Correzione di Esercizi 3 di Calcolo delle Probabilità e Statistica. Mercoledì 20 aprile 2016

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1. Answer of Exercise 1

1.1. **(a) la funzione di ripartizione di X.** Si ricorda che la distribuzione gamma ha la densità:

$$g_{\alpha,\lambda}(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \tag{1.1}$$

quando $\alpha = 2$ e $\lambda = 1$, la densità di X è

$$p(x) = \begin{cases} xe^{-x} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \tag{1.2}$$

Sia $F(x)$ la funzione di ripartizione di X. Quando $x < 0$, $F(x) \equiv 0$. Quando $x \geqslant 0$,

$$\begin{aligned} F(X) &= \int_0^x p(t)dt = \int_0^x te^{-t}dt = -\int_0^x td(e^{-t}) \\ &= -\left[te^{-t}\Big|_0^x - \int_0^x e^{-t}dt\right] = -\left[xe^{-x} + e^{-t}\Big|_0^x\right] \\ &= -xe^{-x} - e^{-x} + 1 = 1 - e^{-x}(x+1) \end{aligned}$$

quindi la funzione di ripartizione di X è

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x}(x+1) & x \geqslant 0 \end{cases} \tag{1.3}$$

1.2. **(b) la probabilità** $\mathrm{P}(\frac{1}{2} \leqslant X \leqslant \frac{3}{2})$**.**

$$\begin{aligned} \mathrm{P}\Big(\frac{1}{2} \leqslant X \leqslant \frac{3}{2}\Big) &= \mathrm{P}\Big(X \leqslant \frac{3}{2}\Big) - \mathrm{P}\Big(X \leqslant \frac{1}{2}\Big) = F\Big(\frac{3}{2}\Big) - F\Big(\frac{1}{2}\Big) \\ &= \Big[1 - e^{-\frac{3}{2}}(\frac{3}{2}+1)\Big] - \Big[1 - e^{-\frac{1}{2}}(\frac{1}{2}+1)\Big] \\ &= \frac{3}{2}e^{-\frac{1}{2}} - \frac{5}{2}e^{-\frac{3}{2}} \simeq 0.35 \end{aligned}$$

1.3. **(c)** $\mathrm{P}(X)$ **e** $\sigma^2(X)$**.** Uso la formula di distribuzione gamma,

$$\mathrm{P}(X) = \frac{\alpha}{\lambda} = \frac{2}{1} = 2$$
$$\sigma^2(X) = \frac{\alpha}{\lambda^2} = \frac{2}{1^2} = 2$$

Date: April 26, 2016, Version 2.0.

2. Answer of Exercise 2

2.1. **(a) la constante K.** Si vede facilmente che l'area di $\{(x,y)| -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x+1\}$ è 2, quindi $K = \frac{1}{2}$. Si nota che, fissato x, y va da 0 a $x+1$. Invece

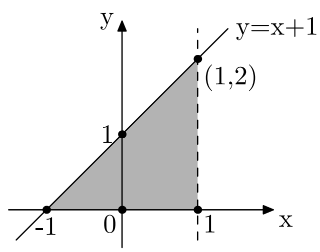

Figure 1. L'area di $p(x,y)$

quando fissato y, x va da $y-1$ a 1, per una desità positiva.

2.2. **(b) le densità di probabilità marginali $p_1(x), p_2(y)$ di X e Y.** Quando $-1 \leqslant x \leqslant 1$,

$$p_1(x) = \int_{-\infty}^{\infty} p(x,y)dy = \int_0^{x+1} \frac{1}{2} dy = \frac{1}{2}(x+1)$$

quindi

$$p_1(x) = \begin{cases} \frac{1}{2}(x+1) & -1 \leqslant x \leqslant 1 \\ 0 & \text{altrimenti} \end{cases} \tag{2.1}$$

Quando $0 \leqslant y \leqslant 2$,

$$p_2(y) = \int_{-\infty}^{\infty} p(x,y)dx = \int_{y-1}^{1} \frac{1}{2} dx = \frac{1}{2}(2-y)$$

quindi

$$p_2(y) = \begin{cases} \frac{1}{2}(2-y) & 0 \leqslant y \leqslant 2 \\ 0 & \text{altrimenti} \end{cases} \tag{2.2}$$

2.3. **(c).**

$$\begin{aligned}
\mathrm{P}(X) &= \int_{-\infty}^{\infty} x p_1(x) dx = \int_{-1}^{1} x \cdot \frac{1}{2}(x+1) dx = \frac{1}{3} \\
\mathrm{P}(Y) &= \int_{-\infty}^{\infty} y p_2(y) dy = \int_{0}^{2} y \cdot \frac{1}{2}(2-y) dy = \frac{2}{3} \\
\mathrm{P}(X^2) &= \int_{-\infty}^{\infty} x^2 p_1(x) dx = \int_{-1}^{1} x^2 \cdot \frac{1}{2}(x+1) dx = \frac{1}{3} \\
\mathrm{P}(Y^2) &= \int_{-\infty}^{\infty} y^2 p_2(y) dy = \int_{0}^{2} y^2 \cdot \frac{1}{2}(2-y) dy = \frac{2}{3} \\
\sigma^2(X) &= \mathrm{P}(X^2) - \mathrm{P}(X)^2 = \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{2}{9} \\
\sigma^2(Y) &= \mathrm{P}(Y^2) - \mathrm{P}(Y)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}
\end{aligned}$$

$$
\begin{aligned}
P(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot p(x,y)dxdy = \frac{1}{2}\int_{-1}^{1} xdx\Big(\int_{0}^{x+1} ydy\Big) \\
&= \frac{1}{2}\int_{-1}^{1} xdx\Big(\frac{1}{2}(x+1)^2\Big) = \frac{1}{4}\int_{-1}^{1} x(x+1)^2dx = \frac{1}{3} \\
\mathrm{Cov}(X,Y) &= \mathrm{P}(XY) - \mathrm{P}(X)\mathrm{P}(Y) = \frac{1}{3} - \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}
\end{aligned}
$$

3. Answer of Exercise 3

3.1. P(X), P(Y).

$$p(x,y) = K\exp(-\frac{1}{2}(4x^2+2y^2+xy+x-y)) = K\exp(-\frac{1}{2}(4x^2+2y^2+xy) - \frac{1}{2}x + \frac{1}{2}y)$$

quindi abbiamo

$$A = \begin{bmatrix} 4 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix} \quad b = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = 4\cdot 2 - \frac{1}{2}\cdot\frac{1}{2} = 8 - \frac{1}{4} = \frac{31}{4}$$

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{4}{31}\begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 4 \end{bmatrix} = \begin{bmatrix} \frac{8}{31} & -\frac{2}{31} \\ -\frac{2}{31} & \frac{16}{31} \end{bmatrix}$$

$$A^{-1}b = \begin{bmatrix} \frac{8}{31} & -\frac{2}{31} \\ -\frac{2}{31} & \frac{16}{31} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{31} \\ \frac{9}{31} \end{bmatrix}$$

quindi $P(X) = -\frac{5}{31}$, $P(Y) = \frac{9}{31}$.

4. Answer of Exercise 4

4.1. La matrice di covarianza.

$$C = \begin{bmatrix} \sigma^2(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \sigma^2(Y) \end{bmatrix}$$

Dato che $C = A^{-1}$, allora abbiamo

$$C = \begin{bmatrix} \sigma^2(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \sigma^2(Y) \end{bmatrix} = \begin{bmatrix} \frac{8}{31} & -\frac{2}{31} \\ -\frac{2}{31} & \frac{16}{31} \end{bmatrix}$$

in un'altra parola,

$$
\begin{aligned}
\sigma^2(X) &= \frac{8}{31} \\
\sigma^2(Y) &= \frac{16}{31} \\
\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X) &= -\frac{2}{31}
\end{aligned}
$$

5. Answer of Exercise 5

5.1. **(a)** $\mathrm{P}(Y)$ **e** $\sigma^2(Y)$**.** Si vede facilmente che la lunghezza dell'intervallo $[-1,2]$ è 3, quindi la densità di X è

$$p(x) = \begin{cases} \frac{1}{3} & -1 \leqslant x \leqslant 2 \\ 0 & \text{altrimenti} \end{cases} \tag{5.1}$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{3} & -1 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases} \tag{5.2}$$

qundi

$$\mathrm{P}(Y) = \mathrm{P}(X^3) = \int_{-\infty}^{\infty} x^3 p(x) dx = \frac{1}{3}\int_{-1}^{2} x^3 dx = \frac{5}{4}$$

$$\mathrm{P}(Y^2) = \mathrm{P}(X^6) = \int_{-\infty}^{\infty} x^6 p(x) dx = \frac{1}{3}\int_{-1}^{2} x^6 dx = \frac{43}{7}$$

$$\sigma^2(Y) = \mathrm{P}(Y^2) - \mathrm{P}(Y)^2 = \frac{43}{7} - \left(\frac{5}{4}\right)^2 = \frac{513}{112} = \frac{3^3 \cdot 19}{2^4 \cdot 7}$$

5.2. $\mathrm{Cov}(X,Y)$**.**

$$\mathrm{P}(X) = \int_{-\infty}^{\infty} x p(x) dx = \frac{1}{3}\int_{-1}^{2} x dx = \frac{1}{2}$$

$$\mathrm{P}(XY) = \mathrm{P}(X^4) = \int_{-\infty}^{\infty} x^4 p(x) dx = \frac{1}{3}\int_{-1}^{2} x^4 dx = \frac{11}{5}$$

$$\mathrm{Cov}(X,Y) = \mathrm{P}(XY) - \mathrm{P}(X)\mathrm{P}(Y) = \frac{11}{5} - \frac{1}{2}\cdot\frac{5}{4} = \frac{63}{40}$$

5.3. **Il coefficiente di correlazione** $\mathrm{r}(X,Y)$**.**

$$\mathrm{P}(X^2) = \int_{-\infty}^{\infty} x^2 p(x) = \frac{1}{3}\int_{-1}^{2} x^2 = 1$$

$$\sigma^2(X) = \mathrm{P}(X^2) - \mathrm{P}(X)^2 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

$$\mathrm{r}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma(X)\sigma(Y)} = \frac{\frac{63}{40}}{\sqrt{\frac{3}{4}}\sqrt{\frac{513}{112}}} = \frac{64}{45}\sqrt{\frac{7}{19}} \simeq 0.86$$

6. Answer of Exercise 6

(La versione precedente è sbagliata.)

Il numero aleatorio X soddisfa la distribuzione *ipergeometrica* con parametri $N = 20$, $n = 6$, e $b = 8$.

6.1. **(a) La probabilità** $\mathrm{P}(X = 4)$**.**

$$\mathrm{P}(X=4) = \frac{\binom{b}{k}\binom{N-b}{n-k}}{\binom{N}{n}} = \frac{\binom{6}{4}\binom{12}{2}}{\binom{20}{6}} = \frac{33}{1292} = \frac{3\cdot 11}{2^2\cdot 17\cdot 19}$$

6.2. **(b) La previsione** $\mathrm{P}(X)$**.**

$$\mathrm{P}(X) = n\frac{b}{N} = 6 \cdot \frac{8}{20} = \frac{12}{5}$$

6.3. **(c) La varianza** $\sigma^2(X)$**.** (È un po' difficile memorizzare la formula di varianza, anche difficile farla a mano...)

$$\sigma^2(X) = n\Big(\frac{N-n}{N-1}\Big)\frac{b}{N}\Big(1-\frac{b}{N}\Big) = 6 \cdot \frac{14}{19} \cdot \frac{6}{20} \cdot \Big(1-\frac{8}{20}\Big) = \frac{378}{475} = \frac{2 \cdot 3^3 \cdot 7}{5^2 \cdot 19}$$

References

[1] F. Biagini and M. Campanino, *Elementi di Probabilità e Statistica*, Springer-Verlag Italia, Milano, 2006.

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```
Integrate[t Exp[-t], {t, 0, x}]
```

$1 - e^{-x} (1 + x)$

```
3 / 2 Exp[-1 / 2] - 5 / 2 Exp[-3 / 2]
```

$-\frac{5}{2 e^{3/2}} + \frac{3}{2 \sqrt{e}}$

```
N[%]
```

0.351971

```
Integrate[t^2 Exp[-t], {t, 0, Infinity}]
```

2

```
Integrate[x (x + 1) / 2, {x, -1, 1}]
```

$\frac{1}{3}$

```
Integrate[y (2 - y) / 2, {y, 0, 2}]
```

$\frac{2}{3}$

```
Integrate[x^2 (x + 1) / 2, {x, -1, 1}]
```

$\frac{1}{3}$

```
Integrate[y^2 (2 - y) / 2, {y, 0, 2}]
```

$\frac{2}{3}$

```
Integrate[x (x + 1)^2, {x, -1, 1}] / 4
```

$\frac{1}{3}$

```
A = {{4, 1 / 2}, {1 / 2, 2}}
```

$\{\{4, \frac{1}{2}\}, \{\frac{1}{2}, 2\}\}$

```
Inverse[A]
```

$\{\{\frac{8}{31}, -\frac{2}{31}\}, \{-\frac{2}{31}, \frac{16}{31}\}\}$

```
b = {{-1 / 2}, {1 / 2}}
```

$\{\{-\frac{1}{2}\}, \{\frac{1}{2}\}\}$

```
Inverse[A] .b
```

$\{\{-\frac{5}{31}\}, \{\frac{9}{31}\}\}$

```
Integrate[x^3, {x, -1, 2}] / 3
```

$\frac{5}{4}$

```
Integrate[x^6, {x, -1, 2}] / 3
```

$\frac{43}{7}$

```
43 / 7 - (5 / 4) ^2
```

$\frac{513}{112}$

```
Integrate[x^2, {x, -1, 2}] / 3
```

1

```
Integrate[x, {x, -1, 2}] / 3
```

$\frac{1}{2}$

```
Integrate[x^4, {x, -1, 2}] / 3
```

$\frac{11}{5}$

```
11 / 5 - 1 / 2 * 5 / 4
```

$\frac{63}{40}$

```
64 / 40 / ( Sqrt[3 / 4] Sqrt[513 / 112] )
```

$$\frac{64\sqrt{\frac{7}{19}}}{45}$$

```
N[%]
```

0.863256

```
FactorInteger[513]
```

{{3, 3}, {19, 1}}

```
3^3 × 19
```

513

```
FactorInteger[112]
```

{{2, 4}, {7, 1}}

```
Binomial[6, 4]
```

15

```
15 (2 / 5) ^4 (3 / 5) ^2
```

$\frac{432}{3125}$

```
FactorInteger[432]
{{2, 4}, {3, 3}}

FactorInteger[3125]
{{5, 5}}

In[1]:= Binomial[6, 4] Binomial[12, 2] / Binomial[20, 6]
Out[1]= 33/1292

In[2]:= FactorInteger[1292]
Out[2]= {{2, 2}, {17, 1}, {19, 1}}

In[3]:= 6 * 8 / 20
Out[3]= 12/5

In[4]:= 6 * 14 / 19 * 6 / 20 * (1 - 8 / 20)
Out[4]= 378/475

In[5]:= FactorInteger[378]
Out[5]= {{2, 1}, {3, 3}, {7, 1}}

In[6]:= FactorInteger[475]
Out[6]= {{5, 2}, {19, 1}}
```