MacroEconomia Avanzata
Esercitazione 4
Correzione.

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1 Question 1.

1. As in the Ramsey model, we assume infinite-lived households maximising utility, given by

$$U = \int_{t=0}^{\infty} e^{-(\rho-n)t} \left[\frac{c^{(1-\theta)} - 1}{1-\theta} \right] \mathrm{d}t, \tag{1}$$

subject to the constraint

$$\dot{a} = (r-n)a + w - c, \tag{2}$$

where a are assets per capita, r is the interest rate, w is the wage per capita. The trasversality condition is assumed to hold.

Firms have production function defined by

$$y = f(k) = Ak \qquad \text{with } A > 0,\ f'' = 0. \tag{3}$$

Inada conditions are violated. In particular:

$$\lim_{k \to \infty, 0} f'(k) = A. \tag{4}$$

Capital's definition is broader, as it encompasses human capital, knowledge, public infrastructure, and so on, while labour's definition is stricter, not being augmented by human capital.

2. Consumer's maximum problem is defines as

$$\max_{c_t, a} H = u(c_t) + \lambda \Big[w_t + (r-n)a - c_t \Big]. \tag{5}$$

FOCs are

$$\frac{\partial H}{\partial c_t} = u'(c_t) - \lambda = 0;$$

$$c^{-\theta} = \lambda; \tag{6}$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = (\rho - n)\lambda - \dot{\lambda}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - r. \tag{7}$$

Taking the derivative of (6) w.r.t. t and substituting (7) into (8) yields

$$\begin{aligned}\frac{\partial u'(c_t)}{t} = \frac{u''(c_t)\dot{c}}{\lambda} &= \frac{\dot{\lambda}}{\lambda};\\ -\theta\frac{c^{-\theta-1}}{c^{-\theta}}\dot{c} &= \frac{\dot{\lambda}}{\lambda};\\ \frac{\dot{c}}{c} &= \frac{r-\rho}{\theta}.\end{aligned} \tag{8}$$

Firms' maximum problem is given by

$$\max_{k_t} \ \Pi = Ak_t - w_t L_t - (r+\delta)k_t. \tag{9}$$

FOC is

$$\begin{aligned}\frac{\partial \Pi}{\partial k_t} &= A - (r+\delta) = 0\\ r &= A - \delta.\end{aligned} \tag{10}$$

In a closed economy, $a = k$ and $w = 0$. Substituting (10) into the households' constraint (2), we obtain

$$\begin{aligned}\dot{a} &= (r-n)a + w - c,\\ \dot{k} &= (A-\delta-n)k - c.\end{aligned} \tag{11}$$

(8),(11) are the equations that describe the dynamics of the economy. It can be shown that

$$\frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \frac{\dot{y}}{y} = \frac{A-\delta-\rho}{\theta}. \tag{12}$$

In this model there is no transitional dynamics: growth rates of consumption, capital and output depend upon parameters (and not variables), such as saving rate and capital productivity. Conversely, the Ramsey's framework assumed that variations in parameters would change volumes, rather than growth rates as in the AK model. This is due to the fact that this model does not assume diminishing returns, which seems to be a rather appropriate assumption for describing short-run features of the economy.

2 Question 2.

1. Romer's model uses Arrow's setup to eliminate the tendency for diminishing returns to capital accumulation by assuming that knowledge creation is a side product of investment. A firm that increases its physical capital learns simultaneously how to produce more efficiently (learning-by-investing).

   Production function for the i-th firm is given by $Y_i = F(K_i, A_i L_i)$, which is characterised by diminishing marginal products of each input, constant returns to scale, Inada conditions. However, A_i's growth rate is not exogenous. $L_t = L$, hence labour is constant over time.

   Spillovers are modelled as follows: once discovered, each piece of knowledge becomes accessible to all firms at zero cost. Hence, $\dot{A}_i = \dot{K}$. We can then rewrite firm's production function as

$$Y_i = F(K_i, KL_i). \tag{13}$$

This means that, holding fixed K and L_i, each firms faces diminishing returns to K_i; however, if producer expands K_i, K rises accordingly and provides a spillover benefit that raises the productivity of all firms. Furthermore, the production function is homogeneous of degree one in K_i and K: that is, there are constant returns to capital at the social level.

2. Households' maximum problem is the same as in Question 2. FIrms' is defined as

$$\max_{k_i, L_i} \Pi = L_i \Big[F(k_i, K) - (r+\delta)k_i - w \Big]. \tag{14}$$

FOCs are:

$$\begin{aligned} \frac{\partial \Pi}{\partial k_i} =& L_i \frac{\partial F(k_i, K)}{\partial k_i} = (r+\delta)L_i \\ r =& \frac{\partial F(k_i, K)}{\partial k_i} - \delta; \qquad (15) \\ \frac{\partial \Pi}{\partial L_i} =& F(k_i, K) - (r+\delta)k_i = w. \qquad (16) \end{aligned}$$

Since firms are symmetric, $k_i = k$ and $K = kL$. Since $F(k_i, K)$ is homogeneous(1) in k_i, K, we can define the average and marginal product of capital as

$$\text{APK} = \frac{F(k_i, K)}{k_i} = f\left(\frac{K}{k_i}\right) = f(L); \tag{17}$$

$$\text{MPK} = \frac{\partial F(k_i, K)}{\partial k_i} = f(L) - Lf'(L). \tag{18}$$

Substituting (15) and (16) into the dynamic equations of consumption and capital respectively yields

$$\frac{\dot{c}}{c} = \frac{f(L) - Lf'(L) - \delta - \rho}{\theta}; \tag{19}$$

$$\dot{k} = f(L)k - c - \delta k. \tag{20}$$

This model also does not have transitional dynamics.

3 Question 3.

1. Accumulation of human capital is a direct consequence of physical capital accumulation, hence this economy is characterised by the learning-by-investing effect.

   Since human capital is an externality, generated by the creation of capital, we expect the decentralised equilibrium to be non-optimal.

2. Exercise 3.2. is substituted by the following. Consider the social planner's maximum problem. Show that the decentralised equilibrium in exercise 2.2. is sub-optimal.

   Social planner maximises social welfare (consumers' utility and firms' profit) subject to the constraint given in (20).

$$\dot{k} = f(L)k - c - \delta k. \tag{21}$$

This is different from the decentralised constraint: here the social planner internalises the technological externality of investment by taking into account the fact that each firm's investment adds on to the total stock of the economy, increasing other firms' productivity.

The maximum problem is then

$$\max_{c_t,a} \; H = u(c_t) + \lambda \Big[f(L)k - c - \delta k - \dot{k} \Big]. \tag{22}$$

FOCs are

$$\frac{\partial H}{\partial c_t} = c^{-\theta} = \lambda; \tag{23}$$

$$\frac{\partial H}{\partial k} = \lambda(f(L) - \delta) = (\rho)\lambda - \dot{\lambda}. \tag{24}$$

$$\tag{25}$$

Finally, we get

$$\frac{\dot{c}}{c} = \frac{f(L) - \delta - \rho}{\theta}, \tag{26}$$

that is greater than (19).