C, [30]). So far, our attempts of finding such a perturbation have been unfruitful; $-F^k$ is very degenerate and it is not differentiable, since the k -th eigenvalue of a matrix is just a Lipschitz function. Moreover, even though we proved that $A(p)$ is uniformly Lipschitz if p varies in a compact set, the Lipschitz constant of $-F^k$ with respect to the p entry also depends on M , for which we have only partial information in general.

Given a subsolution u and $\phi \in C^2$, setting $u_\epsilon = u + \epsilon\phi$ is the standard perturbation procedure (see [13, 14, 42, 50]). Using non-degenerate ellipticity and Lipschitz regularity one has

$$\begin{aligned} -F(Du + \epsilon D\phi, D^2u + \epsilon D^2\phi) &\leq -F(Du + \epsilon D\phi, D^2u) - \delta(\epsilon D^2\phi) \\ &\leq L|\epsilon D\phi| - \delta(\epsilon D^2\phi) < 0, \end{aligned}$$

where δ is the contribution from ellipticity which is usually (in non-totally degenerate situations) strong enough to beat $L|D\phi|$. In our case L, δ depend on D^2u, Du respectively, for which we have no a-priori control, hence this method fails.

Our strategy will be rather to perturb *locally* a subsolution u , around a point for which $u - v$ (where v is a supersolution of (6.4)) has a maximum. In such a point, we can control $\lambda_1(D^2u)$ up to $\lambda_k(D^2u)$ using C -semiconvexity of u and the fact that v is a supersolution. We shall show now that this information is *sufficient* to obtain a bound on the Lipschitz constant of F^k with respect to Du uniform in D^2u , namely

Proposition 7.1. *Let $k = 1, \dots, d$, $M \in \mathcal{S}$, $\mathcal{K} \subset \mathbb{R}^d$ and $A : \mathcal{K} \rightarrow \mathcal{S}$ such that*

$$i) \quad -\Lambda \leq \lambda_1(M) \leq \dots \leq \lambda_k(M) \leq \Lambda,$$

ii) $(1/\alpha)I \leq A(p) \leq \alpha I$ for all $p \in \mathcal{K}$,

iii) $\|A(p) - A(r)\| \leq L|p - r|$ for all $p, r \in \mathcal{K}$

for some $\alpha, \Lambda, L > 0$. Then there exists $L' = L'(d, k, \alpha, \Lambda, L) > 0$ such that

$$\lambda_k(A(r)M) \geq \lambda_k(A(p)M) - L'|p - r|$$

for all $p, r \in \mathcal{K}$, $|p - r| < 1$.

Proof. Step 1. We may suppose that M is diagonal; indeed, there exist some orthogonal \mathcal{O} such that

$$M = \mathcal{O}^{-1}\Delta\mathcal{O}, \quad \Delta = \text{diag}[\mu_1, \dots, \mu_d]$$

where $-\Lambda \leq \mu_1 \leq \dots \leq \mu_k \leq \Lambda$, then

$$\lambda_k(A(p)M) = \lambda_k(\mathcal{O}A(p)\mathcal{O}^{-1}\Delta) = \lambda_k(B(p)\Delta B(p)), \quad (7.1)$$

with $B^2(p) = \mathcal{O}A(p)\mathcal{O}^{-1} > 0$; note that B has the same properties as A , that is there exist $M, L > 0$ such that

- $(1/M)I \leq B(p) \leq M I$, for all $p \in \mathcal{K}$,
- $\|B(p) - B(r)\| \leq L|p - r|$ for all $p, r \in \mathcal{K}$.

Step 2. Thanks to the variational characterization of eigenvalues (Theorem B.1) and (7.1) we may write

$$\begin{aligned} \lambda_k(A(p)M) &= \max_{S \in \mathcal{S}_{d-k+1}} \min_{|\xi|=1, \xi \in S} \xi^T B(p)\Delta B(p)\xi \\ &= \max_{S \in \mathcal{S}_{d-k+1}} \min_{v \in V_p(S)} v^T \Delta v = \max_{S \in \mathcal{S}_{d-k+1}} \min_{v \in V_p(S)} \sum_{i=1}^n \mu_i v_i^2, \end{aligned}$$

where \mathcal{S}_{d-k+1} indicates the family of $(d - k + 1)$ -dimensional subspaces of \mathbb{R}^d and $V_p(S) = \{B(p)\xi : |\xi| = 1, \xi \in S\}$ for all $S \subseteq \mathbb{R}^d$, $p \in \mathcal{K}$.

Step 3. Let $S \in \mathcal{S}_{d-k+1}$ be fixed, we prove that

$$\min_{v \in V_p(S)} \sum_{i=1}^d \mu_i v_i^2 = \min_{v \in W_p(S)} \sum_{i=1}^d \mu_i v_i^2, \quad (7.2)$$

for some suitable $W_p(S) \subset V_p(S)$. Indeed, let $\hat{\xi} \in S$, $|\hat{\xi}| = 1$ be such that

$$\hat{v} = B(p)\hat{\xi} = (\dots, \underbrace{0, 0, \dots, 0}_{d-k}, \dots). \quad (7.3)$$

In order to prove the existence of such $\hat{\xi}$, we consider an orthonormal basis $\{\xi^i\}$ of S and $\hat{\xi} = \sum_{i=1}^{d-k+1} \alpha_i \xi^i$, such that $\alpha \in \mathbb{R}^{d-k+1}$ that will be determined later. Then

$$(7.3) \Leftrightarrow \sum_{i=1}^{d-k+1} \alpha_i \underbrace{B(p)\xi^i}_{\eta^i \in \mathbb{R}^d} = (\dots, 0, 0, \dots, 0)$$

$$\Leftrightarrow \begin{cases} \alpha_1 \eta_{k+1}^1 + \alpha_2 \eta_{k+1}^2 + \dots + \alpha_{d-k+1} \eta_{k+1}^{d-k+1} = 0 \\ \vdots \\ \alpha_1 \eta_d^1 + \alpha_2 \eta_d^2 + \dots + \alpha_{d-k+1} \eta_d^{d-k+1} = 0. \end{cases}$$

The (homogeneous) system has $d - k$ equations and $d - k + 1$ unknowns α_i , so there exists a linear non-trivial subspace of solutions; we pick a solution α in order to have $|\hat{\xi}| = 1$, and so $\hat{v} \in V_p(S)$. Due to the particular choice of \hat{v} we deduce

$$\min_{v \in V_p(S)} \sum_{i=1}^d \mu_i v_i^2 \leq \sum_{i=1}^d \mu_i \hat{v}_i^2 = \sum_{i=1}^k \mu_i \hat{v}_i^2 \leq k\Lambda M^2,$$

since $|\hat{v}_i| \leq |\hat{v}| = |B(p)\hat{\xi}| \leq \|B(p)\| \leq M$. Let now $v \in V_p(S)$; se $\mu_j v_j^2 > k\Lambda M^2 + (d-1)\Lambda M^2$ for some $j = d - k + 1, \dots, n$, then

$$\sum_{i=1}^d \mu_i v_i^2 = \sum_{i \neq j} \mu_i v_i^2 + \mu_j v_j^2 > -(d-1)\Lambda M^2 + k\Lambda M^2 + (d-1)\Lambda M^2,$$

and (7.2) holds setting

$$W_p(S) = V_p(S) \cap \{v : \mu_j v_j^2 \leq k\Lambda M^2 + (d-1)\Lambda M^2 \text{ per ogni } j = d - k + 1, \dots, d\}.$$

Step 4. We want to compare $\lambda_k(A(p)M)$ and $\lambda_k(A(p+q)M)$, $q \in \mathbb{R}^d$. We fix $S' \in \mathcal{S}_{d-k+1}$; let $S = B^{-1}(p+q)B(p)S' \in \mathcal{S}_{d-k+1}$ and $v \in W_{p+q}(S)$ (that is $v = B(p+q)\xi$ for some $\xi \in S$, $|\xi| = 1$). Set

$$\xi' = \frac{B^{-1}(p)B(p+q)\xi}{|B^{-1}(p)B(p+q)\xi|}, \quad w = B(p)\xi';$$

we have $|\xi'| = 1$ and $\xi' \in S'$, so $w \in V_p(S')$. Hence

$$w = \frac{1}{|B^{-1}(p)B(p+q)\xi|} B(p+q)\xi = (1 + \delta)v$$

for some $\delta = \delta(q, \xi) \in \mathbb{R}$. It then follows that

$$\begin{aligned} \sum_{i=1}^d \mu_i v_i^2 - \sum_{i=1}^d \mu_i w_i^2 &= \sum_{i=1}^d \mu_i (v_i - w_i)(v_i + w_i) = \sum_{i=1}^d -\mu_i \delta v_i (2 + \delta) v_i \\ &= -\delta(2 + \delta) \sum_{i=1}^d \mu_i v_i^2 \geq -C(d, k, \Lambda, M)\delta \end{aligned} \quad (7.4)$$

if $|\delta| < 1$, since $v \in W_{p+q}(S)$, so

$$\sum_{i=1}^d \mu_i v_i^2 = \sum_{i=1}^k \mu_i v_i^2 + \sum_{i=k+1}^d \mu_i v_i^2 \leq k\Lambda M^2 + (d-k)(k\Lambda M^2 + (d-1)\Lambda M^2).$$

We have then, by (7.4),

$$\sum_{i=1}^d \mu_i v_i^2 \geq \sum_{i=1}^d \mu_i w_i^2 - C\delta \geq \min_{w \in V_p(S')} \sum_{i=1}^d \mu_i w_i^2 - C\delta,$$

and since $v \in W_{p+q}(S)$ was arbitrarily chosen,

$$\min_{v \in W_{p+q}(S)} \sum_{i=1}^d \mu_i v_i^2 \geq \min_{w \in V_p(S')} \sum_{i=1}^d \mu_i w_i^2 - C\delta,$$

therefore

$$\max_{S \in \mathcal{S}_{d-k+1}} \min_{v \in W_{p+q}(S)} \sum_{i=1}^d \mu_i v_i^2 \geq \min_{w \in V_p(S')} \sum_{i=1}^d \mu_i w_i^2 - C\delta.$$

Being also $S' \in \mathcal{S}_{d-k+1}$ arbitrarily chosen we conclude

$$\max_{S \in \mathcal{S}_{d-k+1}} \min_{v \in W_{p+q}(S)} \sum_{i=1}^d \mu_i v_i^2 \geq \max_{S' \in \mathcal{S}_{d-k+1}} \min_{w \in V_p(S')} \sum_{i=1}^d \mu_i w_i^2 - C\delta,$$

and the inequality $\lambda_k(A(p+q)M) \geq \lambda_k(A(p)M) - C\delta$ using steps 2 and 3.

Step 5. It remains to estimate the constant δ . By the hypothesis of Lipschitz continuity of $B(p)$,

$$B(p+q) = B(p) + R, \quad \|R\| \leq L|q|$$

if $p+q \in \mathcal{K}$. Hence, for all ξ , $|\xi| = 1$

$$\begin{aligned} |B^{-1}(p)B(p+q)\xi| &= |B^{-1}(p)(B(p) + R)\xi| = |\xi + B^{-1}(p)R\xi| \\ &\geq 1 - |B^{-1}(p)R\xi| \geq 1 - \|B^{-1}(p)R\| \geq 1 - \frac{L}{M}|q|. \end{aligned}$$

Setting $x = -\frac{L}{M}|q| \geq -\frac{1}{2}$ (possibly increasing M) we finally obtain

$$1 + \delta \leq (1+x)^{-1} \leq 1 - 2x,$$

so $\delta(q, \xi) \leq \frac{2L}{M}|q|$, which implies

$$\lambda_k(A(p+q)M) \geq \lambda_k(A(p)M) - \frac{2CL}{M}|q|.$$

□

Proposition 7.1 gives a fine estimate which is the lynchpin of the comparison principle we are going to show. We also follow the ideas of [15], suggested by [30] (remark 5.A), also exploiting a technical result of [66].

The main assumption is

(h0) $f \in C(\bar{\Omega})$, Ω is a bounded domain of class C^2 .

Theorem 7.2. *Suppose that (h0) holds. If u is a C -semiconvex subsolution of $-F^k(Du, D^2u) + f \leq 0$ and v is a C -supersolution of $-F^k(Du, D^2u) + f \geq 0$ in Ω such that $u \leq v$ on $\partial\Omega$, then*

$$u \leq v \quad \text{in } \bar{\Omega}.$$

Proof. Step 1. Suppose by contradiction that $\bar{x} \in \Omega$ is a maximum point of $u - v$. We choose $\Omega' \subset\subset \Omega$ such that $\bar{x} \in \Omega'$; u is bounded and C -semiconvex, so there exists a constant $D > 0$ such that

$$|Du(x)| \leq D \quad \forall x \in \Omega', \tag{7.5}$$

almost everywhere (see, for example, Remark 2.1.8 [27]), where D depends on $C, \|u\|_\infty, \text{dist}(\Omega', \partial\Omega)$. Moreover, by Lemma 6.2, i)

$$(1/\alpha)I \leq A(p) \leq \alpha I \quad \forall |p| \leq D + 1 \tag{7.6}$$

for some $\alpha = \alpha(D) > 0$. Being $A(p)$ Lipschitz uniformly on compact sets (Lemma 6.2, ii)), it is true by Proposition 7.1 that for all $p, q \in \mathbb{R}^d$, $M \in \mathcal{S}$ satisfying

$$|p|, |q| \leq D + 1 \quad -\Lambda \leq \lambda_1(M) \leq \dots \leq \lambda_k(M) \leq \Lambda,$$

where $\Lambda = \max\{C, \alpha \max_{\bar{\Omega}} |f| + 1\}$, there exists L' depending on D, C and $\alpha \max_{\bar{\Omega}} |f|$ only (so depending on u ed f) such that

$$\lambda_k(A(r)M) \geq \lambda_k(A(p)M) - L'|p - r|$$

for all $|p|, |r| \leq D + 1$, $|p - r| < 1$.

Moreover, [66], Lemma 4.1 states that for all $\epsilon > 0$ there exists $\psi \in C^\infty(\mathbb{R}^d)$ such that

- i) $u - v - \psi$ has a maximum in $\tilde{x} \in \Omega$, con $|\tilde{x} - \bar{x}| < \epsilon$.
- ii) $D^2\psi(\tilde{x})$ is negative defined and its eigenvalues satisfy $-\epsilon < \lambda_i(D^2\psi(\tilde{x})) < 0$ for all i .
- iii) $|D\psi(\tilde{x})| < \epsilon \min_i |\lambda_i(D^2\psi(\tilde{x}))|$.

Pick $\epsilon > 0$ sufficiently small to have $\tilde{x} \in \Omega'$ and $\epsilon < 1/(16\alpha L')$. Setting $\phi(x) = \psi(x) + |x - \tilde{x}|^4$, the function $u - v - \phi$ has \tilde{x} as the unique strict maximum in $\bar{\Omega}'$. Let $\hat{\sigma} = \min_i |\lambda_i(D^2\psi(\tilde{x}))|/2$, so

$$\text{ii')} \quad D^2\psi(\tilde{x}) < -\hat{\sigma}I.$$

Let now \underline{v}_μ be the inf-convolution of v :

$$\underline{v}_\mu(x) = \inf_{\xi \in \mathbb{R}^d} \{v(\xi) + (\mu/2)|x - \xi|^2\}.$$

By standard arguments \underline{v}_μ is μ -semiconcave; moreover, there exists a sequence $\mu_j \rightarrow +\infty$ such that the maximum points x_{μ_j} of $u - \underline{v}_{\mu_j} - \phi$ satisfy $x_{\mu_j} \rightarrow \tilde{x}$. By continuity of $D\phi$ and $D^2\phi$ we may choose j large such that properties ii') and iii) are true also for $D\phi(x_{\mu_j})$ e $D^2\phi(x_{\mu_j})$; we may increase j so that,

$$\omega(1/\mu_j) < \frac{\hat{\sigma}}{4\alpha}, \tag{7.7}$$

where ω is a continuity modulus of f (on $\bar{\Omega}'$). For simplicity we set $\underline{v} = \underline{v}_{\mu_j}$ e $\hat{x} = x_{\mu_j}$.

Step 2. There exists a sequence $t_m \rightarrow 0$ such that the function

$$x \mapsto u(x) - \underline{v}(x) - \phi(x) - \langle t_m, x \rangle$$

has a strict maximum at x_m and $x_m \rightarrow \hat{x}$. By Jensen's Lemma, we know that if $r > 0$ is sufficiently small there exists $\bar{\rho} > 0$ such that the set of maxima of

$$x \mapsto u(x) - \underline{v}(x) - \phi(x) - \langle t_m, x \rangle - \langle q, x \rangle \tag{7.8}$$

in $B_r(x_m)$, $q \in B_\rho(0)$ with $\rho < \bar{\rho}$ contains a set of positive measure. Due to Aleksandrov's Theorem u e \underline{v} are twice differentiable almost everywhere, therefore, for all $\rho, r > 0$ (small) we may choose $z = z_m \in B_r(x_m)$ e $q \in B_\rho(0)$ such that z is a maximum of (7.8) and u, \underline{v} are twice differentiable. Note that $Du(z) - D\underline{v}(z) - D\phi(z) - t_m - q = 0$ holds, so (for ϵ small enough)

$$|D\underline{v}(z)| \leq D + \max_{B_r(x_m)} |D\phi| + \bar{\rho} + |t_0| \leq D + 1.$$

Moreover

$$D^2u(z) \leq D^2\underline{v}(z) + D^2\phi(z),$$

so $-CI \leq D^2u(z) < -D^2\phi(z) + D^2u(z) \leq D^2\underline{v}(z)$ by ii') and C -semiconvexity of u . Moreover, v is a C -supersolution of $-F^k + f \geq 0$, and it is standard that \underline{v} is a C -supersolution of

$$-F^k(D\underline{v}, D^2\underline{v}) + f(x) \geq \omega(1/\mu_k),$$

therefore by the estimate on $|D\underline{v}(z)|$, (7.6) and Theorem B.3,

$$\begin{aligned} \max_{\Omega} |f| &\geq f(z) - \omega(1/\mu_k) \\ &\geq F^k(D\underline{v}(z), D^2\underline{v}(z)) = \lambda_k(A(D\underline{v}(z)) D^2\underline{v}(z)) \geq (1/\alpha)\lambda_k(D^2\underline{v}(z)). \end{aligned}$$

We obtain, again using $D^2u(z) \leq D^2\underline{v}(z) + D^2\phi(z)$ and Theorem B.2, that

$$\begin{aligned} -C &\leq \lambda_1(D^2u(z)) \leq \lambda_i(D^2u(z)) \leq \lambda_i(D^2\underline{v}(z) + D^2\phi(z)) \\ &\leq \lambda_k(D^2\underline{v}(z) + D^2\phi(z)) \leq \alpha \max_{\Omega} |f| + \lambda_d(D^2\phi(z)) \quad \forall i = 1, \dots, k, \end{aligned}$$

which implies $-\Lambda \leq \lambda_i(D^2u(z)) \leq \Lambda$ for all $\forall i = 1, \dots, k$.

Step 3. Finally, we let $\rho, r \rightarrow 0$ and $m \rightarrow +\infty$; being $|Du(z_m)|$ and $|D\underline{v}(z_m)|$ bounded, we extract a subsequence such that $z_m \rightarrow \hat{x}$ and $p_m = Du(z_m) \rightarrow \bar{p}$, $q_m = D\underline{v}(z_m) \rightarrow \bar{q}$. Setting $X_m = D^2u(z_m)$ and $Y_m = D^2\underline{v}(z_m)$, we have

$$\begin{aligned} (p_m, X_m) &\in \bar{J}_{\Omega}^{2,+} u(z_m), (q_m, Y_m) \in \bar{J}_{\Omega}^{2,-} \underline{v}(z_m) \\ p_m &= q_m + D\phi(z_m) + o(1) \\ X_m &\leq Y_m + D^2\phi(z_m) \\ -\Lambda &\leq \lambda_1(X_m) \leq \dots \leq \lambda_k(X_m) \leq \Lambda, \end{aligned}$$

hence

$$\begin{aligned} F^k(p_m, X_m) &\geq f(z_m) \\ F^k(q_m, Y_m) &\leq f(z_m) + \omega(1/\mu_k). \end{aligned}$$

Therefore

$$\begin{aligned} f(z_m) + \omega(1/\mu_k) &\geq F^k(q_m, Y_m) = \lambda_k(A(q_m)Y_m) \geq \lambda_k(A(q_m)(X_m - D^2\phi(z_m))) \\ &\geq \lambda_k(A(q_m)X_m) + \lambda_1(-A(q_m)D^2\phi(z_m)) \\ &\geq \lambda_k(A(p_m)X_m) + \beta_m = F^k(p_m, X_m) + \beta_m \geq f(z_m) + \beta_m. \end{aligned}$$

where $\beta_m = -L'|D\phi(z_m) + o(1)| + \lambda_1(-A(q_m)D^2\phi(z_m))$. By the choice of ϵ

$$\begin{aligned} |D\phi(z_m) + o(1)| &\rightarrow |D\phi(\hat{x})| < \epsilon(2\hat{\sigma}) < \frac{\hat{\sigma}}{8\alpha L'}, \\ D^2\phi(z_m) &\rightarrow D^2\phi(\hat{x}) < -\hat{\sigma}I \Rightarrow -A(q_m)D^2\phi(z_m) \rightarrow -A(\bar{q})D^2\phi(\hat{x}) > \frac{\hat{\sigma}}{\alpha}I, \end{aligned}$$

simplifying we have

$$\omega(1/\mu_j) \geq \beta_m \geq -\frac{\hat{\sigma}}{4\alpha} + \frac{\hat{\sigma}}{2\alpha} = \frac{\hat{\sigma}}{4\alpha},$$

for m large enough, which contradicts (7.7). \square

7.2 Existence for the Dirichlet problem.

We first state a result that assures that some inequality is satisfied if loss of boundary data for the Dirichlet problem $(G^k)_* \leq 0$ happens. We borrow this result from [32], adapting the proof for our particular definition of supersolution.

Proposition 7.3. *Let Ω be a strictly convex C^2 domain, $g \in C(\overline{\Omega})$ and $u \in \text{USC}(\overline{\Omega})$ a viscosity subsolution of $(G^k)_* \leq 0$ in $\overline{\Omega}$ (resp. $\text{LSC}(\overline{\Omega})$ supersolution of $(G^k)^* \geq 0$) and suppose that $u(x_0) > g(x_0)$ at $x_0 \in \partial\Omega$ (resp. $u(x_0) < g(x_0)$). Then the following two conditions hold:*

$$\begin{cases} \liminf_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2} Dd(y) \otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2} \right) + f(y) \right] \leq 0 \\ \liminf_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha} \right) + f(y) \right] \leq 0, \end{cases}$$

(resp.

$$\begin{cases} \limsup_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{-Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha^2} Dd(y) \otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2} \right) + f(y) \right] \geq 0 \\ \limsup_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{-Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha} \right) + f(y) \right] \geq 0, \end{cases}$$

where $o_\alpha(1) \rightarrow 0$ as $\alpha \searrow 0$.

Proof. This is Proposition 3.1 of [32] if $u \in \text{USC}(\overline{\Omega})$ is a subsolution of $(G^k)_* \leq 0$.

If u is a supersolution the proof goes along the same lines, but since the space of test functions is restricted we have to check that $\Psi(x) := -\frac{|x-x_0|^4}{\epsilon^4} + \psi\left(\frac{d(x)}{\alpha}\right)$ is semiconvex, at least for $\alpha \ll \epsilon$ and in a neighborhood of x_0 ; here $\alpha, \epsilon > 0$ and ψ is a negative smooth function such that $\psi(t) \rightarrow -\infty$ as $t \rightarrow \infty$, $\psi(0) = 0$, $\psi'(0) = -1$, $\psi''(0) = 1$. We take a minimum point $x = x_{\epsilon, \alpha}$ of $u - \Psi$ on $\overline{\Omega}$. By direct computations

$$D^2\Psi(x) = \psi' \left(\frac{d(x)}{\alpha} \right) \frac{D^2 d(x)}{\alpha} + \psi'' \left(\frac{d(x)}{\alpha} \right) \frac{Dd(x)}{\alpha} \otimes \frac{Dd(x)}{\alpha} + \frac{o_\epsilon(1)}{\epsilon^2} \quad \text{as } \epsilon \rightarrow 0.$$

α may be chosen small enough compared to ϵ so that $(o_\epsilon(1)/\epsilon^2)$ is also $(o_\alpha(1)/\alpha)$ as $\alpha \searrow 0$. Moreover $d(x)/\alpha \rightarrow 0$ and in particular $\psi'(x) = -1 + o_\alpha(1)$, $\psi''(x) = 1 + o_\alpha(1)$, hence

$$D^2\Psi(x) = \frac{1}{\alpha^2} [(1 + o_\alpha(1))Dd(x) \otimes Dd(x) + \alpha(-1 + o_\alpha(1))D^2 d(x) + o_\alpha(\alpha)].$$

We recall that by a change of coordinates (see [37], Lemma 14.17), Dd , $D^2 d$ have the form

$$Dd(x) = (0, \dots, 0, 1) \quad D^2 d(x) = \text{diag} \left[\frac{-k_1}{1 - k_1 d(x)}, \dots, \frac{-k_{n-1}}{1 - k_{n-1} d(x)}, 0 \right],$$

where k_1, \dots, k_{n-1} are the principal curvatures of $\partial\Omega$ at a point $y_0 \in \partial\Omega$ such that $d(x) = |x - y_0|$; therefore

$$D^2\Psi(x) = \frac{1}{\alpha^2} \text{diag}[\alpha(k_1 + o_\alpha(1)), \dots, \alpha(k_{n-1} + o_\alpha(1)), 1 + o_\alpha(1)] \geq 0$$

if α is small enough, as the curvatures k_1, \dots, k_{n-1} are positive.

By hypothesis we have $u(x_0) > g(x_0)$, g is continuous and $u(x) \rightarrow u(x_0)$, so $u(x) > g(x)$ if $x \in \partial\Omega$ if α is small; Ψ is an admissible test function, being convex in a neighborhood of x , hence

$$-F^k(D\Psi(x), D^2\Psi(x)) + f(x) \geq 0.$$

Substituting $D\Psi, D^2\Psi$ we take the limit as $\alpha \searrow 0$ and use the ellipticity of $-F^k$ to obtain the assertion (as in [32]). □

We now state a general existence result for the Dirichlet problem for (6.4). Our abstract assumptions will be

(h1) There exist a C -semiconvex subsolution \underline{u} of $(G^k)_* \leq 0$ in $\overline{\Omega}$ and a bounded C -supersolution \bar{u} of $(G^k)^* \geq 0$ in $\overline{\Omega}$ such that $\underline{u} \leq \bar{u}$ on $\overline{\Omega}$.

(h2) For all $x \in \partial\Omega$

$$i) \left\{ \begin{array}{l} \liminf_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha^2} Dd(y) \otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2} \right) + f(y) \right] > 0 \\ \text{or} \\ \liminf_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha} \right) + f(y) \right] > 0, \end{array} \right.$$

and

$$ii) \left\{ \begin{array}{l} \limsup_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{-Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha^2} Dd(y) \otimes Dd(y) + \frac{o_\alpha(1)}{\alpha^2} \right) + f(y) \right] < 0 \\ \text{or} \\ \limsup_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{-Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha} \right) + f(y) \right] < 0, \end{array} \right.$$

where $o_\alpha(1) \rightarrow 0$ as $\alpha \searrow 0$ and d denotes the distance function from $\partial\Omega$ (which is C^2 in a neighborhood of $\partial\Omega$).

We need (h1) to have a well-defined Perron family and (h2) to guarantee that there is no loss of boundary data for the solution of the Dirichlet problem, that we will find in a generalized (viscosity) sense (Definition 6.13). We point out that the boundary function g does not enter in (h2), but it is just a condition on the geometry of the boundary Ω and the datum f , coupled via the elliptic operator $-F^k$.

Theorem 7.4. *Suppose that (h0) and (h1) hold. Then, for every $g \in C(\partial\Omega)$ there exists a discontinuous solution u of $G^k = 0$ in $\overline{\Omega}$. If also (h2) holds, then u is the unique solution of the Dirichlet problem $G^k = 0$ in $\overline{\Omega}$.*

Proof. As in [15], we implement the Perron's method for viscosity solutions with the variant of Da Lio [32], considering the boundary conditions in the generalized viscosity sense. We consider the Perron family

$$\mathcal{W} := \{w : \underline{u} \leq w \leq \bar{u}, w \text{ is a } C\text{-semiconvex subsolution of } (G^k)_* \leq 0\},$$

which is non-empty by (h1), and define a candidate solution as

$$u(z) := \sup_{w \in \mathcal{W}} w(z), \quad z \in \overline{\Omega}.$$

The fact that u^* is C -semiconvex is standard, as it the supremum of a family of C -convex functions. Then $u^* \in C(\Omega)$, so $u = u^*$ in Ω , but no continuity is assured up to the boundary. Moreover, u^* is a subsolution of $(G^k)_* \leq 0$ (see [32] Lemma 6.1).

In order to prove that u_* is a C -supersolution we use the usual method of "bump" functions, arguing by contradiction: if u_* fails to be a C -supersolution at some point $\bar{z} \in \overline{\Omega}$,

then it is possible to construct a function $V_\epsilon \in \mathcal{W}$ such that $V_\epsilon > u$ at some point in $\bar{\Omega}$. The proof is similar to the one of Theorem 6.1 [32], but some care has to be taken since we have restricted the class of test functions for supersolutions. Let then $\phi \in C^2(\bar{\Omega})$ be such that $u_* - \phi$ has a global minimum at \bar{z} , $-C < \lambda_1(D^2\phi(\bar{z}))$, and

$$(G^k)^*(\bar{z}, u_*(\bar{z}), D\phi(\bar{z}), D^2\phi(\bar{z})) < 0,$$

and assume without loss of generality that $u_*(\bar{z}) = \phi(\bar{z})$ and $u_*(z) \geq \phi(z)$ for all $z \in \bar{\Omega}$. We consider for all $\epsilon > 0$

$$V_\epsilon(z) = \max\{u(z), \phi_\epsilon(z)\}, \quad \phi_\epsilon(z) := \phi(z) + \epsilon - |z - \bar{z}|^4.$$

In order to conclude, we just have to show that $V_\epsilon(z)$ is C -semiconvex. We have that $V_\epsilon = u$ except perhaps in $B(\bar{z}, \epsilon^{1/4})$ and u is C -semiconvex. Moreover, $D^2(|z - \bar{z}|^4) \rightarrow 0$ as $z \rightarrow \bar{z}$, and $-C < \lambda_1(D^2\phi(z))$ for all $z \in B(\bar{z}, 2\epsilon^{1/4})$ if ϵ is small enough by continuity of the second derivatives of ϕ , so ϕ_ϵ is C -semiconvex in $B(\bar{z}, 2\epsilon^{1/4})$, possibly reducing ϵ . It is then standard that the maximum (considered in $B(\bar{z}, 2\epsilon^{1/4})$) between C -semiconvex functions is C -semiconvex. Therefore u_* is a C -supersolution of $(G^k)^* \geq 0$, and the first assertion of the theorem is proved.

If also (h2) holds, then (6.9) and (6.10) reduce to $u^* \leq g$ and $u_* \geq g$ on the boundary $\partial\Omega$ by Proposition 7.3, so there is no loss of boundary data. By Theorem 7.2 we can conclude that $u^* \leq u_*$ on $\bar{\Omega}$, and so $u = u^* = u_*$ by the definition of envelopes. Then $u \in C(\bar{\Omega})$ and $u = g$ on $\partial\Omega$.

Through a standard argument, uniqueness for the Dirichlet problem follows using the comparison principle stated in Theorem 7.2. □

7.3 Sufficient conditions for Existence and Uniqueness

We shall look now for sufficient conditions for (h1), (h2). We first define the sets

$$\Gamma_k^R := \left\{ x \in \mathbb{R}^d : \sum_{i=1}^k x_i^2 < R^2 \right\},$$

$R > 0$ and $k = 1, \dots, d$. Note that

$$\Gamma_d^R \subset \Gamma_{d-1}^R \subset \dots \subset \Gamma_1^R,$$

and that only Γ_d^R is bounded.

The next Proposition shows that if Ω is contained in Γ_k^R , for some suitable k, R depending on f , then it is possible to write explicitly a subsolution and a supersolution to $(G^k)_* = 0$.

Proposition 7.5. *Suppose that $|f(x)| \leq M$ for some $M > 0$ and $\forall x \in \bar{\Omega}$, g is bounded on $\partial\Omega$ and Ω is a domain such that*

$$\Omega \subset\subset \Gamma_{\tilde{k}}^R, \tag{7.9}$$

where $R = M^{-1}$ and $\tilde{k} = \max\{k, d - k + 1\}$. Then (h1) holds.

Proof. As a subsolution, we take

$$\underline{u} := -\sqrt{R^2 - \sum_{i=1}^{d-k+1} x_i^2} - \underline{C},$$

which is well-defined since (7.9) holds; we set $\underline{C} = \inf_{\partial\Omega} g$. Then $\underline{u} \in C^2(\Omega)$, and a computation yields

$$-F^k(D\underline{u}(x), D^2\underline{u}(x)) + f(x) = -\frac{1}{R} + f(x) \leq 0$$

for all $x \in \Omega$, so \underline{u} is a (classical) subsolution of $-F^k(Du, D^2u) + f$ in Ω . Moreover it is convex, so it is C -semiconvex for all $C \geq 0$. Finally,

$$\underline{u}(x) \leq -\underline{C} = \inf_{\partial\Omega} g \leq g(x)$$

for all $x \in \partial\Omega$, so \underline{u} is a C -semiconvex subsolution of $(G^k)_* \leq 0$.

As a supersolution,

$$\bar{u} := \sqrt{R^2 - \sum_{i=1}^k x_i^2} + \bar{C},$$

$\bar{C} = \sup_{\partial\Omega} g$. Similarly,

$$-F^k(D\bar{u}(x), D^2\bar{u}(x)) + f(x) = \frac{1}{R} + f(x) \geq 0.$$

for all $x \in \Omega$, so \bar{u} is a classical supersolution of $-F^k(Du, D^2u) + f$ in Ω , and thus it is a bounded C -supersolution of $(G^k)^* \geq 0$ in $\bar{\Omega}$, satisfying also $\underline{u} \leq \bar{u}$. \square

We state now a sufficient condition on f and the principal curvatures of $\partial\Omega$ for (h2).

Proposition 7.6. *Suppose that Ω is a strictly convex C^2 domain, let*

$$\kappa_{\Omega,1}(x), \dots, \kappa_{\Omega,d-1}(x)$$

be the principal curvatures of $\partial\Omega$ at $x \in \partial\Omega$ and let $\kappa_{\Omega,0} = 0$. If

$$-\kappa_{\Omega,d-k}(x) < f(x) < \kappa_{\Omega,k-1}(x) \quad \forall x \in \partial\Omega, \quad (7.10)$$

then (h2) holds.

Proof. We recall that principal curvatures are rotationally invariant, and by a change of coordinates ([37], Lemma 14.17), Dd , D^2d have the form

$$\begin{aligned} Dd(y) &= (0, \dots, 0, 1), \\ D^2d(y) &= \text{diag} \left[\frac{-\kappa_{\Omega,1}(\bar{y})}{1 - \kappa_{\Omega,1}(\bar{y})d(y)}, \dots, \frac{-\kappa_{\Omega,d-1}(\bar{y})}{1 - \kappa_{\Omega,d-1}(\bar{y})d(y)}, 0 \right], \end{aligned}$$

where $\bar{y} \in \partial\Omega$ is such that $|\bar{y} - y| = d(y)$. By computation,

$$\begin{aligned} F^k \left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha} D^2d(y) + \frac{o_\alpha(1)}{\alpha} \right) \\ = \lambda_k \left(\frac{I - Dd(y) \otimes Dd(y) + o_\alpha(1)}{1 + o_\alpha(1)} (D^2d(y) + o_\alpha(1)) \right) \\ \rightarrow \lambda_k(\text{diag}[-\kappa_{\Omega,1}(x), \dots, -\kappa_{\Omega,d-1}(x), 0]) = -\kappa_{\Omega,d-k}(x) \end{aligned}$$

as $\alpha \rightarrow 0$ and $y \rightarrow x$ (so $\bar{y} \rightarrow x$). Hence

$$\lim_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{Dd(y) + o_\alpha(1)}{\alpha}, \frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha} \right) + f(y) \right] = \kappa_{\Omega, d-k}(x) + f(x) > 0,$$

hence (h2), *i*) is proved.

Similarly,

$$\begin{aligned} F^k \left(-\frac{Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha} \right) \\ = \lambda_k \left(\frac{I - Dd(y) \otimes Dd(y) + o_\alpha(1)}{1 + o_\alpha(1)} (-D^2 d(y) + o_\alpha(1)) \right) \\ \rightarrow \lambda_k(\text{diag}[0, \kappa_{\Omega, 1}(x), \dots, \kappa_{\Omega, d-1}(x)]) = \kappa_{\Omega, k-1}(x), \end{aligned}$$

so

$$\lim_{y \rightarrow x, \alpha \searrow 0} \left[-F^k \left(\frac{-Dd(y) + o_\alpha(1)}{\alpha}, -\frac{1}{\alpha} D^2 d(y) + \frac{o_\alpha(1)}{\alpha} \right) + f(y) \right] = -\kappa_{\Omega, k-1}(x) + f(x) < 0,$$

that leads to (h2), *ii*).

□

Remark 7.7. Condition (7.10) can be satisfied only if $d - k \neq k - 1$, i.e. $k \neq \frac{d+1}{2}$.

By the two propositions we have proved and Theorem 7.4 we are able to state the

Corollary 7.8. *Suppose that Ω is a C^2 strictly convex bounded domain and $\Omega, f \in C(\Omega)$ satisfy the hypotheses of Proposition 7.5 and Proposition 7.6. Then, for all $g \in C(\partial\Omega)$ there exists a unique $u \in C(\bar{\Omega})$ C -solution of (6.4) (for some $C \geq 0$) such that $u|_{\partial\Omega} = g$.*

We now show that, for the case $k = 1$ condition (7.10), which becomes $-\kappa_{\Omega, d-1} < f < 0$, seems to be almost optimal for the solvability of the Dirichlet problem, at least considering classical solutions. Indeed, it is solvable if f is negative at the boundary and does not admit any solution if f is positive or going to zero slowly enough.

Proposition 7.9. *Let Ω be a C^2 uniformly convex domain in \mathbb{R}^d and f a positive function on Ω satisfying*

$$f(x) \geq \epsilon d^b(x) \quad \forall x \in \mathcal{N}_y,$$

where \mathcal{N}_y is a neighborhood of some point $y \in \partial\Omega$, $\epsilon > 0$ and $b < 1/d$. Then, there exists a function $g \in C^\infty(\partial\Omega)$ for which the Dirichlet problem for $F^1(Du, D^2u) = f$ is not solvable for convex $u \in C(\bar{\Omega}) \cap C^2(\Omega)$.

Proof. Suppose that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ solves $\kappa_1[u] = F^1(Du, D^2u) = f$ (recall that $\kappa_j[u](x)$ denotes the j -th curvature of the graph of u at a point x) in \mathcal{N}_y and $u = g$ for some g that will be constructed later. Then u is convex, since $f \geq 0$, and therefore D^2u has to be non-negative on Ω . Moreover,

$$\begin{aligned} \frac{\det D^2u(x)}{(1 + |Du(x)|^2)^{(d+2)/2}} &= \kappa_1[u](x) \kappa_2[u](x) \cdots \kappa_d[u](x) \\ &\geq (\kappa_1[u](x))^d = (f(x))^d \geq \left(\frac{\epsilon}{2}\right)^d d^{bd}(x), \end{aligned} \quad (7.11)$$

for all $x \in \mathcal{N}(y)$, so

$$\det D^2u(x) =: \bar{F}(x, Du) \geq \left(\frac{\epsilon}{2}\right)^d d^{bd}(x) (1 + |Du(x)|^2)^{(n+2)/2}. \quad (7.12)$$

We now exploit a non-existence result for the Dirichlet problem for the prescribed Gaussian curvature equation. Theorem 1.3 of [75] states that if $bd < 1$ (that is true by hypothesis) then there exist $g \in C^\infty(\partial\Omega)$ for which the Dirichlet problem for (7.12) is not solvable for convex $u \in C(\bar{\Omega}) \cap C^2(\Omega)$. For that boundary data g is then impossible to solve the Dirichlet problem for $\kappa_1[u] = f$. \square

Remark 7.10. In the last proposition, we used the observation that if u is a subsolution of the prescribed first curvature equation, i.e. $-\kappa_1[u] = -F^1(Du, D^2) + f \leq 0$ and $f \geq 0$, then u is also a subsolution of a similar prescribed *gaussian* curvature equation

$$\det D^2u \geq f^d (1 + |Du|^2)^{(d+2)/2}.$$

If $k = 1$ it is easy to derive a necessary condition for existence of a solution of (6.4), at least if $f > 0$, using standard knowledge on the gaussian curvature equation. Indeed, suppose that u is a viscosity solution of $\kappa_1[u] = f$ on Ω , then u has to be (strictly) convex on Ω since $D^2u > 0$ (in viscosity sense). Therefore

$$\frac{\det D^2u(x)}{(1 + |Du(x)|^2)^{(d+2)/2}} = \kappa_1[u](x) \cdots \kappa_n[u](x) \geq (\kappa_1[u](x))^d = (f(x))^d$$

a.e. on Ω . By integrating and through the change of variables $z = Du(x)$ (Du is one-to-one) we get

$$\int_{\Omega} (f(x))^d \leq \int_{\mathbb{R}^d} \frac{1}{(1 + |z|^2)^{(d+2)/2}} dz. \quad (7.13)$$

This shows that, in space dimension one, the geometric condition (7.9) on Ω becomes (nearly) optimal: let $d = 1$ and $\Omega = (-a, a)$ for some $a > 0$ and $f \equiv M > 0$. Then (7.9) is

$$(-a, a) \subset\subset \Gamma_1^{\frac{1}{M}} = \left(-\frac{1}{M}, \frac{1}{M}\right),$$

i.e. $a < 1/M$, and the necessary condition (7.13) becomes

$$2Ma = \int_{-a}^a M \leq \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^{3/2}} dx = 2,$$

so $a \leq 1/M$.

We end the section with an example which shows that there are standard viscosity solutions and C -solutions of the Dirichlet problem for (6.4) that do not coincide, so existence (and uniqueness?) of solutions in the standard viscosity sense and for general u (not necessarily semiconvex), is an open problem.

Example 7.11. Suppose now that $d = 4$, $k = 2$, $\Omega = B_1(0)$, $f \equiv 0$ and

$$g(x) = (\text{sgn}(x_1) \sqrt{1 - (|x_1| - 1)^2})|_{\partial B_1(0)}.$$

Theorem 7.4 guarantees the existence of a convex solution of (6.4), indeed $\bar{u} = 1$ and $\underline{u} = -1$ satisfy (h1), (h2) holds because of Proposition 7.6 ($-\kappa_{\Omega,2} < 0 < \kappa_{\Omega,1}$ and $\kappa_{\Omega,1} = \kappa_{\Omega,2} = \kappa_{\Omega,3} = 1$), and (h0) is satisfied.

Consider now $u(x) = \operatorname{sgn}(x_1)\sqrt{1 - (|x_1| - 1)^2}$. It satisfies the Dirichlet boundary conditions and it is a standard viscosity solution of (6.4), but it is *not* convex (nor C -semiconvex for all $C \geq 0$); indeed, if $x_1 \neq 0$ then u is twice differentiable at x and $F^2(Du(x), D^2u(x)) = \lambda_2(\operatorname{diag}[c, 0, 0, 0]) = 0$, where $c = -1$ if $x_1 > 0$ and $c = 1$ if $x_1 < 0$. If $x_1 = 0$ there are no test functions ϕ such that $u - \phi$ has a maximum or a minimum at x , so the definition of viscosity sub/supersolution is satisfied trivially.

Appendix A

Some Elliptic Estimates

In this appendix we will state some a-priori estimates for Hamilton-Jacobi equations.

Theorem A.1. *Let Ω be a C^2 bounded domain, $f \in C^0(\overline{\Omega})$, $H \in C^1(\overline{\Omega} \times \mathbb{R}^d)$ such that*

$$\|u\|_{C^0(\overline{\Omega})} \leq M, \quad \|f\|_{C^0(\overline{\Omega})} \leq F, \quad H(x, p) \leq \mu(1 + |p|)^m$$

for some $M, F, \mu > 0$, $1 < m \leq 2$. Suppose that $u \in C^2(\overline{\Omega})$ is a solution of

$$\begin{cases} -\nu\Delta u(x) + H(x, Du(x)) = f(x) & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{A.1})$$

Then,

$$\|u\|_{C^{1,\delta}(\overline{\Omega})} \leq C,$$

for some $0 < \delta < 1$, $C = C(\Omega, \nu, M, F, \mu)$.

Proof. See [59], Lemma 6 and Lemma 7. □

Theorem A.2. *Let D, D' be bounded domains in \mathbb{R}^d with C^3 boundary such that $D' \subset\subset D$. Let H satisfy the properties stated in Proposition 2.1 and suppose that $u \in C^2(\Omega)$ solves on D*

$$-\nu\Delta u - b(x) \cdot Du + H(Du) + \lambda = f(x)$$

for some $\lambda \in \mathbb{R}$. Then there exists $K > 0$ depending on $d, \text{dist}(D', \partial D), m, C_1, \sup_D (f - \lambda)_+, \sup_D |Df|, \sup_D |b|, \sup_D |Db|$ such that

$$\sup_{D'} |Du| \leq K.$$

Proof. The estimate is obtained through Bernstein methods, see in particular [47], Theorem A.1. The dependence of K on $\sup_D |b|, \sup_D |Db|$ comes from the lower bounds on $\tilde{H}(x, p) := -b(x) \cdot p + H(p)$. □

Theorem A.3. *Let Ω be convex, $f \in L^q(\Omega)$ with $q > d$, $\gamma > 1$ and $u \in C^2(\overline{\Omega})$ a solution of*

$$\begin{cases} -\nu\Delta u + R|Du|^\gamma + \lambda = f(x) & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \quad (\text{A.2})$$

Let $r \geq 1$ be fixed. If $\lambda \geq \lambda_0$, then

$$\|Du\|_{L^r(\Omega)} \leq C, \quad (\text{A.3})$$

with $C = C(\nu, R, \gamma, r, n, \lambda_0)$

Proof. We derive the estimates by applying the integral Bernstein method, introduced in [60]. We suppose that $u \in C^3(\Omega)$, otherwise through a standard argument we regularize f and obtain the estimate by taking the limit of approximating problems.

Set $w = |Du|^2$, so

$$D_j w = 2 \sum_i D_i u D_{ij} u, \quad D_{jj} w = 2 \sum_i ((D_{ij} u)^2 + D_i u D_{ijj} u).$$

It holds that

$$|Dw| \leq 2d^2 |Du| |D^2 u|. \quad (\text{A.4})$$

By differentiating the equation (A.2) in Ω with respect to D_j and taking the sums for $j = 1, \dots, n$ one obtains

$$-\nu \Delta w + R\gamma |Du|^{\gamma-2} Du \cdot Dw + 2|D^2 u|^2 = 2Df \cdot Du.$$

We multiply it by w^p , with $p \geq 1$ that will be fixed later; through all the proof we will denote by C a constant that depends upon ν, R, γ, r, d and emphasize with C_p the dependance upon p . One has

$$-\nu \int_{\Omega} \Delta w w^p + 2 \int_{\Omega} |D^2 u|^2 w^p = -R\gamma \int_{\Omega} |Du|^{\gamma-2} Du \cdot Dw w^p + 2 \int_{\Omega} Df \cdot Du w^p$$

We are going to estimate separately each of the four terms appearing in the equation. Since $Dw \cdot n \leq 0$ (p. 236 [60]),

$$\begin{aligned} -\nu \int_{\Omega} \Delta w w^p &= \nu \int_{\Omega} Dw \cdot D(w^p) - \nu \int_{\partial\Omega} w^p Dw \cdot n \geq \frac{4p\nu}{(p+1)^2} \int_{\Omega} |D(w^{(p+1)/2})|^2 \\ &\geq C \frac{4p}{(p+1)^2} \left(\int_{\Omega} |w|^{\frac{(p+1)d}{d-2}} \right)^{\frac{d-2}{d}} - C \frac{4p}{(p+1)^2} \int_{\Omega} w^{p+1} \\ &\geq C \frac{4p}{(p+1)^2} \left(\int_{\Omega} |w|^{\frac{(p+1)d}{d-2}} \right)^{\frac{d-2}{d}} - C \frac{4p}{(p+1)^2} \left(\int_{\Omega} w^{p+\gamma} \right)^{\frac{p+1}{p+\gamma}} \end{aligned}$$

by the Sobolev embedding theorem and Holder inequality also. Then,

$$\begin{aligned} 2 \int_{\Omega} |D^2 u|^2 w^p &\geq \\ &\int_{\Omega} |D^2 u|^2 w^p + \int_{\Omega} \sum_i (D_{ii} u)^2 w^p \geq \int_{\Omega} |D^2 u|^2 w^p + \frac{1}{d} \int_{\Omega} (\Delta u)^2 w^p \\ &\geq \int_{\Omega} |D^2 u|^2 w^p + \frac{R^2}{2d\nu^2} \int_{\Omega} |Du|^{2\gamma} w^p + \frac{\lambda^2}{2d\nu^2} \int_{\Omega} w^p - \frac{2}{d\nu^2} \int_{\Omega} f^2 w^p, \end{aligned}$$

using the equation (A.2) and the fact that $(a-b)^2 \geq \frac{a^2}{2} - 2b^2$ for every $a, b \in \mathbb{R}$. For the third term we have that

$$\begin{aligned} -R\gamma \int_{\Omega} |Du|^{\gamma-2} Du \cdot Dw w^p &= -R\gamma \int_{\Omega} |Du|^{\gamma-2} Du \cdot D \left(\frac{1}{p+1} w^{p+1} \right) \\ &= \frac{R\gamma}{p+1} \int_{\Omega} \operatorname{div}(|Du|^{\gamma-2} Du) w^{p+1} - \frac{R\gamma}{p+1} \int_{\partial\Omega} |Du|^{\gamma-2} w^{p+1} Du \cdot n \\ &= \frac{R\gamma}{p+1} \int_{\Omega} D(|Du|^{\gamma-2}) \cdot Du w^{p+1} + \frac{R\gamma}{p+1} \int_{\Omega} |Du|^{\gamma-2} \Delta u w^{p+1} \\ &\leq \frac{2d^2 R\gamma(\gamma-1)}{p+1} \int_{\Omega} |Du|^{\gamma-2} |D^2 u| w^{p+1} \leq \frac{C}{p+1} \int_{\Omega} w^{p+\gamma} + \frac{C}{p+1} \int_{\Omega} |D^2 u|^2 w^p, \end{aligned}$$

by the estimate (A.4). Finally,

$$\begin{aligned}
2 \int_{\Omega} Df \cdot Du w^p &= -2 \int_{\Omega} f \operatorname{div}(Du w^p) + 2 \int_{\partial\Omega} f w^p Du \cdot n = \\
&- 2 \int_{\Omega} f \Delta u w^p - 2 \int_{\Omega} f Du \cdot D(w^p) \leq 2d \int_{\Omega} |f| |D^2 u| w^p + 4d^2(p-1) \int_{\Omega} |f| |D^2 u| w^p \\
&\leq \frac{1}{2} \int_{\Omega} |D^2 u|^2 w^p + C_p \int_{\Omega} |f|^2 w^p,
\end{aligned}$$

and by putting all the estimates together

$$\begin{aligned}
&C \frac{4p}{(p+1)^2} \left(\int_{\Omega} |w|^{\frac{(p+1)d}{d-2}} \right)^{\frac{d-2}{d}} + \frac{1}{2} \int_{\Omega} |D^2 u|^2 w^p + \\
&\quad + \frac{R^2}{2d\nu^2} \int_{\Omega} w^{p+\gamma} + \frac{\lambda^2}{2d\nu^2} \int_{\Omega} w^p - \frac{2}{d\nu^2} \int_{\Omega} f^2 w^p \\
&\leq \frac{C}{p+1} \int_{\Omega} w^{p+\gamma} + \frac{C}{p+1} \int_{\Omega} |D^2 u|^2 w^p + C_p \int_{\Omega} f^2 w^p + C \frac{4p}{(p+1)^2} \left(\int_{\Omega} w^{p+\gamma} \right)^{\frac{p+1}{p+\gamma}}.
\end{aligned}$$

One may choose p sufficiently large in order to have

$$\begin{aligned}
&C \frac{4p}{(p+1)^2} \left(\int_{\Omega} |w|^{\frac{(p+1)d}{d-2}} \right)^{\frac{d-2}{d}} + \frac{R^2}{4d\nu^2} \int_{\Omega} w^{p+\gamma} \\
&\leq C_p \int_{\Omega} f^2 w^p + C \frac{4p}{(p+1)^2} \left(\int_{\Omega} w^{p+\gamma} \right)^{\frac{p+1}{p+\gamma}} \\
&\leq C_p \left(\int_{\Omega} |w|^{\frac{(p+1)d}{d-2}} \right)^{\frac{(d-2)p}{(p+1)d}} \left(\int_{\Omega} |f|^{2\beta} \right)^{\frac{1}{\beta}} + C \frac{4p}{(p+1)^2} \left(\int_{\Omega} w^{p+\gamma} \right)^{\frac{p+1}{p+\gamma}} \quad (\text{A.5})
\end{aligned}$$

using Holder inequality, with $\beta = \alpha'$ and $\alpha = (p+1)d/(d-2)p$. Since $2\beta \rightarrow d$, choosing p large enough we conclude that (A.3) holds. \square

Appendix B

Some Matrix Analysis

In this Appendix we will collect some facts on eigenvalues of matrices.

Theorem B.1 (Courant-Fischer). *Suppose that $A \in \text{Sym}$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d$ and k be a given integer with $1 \leq k \leq d$. Then*

$$\lambda_k = \min_{S \in \mathcal{S}_k} \max_{|\xi|=1, \xi \in S} \xi^T A \xi$$

and

$$\lambda_k = \max_{S \in \mathcal{S}_{d-k+1}} \min_{|\xi|=1, \xi \in S} \xi^T A \xi,$$

where \mathcal{S}_j denotes the family of j -dimensional subspaces of \mathbb{R}^d .

Proof. See Theorem 4.2.11 [43]. □

Theorem B.2 (Weyl). *Let $A, B \in \mathcal{S}$. For each $k = 1, \dots, d$ we have*

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_d(B).$$

Proof. See Theorem 4.3.1 [43]. □

Theorem B.3 (Ostrowski). *Let $A, S \in \mathcal{S}$ and S be non-singular. For each $k = 1, \dots, d$, there exists a positive real number θ_k such that $\lambda_1(SS^T) \leq \theta_k \leq \lambda_d(SS^T)$ and*

$$\lambda_k(SAS^T) = \theta_k \lambda_k(A).$$

Proof. See Theorem 4.5.9 [43]. □

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