MacroEconomia Avanzata
Esercitazione 1
Correzione.

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Remember:

- Properties of derivatives;
- Properties of logarithms;
- We denote the derivative of a variable $X(t)$ or a function $f(X(t))$ with respect to t as $\dot{X}$ or $\dot{f(X)}$;
- We denote the derivative of a function $f(X)$ with respect to its argument X as $f'(X)$;
- The elasticity of Z with respect to X is

$$\varepsilon_X = \frac{\partial Z}{\partial X}\frac{X}{Z}.$$

t is the argument of all functions. At some point I will omit such argument (t) when not essential to the demonstration.

1 Question 1.

We define the growth rate of a variable Z as the derivative of $\ln Z$ with respect to time t:

$$\frac{\partial \ln Z(t)}{\partial t} = \frac{1}{Z(t)}\frac{\partial Z(t)}{\partial t} = \frac{\dot{Z}}{Z(t)}. \tag{1}$$

Hence:

1. If $Z(t) = X(t)Y(t)$, its growth rate is:

$$\frac{\partial \ln X(t)Y(t)}{\partial t} = \frac{\partial \ln X(t)}{\partial t} + \frac{\partial \ln Y(t)}{\partial t} = \frac{\dot{X}}{X(t)} + \frac{\dot{Y}}{Y(t)}. \tag{2}$$

2. Similarly, if $Z(t) = \frac{X(t)}{Y(t)}$, its growth rate is:

$$\frac{\partial \ln X(t)/Y(t)}{\partial t} = \frac{\partial \ln X(t)}{\partial t} - \frac{\partial \ln Y(t)}{\partial t} = \frac{\dot{X}}{X(t)} - \frac{\dot{Y}}{Y(t)}. \tag{3}$$

2 Question 2.

1. Defined the production function as $Y(t) = F[K(t), A(t)L(t)]$, the model's assumptions are:

   (a) Constant returns to scale:

   $$F[cK, cAL] = cF(K, AL). \tag{4}$$

   This implies that, if $y = \frac{Y}{AL}$, $k = \frac{K}{AL}$, then $y = f(k)$ and $Y = ALf(k)$.

   (b) The production function has:

   $$f(0) = 0, \qquad f'(X) > 0, \qquad f''(X) < 0. \tag{5}$$

   This implies, considering $Y = AL(f(K/AL))$:

   $$\frac{\partial Y}{\partial K} = ALf'(k)\frac{1}{AL} = f'(k) > 0, \tag{6}$$

   $$\frac{\partial Y}{\partial AL} = f(k) + ALf'(k)\left[\frac{-K}{(AL)^2}\right] = f(k) - kf'(k) < 0. \tag{7}$$

   (c) Inada conditions:

   $$\lim_{k\to\infty} f'(k) = 0, \qquad \lim_{k\to 0} f'(k) = \infty. \tag{8}$$

   (d) Growth rates of L, A are constant and exogenous:

   $$\dot{L} = nL(t), \quad \frac{\dot{L}}{L(t)} = n \quad \ln L(t) = \ln L(0) + nt, \quad L(t) = L(0)e^{nt}; \tag{9}$$

   $$\dot{A} = gA(t), \quad \frac{\dot{A}}{A(t)} = g \quad \ln A(t) = \ln A(0) + gt, \quad A(t) = A(0)e^{gt}. \tag{10}$$

   (e) The saving rate s expresses the share of output dedicated to investment. The depreciation rate δ expresses the rate at which capital loses value over time. Both s and δ are constant and exogenous. Hence, the dynamics of capital can be expressed as:

   $$\dot{K} = sY(t) - \delta K(t). \tag{11}$$

2. The Cobb-Douglas production function can be expressed in the reduced form $y = f(k) = k^\alpha$ with $0 < \alpha < 1$ and meets the assumptions of the Solow model:

   (a) Constant returns to scale:

   $$F[cK, cAL] = (cK)^\alpha(cAL)^{1-\alpha} = c[K^\alpha(AL)^{1-\alpha}] = cF[K, AL]. \tag{12}$$

(b) The Cobb-Douglas production function has:

$$f(0) = 0; \tag{13}$$
$$f'(k) = \alpha k^{\alpha-1} > 0; \tag{14}$$
$$f''(k) = \alpha(\alpha-1)k^{\alpha-2} < 0. \tag{15}$$

(c) Inada conditions ($f'(k)$ is a negative exponential):

$$\lim_{k\to\infty} \alpha k^{\alpha-1} = 0, \qquad \lim_{k\to 0} \alpha k^{\alpha-1} = \infty. \tag{16}$$

The dynamics of capital is derived as follows:

$$\begin{aligned} \dot{k} &= \frac{\dot{K}}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2}[\dot{A}L(t) + A(t)\dot{L}] = \\ &= \frac{\dot{K}}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\left[\frac{\dot{A}}{A(t)}\frac{\dot{L}}{L(t)}\right] = \\ &= sk^{\alpha} - \delta k(t) - nk(t) - gk(t) = \\ &= sk^{\alpha} - (\delta + n + g)k(t). \end{aligned} \tag{17}$$

3. See Figure 1.

3 Question 3.

Method: given the dynamic equation of capital

$$\dot{k} = \text{share of Input Invested} - \text{Break Even capital}, \tag{18}$$

explain which of its components moves and how. Thence, find the new (if any) equilibrium on the balanced growth path. If not stated otherwise assume that the economy starts at equilibrium $k^\star$. We will denote new curves and equilibria as $(\cdot)_N$.

1. $\delta \uparrow$ implies $BE \downarrow$. At $k^\star$, $II > BE_N$, hence $\dot{k} > 0$. Capital grows until $II = BE_N$, which is the new equilibrium $k^\star_N > k^\star$.

2. $n \downarrow$ implies $BE \uparrow$. At $k^\star$, $II < BE_N$, hence $\dot{k} < 0$. Capital diminishes until $II = BE_N$, which is the new equilibrium $k^\star_N < k^\star$.

3. If $f(k) \to \gamma f(k) > f(k)$, then $y \uparrow$ implies $II \uparrow$. At $k^\star$, $II_N > BE$, hence $\dot{k} > 0$. Capital grows until $II_N = BE$, which is the new equilibrium $k^\star_N > k^\star$.

4. If $n = 0$ and $L \uparrow$, this implies $f(k) = f(K/AL) \downarrow$. Yet, the curves do not shift. The economy jumps to a new point $k_N < k^\star$ which is not on the balanced grow path. Since, at k_N, $II > BE$, $\dot{k} > 0$, capital grows until $II = BE$, which is the starting point $k^\star$.

5. $s \uparrow$ implies $II \uparrow$. At $k^\star$, $II_N > BE$, hence $\dot{k} > 0$. Capital grows until $II_N = BE$, which is the new equilibrium $k^\star_N > k^\star$.

   (Optional) Consumption is defined as $c = f(k) - sf(k)$. On the balanced growth path:

$$\begin{aligned} c^\star &= (1-s)f(k^\star) = \\ &= f(k^\star) - (n+g+\delta)k^\star. \end{aligned} \tag{19}$$

   Taking the derivative of $c^\star$ with respect to s gives:

$$\begin{aligned} \frac{\partial c^\star}{\partial s} &= \frac{\partial c^\star}{\partial k^\star}\frac{\partial k^\star}{\partial s} = \\ &= \Big[f'(k^\star) - (n+g+\delta)\Big]\frac{\partial k^\star}{\partial s}. \end{aligned} \tag{20}$$

   Since $\partial k^\star/\partial s > 0$, the sign of the derivative depends upon the term in parenthesis:

   - If $f'(k) < n+g+\delta$, then $\partial c^\star/\partial s < 0$ and $c^\star_N < c^\star$;
   - If $f'(k) > n+g+\delta$, then $\partial c^\star/\partial s > 0$ and $c^\star_N > c^\star$;
   - If $f'(k) = n+g+\delta$, then $\partial c^\star/\partial s = 0$ and $c^\star_N = c^\star$.

   When the latter case is true, $k^\star$ is called capital *of Golden Rule*, $k^\star_{GR}$. At this level of capital, consumption is maximised. See figure 2 (e,f).

4 Question 4.

Considering a generic production function $Y = F(K, A, L)$, one can analyse output growth with respect to its inputs:

$$\begin{aligned} \dot{Y} &= \frac{\partial Y(t)}{\partial t} = \\ &= \frac{\partial Y(t)}{\partial K(t)}\dot{K} + \frac{\partial Y(t)}{\partial L(t)}\dot{L} + \frac{\partial Y(t)}{\partial A(t)}\dot{A} = \qquad \left[\frac{1}{Y(t)}; \frac{K, L, A}{K, L, A}\right] \\ \frac{\dot{Y}}{Y(t)} &= \frac{K(t)}{Y(t)}\frac{\partial Y(t)}{\partial K(t)}\frac{\dot{K}}{K(t)} + \frac{L(t)}{Y(t)}\frac{\partial Y(t)}{\partial L(t)}\frac{\dot{L}}{L(t)} + \frac{A(t)}{Y(t)}\frac{\partial Y(t)}{\partial A(t)}\frac{\dot{A}}{A(t)} = \\ &= \varepsilon_K\frac{\dot{K}}{K(t)} + \varepsilon_L\frac{\dot{L}}{L(t)} + R(t). \end{aligned} \tag{21}$$

where $\varepsilon_{K,L}$ represents the elasticity of output with respect to K, L. An extensive commentary of empirical evidence and Convergence is reported in the text book.

5 Question 5 (Optional).

1. The production function is $Y = F(K, AL)$. Under the hypothesis of constant returns to scale, we can write $Y = ALf(K/AL) = ALf(k)$. Taking the derivative with respect to L gives:

$$w = \frac{\partial Y}{\partial L} = Af(k) + ALf'(k)\left[\frac{-K}{(AL)^2}\right] = A\big[f(k) - kf'(k)\big]. \tag{22}$$

Note: this can be seen as the derivative of $f(x)g(f(x))$ w.r.t. x.

2. Taking the derivative of Y with respect to K and subtracting δ gives:

$$r = \frac{\partial Y}{\partial K} - \delta = ALf'(k)\left[\frac{1}{AL}\right] - \delta = f'(k) - \delta. \tag{23}$$

Substitution of r, w into $wL + rK$ gives:

$$\begin{aligned} wL + rK &= A\big[f(k) - kf'(k)\big]L + (f'(k) - \delta)K = \\ &= ALf(k) - \frac{K}{AL}f'(k)AL + f'(k)K - \delta K = \\ &= ALf(k) - \delta K = \qquad \Big[\text{since } cf(k) = f(ck)\Big], \\ &= f(ALK/AL, AL) - \delta K = \\ &= f(K, AL) - \delta K. \end{aligned} \tag{24}$$

3. (a) Given $r = f'(k) - \delta$, we know that, on a balanced growth path, δ, $\dot{k}$, $f'(k)$ are constant. Hence, r is constant, which implies that its growth rate $\dot{r}/r = 0$.

   (b) Given $w = A\big[f(k) - kf'(k)\big]$, its growth rate is

$$\begin{aligned} \frac{\dot{w}}{w} &= \frac{\dot{A}}{A}\frac{\dot{\big[f(k) - kf'(k)\big]}}{f(k) - kf'(k)} \\ &= g + \frac{f'(k)\dot{k} - \dot{k}f'(k) - kf''(k)\dot{k}}{f(k) - kf'(k)} = \qquad \Big[\text{since } \dot{k} = 0\Big], \\ &= g. \end{aligned} \tag{25}$$

   Note: The derivative of the numerator $f(k(t)) - k(t)f'(k(t))$ w.r.t. t can be seen as the derivative of $f(g(x)) - g(x)h(g(x))$ w.r.t. x.

4. (a) The growth rate of r is defined as

$$\frac{\dot{r}}{r} = \frac{\dot{\big[f'(k)\big]}}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)} < 0. \tag{26}$$

   Hence, r decreases as k moves toward $k^{\star}$.

   (b) The growth rate of w is defined as

$$\frac{\dot{w}}{w} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}. \tag{27}$$

   Since k, $\dot{k}$ are both positive, while $f''(k)$ is negative, the second term on the right-hand side is positive. Hence $\dot{w}/w > g$.

6 Graphs.

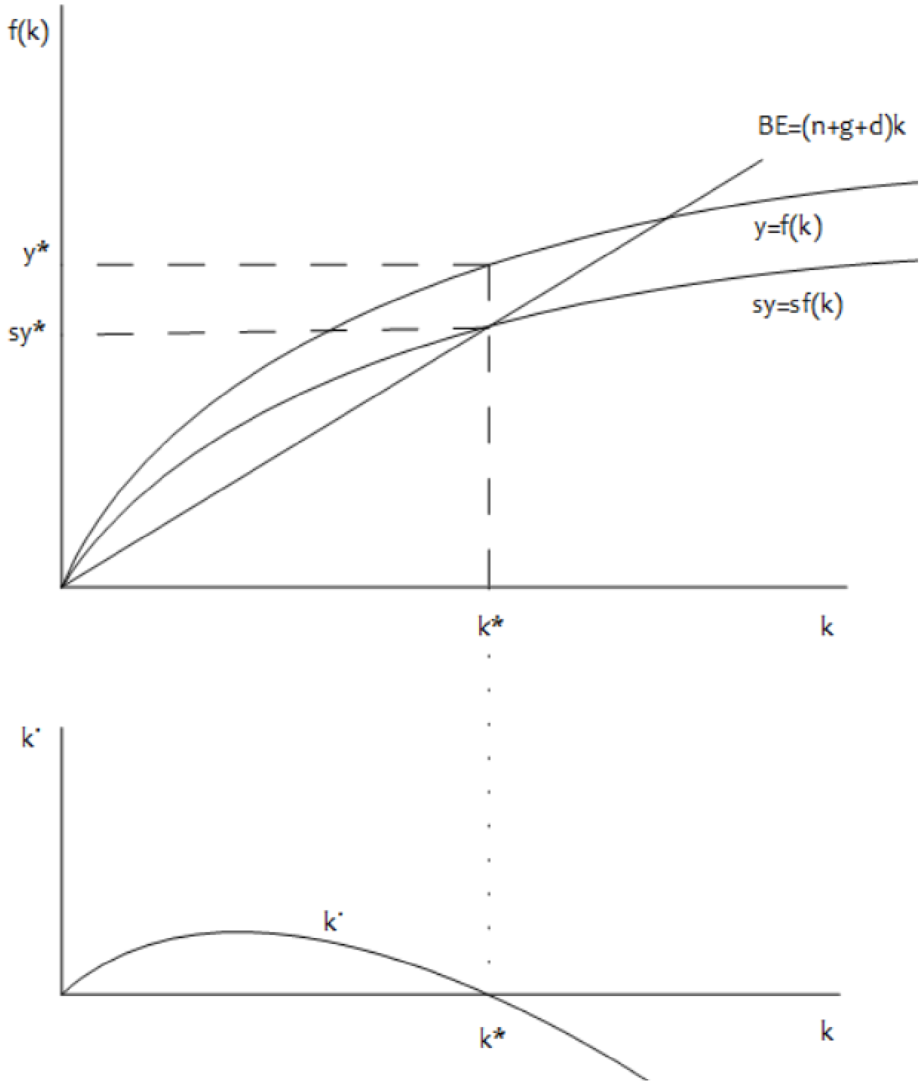
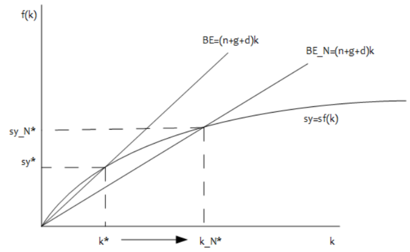
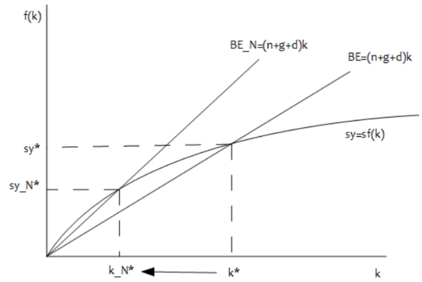


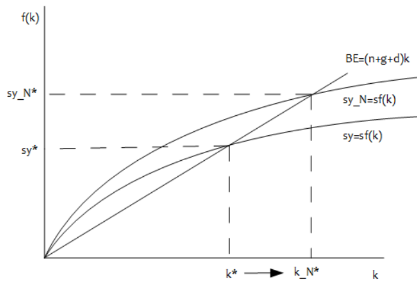
Figure 1: Exercise 2.3.



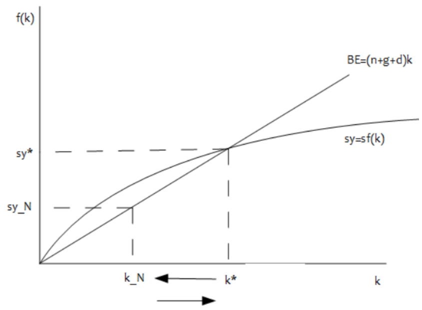
(a) Exercise 3.1.



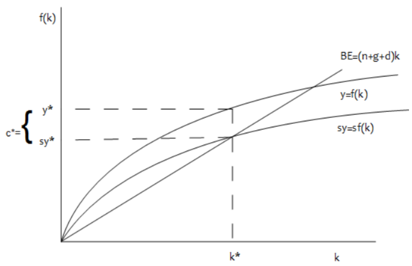
(b) Exercise 3.2.



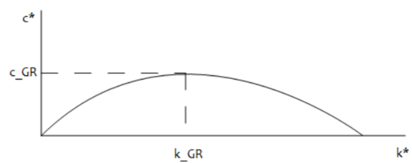
(c) Exercise 3.3.



(d) Exercise 3.4.



(e) Exercise 3.5.a.



(f) Exercise 3.5.b.

Figure 2: Exercise 3.