MacroEconomia Avanzata
Esercitazione 2
Correzione.

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1 Question 1.

1. The maximum problem is:

$$\max_{C_1,C_2} L = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_2^{1-\theta}}{1-\theta} - \lambda\big[P_1C_1 + P_2C_2 - W\big]. \tag{1}$$

The First Order Conditions are:

$$\frac{\partial L}{\partial C_1} = C_1^{-\theta} - \lambda P_1 = 0; \tag{2}$$

$$\frac{\partial L}{\partial C_2} = \frac{1}{1+\rho}C_2^{-\theta} - \lambda P_2 = 0; \tag{3}$$

$$\frac{\partial L}{\partial \lambda} = P_1C_1 + P_2C_2 - W = 0$$

$$= C_2 = \frac{W}{P_2} - \frac{P_1}{P_2}C_1. \tag{4}$$

Dividing (2) by (3) yields

$$\left(\frac{C_1}{C_2}\right)^{-\theta} = \frac{1}{(1+\rho)}\frac{P_1}{P_2}; \tag{5}$$

and, solving for C_1,

$$C_1 = C_2\left[(1+\rho)\frac{P_2}{P_1}\right]^{\frac{1}{\theta}}. \tag{6}$$

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Substituting (6) in (4) yields

$$\begin{aligned} C_2 &= \frac{W}{P_2} - \frac{P_1}{P_2} C_2 \left[(1+\rho) \frac{P_2}{P_1} \right]^{\frac{1}{\theta}} \\ &= \frac{W}{P_2} - C_2 (1+\rho)^{\frac{1}{\theta}} \left(\frac{P_2}{P_1} \right)^{\frac{1-\theta}{\theta}} \\ &= \frac{W/P_2}{1 + (1+\rho)^{\frac{1}{\theta}} (P_2/P_1)^{\frac{1-\theta}{\theta}}}. \end{aligned} \tag{7}$$

Substituting (7) in (6), gives

$$C_1 = \frac{W/P_2}{1 + (1+\rho)^{\frac{1}{\theta}} (P_2/P_1)^{\frac{1-\theta}{\theta}}} \left[(1+\rho) \frac{P_2}{P_1} \right]^{\frac{1}{\theta}}. \tag{8}$$

2. Dividing (8) by (7) yields:

$$\frac{C_1}{C_2} = \left[(1+\rho) \frac{P_2}{P_1} \right]^{\frac{1}{\theta}}. \tag{9}$$

Taking logs gives:

$$\ln \frac{C_1}{C_2} = \left(\frac{1}{\theta} \right) \left[\ln \left(\frac{P_2}{P_1} \right) + \ln(1+\rho) \right]. \tag{10}$$

Finally, the derivative w.r.t. (P_2/P_1) defines the (positive) elasticity:

$$-\frac{\partial \ln(C_1/C_2)}{\partial P_1/P_2} = \frac{\partial \ln(C_1/C_2)}{\partial P_2/P_1} = \frac{1}{\theta}. \tag{11}$$

3. Consider $\Delta t = (t_0 + \varepsilon) - (t_0 - \varepsilon)$. At $(t_0 - \varepsilon)$, consumption falls by Δc, leading to an increase in savings of Δs that will increase c at $(t_0 + \varepsilon)$ by the instantaneous rate of return $r(t)$. If the individual is maximising its utility, the impact of this change must be zero. Let us define c_b the consumption before t_0, and c_a the consumption after t_0. Then, utilities in the two moments are defined as:

$$U_b = u'(c_b) \Delta c; \tag{12}$$

$$U_a = (1/2) u'(c_a) \Delta c \; e^{[r(t) - g - n] \Delta t}. \tag{13}$$

Imposing

$$\begin{aligned} U_b &= U_a \\ u'(c_b) \Delta c &= (1/2) u'(c_a) \Delta c \; e^{[r(t) - g - n] \Delta t}, \end{aligned} \tag{14}$$

and letting $\Delta t \to 0$, we obtain

$$u'(c_b) = (1/2) u'(c_a). \tag{15}$$

Hence, will jump down at t_0. Intuitively, households will consume more before t_0, as they have an incentive not to save so as to avoid the wealth confiscation.

4. In this case, each household's behaviour will not affect the amount of wealth that is confiscated: a predetermined amount will be confiscated at t_0, and households' optimisation will take this into account. Hence, individuals will keep smoothing consumption over time, and there will be no discontinuous jump in consumption at t_0.

2 Question 2.

1. The assumptions of the Ramsey Model are:
   (a) The production function $Y = F(K, AL)$ has the same properties as in the Solow Model;
   (b) Markets are competitive, thus the (symmetric) firms will have profits $\pi = 0$;
   (c) Households' utility is given by
   $$U = B \int_{t=0}^{\infty} e^{\beta t} u(c(t))\mathrm{d}t, \tag{16}$$
   where $B \equiv A(0)^{1-\theta} \frac{L(0)}{H}$, $\beta \equiv \rho - n - (1-\theta)g$.
   (d) Trasversality condition:
   $$\lim_{s\to\infty} e^{-R(s)} \frac{K(s)}{H} \geq 0. \tag{17}$$
   (e) Utility cannot grow boundlessly:
   $$\rho - n - (1-\theta)g > 0. \tag{18}$$

2. Since in this case $H = 1$, $L(0) = 1$, $A(t) = 1$, the optimisation problem is given by:
$$\max_{c(t)} L = \int_{t=0}^{\infty} e^{(n-\rho)t} \ln(c(t))\mathrm{d}t - \lambda \left[\int_{t=0}^{\infty} e^{-R(t)+nt} c(t)\mathrm{d}t - W(t) \right]. \tag{19}$$
The First Order Condition for $c(t)$ is:
$$\frac{\partial L}{\partial c(t)} = c(t)^{-1} e^{(n-\rho)t} - \lambda e^{nt-R(t)} = 0. \tag{20}$$
Taking logs yields
$$\begin{aligned} &-\ln c(t) - \rho t - \ln \lambda - R(t) = 0 \\ &-\ln c(t) - \rho t - \ln \lambda - \int_{\tau=0}^{t} r(\tau)\mathrm{d}t = 0, \end{aligned} \tag{21}$$
and deriving w.r.t. t and rearranging terms yields
$$\frac{\dot{c(t)}}{c(t)} = r(t) - \rho. \tag{22}$$

Since a Cobb-Douglas production function implies $r(t) = f'(k) = \alpha k(t)^{\alpha-1}$, the dynamics of consumption given in (20) is

$$\frac{\dot{c(t)}}{c(t)} = \alpha k(t)^{\alpha-1} - \rho = 0 \quad \text{iff } k^\star(t) = \left(\frac{\rho}{\alpha}\right)^{\frac{1}{\alpha-1}}. \tag{23}$$

The dynamics of capital is

$$\dot{k(t)} = f(k(t)) - c(t) - (n+g)k(t), \tag{24}$$

which, in our case, becomes

$$\dot{k(t)} = k(t)^\alpha - c(t) - nk(t) = 0 \quad \text{iff } c^\star(t) = k(t)^\alpha - nk(t). \tag{25}$$

3. Questo esercizio verrà ripreso prima del primo parziale.

3 Question 3.

The dynamics of the economy are described by:

$$\frac{\dot{c(t)}}{c(t)} = \frac{r(t) - \rho - \theta g\ [-\delta]}{\theta} = 0 \tag{26}$$
$$\text{iff } r(t) \equiv f'(k^\star(t)) = \rho + \theta g\ [+\delta];$$
$$\dot{k(t)} = f(k(t)) - c(t) - (n + g\ [+\delta])k(t) = 0 \tag{27}$$
$$\text{iff } c^\star(t) = f(k(t)) - (n + g\ [+\delta])k(t).$$

Method: describe how and why the change shifts the curves. Thence, find the new equilibrium. Remember that k cannot jump discontinuously, since it is determined by the history of the economy.

1. If $g \downarrow$, then the level of c consistent with $\dot{k(t)} = 0$ is higher for a given k. Hence, the locus of $\dot{k(t)} = 0 \uparrow$. After the fall in g, the level of $f'(k(t))$ consistent with $\dot{c(t)}/c(t) = 0$ is lower. Since $f''(k(t)) < 0$, $k(t) \uparrow$. Hence, the locus of $\dot{c(t)}/c(t) = 0 \uparrow$.

   At t_0, when the change happens, c jumps upwards to the new saddle path, and the economy converges at the new equilibrium, with $k^\star_N > k^\star$ and $c^\star_N > c^\star$.

2. If $f(k(t)) \downarrow$ (and $f'(k(t)) \downarrow$), then the level of c consistent with $\dot{k(t)} = 0$ is lower for a given k. Hence, the locus of $\dot{k(t)} = 0 \downarrow$. After the fall in $f(k(t))$ (and $f'(k(t))$), the level of $k(t)$ consistent with $\dot{c(t)}/c(t) = 0$ is lower. Hence, the locus of $\dot{c(t)}/c(t) = 0 \downarrow$.

   At t_0, when the change happens, c jumps downwards to the new saddle path, and the economy converges at the new equilibrium, with $k^\star_N < k^\star$ and $c^\star_N < c^\star$.

3. If $\delta > 0$, then the level of c consistent with $\dot{k(t)} = 0$ is lower for a given k. Hence, the locus of $\dot{k(t)} = 0 \downarrow$. After the rise in δ, the level of $f'(k(t))$

consistent with $\dot{c(t)}/c(t) = 0$ is higher. Since $f''(k(t)) < 0$, $k(t) \downarrow$. Hence, the locus of $\dot{c(t)}/c(t) = 0 \downarrow$.

At t_0, when the change happens, c jumps downwards to the new saddle path, and the economy converges at the new equilibrium, with $k_N^\star < k^\star$ and $c_N^\star < c^\star$.

4. The fact that, now, $r(t) = (1 - \tau)f'(k(t))$ does not affect the $\dot{k(t)} = 0$ locus. On the other hand, the the locus of $\dot{c(t)}/c(t) = 0 \downarrow$.

   (a) If the tax is unanticipated (i.e. the change happens at t_0), c jumps up to the new saddle path, and the economy converges at the new equilibrium, with $k_N^\star < k^\star$ and $c_N^\star < c^\star$.

   (b) If the tax is announced at t_0 and implemented at t_1,

   - At t_0, c jumps up to a new point A that is in between the current saddle path and the future one. At A, though, c is too high to maintain the capital stock at $k^\star$, so $k \downarrow$. At this point, the dynamics of the system are still governed by the original $\dot{c(t)} = 0$ locus, so c rises further, as the economy moves north-west.
   - At t_1 the tax is implemented, and the system is governed by the new $\dot{c(t)} = 0$ locus. Thus c begins falling, as the economy moves on the new saddle path to the new equilibrium, with $k_N^\star < k^\star$ and $c_N^\star < c^\star$.

5. The dynamic equations of the economy are the same as above, but now the change is temporary, hence the economy eventually goes back to the original $\dot{c(t)} = 0$ locus.

   (a) If the tax is unanticipated (i.e. the change happens at t_0 and lasts until t_1),

   - At t_0, c jumps up towards the new saddle path to some point A (it does not reach the new saddle path, since households know that the tax will end). The system is governed by the new locus of $\dot{c(t)} = 0$. At this point, c is too high and begins falling together with k. Eventually the economy crosses the $\dot{k(t)} = 0$ locus, and k starts rising again.
   - At t_1, the tax policy is withdrawn and the economy is governed again by the original $\dot{c(t)} = 0$ locus. The economy then is on the old saddle path and converges to the original equilibrium.

   (b) If the tax is anticipated (i.e. the announcement is made at t_0, the change happens at t_0 and lasts until t_1),

   - At t_0, c jumps up to a new point A that is in between the current saddle path and the future one. At A, though, c is too high to maintain the capital stock at $k^\star$, so $k \downarrow$. At this point, the dynamics of the system are still governed by the original $\dot{c(t)} = 0$ locus, so c rises further, as the economy moves north-west.
   - At t_1 the tax is implemented, and the economy is governed by the new $\dot{c(t)} = 0$ locus, so that k keeps falling and c stops rising and begins to fall too. At some point the economy crosses the $\dot{k(t)} = 0$ locus, and k starts rising again.

- At t_2 the tax is withdrawn, the system is again governed by the original $\dot{c}(t) = 0$ locus, so that the economy then is on the old saddle path and converges to the original equilibrium.

6. $n \downarrow$ implies that the $\dot{k}(t) = 0$ locus $\uparrow$. The policy is announced at t_0, and implemented at t_1 until t_2.

   - At t_0, c jumps up to a point such A. Since the system is still governed by the old dynamics, now c is too high, and k falls.
   - At t_1 the policy is implemented and, since now c is too low, thus it rises together with k. Eventually, the economy will cross the $\dot{c}(t) = 0$ and c will begin to fall again.
   - At t_2 the policy is withdrawn, the system is again governed by the original dynamics, so that the economy is on the old saddle path and converges to the original equilibrium.

4 Question 4 (Optional).

For a constant $G(t) = G$ (N.B. without loss of generality we will derive the dynamics of consumption with the functions given in Exercise (2)), the new optimisation problem is given by:

$$\max_{c(t)} L = \int_{t=0}^{\infty} e^{(n-\rho)t} \ln(c(t) + G)\mathrm{d}t - \lambda \left[\int_{t=0}^{\infty} e^{-R(t)+nt} c(t)\mathrm{d}t - W(t) \right]. \quad (28)$$

The First Order Condition for $c(t)$ is:

$$\frac{\partial L}{\partial c(t)} = [c(t) + G]^{-1} e^{(n-\rho)t} - \lambda e^{nt-R(t)} = 0. \quad (29)$$

Taking logs yields

$$-\ln[c(t) + G] - \rho t - \ln \lambda - R(t) = 0 \quad (30)$$

and deriving w.r.t. t and rearranging terms yields

$$\frac{\dot{c}(t)}{c(t) - G} = r(t) - \rho. \quad (31)$$

Thus, the (general) condition for constant consumption is still given by $r(t) \equiv f'(k^\star(t)) = \rho + \theta g$. Changes in G will affect the level of c, but will not shift the $\dot{c}(t) = 0$ locus.

The capital accumulation equation is given by $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n + g)k(t)$. Suppose the economy starts on a balanced growth path. At some time t_0, G increases to some level G_H. The $\dot{k}(t) = 0$ locus shifts down by $(G_H - G)$. At time t_1, when government purchases return to G, the $\dot{k}(t) = 0$ locus shifts up to its original place and c jumps back to the original equilibrium. This is determined by the fact that G enters the utility function, so even if households know that the change is only temporary, c must jump down by *no less than* $(G_H - G)$. Otherwise, the economy could not reach equilibrium again.